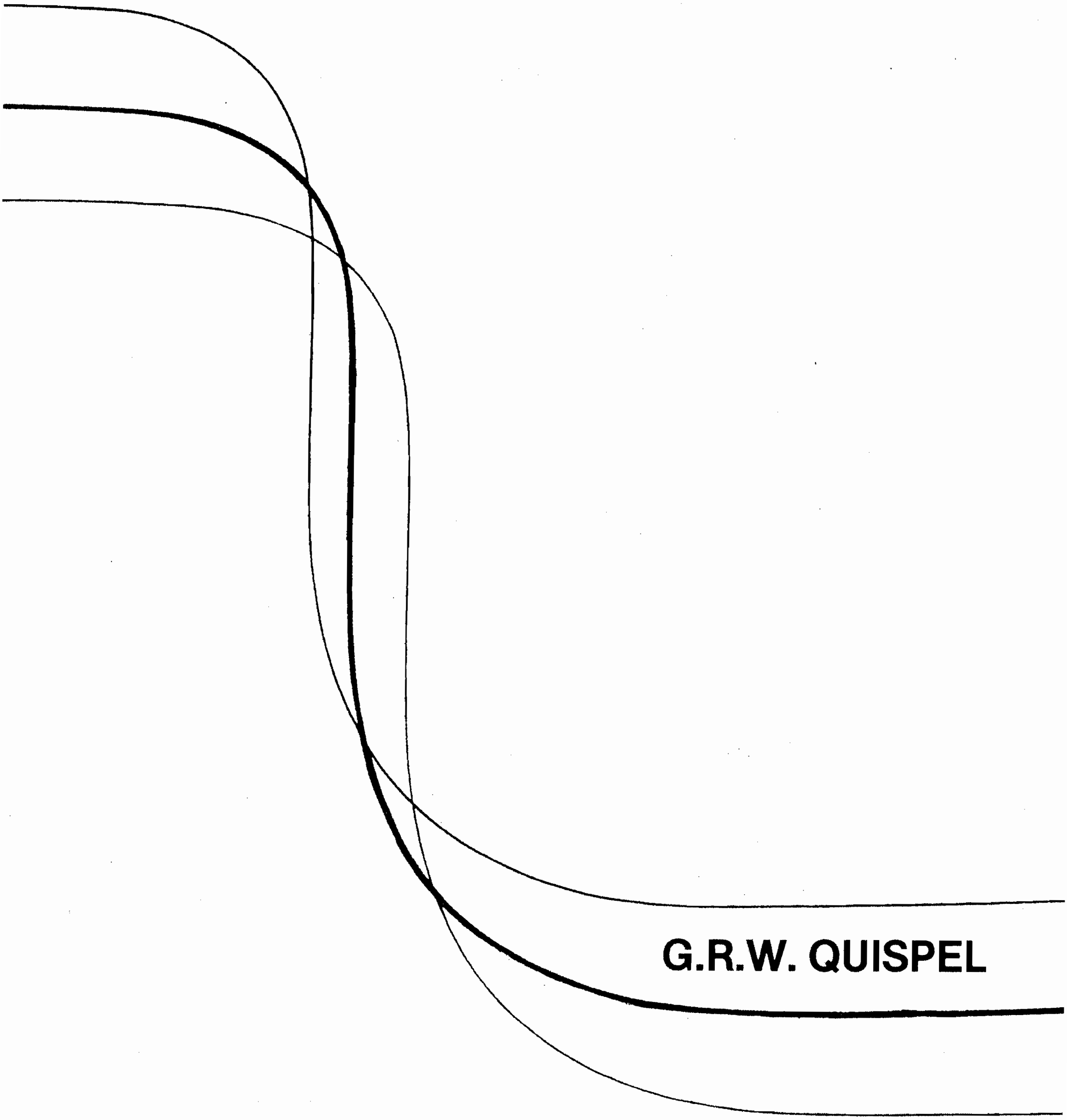
LINEAR INTEGRAL EQUATIONS AND SOLITON SYSTEMS



G.R.W. QUISPEL

LINEAR INTEGRAL EQUATIONS AND SOLITON SYSTEMS

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE
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door

GILLES REINOUT WILLEM QUISPEL

geboren te Bilthoven in 1953

Promotor	Prof. dr. H.W. Capel
Referent	Dr. J. van der Linden
Leden van de promotiecommissie	Prof. dr. P.W. Kasteleyn Prof. dr. P. Mazur Prof. dr. Th.W. Ruijgrok Prof. dr. J.H. van der Waals

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"Do not try to know the truth, for knowledge by the mind is not true knowledge. But you can know what is not true - which is enough to liberate you from the false. The idea that you know what is true is dangerous, for it keeps you imprisoned in the mind. It is when you do not know, that you are free to investigate. And there can be no salvation, without investigation, because non-investigation is the main cause of bondage."

Sri Nisargadatta Maharaj

voor Nel

voor mijn vader en moeder

CONTENTS

CHAPTER I

INTRODUCTION AND SUMMARY

Many phenomena in physics are of an essentially nonlinear nature. For centuries most of these phenomena were studied in linear approximation. Only in recent decades the mathematical methods have begun to be developed, to study certain classes of nonlinear systems exactly.

In 1955 Fermi, Pasta and Ulam $^{1)}$ published a paper in which they performed a numerical investigation of a one-dimensional system of anharmonic oscillators described by differential-difference equations of the type

$$\begin{aligned}\partial_t^2 u(n,t) &= u(n+1,t) - 2u(n,t) + u(n-1,t) \\ &\quad + \alpha\left\{\left(u(n+1,t) - u(n,t)\right)^2 - \left(u(n,t) - u(n-1,t)\right)^2\right\} , \qquad (1)\end{aligned}$$

where $u(n,t)$ denotes the displacement of the n'th mass point at time t. Contrary to what was expected at that time, they found no tendency toward equipartition of energy among the degrees of freedom of the system.

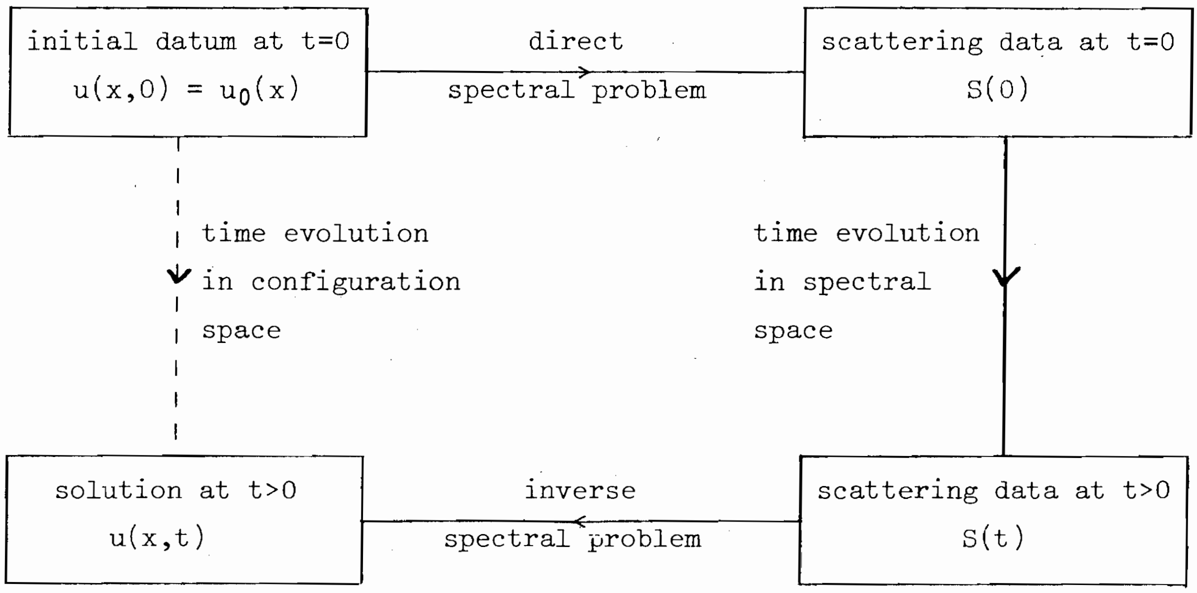
This discovery was an important stimulus for research on dynamical systems, which at present has developed two main branches, the study of integrable nonlinear systems, by which we mean nonlinear systems that can be solved exactly using only linear methods, and the study of nonintegrable systems.

In this thesis a study will be presented of classical integrable dynamical systems in one temporal and one spatial dimension; some general references on this subject are refs. 2-8. For results on integrable systems in higher spatial dimensions see e.g. refs. 9 and 10, and for reviews of integrable quantum systems see refs. 11 and 12. A review of nonintegrable systems can be found in ref. 13.

One of the interesting features of integrable nonlinear dynamical systems is the fact that, under appropriate boundary conditions, they give rise to solitons. Solitons are solitary waves that asymptotically preserve their energy, momentum and amplitude upon collision with other solitary waves. Some examples of integrable dynamical systems which will be discussed in this thesis are the Korteweg-de Vries equation, which describes e.g. waves in shallow water, the nonlinear Schrödinger equation, occurring e.g. in the theoretical description of plasma waves, the sine-Gordon equation, which describes a system of coupled pendula, the equations of motion for the isotropic and the anisotropic Heisenberg spin chain, etc. Of course these equations also have other physical applications, as different physical situations can often be described by the same mathematical model. For an application of soliton systems to field theory see ref. 14, and for an application to solid state physics see ref. 15.

Of course integrable dynamical systems form a small minority, as most systems turn out to be nonintegrable. One can, however, hope that soliton theory can in some sense be regarded as a zero'th order theory 8), that may be used as a starting point for perturbation expansions 16).

There are several approaches that have been used to study soliton systems, using e.g. the inverse scattering transform 2-7), the Riemann-Hilbert method 8,9,17), prolongation structures 18-20), or bilinearization 21). One of the most successful of these methods uses the inverse scattering transform, also called the inverse spectral transform. This method for solving the initial value problem of integrable nonlinear evolution equations under suitable boundary conditions at infinity, can be regarded as the nonlinear analogue of the Fourier transform; a thorough treatment of this analogy is given in ref. 2. A schematic representation of the method is given in the following diagram



The success of the inverse scattering method stems from the fact that the time evolution of the scattering data is governed by linear equations, in contrast to the time evolution in configuration space. One of the key steps in this method is the inverse spectral problem yielding the solution $u(x,t)$ of the nonlinear evolution equation, in terms of the scattering data $S(t)$. The solution of this inverse problem is expressed by a so-called Gel'fand-Levitan equation.

As an example, consider the case that $u(x,t)$ obeys the Korteweg-de Vries equation [22], i.e.

$$\partial_t u = \partial_x^3 u - 6u\partial_x u \ , \tag{2}$$

which first led to the discovery of the inverse scattering transform by Gardner, Greene, Kruskal and Miura [23]. The Gel'fand-Levitan equation for the Korteweg-de Vries equation reads as follows

$$K(x,y,t) + M(x+y,t) + \int_x^\infty dz\ K(x,z,t)M(z+y,t) = 0 \ , \qquad y>x \ , \tag{3}$$

where the kernel M is expressed in terms of the scattering data

$$\begin{aligned} R(k,t) &= R(k,0)\exp(-8ik^3t) \ , \\ p_n(t) &= p_n(0) \equiv p_n \ , \\ \rho_n(t) &= \rho_n(0)\exp(-8p_n^3t) \ , \end{aligned} \tag{4}$$

in the following way:

$$M(x,t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\exp(ikx)dk + \sum_{n=1}^{N} \rho_n(t)\exp(-p_n x) \ . \tag{5}$$

Here $R(k,0)$, p_n, and $\rho_n(0)$ play the role of a reflection coefficient, a discrete eigenvalue, and a normalization factor, respectively, associated with a Schrödinger equation in which $u_0(x)$ is the potential. A more detailed definition of these quantities is given in ref. 2. The Gel'fand-Levitan equation (3) is a linear equation, and from its solution $K(x,y,t)$, the solution $u(x,t)$ of the Korteweg-de Vries equation (2) is obtained as follows

$$u(x,t) = -2\partial_x \lim_{y\downarrow x} K(x,y,t) \ . \tag{6}$$

A convincing application of the inverse scattering transform to the analysis of experimental data has been given by A.R. Osborne et al. in ref. 24.

The treatment of soliton systems that will be presented in this thesis, however, is not based on the inverse scattering transform or on one of the other methods mentioned above, but on a new method that has been introduced by Fokas and Ablowitz, and which we will call the method of direct linearization.

In the method of direct linearization one starts from a singular linear integral equation, involving an arbitrary contour and measure in the complex plane. This singular linear integral equation yields not only very general solutions of the associated nonlinear evolution equations, but also many of the features of these evolution equations, e.g. Miura transformations, linear scattering problems, Bäcklund transformations, and integrable discretizations. The method has the advantage that different evolution equations can be treated in a comprehensive and unifying way. (For some general references on singular integral equations, their connection with Riemann-Hilbert problems and with inverse spectral problems see refs. 9, 21, 26, 27.)

Very briefly, the content of this thesis is the following. In chapter II the direct linearizations are given of several nonlinear partial differential equations, for example the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the sine-Gordon equation, the nonlinear Schrödinger equation, and the equation of motion for the isotropic Heisenberg spin chain; and we also discuss several relations between these equations. In chapter III the Bäcklund transformations of these partial differential equations are treated on the basis of a singular transformation of the measure (or equivalently of the plane-wave factor) occurring in the corresponding linear integral equations, and the Bäcklund transformations are used to derive the direct linearization of a chain of so-called modified partial differential equations; for example, from the Bäcklund transformation of the nonlinear Schrödinger equation the direct linearization of the equation of motion for the anisotropic Heisenberg spin chain is derived. Finally in chapter IV it is shown that the singular linear integral equations lead in a natural way to the direct linearizations of various nonlinear difference-difference equations. These equations for functions of two discrete variables n and m, reduce to the partial differential equations mentioned above, after two successive continuum limits. As an intermediate result we also present the direct linearizations of the differential-difference equations that obtain after one single continuum limit, e.g. the equation of motion for the Toda lattice, the discrete nonlinear Schrödinger equation, the discrete complex sine-Gordon equation, etcetera.

As an illustration, we here summarize some of the main steps of the treatment given in this thesis, using as an example the Korteweg-de Vries equation and its discrete analogues.

The singular integral equation providing the direct linearization of the Korteweg-de Vries equation is the following

$$u_k + i\rho_k \int_C d\lambda(\ell)\, \frac{u_\ell}{k+\ell} = \rho_k \ , \tag{7}$$

where C and $d\lambda(k)$ denote an arbitrary contour and measure in the complex k-plane, and where the plane-wave factor ρ_k is given by

$$\rho_k \equiv \rho_k(x,t) = e^{i(kx-k^3t)}\rho_k(0,0) \ . \tag{8}$$

This equation was introduced in ref. 25, where it was shown that if the solution u_k of equation (7) for a given measure $d\lambda(k)$ and contour C is unique, then the function u which is given by an integration of $u_k = u_k(x,t)$ over the same contour and measure, i.e.

$$u \equiv u(x,t) = \int_C d\lambda(k) u_k \tag{9}$$

obeys the potential Korteweg-de Vries equation

$$\partial_t u = \partial_x^3 u - 3(\partial_x u)^2 \ . \tag{10}$$

(Note that $v \equiv \partial_x u$ obeys the Korteweg-de Vries equation (2).) Explicit solutions describing N solitons, for example, are obtained by choosing a measure containing N simple poles, i.e.

$$d\lambda(k) = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c_n}{k-k_n}\, dk \ , \tag{11}$$

where c_n and k_n are constants, and a contour C that surrounds the poles k_n. For $N=2$ we obtain e.g.

$$u = \frac{c_1\rho_{k_1} + c_2\rho_{k_2} + ic_1c_2\rho_{k_1}\rho_{k_2}\dfrac{(k_1-k_2)^2}{2k_1k_2(k_1+k_2)}}{1 + ic_1\dfrac{\rho_{k_1}}{2k_1} + ic_2\dfrac{\rho_{k_2}}{2k_2} - c_1c_2\rho_{k_1}\rho_{k_2}\dfrac{(k_1-k_2)^2}{4k_1k_2(k_1+k_2)}} , \tag{12}$$

where ρ_{k_1} and ρ_{k_2} are given by equation (8).

If we apply a singular transformation to the plane-wave factor ρ_k in equation (7) of the form

$$\rho_k \to \tilde{\rho}_k = \frac{q+k}{q-k}\,\rho_k , \tag{13}$$

and denote by $\tilde{u}_k$ the solution of equation (7) with $\tilde{\rho}_k$ instead of ρ_k, and define the function

$$\tilde{u} \equiv \tilde{u}(x,t) = \int_C d\lambda(k)\tilde{u}_k , \tag{14}$$

then we can derive the following Bäcklund transformation (cf. refs. 28 and 29)

$$\partial_x\tilde{u} = -\partial_x u + iq(\tilde{u}-u) + \tfrac{1}{2}(\tilde{u}-u)^2, \tag{15}$$

$$\partial_t\tilde{u} = -\partial_t u + iq\partial_x^2\tilde{u} - iq\partial_x^2 u + \tfrac{1}{2}\partial_x^2(\tilde{u}-u)^2 - 3(\partial_x u)^2 - 3(\partial_x\tilde{u})^2 . \tag{16}$$

Given an arbitrary solution u of equation (10), the Bäcklund transformation enables one to obtain another solution $\tilde{u}$ of equation (10) after solving the ordinary Riccati differential equation (15). (The integration constant is determined by eq. (16).) Using equations (13) and (11) it can be shown that the Bäcklund transformation transforms an N-soliton solution u into an (N+1)-soliton solution $\tilde{u}$. From equations (15) and (16) it also follows that $u^- \equiv \tilde{u}-u$ obeys the modified Korteweg-de Vries equation

$$\partial_t u^- = \partial_x^3 u^- - 3iqu^-\partial_x u^- - \frac{3}{2}(u^-)^2\partial_x u^- . \tag{17}$$

Introducing a second singular transformation of the plane-wave factor

$$\rho_k \to \hat{\rho}_k \equiv \frac{p+k}{p-k}\,\rho_k, \quad u_k \to \hat{u}_k , \quad u \to \hat{u} \equiv \int_C d\lambda(k)\hat{u}_k , \tag{18}$$

it is immediately clear that $\tilde{\hat{\rho}}_k = \hat{\tilde{\rho}}_k$, hence $\tilde{\hat{u}} = \hat{\tilde{u}}$. From this, one can derive the following Bianchi-identity 30,31)

$$\frac{\hat{\tilde{u}}-u}{ip+iq} = - \frac{\tilde{u}-\hat{u}}{\tilde{u}-\hat{u}-ip+iq} . \tag{19}$$

Alternatively, we can say that if ρ_k in equation (7) is given by

$$\rho_k \equiv \rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m \rho_k(0,0) \quad , \tag{20}$$

instead of by equation (8), and if the function $u(n,m)$ is defined by

$$u(n,m) \equiv \int_C d\lambda(k) u_k \ , \tag{21}$$

where $u_k = u_k(n,m)$ is the solution of eq. (7) with ρ_k given by (20), then $u(n,m)$ obeys the following difference-difference equation

$$\frac{u(n+1,m+1) - u(n,m)}{ip+iq} = - \frac{u(n,m+1) - u(n+1,m)}{u(n,m+1) - u(n+1,m) - ip + iq} . \tag{22}$$

Hence the linear integral equation (7), with (20), provides a direct linearization of equation (22). The N-soliton solution of equation (22) can be obtained using the measure given in equation (11). The 2-soliton solution e.g. is again given by equation (12), where we should now insert equation (20) for ρ_{k_1} and ρ_{k_2}.

Taking a continuum limit of equation (22) with respect to m, we obtain

$$\partial_t u(n,t) = - \frac{u(n+1,t) - u(n-1,t)}{u(n+1,t) - u(n-1,t) + 2ip} , \tag{23}$$

and taking a second continuum limit, with respect to n, we recover, after some obvious transformations, the potential Korteweg-de Vries equation (10).

More details and results concerning these and other equations are given in the following chapters of this thesis.

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SAMENVATTING

Solitonen zijn gelokaliseerde golven die na een onderlinge botsing hun oorspronkelijke vorm en snelheid behouden. Zij treden op in uiteenlopende gebieden van de natuurkunde, zoals de hydrodynamica, optica, plasmafysica, veldentheorie en vaste-stoffysica.

Lang niet alle fysische systemen vertonen soliton-gedrag. Dit treedt met name op in de zogenaamde integreerbare dynamische systemen, waarmee bedoeld wordt niet-lineaire systemen waarvan de oplossing teruggebracht kan worden tot het oplossen van uitsluitend lineaire problemen. De hoop bestaat dat de eigenschappen van niet-integreerbare systemen, die in de meerderheid zijn, benaderd kunnen worden met behulp van integreerbare systemen.

Een bekende methode om solitonsystemen te bestuderen is de methode van de inverse verstrooiing, ontdekt door Gardner, Greene, Kruskal en Miura. In dit proefschrift worden solitonen echter bestudeerd met de methode van directe linearisatie, ingevoerd door Fokas en Ablowitz. Deze methode gaat uit van een lineaire singuliere integraalvergelijking, met een integraal over een willekeurige contour en maat in het complexe vlak, en leidt hieruit de oplossingen en eigenschappen van het niet-lineaire solitonsysteem af. De methode heeft het voordeel dat fysisch zeer verschillende solitonsystemen op een unificerende manier behandeld worden.

De inhoud van dit proefschrift is in het kort als volgt. Hoofdstuk I behelst een inleiding tot dit proefschrift en een samenvatting van de belangrijkste resultaten. In hoofdstuk II wordt de directe linearisatie van verschillende partiële differentiaalvergelijkingen gegeven, zoals de Korteweg-de Vries vergelijking, de gemodificeerde Korteweg-de Vries vergelijking, de sine-Gordon vergelijking, de niet-lineaire Schrödinger vergelijking en de bewegingsvergelijking voor de klassieke isotrope Heisenberg spinketen; tevens worden verscheidene verbanden tussen deze vergelijkingen uitgewerkt.

In hoofdstuk III worden de Bäcklund transformaties van deze vergelijkingen behandeld op grond van een singuliere transformatie van de maat die in de integraalvergelijking voorkomt en de Bäcklund transformaties worden gebruikt om de directe linearisatie van een keten van zogenaamde gemodificeerde partiële differentiaalvergelijkingen af te leiden. Zo wordt bijvoorbeeld uit de transformatie van de maat in de integraalvergelijking voor de niet-lineaire Schrödinger vergelijking de directe linearisatie van de bewegingsvergelijking voor de klassieke anisotrope Heisenberg spinketen afgeleid.

Tenslotte wordt in hoofdstuk IV aangetoond hoe singuliere lineaire integraalvergelijkingen op een natuurlijke wijze leiden tot de directe linearisatie van verscheidene niet-lineaire differentie-differentievergelijkingen. Deze vergelijkingen voor functies van twee discrete variabelen gaan over in bovengenoemde partiële differentiaalvergelijkingen na twee opeenvolgende continuümlimieten. Als tussenresultaat wordt de directe linearisatie afgeleid van de differentie-differentiaalvergelijkingen die worden verkregen na een enkele continuum-limiet, bijvoorbeeld de bewegingsvergelijking voor het Toda rooster, de discrete niet-lineaire Schrödinger vergelijking en de discrete complexe sine-Gordon vergelijking.

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren op 8 oktober 1953 te Bilthoven. In 1970 behaalde hij het einddiploma Gymnasium β aan het Nieuwe Lyceum te Bilthoven. Hierna behaalde hij aan de Rijksuniversiteit te Utrecht in 1973 het kandidaatsexamen scheikunde, in 1976 het kandidaatsexamen natuurkunde en in 1979 het doctoraalexamen theoretische natuurkunde met bijvakken wiskunde en mathematische fysica met onderwijsbevoegdheid in de wis- en natuurkunde. Zijn doctoraal-scriptie over solitonen in de Heisenberg spinketen schreef hij onder supervisie van Prof.dr. Th.W. Ruijgrok.

In 1979 trad hij in het kader van een beleidsruimteproject toe tot de werkgroep VS-th-L van de Stichting voor Fundamenteel Onderzoek der Materie. Onder leiding van Prof.dr. H.W. Capel verrichtte hij op het Instituut-Lorentz voor Theoretische Natuurkunde te Leiden onderzoek op het gebied van een-dimensionale integreerbare systemen; bovendien vervulde hij enige onderwijstaken. Een aantal resultaten van het bovengenoemde onderzoek zijn in dit proefschrift vastgelegd. Voorts nam hij deel aan de zomerschool "Fundamental Problems in Statistical Mechanics V" in Enschede (1980) en aan de conferenties over niet-lineaire evolutievergelijkingen en dynamische systemen in Trieste (1981), Edinburgh (1982) en Chania (1983). Vanaf 1 oktober 1983 is hij verbonden aan de vakgroep theoretische natuurkunde van de Technische Hogeschool Twente, als wetenschappelijk medewerker van Dr. R.H.G. Helleman.

LIST OF PUBLICATIONS

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Apart from minor modifications chapters II, III and IV of this thesis are contained in the publications 9, 11 and 15, respectively.

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