INTEGRABLE LATTICE VERSION OF THE MASSIVE THIRRING MODEL AND ITS LINEARIZATION

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A linear integral equation is proposed, yielding an exact linearization of a discretized version of the classical massive Thirring model on a two-dimensional lattice.

1. The massive Thirring model (MTM), which is a two-dimensional field-theoretical model for a self-coupled massive spinor field $\psi = (\psi_1, \psi_2)$, has received a great deal of interest in the past [1–9]. On the classical level the model has been solved exactly by means of the inverse-scattering formalism [3–5]. On the quantum level the associated hamiltonian has been diagonalized via Bethe's ansatz [7], and the exact S-matrix has been constructed [8,9]. In contrast to the continuum model, very little attention has been paid to the formulation of exactly integrable lattice versions of the MTM. A natural discretization has been obtained in ref. [10] in the context of the XYZ Heisenberg spin chain, cf. also [11]. Exact results on the latter model [12–14] give an indication that this discretization of the MTM is exactly integrable, but no inverse-scattering scheme has been found.

In this letter we propose a two-dimensional lattice version of the classical massive Thirring model, which is shown to be exactly integrable in the sense that solutions can be found from a linear integral equation. The continuum limits, which can be taken by letting either one or two of the lattice parameters a and b go to zero, will be discussed.

2. Consider the following linear integral equation

$$\boldsymbol{\phi}_k(n,m) + k \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k(n,m)\rho_{l'}^*(n,m)}{(k-l')(l'-l)} \boldsymbol{\phi}_l(n,m) = \rho_k(n,m)\ \mathbf{c}_k\ . \tag{1}$$

In (1) n and m label the sites of an infinite two-dimensional lattice, and the solutions $\boldsymbol{\phi}_k(n,m)$ are vectors with components $\phi_k^{(i)}(n,m)$, associated with different powers $c_k^{(i)} \equiv k^{-i}$ in the source term on the right-hand side of (1). The integrations in (1) are performed over an arbitrary contour C and its complex conjugate C*, with arbitrary measure $\mathrm{d}\lambda$ and its conjugate $\mathrm{d}\lambda^*$. For a given choice of the measure and contour, $\boldsymbol{\phi}_k$ is to be solved from (1) as a function of the variable k. We restrict ourselves to contours and measures for which the solution of (1) is unique, cf. refs. [15–18]. Eq. (1) is similar to the integral equation presented in ref. [18] for the continuum MTM, except that we now impose the quantities $\rho_k(n,m)$ to obey linear difference equations instead of linear differential equations, i.e.

$$\rho_k(n+1,m) = \theta[(p-k)/(p^*-k)]\,\rho_k(n,m)\,,\quad \rho_k(n,m+1) = \theta'[(q-k)/(q^*-k)]\,\rho_k(n,m)\,,\ |\theta| = |\theta'| = 1\,, \tag{2}$$

where the complex parameters p and q can be specified to obtain various dispersions in the continuum limit. More specifically, if $p = (\mathrm{i}a)^{-1}$, $\theta = -1$, and $q = \mathrm{i}b$, $\theta' = 1$, a and b being the lattice parameters, it is easy to note that if $a, b \to 0$, $n, m \to \infty$, such that $na \to -\frac{1}{2}x$, $mb \to \frac{1}{2}t$, x and t fixed, we recover the dispersion for the continuous case, i.e. $\rho_k(x,t) \propto \exp[\mathrm{i}(kx - k^{-1}t)]$ [18].

We introduce the (dyadic) matrices

$$\mathbf{\Phi}(n,m) = \int_C d\lambda(k)\, \boldsymbol{\phi}_k(n,m) \mathbf{c}_k\,, \quad \mathbf{\Psi}(n,m) = \int_C d\lambda(k)\, \boldsymbol{\psi}_k(n,m)\mathbf{c}_k\,, \tag{3}$$

where

$$\boldsymbol{\psi}_k(n,m) = \int_{C^*} d\lambda^*(l')\, \frac{\rho_k(n,m)}{k-l'}\, \boldsymbol{\phi}_{l'}(n,m)\,. \tag{4}$$

From the integral equation (1) and a similar equation for $\boldsymbol{\psi}_k$, which follows directly from (4) and (1), several relations can be derived, in the same way as in [18], making use of the uniqueness of the solutions of the integral equation, for instance

$$k^p \boldsymbol{\phi}_k = (\mathbf{J}^\mathrm{T})^p \cdot \boldsymbol{\phi}_k - \mathbf{\Psi}^* \cdot \mathbf{Q}_p \cdot \mathbf{J}^\mathrm{T} \cdot \boldsymbol{\phi}_k - \mathbf{\Phi} \cdot \mathbf{Q}_p \cdot (\mathbf{J}^\mathrm{T} \cdot \boldsymbol{\psi}_k + \mathbf{\Phi}^\dagger \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k)\,, \tag{5a}$$

$$k^p \boldsymbol{\psi}_k = (\mathbf{J}^\mathrm{T})^p \cdot \boldsymbol{\psi}_k - \mathbf{\Psi} \cdot \mathbf{Q}_p \cdot \mathbf{J}^\mathrm{T} \cdot \boldsymbol{\psi}_k + \mathbf{\Phi}^* \cdot \mathbf{Q}_p \cdot (\mathbf{J}^\mathrm{T} k^{-1} \boldsymbol{\phi}_k - \mathbf{\Phi}^\mathrm{T} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k)\,, \tag{5b}$$

$$\mathbf{\Psi} \cdot \mathbf{J}^p + (\mathbf{J}^\mathrm{T})^p \cdot \mathbf{\Psi}^\dagger = \mathbf{\Phi}^* \cdot \mathbf{Q}_p \cdot \mathbf{\Phi}^\mathrm{T} + \mathbf{\Psi} \cdot \mathbf{Q}_{p+1} \cdot \mathbf{\Psi}^\dagger\,, \tag{5c}$$

$$\mathbf{\Phi} \cdot (\mathbf{J}^p - \mathbf{Q}_{p+1} \cdot \mathbf{\Psi}^\dagger) = [(\mathbf{J}^\mathrm{T})^p - \mathbf{\Psi}^* \cdot \mathbf{Q}_{p+1}] \cdot \mathbf{\Phi}^\mathrm{T} \quad (\forall p \text{ integer})\,, \tag{5d}$$

where all the fields $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are taken at the same lattice point (n, m). In (5) the matrix $\mathbf{Q}_p$ is given by

$$\mathbf{Q}_p = \operatorname{sgn} p \sum_{j=0}^{|p|-1} \mathbf{J}^{(p+|p|)/2-1-j} \cdot \mathbf{O} \cdot (\mathbf{J}^\mathrm{T})^{j-(|p|-p)/2}\,,$$

$\mathbf{O}$ is a matrix with elements $O_{ij} = \delta_{i,0}\delta_{j,0}$, and $\mathbf{J}$ and $\mathbf{J}^\mathrm{T}$ are index-raising and -lowering matrices with components $J_{ij} = \delta_{j,j+1}$ and $J^\mathrm{T}_{ij} = \delta_{i,j+1}$ respectively. In addition to (5), one can also derive expressions which relate the solutions of (1) at different lattice points, for example

$$(p^* - k)\boldsymbol{\phi}_k(n+1,m) = \theta\,[p - \mathbf{J}^\mathrm{T} + \mathbf{\Psi}^*(n+1,m) \cdot \mathbf{O} \cdot \mathbf{J}^\mathrm{T}] \cdot \boldsymbol{\phi}_k(n,m)$$

$$+ \mathbf{\Phi}(n+1,m) \cdot \mathbf{O} \cdot [\mathbf{J}^\mathrm{T} \cdot \boldsymbol{\psi}_k(n,m) + \mathbf{\Phi}^\dagger(n,m) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k(n,m)]\,, \tag{6a}$$

and

$$(p^* - k)\,\boldsymbol{\psi}_k(n+1,m) = [p^* - \mathbf{J}^\mathrm{T} + \mathbf{\Psi}(n+1,m) \cdot \mathbf{O} \cdot \mathbf{J}^\mathrm{T}] \cdot [\boldsymbol{\psi}_k(n,m) + \mathbf{\Phi}^*(n,m) \cdot \mathbf{J}^\mathrm{T} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k(n,m)]$$

$$- p\theta\, \mathbf{\Phi}^*(n+1,m) \cdot \mathbf{J}^\mathrm{T} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k(n,m)\,. \tag{6b}$$

The inverse relations of (6) can be obtained by interchanging in (6a) and (6b) $p \leftrightarrow p^*$, $\theta \leftrightarrow \theta^*$ and the lattice points $(n, m) \leftrightarrow (n + 1, m)$. Furthermore similar equations relating lattice points $(n, m + 1)$ and (n, m) can be obtained from (6) by replacing

$$p \to q\,, \quad \theta \to \theta'\,, \quad (n+1,m) \to (n,m+1)\,. \tag{7}$$

3. We shall now derive two coupled difference–difference equations for the quantities

$$\psi_1(n,m) \equiv [1 - \psi_{0,0}(n,m)]\phi_{0,1}(n,m)\,, \quad \psi_2(n,m) \equiv \phi_{0,0}(n,m)\,, \tag{8}$$

where $\phi_{i,j}$ and $\psi_{i,j}$ are the elements of the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ defined in (3). From (6a) together with the relations

$\phi_{0,0}=\phi_{-1,1}$ and $\psi_{-1,1}+\psi^*_{0,0}=-\phi^*_{0,0}\phi_{0,1}$, cf. (5c) and (5d), we can derive

$$p^*\phi_{0,1}(n+1,m)-\theta p\phi_{0,1}(n,m)=[1-\psi^*_{0,0}(n,m)]\,\phi_{0,0}(n+1,m)-\theta[1-\psi^*_{0,0}(n+1,m)]\,\phi_{0,0}(n,m)\,. \quad (9)$$

Similarly from (6b) and the relations $\psi_{0,-1}+(1-\psi_{0,0})\psi^*_{-1,0}=0$ and $2\,\mathrm{Re}\,\psi_{-1,0}+|\phi_{0,0}|^2=0$, we derive

$$|p^*\phi_{0,1}(n+1,m)-\theta p\phi_{0,1}(n,m)|^2+p^*[1-A^*(n,m)]+p[1-A(n,m)]=0\,, \quad (10)$$

where

$$A(n,m)\equiv[1-\psi_{0,0}(n+1,m)]\,[1-\psi_{0,0}(n,m)]^{-1}\,,\quad |1-\psi_{0,0}(n,m)|^2=1\,. \quad (11)$$

From (10), $A(n,m)$ can be solved either in terms of $\phi_{0,1}$ or in terms of ψ_1 or ψ_2, by inserting (9). Furthermore (9) can be rewritten in terms of ψ_1 and ψ_2, using (8), and symmetrized by using also the inverse relation with $(n+1,m)\leftrightarrow(n,m)$, $\theta\leftrightarrow\theta^*$, $p\leftrightarrow p^*$. The result is:

$$p^*\psi_1(n+1,m)-\theta p\psi_1(n,m)=\tfrac{1}{2}[\psi_2(n+1,m)-\theta\psi_2(n,m)]-\tfrac{1}{2}p^*[A^*(n,m)-1]\,\psi_1(n+1,m)$$
$$+\tfrac{1}{2}\theta p[A(n,m)-1]\,\psi_1(n,m)+\tfrac{1}{2}A(n,m)\psi_2(n+1,m)-\tfrac{1}{2}\theta A^*(n,m)\psi_2(n,m)\,, \quad (12)$$

where $A(n,m)$ is given by

$$A(n,m)=\tfrac{1}{2}[p+\theta^*\psi^*_2(n,m)\psi_2(n+1,m)]^{-1}\,\{[p+p^*+|\psi_2(n+1,m)|^2+|\psi_2(n,m)|^2]$$
$$\pm\,\mathrm{i}[4|p+\theta^*\psi^*_2(n,m)\psi_2(n+1,m)|^2-(p+p^*+|\psi_2(n+1,m)|^2+|\psi_2(n,m)|^2)^2]^{1/2}\}\,, \quad (13)$$

or by a similar expression which can be obtained from (13) by replacing $\theta\to\theta^*$, $\psi^*_2(n,m)\to p\psi_1(n,m)$, $\psi_2(n+1,m)\to p\psi^*_1(n+1,m)$. From (12) one can also derive an equation for $\psi_2(n+1,m)-\theta\psi_2(n,m)$ and with the replacements (7) one obtains

$$\psi_2(n,m+1)-\theta'\psi_2(n,m)=\tfrac{1}{2}q^*[1+B^*(n,m)]\,\psi_1(n,m+1)-\tfrac{1}{2}\theta' q[1+B(n,m)]\,\psi_1(n,m)$$
$$-\tfrac{1}{2}[B(n,m)-1]\,\psi_2(n,m+1)+\tfrac{1}{2}\theta'[B^*(n,m)-1]\,\psi_2(n,m)\,, \quad (14)$$

where

$$B(n,m)\equiv[1-\psi_{0,0}(n,m+1)]\,[1-\psi_{0,0}(n,m)]^{-1} \quad (15)$$

can be inferred from the expression for $A(n,m)$ in terms of ψ_1 and using the replacements (7). Eqs. (12) and (14), together with (13) and (15), form an exactly integrable set of coupled difference–difference equations. In a continuum limit with $p=\pm\mathrm{i}|p|=(\mathrm{i}a)^{-1}$, $\theta=-1$, $q=\pm\mathrm{i}|q|=\mathrm{i}b$, $\theta'=1$, $a,b\to0$, $m,n\to\infty$, $na\to-\tfrac{1}{2}x$, $mb\to\tfrac{1}{2}t$, eqs. (12) and (14) reduce to $-\mathrm{i}\partial_x\psi_1=\psi_2-|\psi_2|^2\psi_1$, $\mathrm{i}\partial_t\psi_2=\psi_1-|\psi_1|^2\psi_2$, which are the equations of motion of the MTM, cf. ref. [18].

4. As was shown in ref. [18], cf. also ref. [5], there exists an equivalent formulation of the MTM in terms of only one complex field. This property is also true, if we consider the lattice version of the MTM. In fact, from (9), (11) and (7) it is straightforward to derive a difference–difference equation for $\phi_{0,1}(n,m)$ namely

$$B(n,m)[p^*\phi_{0,1}(n+1,m+1)-\theta p\phi_{0,1}(n,m+1)]-\theta' B^*(n+1,m)[p^*\phi_{0,1}(n+1,m)-\theta p\phi_{0,1}(n,m)]$$
$$=A(n,m)[q^*\phi_{0,1}(n+1,m+1)-\theta' q\phi_{0,1}(n+1,m)]-\theta A^*(n\ m+1)[q^*\phi_{0,1}(n,m+1)-\theta' q_{0,1}(n,m)]\,, \quad (16)$$

where

$$A(n,m)=(1/2p)\{[p+p^*+|p^*\phi_{0,1}(n+1,m)-\theta p\phi_{0,1}(n,m)|^2]$$
$$\pm\,\mathrm{i}\{4|p|^2-[p+p^*+|p^*\phi_{0,1}(n+1,m)-\theta p\phi_{0,1}(n,m)|^2]^2\}^{1/2}\}\,, \quad (17)$$

and the expression for $B(n, m)$ can be obtained from (17) using the replacements (7). In a continuum limit with $p = (ia)^{-1}$, $\theta = -1$, $q = ib$, $\theta' = 1$, $b \to 0$, $m \to \infty$, $mb \to \frac{1}{2}t$, $\phi_{0,1}(n, m) \to \phi_n(t)$, we obtain the integrable differential–difference equation

$$\partial_t(\phi_{n+1} - \phi_n) = i(|\phi_{n+1}|^2 + |\phi_n|^2)(\phi_{n+1} - \phi_n) - \tfrac{1}{2}i|\phi_{n+1} - \phi_n|^2(\phi_{n+1} - \phi_n)$$
$$- a(\phi_{n+1} + \phi_n)[1 - (1/4a^2)|\phi_{n+1} - \phi_n|^4]^{1/2} . \quad (18)$$

Taking a second continuum limit with $a \to 0$, $n \to \infty$, $na \to -\frac{1}{2}x$, $\phi_n(t) \to \phi(x, t)$, we obtain from (18) $\partial_x \partial_t \phi = \phi + 2i|\phi|^2 \partial_x \phi$, which is the equivalent formulation of the MTM in terms of one complex field, cf. ref. [18]. Taking a different continuum limit with $q \to -p$, $n' = n - m$, $m \to \infty$, $(p + q)m \to -2ipt$, $\phi(n, m) \to \phi(n', t)$, $\phi(n + 1, m) \to \phi(n' + 1, t)$, $\phi(n, m + 1) \to \phi(n' - 1, t) + \frac{1}{2}ip^{-1}(p + q)\partial_t\phi(n' - 1, t)$, $\phi(n + 1, m + 1) \to \phi(n', t) + \frac{1}{2}ip^{-1}(p + q)\partial_t \phi(n', t)$, as in ref. [19] where we studied the lattice nonlinear Schrödinger equation, we obtain from (16) a discrete version of the (potential) derivative nonlinear Schrödinger equation (DNLS), cf. eq. (8) of [18], but we shall not give the result here.

Another difference–difference version of the DNLS, which is different from (18) has been given in [20].

5. In this letter we have presented an exactly integrable version of the massive Thirring model on a two-dimensional lattice. The solutions of the classical equations of motion can be found by choosing a measure $d\lambda$ and a contour C and solving the linear integral equation (1). Multisoliton solutions e.g. can be found by choosing $d\lambda(k) = \Sigma_{i=1}^{N} c_i \delta(k - k_i)\, dk$ and a contour through the poles of the delta functions. Additional results, such as the derivation of the associated linear spectral problem and the Bäcklund transformations, can be derived from (1), but we do not present them here. The case of fermions, where the fields take on values in a Grassman algebra, cf. ref. [6], can be treated also by imposing that the factors ρ_k obey anticommutation relations, but it does not seem easy to extract nontrivial results from an integral equation as given in (1).

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