DIRECT LINEARIZATION OF NONLINEAR DIFFERENCE–DIFFERENCE EQUATIONS

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Starting from the linear integral equation for the solutions of the Korteweg–de Vries (KdV) equation, we obtain the direct linearization of a general nonlinear difference–difference equation. In a continuum limit this equation reduces to a general integrable differential–difference equation which contains e.g. the Toda equation and the discrete KdV and MKdV as special cases.

1. Consider the following linear integral equation

$$s_k(n,m;\alpha) + \mathrm{i}\rho_k(n,m)\int_C \mathrm{d}\lambda(l)\,\frac{s_l(n,m;\alpha)}{k+l} = \frac{\rho_k(n,m)}{k+\alpha}. \qquad (1)$$

In (1) n and m label the sites of an infinite two-dimensional lattice, and the integration is performed over an arbitrary contour C in the complex k-plane, with arbitrary measure $\mathrm{d}\lambda(k)$. For a given choice of contour and measure $s_k(n,m;\alpha)$, in which α is a free parameter, is to be solved from (1) as a function of the variable k. We restrict ourselves to choices of contour and measure, for which the solution of (1) is unique, cf. refs. [1–4]. For $\rho_k(n,m)$ we impose the linear difference equations

$$\rho_k(n+1,m) = [(p+k)/(p-k)]\,\rho_k(n,m)\,,$$

$$\rho_k(n,m+1) = [(q+k)/(q-k)]\,\rho_k(n,m)\,. \qquad (2)$$

We will show that the function $s(n,m)$ defined by

$$s(n,m) \equiv \mathrm{i}\int_C \mathrm{d}\lambda(k)\,\frac{s_k(n,m;\alpha)}{k+\beta} = \mathrm{i}\int_C \mathrm{d}\lambda(k)\,\frac{s_k(n,m;\beta)}{k+\alpha} \qquad (3)$$

obeys the following nonlinear difference–difference equation

$$\begin{aligned}&[(p+\alpha)s(n+1,m+1)-(p-\beta)s(n,m+1)-1]\\ &\quad\times[(p+\beta)s(n+1,m)-(p-\alpha)s(n,m)-1]\\ &=[(q+\alpha)s(n+1,m+1)-(q-\beta)s(n+1,m)-1]\\ &\quad\times[(q+\beta)s(n,m+1)-(q-\alpha)s(n,m)-1]\,, \qquad (4)\end{aligned}$$

where p, q, α and β are free parameters.

2. In order to prove (4) we consider the integral equation for the KdV [1–3], i.e.

$$\boldsymbol{u}_k + \mathrm{i}\rho_k\int_C \mathrm{d}\lambda(l)\,\frac{\boldsymbol{u}_l}{k+l} = \rho_k\boldsymbol{c}_k\,. \qquad (5)$$

The notation is the same as in (1), but here the solutions $\boldsymbol{u}_k$ are vectors with components $u_k^{(i)}$ corresponding to a factor $c_k^{(i)} \equiv k^{-i}$ on the right-hand side. In the case of the KdV the ρ_k are plane-wave factors, but here no specification of ρ_k is required. We apply a Bäcklund transformation (BT) $\rho_k \to \tilde{\rho}_k = (p+k)\times(p-k)^{-1}\rho_k$, which is equivalent to a transformation of the measure $\mathrm{d}\lambda(k) \to \mathrm{d}\tilde{\lambda}(k) = (p+k)(p-k)^{-1}\times\mathrm{d}\lambda(k)$ as treated in ref. [4]. From (5) and the uniqueness of the solution we have

$$(p-k)\tilde{\boldsymbol{u}}_k = (p+\mathbf{J}^{\mathrm{T}})\cdot\boldsymbol{u}_k - \tilde{\mathbf{U}}\cdot\mathbf{0}\cdot\boldsymbol{u}_k\,, \qquad (6)$$

where $\tilde{\boldsymbol{u}}_k$ is the solution of (5) with $\tilde{\rho}_k$ instead of ρ_k, $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{0}$ are matrices with components J_{ij}^{T}

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$= \delta_{i,j+1}$, $0_{ij} = \delta_{i,0}\delta_{j,0}$ respectively, and the (symmetric) dyadics U and $\widetilde{\mathsf{U}}$, with components $u^{(i,j)}$ and $\tilde{u}^{(i,j)}$, are given by

$$\mathsf{U} = \mathrm{i} \int_C \mathrm{d}\lambda(k)\, u_k c_k, \quad \widetilde{\mathsf{U}} = \mathrm{i} \int_C \mathrm{d}\lambda(k)\, \tilde{u}_k c_k . \tag{7}$$

We divide (6) by $(\mathsf{J}^{\mathrm{T}} + \alpha)\,(k + \beta)$, and using the relations

$$s_k = [(\mathsf{J}^{\mathrm{T}} + \alpha)^{-1} \cdot u_k]^{(0)},$$

$$u(\alpha) \equiv \mathrm{i} \int_C \mathrm{d}\lambda(k)\, s_k = \mathrm{i} \int_C \mathrm{d}\lambda(k) \frac{u_k^{(0)}}{k+\alpha}, \tag{8}$$

cf. (1), (3) and (5), one finds after integration over C

$$1 - (p+\beta)\tilde{s} + (p-\alpha)s = [1 - \tilde{u}(\alpha)]\,[1 - u(\beta)] , \tag{9}$$

in which $\tilde{s}$ is given by (3), (1) with $\tilde{\rho}_k$ instead of ρ_k, and $\tilde{u}(\alpha)$ is given by (8) with $\tilde{u}_k$ instead of u_k. The inverse BT can be obtained by replacing p by $-p$, cf. (2), and interchanging the quantities with and without wiggles, which amounts to an interchange of α and β. We have

$$1 + (p-\beta)s - (p+\alpha)\tilde{s} = [1 - u(\alpha)]\,[1 - \tilde{u}(\beta)] . \tag{10}$$

Applying a second BT, $\rho_k \to \hat{\rho}_k = (q+k)\,(q-k)^{-1} \times \rho_k$, we obtain two further relations (9′) and (10′) which can be inferred from (9) and (10) replacing p by q and $\tilde{s}$ and $\tilde{u}$ by $\hat{s}$ and $\hat{u}$ respectively. From the four relations (9), (10), (9′) and (10′), with repeated use of the BT, one can eliminate all the functions u and obtain the relation

$$\frac{1 - (p+\beta)\tilde{s} + (p-\alpha)s}{1 - (q+\beta)\hat{s} + (q-\alpha)s} = \frac{1 - (q+\alpha)\hat{\tilde{s}} + (q-\beta)\tilde{s}}{1 - (p+\alpha)\hat{\tilde{s}} + (p-\beta)\hat{s}}, \tag{11}$$

which, in fact, is a Bianchi-identity [5,6] expressing the commutativity of the BTs $\rho_k \to \tilde{\rho}_k$ and $\rho_k \to \hat{\rho}_k$, cf. also refs. [7–9]. The equivalence of the Bianchi-identity (11) and the difference–difference equation (4) can now be easily established by introducing a function $\rho_k(n,m)$ of the lattice sites (n,m) such that $\tilde{\rho}_k(n,m) = \rho_k(n+1,m)$, $\hat{\rho}_k(n,m) = \rho_k(n,m+1)$, $\hat{\tilde{\rho}}_k(n,m) = \rho_k(n+1,m+1)$, in accordance with (2). This implies $s = s(n,m)$, $\tilde{s}(n,m) = s(n+1,m)$, $\hat{s}(n,m) = s(n,m+1)$, $\hat{\tilde{s}}(n,m) = s(n+1,m+1)$, which inserted in (11) yields the integrable difference–difference equation (4). (A similar connection between BTs and differential difference equations has been discussed in ref. [10]).

3. We now derive a differential–difference equation which can be obtained from (4) in a continuum limit $m \to \infty$, $b \to 0$, $mb \to t$, where b is a lattice parameter and t the continuous time. From (4) it is clear that we can take $p + q = b$ as a small parameter provided that $s(n+1,m+1) - s(n,m) \to 0$. This means that we have to relabel (n,m) as (n',m), $n' \equiv n - m$ and then take the continuum limit with respect to m. Accordingly we have

$$\begin{aligned} &s(n,m) \to s(n',t), \quad s(n+1,m) \to s(n'+1,t) ,\\ &s(n,m+1) \to s(n'-1,t) + (p+q)\,\partial_t s(n'-1,t) ,\\ &s(n+1,m+1) \to s(n',t) + (p+q)\partial_t s(n',t) . \end{aligned} \tag{12}$$

In this limit (4) reduces to the following differential–difference equation

$$\begin{aligned} &\partial_t s(n',t)\,[2p - (p+\alpha)\,(p+\beta)s(n'+1,t)\\ &\quad + (p-\alpha)\,(p-\beta)s(n'-1,t)]\\ &\quad + \{s(n'+1,t) - s(n'-1,t)\\ &\quad + 2p\,[s(n'+1,t)s(n'-1,t) + s^2(n',t)]\\ &\quad - (2p+\alpha+\beta)\,s(n'+1,t)s(n',t)\\ &\quad - (2p-\alpha-\beta)s(n'-1,t)s(n',t)\} = 0 . \end{aligned} \tag{13}$$

The continuum limit (12) is consistent with the integral equation (1), in which we have

$$\begin{aligned} \rho_k(n,m) &= \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m\\ &= \left(\frac{p+k}{p-k}\right)^{n'} \left[1 + \frac{2(p+q)k}{(p-k)\,(q-k)}\right]^m\\ &\to \rho_k(n',t) \equiv \left(\frac{p+k}{p-k}\right)^{n'} \exp\left(\frac{-2kt}{p^2-k^2}\right), \end{aligned} \tag{14}$$

and the solution of (13) can be obtained from the integral equation

$$\begin{aligned} &s_k(n',t) + \mathrm{i}\rho_k(n',t) \int_C \mathrm{d}\lambda(l) \frac{s_l(n',t)}{k+l}\\ &\quad = \rho_k(n',t)/(k+\alpha) , \end{aligned} \tag{15}$$

using also

$$s(n', t) = \mathrm{i} \int_C \mathrm{d}\lambda(k)\, \frac{s_k(n', t)}{k + \beta} . \tag{16}$$

Eq. (13) therefore is a general integrable differential–difference equation. In order to discuss some special cases we first note that the quantity $s_n = (2p)^{-1} \times (p + \alpha)(p + \beta) s(n, t)$ obeys

$$\begin{aligned} \partial_t s_n = [& s_{n-1} - s_{n+1} + (2AB + A + B) s_n s_{n-1} \\ & + (2 + A + B) s_{n+1} s_n \\ & - (1 + A)(1 + B)(s_{n+1} s_{n-1} + s_n^2)] \\ & \times (1 - s_{n+1} + AB s_{n-1})^{-1} \end{aligned} \tag{17}$$

with $A = (p - \alpha)/(p + \alpha)$, $B = (p - \beta)/(p + \beta)$, and where $t/(2p) \to t$. An integral equation similar to (15), in terms of $z = (p + k)/(p - k)$ has been investigated in ref. [11] for the discrete versions of the nonlinear Schrödinger equation, the isotropic Heisenberg spin chain and the complex sine-Gordon equation.

We now list some special cases contained in (17).

(i) $A, B \to 0$. In this case (17) reduces to

$$\partial_t y_n = 2 - \exp(y_n - y_{n-1}) - \exp(y_{n+1} - y_n),$$

$$y_n \equiv -\ln(1 - s_n) , \tag{18}$$

which after differentiation yields the equation of motion for the Toda lattice [12], i.e.

$$\partial_t^2 r_n = 2 \exp(-r_n) - \exp(-r_{n+2}) - \exp(-r_{n-2}) ,$$

$$r_n \equiv y_{n-2} - y_n . \tag{19}$$

(ii) $A s_n \to s_n$, $A \to \infty$, $B \to 0$ yields

$$\partial_t s_n = (s_n - s_{n-1} - 1)(s_{n+1} - s_n - 1) - 1 , \tag{20}$$

which with $a_n \equiv s_n - s_{n-1}$ reduces to the Kac–van Moerbeke equation or discrete KdV [13,14], i.e. $\partial_t a_n = (1 - a_n)(a_{n-1} - a_{n+1})$. The discrete KdV can also be obtained from (18) using $y_n - y_{n-1} = \ln(a_n - 1)$.

(iii) $A = -1$, $B = 1$, $z_n = \frac{1}{2}\ln(2s_n - 1)$ gives

$$\partial_t z_n = -\tanh(z_{n+1} - z_{n-1}) , \tag{21}$$

which is a potential equation for the discrete MKdV, i.e.

$$\partial_t b_n = (1 - b_n^2)(b_{n-1} - b_{n+1}) . \tag{22}$$

Eq. (22) can also be inferred from (17) in the limit $A s_n \to s_n$, $-A \to \infty$, $B = 1$, $b_n = (s_n - s_{n-1})(s_{n-1} + 1)^{-1}$.

We conclude that (4) can be considered as a difference–difference version of eqs. (17)–(22). Hence we can regard (4) as a generalized discrete time Toda equation, cf. [15], with the corresponding linearization (1).

4. Finally we derive the lattice sine-Gordon equation starting from (5) with

$$\rho_k(n + 1, m) = [(p + k)/(p - k)]\ \rho_k(n, m),$$

$$\rho_k(n, m + 1) = [(q + k^{-1})/(q - k^{-1})]\ \rho_k(n, m) \tag{23}$$

instead of (2). In this case we can consider the BT $\rho_k \to \tilde{\rho}_k$ as before, leading again to (6), and the BT $\rho_k \to \hat{\rho}_k = (q + k^{-1})/(q - k^{-1})\rho_k$, which yields the relations

$$(q - k^{-1})\hat{u}_k = (q + \mathbf{J}) \cdot u_k - \hat{\mathbf{U}} \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot u_k , \tag{24a}$$

$$(q + k^{-1})u_k = (q - \mathbf{J}) \cdot \hat{u}_k + \mathbf{U} \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J} \cdot \hat{u}_k , \tag{24b}$$

where $\hat{\mathbf{U}} = \mathrm{i}\int_C \mathrm{d}\lambda(k)\, u_k c_k$, u_k is the solution of (5) with $\hat{\rho}_k$ instead of ρ_k, $\mathbf{J}$ is the matrix with components $J_{ij} = \delta_{j,i+1}$. From (6) we have

$$p(\tilde{v} - v) = u^{(-1,1)} - v\tilde{u} = \tilde{u}^{(-1,1)} - \tilde{v}u , \tag{25}$$

with $u \equiv u^{(0,0)}$, $v \equiv u^{(0,1)} - 1$. From (24) we find

$$q(\hat{u} - u) = 1 - v\hat{v} \tag{26}$$

and

$$\hat{u} - u = -v\hat{u}^{(-1,1)} + \hat{v}u^{(-1,1)} . \tag{27}$$

Eliminating $u^{(-1,1)}$ from (25) and (27) and using (26) we can derive the following Bianchi-identity in terms of v, i.e.

$$v\,\hat{v}\,\tilde{v}\,\hat{\tilde{v}} - pq\,(v\hat{\tilde{v}} - \hat{v}\tilde{v}) = 1 . \tag{28}$$

Identifying $v = v(n, m)$, $\hat{v} = v(n, m + 1)$, $\tilde{v} = v(n + 1, m)$, $\hat{\tilde{v}} = v(n + 1, m + 1)$ as before and substituting $v = \exp 2\mathrm{i}w$, we find the lattice sine-Gordon equation [16,17]

$$\sin[w(n,m)+w(n+1,m)+w(n,m+1)$$
$$+w(n+1,m+1)]-pq\sin[w(n,m)-w(n+1,m)$$
$$-w(n,m+1)+w(n+1,m+1)]=0\,, \qquad (29)$$

and solutions of (29) can be found from (5) together with (23).

In conclusion, we have given the direct linearization for the nonlinear difference–difference equations (4) and (29). A similar treatment has also been given starting from other linear integral equations, such as eq. (1.3) of [3] with suitable $\rho_k(n,m)$, leading to difference–difference versions of e.g. the nonlinear Schrödinger equation, the complex sine-Gordon equation and the isotropic Heisenberg spin chain, cf. also ref. [18]. These results will be published in the near future [19].

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