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LINEARIZATION OF NONLINEAR DIFFERENTIAL–DIFFERENCE EQUATIONS

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A new linear integral equation is proposed, yielding an exact linearization of the integrable discrete versions of e.g. the nonlinear Schrödinger equation, the isotropic Heisenberg spin chain and the complex sine-Gordon equation. The Bäcklund transformation is discussed, as well as a discrete version of the anisotropic Heisenberg spin chain.

1. Consider the following linear integral equation

$$\phi_n^{(i)}(z,t) + \int_C \mathrm{d}\lambda(w) \int_{C^*} \mathrm{d}\lambda^*(w') \frac{z^n \mathrm{e}^{\mathrm{i}\omega(z)t} w'^n \mathrm{e}^{-\mathrm{i}\omega^*(w')t}}{(zw'-1)(w'w-1)} \phi_n^{(i)}(w,t) = z^{n+i} \mathrm{e}^{\mathrm{i}\omega(z)t} . \tag{1}$$

In (1) n labels the sites of an infinite chain, t is the time, and the integer i labels solutions $\phi_n^{(i)}$, belonging to different powers z^i on the right-hand side. The integrations in (1) are performed over an arbitrary contour C and its complex conjugate C*, with arbitrary measure $\mathrm{d}\lambda(z)$ and its conjugate $\mathrm{d}\lambda^*(z)$. For a given choice of the measure and contour, $\phi_n^{(i)}(z,t)$ is to be solved from (1) as a function of the variable z. We restrict ourselves to contours and measures for which the solution of eq. (1) is unique, cf. refs. [1–4]. The dispersion $\omega(z)$ has the general form $\omega(z) = \Sigma_{r\geqslant 0} (\gamma_r z^r + \gamma_r^* z^{-r})$, in which the coefficients γ_r can be specified for different cases. Eq. (1) can be considered to be the discrete analogue of the integral equation investigated in refs. [2–4]. The continuum limit of (1) can be obtained by taking $z = \mathrm{e}^{ika}$, $w' = \mathrm{e}^{-il'a}$, $w = \mathrm{e}^{ila}$, $n \to \infty$, $a \to 0$, such that $na \to x$, $a^{-1}\mathrm{d}\lambda(\mathrm{e}^{ila}) \to \mathrm{d}\lambda(l)$ and by a proper rescaling of the time.

The following quantities will be useful

$$\phi_n^{(i,j)}(t) \equiv \int_C \mathrm{d}\lambda(z) z^j \phi_n^{(i)}(z,t) , \quad \psi_n^{(i,j)}(t) \equiv \int_C \mathrm{d}\lambda(z) z^j \psi_n^{(i)}(z,t) , \tag{2}$$

where

$$\psi_n^{(i)}(z,t) = \int_{C^*} \mathrm{d}\lambda^*(w') \frac{z^n \mathrm{e}^{\mathrm{i}\omega(z)t}}{zw'-1} \phi_n^{(i)*}(w',t) . \tag{3}$$

From (1) and a similar integral equation for $\psi_n^{(i)}(z,t)$, which follows directly from (1) and (3), one can derive the relations

$$\boldsymbol{\phi}_{n+m}(z,t) = \mathbf{J}^m \cdot \boldsymbol{\phi}_n(z,t) - \boldsymbol{\Psi}_{n+m}^*(t) \cdot \boldsymbol{\Omega}_m^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) , \tag{4a}$$

$$\boldsymbol{\psi}_{n+m}(z,t) = \boldsymbol{\psi}_n(z,t) + \boldsymbol{\Phi}_{n+m}^*(t) \cdot \boldsymbol{\Omega}_m^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) \quad (m \text{ integer}) , \tag{4b}$$

$$z^p \boldsymbol{\phi}(z,t) = \mathbf{J}^p \cdot \boldsymbol{\phi}_n(z,t) + \boldsymbol{\Phi}_n(t) \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\psi}_n(z,t) - \boldsymbol{\Psi}_n^*(t) \cdot \boldsymbol{\Omega}_p^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) , \tag{4c}$$

$$z^p \boldsymbol{\psi}_n(z,t) = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_n(z,t) + \boldsymbol{\Psi}_n(t) \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\psi}_n(z,t) + \boldsymbol{\Phi}_n^* \cdot \boldsymbol{\Omega}_p^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) \quad (p \text{ integer}) , \tag{4d}$$

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independent of the choice of the dispersion $\omega(z)$, and the relations

$$\mathrm{i}\partial_t \boldsymbol{\phi}_n(z,t) = -\sum_{r\geqslant 0} [\gamma_r \mathbf{J}^r \cdot \boldsymbol{\phi}_n(z,t) + \gamma_r^* \mathbf{J}^{\mathrm{T}r} \cdot \boldsymbol{\phi}_n(z,t) - \gamma_r \boldsymbol{\Psi}_n^*(t) \cdot \mathbf{Q}_r^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) + \gamma_r^* \boldsymbol{\Psi}_n^*(t) \cdot \mathbf{Q}_r \cdot \boldsymbol{\phi}_n(z,t)] \,, \tag{5a}$$

$$\mathrm{i}\partial_t \boldsymbol{\psi}_n(z,t) = -\sum_{r\geqslant 0} [\gamma_r \boldsymbol{\Phi}_n^*(t) \cdot \mathbf{Q}_r^{\mathrm{T}} \cdot \boldsymbol{\phi}_n(z,t) - \gamma_r^* \boldsymbol{\Phi}_n^*(t) \cdot \mathbf{Q}_r \cdot \boldsymbol{\phi}_n(z,t)] \tag{5b}$$

for the time-derivatives. In eqs. (4) and (5) we have introduced the following matrix notation; $\boldsymbol{\phi}_n$ and $\boldsymbol{\psi}_n$ denote the vectors with components $(\boldsymbol{\phi}_n)_i = \phi_n^{(i)}$ and $(\boldsymbol{\psi}_n)_i = \psi_n^{(i)}$ respectively, the matrices $\boldsymbol{\Phi}_n$ and $\boldsymbol{\Psi}_n$ are defined by their elements $(\boldsymbol{\Phi}_n)_{ij} = \phi_n^{(i,j)}$ and $(\boldsymbol{\Psi}_n)_{ij} = \psi_n^{(i,j)}$, and $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ are the index-raising and lowering operators, $(\mathbf{J})_{ij} = \delta_{j,i+1}$, $(\mathbf{J}^{\mathrm{T}})_{ij} = \delta_{i,j+1}$ (T denotes matrix transposition). The matrices $\mathbf{Q}_p$ are defined by

$$\mathbf{Q}_p \equiv \sum_{j=0}^{|p|-1} (\operatorname{sgn} p)\, \mathbf{J}^{\mathrm{T}\, j+(p-|p|)/2} \cdot \mathbf{O} \cdot \mathbf{J}^{j-(p+|p|)/2} \,, \qquad (\mathbf{O})_{ij} = \delta_{i,0}\delta_{j,0} \,.$$

It can be easily shown that $\boldsymbol{\Phi}$ is symmetric, $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}}$, and that $\boldsymbol{\Psi}$ is hermitean, $\boldsymbol{\Psi} = \boldsymbol{\Psi}^{\dagger}$. In the following we shall work out the relations (4) and (5) and derive closed differential–difference equations for two specific choices of the dispersion.

2. As a first example we consider the case that $-\gamma_0 = \gamma_1 = 1$, $\gamma_r = 0$ for $r \geqslant 2$. Using eqs. (4) and (5) one can derive for the $(0,-1)$ element of $\boldsymbol{\Phi}$ the discrete nonlinear Schrödinger equation (dNLS) [5], i.e.

$$\mathrm{i}\partial_t \phi_n + \phi_{n+1} + \phi_{n-1} - 2\phi_n = -|\phi_n|^2(\phi_{n-1} + \phi_{n+1}) \,, \qquad \phi_n \equiv \phi_n^{(0,-1)} \,, \tag{6}$$

as well as the associated 2 X 2 linear spectral problem

$$\chi_{n+1}(z,t) = \mathcal{L}_n \cdot \chi_n(z,t) \,, \quad \mathrm{i}\partial_t \chi_n(z,t) = \mathcal{M}_n \cdot \chi_n(z,t) \,, \quad \chi_n(z,t) \equiv (\phi_n^{(0)}(z,t), z\psi_n^{(0)}(z,t)) \,,$$

with

$$\mathcal{L}_n = \begin{pmatrix} z & -\phi_{n+1} \\ z\phi_{n+1}^* & 1 \end{pmatrix}, \qquad \mathcal{M}_n = \begin{pmatrix} 2 - z - z^{-1} - \phi_{n+1}\phi_n^* & \phi_{n+1} - z^{-1}\phi_n \\ -z\phi_n^* + \phi_{n+1}^* & \phi_{n+1}^*\phi_n \end{pmatrix}. \tag{7}$$

In order to show that (1) also provides solutions of a discrete isotropic Heisenberg spin chain (dIHSC) we introduce the quantities

$$q_n \equiv \int_C \mathrm{d}\lambda(z) \frac{\phi_n^{(0)}}{z-1}, \qquad p_n \equiv \int_C \mathrm{d}\lambda(z) \frac{\psi_n^{(0)}}{z-1}, \tag{8}$$

$$S_n \equiv -\sum_{i=0}^{\infty} \int_C \mathrm{d}\lambda(z) \frac{\phi_n^{(i)}}{z-1}, \qquad R_n \equiv -\sum_{i=0}^{\infty} \int_C \mathrm{d}\lambda(z) \frac{\psi_n^{(i)}}{z-1}. \tag{9}$$

Note that the quantities S_n and R_n can be rewritten as

$$S_n = \int_C \mathrm{d}\lambda(z)\,(z-1)^{-1} \cdot \hat{\phi}_n(z,t) \,, \qquad R_n = \int_C \mathrm{d}\lambda(z)\,(z-1)^{-1}\,\hat{\psi}_n(z,t) \,,$$

where $\hat{\phi}_n(z,t)$ is the solution of the integral equation obtained from (1) by replacing z^i on the right-hand side by $(z-1)^{-1}$, and where $\hat{\psi}_n(z,t)$ is given by (3) with $\phi_n^{(i)*}$ replaced by $\hat{\phi}_n^*$. From (4), (8) and (9) one can find

$$p_{n+1} - p_n = \phi^*_{n+1} q_n\,, \quad q_n - q_{n-1} = \phi_n(1 - p_{n-1})\,, \quad |1 - p_n|^2 + |q_n|^2 = 1 + \psi_n\,, \quad \psi_n \equiv \psi_n^{(0,0)}\,, \tag{10}$$

and

$$(1 + \psi_n)(S_{n+1} - S_n) = (1 - p_n) q_n\,, \quad (1 + \psi_n)(R_{n+1} - R_n) = |q_n|^2\,. \tag{11}$$

From (5a) and (5b) we find the equations for the time derivatives of S_n and R_n

$$\begin{aligned} &\mathrm{i}\partial_t S_n + S_{n+1} + S_{n-1} - 2S_n = -[q_n^2(1 + \psi_n)^{-1} + q_{n-1}^2(1 + \psi_{n-1})^{-1}]\,\phi_n^*\,, \\ &\quad \mathrm{i}\partial_t R_n = (1 + \psi_{n-1})^{-1}(q_n^* q_{n-1} - q_{n-1}^*). \end{aligned} \tag{12}$$

From (12) together with the relations (10) and (11) we find after a lengthy calculation the (dIHSC) [6,7]

$$\partial_t S_n = 2(S_{n+1} \times S_n)/(1 + S_{n+1} \cdot S_n) - 2(S_n \times S_{n-1})/(1 + S_n \cdot S_{n-1})\,, \quad S_n \cdot S_n = 1\,, \tag{13}$$

with the identification for the spin vector $S_n = (2\,\mathrm{Re}(S_{n+1} - S_n), -2\,\mathrm{Im}(S_{n+1} - S_n), 1 - 2(R_{n+1} - R_n))$. In the derivation of (13) we have used the fact that $(1 + \psi_n)(1 + \psi_{n-1})^{-1} = 1 + |\phi_n|^2 = 2(1 + S_n \cdot S_{n-1})^{-1}$. In the continuum limit eq. (13) reduces to the IHSC discussed in ref. [3]. We have derived the Lax representation of the dIHSC in terms of the quantities $\hat{\phi}_n(z, t)$ and $\hat{\psi}_n(z, t)$, defined above. The time-independent part, which is also valid for the discrete complex sine-Gordon equation (dCSG, see below), is given by

$$\hat{\chi}_{n+1}(z, t) = [\tfrac{1}{2}(z + 1)\mathbf{1} + \tfrac{1}{2}(z - 1) S_n \cdot \boldsymbol{\sigma}] \cdot \hat{\chi}_n(z, t)\,, \tag{14}$$

where $\hat{\boldsymbol{\chi}}_n(z, t) = (\hat{\phi}_n(z, t), \hat{\psi}_n(z, t))$. The gauge-equivalence [7] between the Lax representations of the dNLS and dIHSC can be expressed by

$$\boldsymbol{\chi}_n(z, t) = (z - 1)\begin{pmatrix} 1 - p_n^* & q_n \\ q_n^* & -(1 - p_n) \end{pmatrix} \cdot \hat{\boldsymbol{\chi}}_n(z, t)\,, \tag{15}$$

supplemented by eqs. (10) and (11) for the potentials. The Miura transformation between the dNLS and the dIHSC can be inferred from (10) and (11).

3. We now consider the dispersion $\mathrm{i}\omega(z) = (z + 1)(z - 1)^{-1}$, corresponding to $\gamma_0 = 0, \gamma_r = \mathrm{i}$ for $r \geqslant 1$. Either by applying ∂_t directly to (1) or by using (5) and (8) one can derive the relations

$$\partial_t \phi_n^{(0)} = \phi_n^{(0)} + 2(1 - p_n^*)\hat{\phi}_n\,, \quad \partial_t \psi_n^{(0)} = 2 q_n^* \hat{\phi}_n\,, \tag{16}$$

where we have used the same notations as above. From (16) one has

$$\partial_t \psi_n = 2|q_n|^2\,, \quad \partial_t p_n = 2 q_n^* S_n\,, \quad \partial_t q_n = q_n + 2(1 - p_n^*) S_n\,, \tag{17}$$

which yields in combination with (11) the exactly integrable differential–difference equation

$$\partial_t (S_{n+1} - S_n) = (S_{n+1} + S_n)(1 - 4|S_{n+1} - S_n|^2)^{1/2}\,. \tag{18}$$

Eq. (18) is a discrete version of the complex sine-Gordon equation (CSG), i.e. the reduced O(4) σ-model [8], cf. also ref. [4], and the linear spectral problem is given by (14). For real S_n, $\theta_n = \arcsin 2(S_{n+1} - S_n)$ obeys $\partial_t (\theta_{n+1} - \theta_n) = \sin \theta_{n+1} + \sin \theta_n$, which is the discrete sine-Gordon equation [9,10]. Another discrete version of the CSG can also be derived for ϕ_n, namely $\partial_t (\phi_{n+1} - \phi_n) = \alpha_n(\phi_{n+1} + \phi_n)$, where the real function α_n is a solution of $|\partial_t \phi_n + \alpha_n \phi_n|^2 = 1 - \alpha_n^2$.

4. Bäcklund transformations for the nonlinear differential–difference equations can be derived by considering a singular transformation of measures $\mathrm{d}\lambda(z) \rightarrow \mathrm{d}\tilde{\lambda}(z) = (s - z)(s^* - z^{-1})^{-1}\,\mathrm{d}\lambda(z)$. This transformation can be treated as in the continuous case [11], and the result is

$$\frac{s^*\tilde{\phi}_n - s\phi_n}{\tilde{\phi}_{n-1} - \phi_{n+1}} = \frac{-\mathrm{Re}(s\tilde{\phi}_n^*\phi_{n+1}) + \frac{1}{2} + \frac{1}{2}|s|^2}{1 + |\phi_{n+1}|^2} \pm \left[\left(\frac{-\mathrm{Re}(s\tilde{\phi}_n^*\phi_{n+1}) + \frac{1}{2} + \frac{1}{2}|s|^2}{1 + |\phi_{n+1}|^2}\right)^2 - |s|^2 \left(\frac{1 + |\tilde{\phi}_n|^2}{1 + |\phi_{n+1}|^2}\right)\right]^{1/2} . \tag{19}$$

Eq. (19) is the time-independent part of the Bäcklund transformation for the dNLS as well as for other equations that can be found by considering different dispersions. The time-dependent part can be found in general by using the differential–difference equation itself. Other treatments of Bäcklund transformations for differential–difference equations can be found in refs. [12,13].

It is also possible to find a differential–difference equation in terms of the eigenfunctions of the spectral problem. Solving ϕ_n in terms of $u_n \equiv \phi_n^{(0)}/\psi_n^{(0)}$ from the linear spectral problem (7), i.e.

$$\phi_n = [|z|^2 u_{n-1} - z^*(1 + |u_{n-1}|^2)u_n + |u_n|^2 u_{n-1}]/(|z|^2 - |u_{n-1}|^2|u_n|^2) , \tag{20}$$

we find the equation for u_n by inserting (20) into

$$\mathrm{i}\partial_t u_n = -[z + z^{-1} - 2 + (\phi_{n+1}\phi_n^* + \phi_{n+1}^*\phi_n)]u_n + (z\phi_{n+1} - \phi_n) - (z^{-1}\phi_{n+1}^* - \phi_n^*)u_n^2 . \tag{21}$$

In the continuum limit this equation reduces to the equation of motion for the Heisenberg spin chain with unaxial anisotropy (AHSC), as given in ref. [14]. Eq. (20) constitutes the transformation, which maps solutions of the dAHSC onto the dNLS.

5. In this letter we have considered a linear integral equation (1), yielding solutions of the dNLS, the dIHSC, the dCSG and the dAHSC. Solutions of the nonlinear differential–difference equations can be found by choosing a measure and a contour and solving the linear integral equation. Multisoliton solutions for instance can be found by choosing the measure to be $\mathrm{d}\lambda(z) = \Sigma_{i=1}^{N} c_i \delta(z - z_i)\,\mathrm{d}z$, and a contour through the poles of the delta functions. In the case of the dIHSC and the dCSG one can find the solutions by choosing a factor $(z - 1)^{-1}$ on the right-hand side of (1) instead of z^i. A more extended version of this work will appear in the future.

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