

Log-concavity of the cohomology of nilpotent Lie algebras in characteristic two

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Abstract

It is known that the Betti numbers of the Heisenberg Lie algebras are unimodal over fields of characteristic two. This note observes that they are log-concave. An example is given of a nilpotent Lie algebra in characteristic two for which the Betti numbers are unimodal but not log-concave.

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The Heisenberg Lie algebra of dimension $2m + 1$ is the Lie algebra \mathfrak{h}_m having the basis $\{x_1, \dots, x_m, y_1, \dots, y_m, z\}$ and nonzero relations $[x_i, y_i] = z$, $1 \leq i \leq m$. For the cohomology with trivial coefficients, the Betti numbers $b_n = \dim H^n(\mathfrak{h}_m)$, have been explicitly computed in all characteristics [3, 4, 1]. Recall that the Betti numbers are *unimodal* if $b_i \leq b_j$ for all $0 \leq i \leq j \leq m$ and $b_i \geq b_j$ for all $m \leq i \leq j \leq 2m + 1$, and they are *concave* (resp. *log-concave*) if b_i is at least as great as the arithmetic (resp. geometric) mean of the pair b_{i-1}, b_{i+1} , for all $1 \leq i \leq 2m$. So concave implies log-concave implies unimodal. In characteristic zero, unimodality is quite common. The Heisenberg Lie algebras play a key role in the construction all known examples of Lie algebras in characteristic zero where the Betti numbers are not unimodal [2]. In fact, the Betti numbers of \mathfrak{h}_m are unimodal only in characteristic two [1]. On the other hand, we know of no nilpotent Lie algebra in characteristic two whose Betti numbers fail to be unimodal. In [1], the question was posed: in characteristic two, do all nilpotent Lie algebras have unimodal Betti numbers? Since log-concavity is a common route taken to prove unimodality, it is natural to ask whether the Betti numbers of the Heisenberg algebras are unimodal in characteristic 2. We record the following observation as a theorem, though it is really just a corollary of the works [4, 1].

Theorem 0.1. *Over fields of characteristic 2, the Betti numbers of \mathfrak{h}_m are log-concave; i.e., $b_n^2 \geq b_{n-1}b_{n+1}$ for all n .*

Proof. For the rest of this note we fix the characteristic to be two. Emil Sköldbberg showed that the Poincaré polynomial $S_m(t) = \sum_n b_n t^n$ is [4]:

$$S_m(t) = \frac{(1+t^3)(1+t)^{2m} + (t+t^2)(2t)^m}{1+t^2}. \quad (0.1)$$

Though we won't need them, we mention that the individual Betti numbers are given in [1]; for all $i \leq m$:

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{2m}{n-2i} + \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^i \binom{2m}{n-3-2i}.$$

To establish the log-concavity, we observe that the Betti numbers of \mathfrak{h}_{m+1} are essentially determined by those of \mathfrak{h}_m , with a curious correction for the middle two terms. Explicitly,

$$S_{m+1}(t) = (1+t)^2 S_m(t) - 2^m(t^{m+1} + t^{m+2}). \quad (0.2)$$

This relation is easily deduced from (0.1). Using induction, we assume that S_m is log-concave. Since $(1+t)^2$ is log-concave, $(1+t)^2 S_m(t)$ is thus also log-concave (see [5]). So in view of (0.2), to establish the log-concavity of S_{m+1} , it remains to verify it for the middle terms; that is, for \mathfrak{h}_{m+1} we require:

$$b_{m+1}^2 \geq b_m b_{m+2}.$$

But by Poincaré duality, $b_{m+1} = b_{m+2}$, and so we only require $b_{m+1} \geq b_m$, and this is given by the unimodality of the Betti numbers, which was shown in [1]. This completes the proof. \square

The following example shows that, despite the above result, log-concavity is not a route for establishing unimodality in the general setting of nilpotent Lie algebras in characteristic two.

Example 0.2. Let \mathfrak{g} denote the 7-dimensional Lie algebra with basis x_1, \dots, x_7 and defining relations:

$$[x_1, x_i] = x_{i+1}, \quad i = 2, \dots, 6,$$

$$[x_2, x_i] = x_{i+2}, \quad i = 3, 4,$$

$$[x_3, x_4] = x_7.$$

Clearly \mathfrak{g} is nilpotent (and actually graded and filiform). Direct calculations using Mathematica show that in characteristic two, the Betti numbers are:

b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
1	2	3	6	6	3	2	1

As $b_2^2 < b_1 b_3$, the Betti numbers are not log-concave.

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