Topological transitivity of solvable group actions on the line $\mathbb{R}$

Suhua Wang **Enhui Shi**[1] **Lizhen Zhou**[2] and **Grant Cairns**

Abstract. Let $\phi : G \to \mathrm{Homeo}_+(\mathbb{R})$ be an orientation preserving action of discrete solvable group G on $\mathbb{R}$. In this paper, the topological transitivity of ϕ is investigated. In particular, the relations between the dynamical complexity of G and the algebraic structure of G are considered.

1 Introduction and preliminaries

Recently, there is a considerable process in studying the dynamics of group actions on 1-manifolds (see, e.g. [1][3][6][7][9][10] [13][14][15][16][18][21][22]). Topological transitivity is one of the most basic notions in dynamical systems. In this paper, we consider the topological transitivity of solvable group actions on the line $\mathbb{R}$.

We are mainly interested in the following two questions:

(1) *which solvable groups possess a faithful topologically transitive action on the line?*

(2) *what can one say about actions with higher transitivity?*

Before stating the main results in this paper, let us recall some definitions. Let X be a topological space and let $\mathrm{Homeo}(X)$ be the homeomorphism group of X. Suppose that G is a discrete group (i.e., G is a topological group with discrete topology). Recall that a group homomorphism $\phi : G \to \mathrm{Homeo}(X)$ is called an *action* of G on X. The action ϕ is said to be *faithful* if it is injective. For convenience, we always use gx or $g(x)$ to denote $\phi(g)(x)$.

2000 Mathematics Subject Classification: 37B05 (Primary), 57S25 (Secondary).

Key words and phrases: Topological transitivity, solvable group action, ping-pong game.

[1]Supported by the National Natural Science Foundation of China (No. 10801103) and by the Natural Sciences Fund for Colleges and Universities in Jiangsu Province (No. 08KJB110010).

[2]Supported in part by the National Natural Science Foundation of China (No. 10871142).

Recall that the *orbit* of $x \in X$ is the set $Gx = \{gx : g \in G\}$. For a subset $A \subseteq X$, set $GA = \bigcup_{x \in A} Gx$. A subset $A \subseteq X$ is said to be *G-invariant*, if $GA = A$. If A is G-invariant, by the symbol $G|_A$, we mean the restriction to A of the action of G.

The action ϕ is said to be *topologically transitive*, if for any two nonempty open subsets U and V of X, there is some $g \in G$ such that $g(U) \cap V \neq \emptyset$. If there is some point $x \in X$ such that the orbit Gx is dense in X then ϕ is said to be *point transitive* and such x is called a *transitive point*. It is well known that when G is countable and X is a Polish space these two notions are the same and in fact the collection of transitive points form a dense G_δ set in X. If for each $x \in X$, Gx is dense in X, then ϕ is said to be *minimal*. For an integer $k \geq 1$, ϕ is said to be *topologically k-transitive*, if for any two families of nonempty open subsets $U_1, U_2, \cdots, U_k$ and $V_1, V_2, \cdots, V_k$ of X, there is some $g \in G$ such that $g(U_i) \cap V_i \neq \emptyset$ for each $i = 1, 2, \cdots, k$. For commutative group actions, a well known theorem of H. Furstenberg says that topological 2-transitivity implies topological k-transitivity for each $k \geq 2$ (see [8] and [11]). Topological k-transitivity of linear group actions is also studied in [4] and [5].

Denote by $\text{Homeo}_+(\mathbb{R})$ the group of all orientation preserving homeomorphisms on $\mathbb{R}$. Let $\phi : G \to \text{Homeo}_+(\mathbb{R})$ be an orientation preserving group action on $\mathbb{R}$. Since each element of G preserves the orientation of $\mathbb{R}$, ϕ cannot be topologically k-transitive for $k \geq 2$. However, we are mainly interested in orientation preserving group actions on $\mathbb{R}$ in this paper, so we have to give the following definition of pseudo-k-transitivity.

Firstly we introduce an ordering $\preceq$ in the collection of all open intervals contained in $\mathbb{R}$. For any two open intervals (a, b) and (c, d) in $\mathbb{R}$, we say that $(a, b) \preceq (c, d)$ if $a \leq c$. We say ϕ is *pseudo-k-transitive* if for any two families of open intervals $(a_1, b_1) \preceq (a_2, b_2) \preceq \cdots \preceq (a_k, b_k)$ and $(c_1, d_1) \preceq (c_2, d_2) \preceq \cdots \preceq (c_k, d_k)$ there is some $g \in G$ such that $g((a_i, b_i)) \cap (c_i, d_i) \neq \emptyset$ for all $i = 1, 2, \cdots, k$. It is easy to see that ϕ is pseudo-k-transitive if and only if there are $x_1 < x_2 < \cdots < x_k \in \mathbb{R}$ such that for any nonempty open intervals $U_1 \preceq U_2 \preceq \cdots \preceq U_k$ there is a $g \in G$ such that $g(x_i) \in U_i$ for each $i = 1, 2, \cdots, k$.

Now let us recall some definitions in group theory which will be used in the following. Suppose that G is a group with identity e. Let $a, b \in G$. The *commutator* $[a, b]$ is defined by $[a, b] = a^{-1}b^{-1}ab$. For any two subsets A and B of G, define $[A, B]$ to be the subgroup generated by the set $\{[a,b] : a \in A, b \in B\}$. Let $G_0 = G$ and $G_{i+1} = [G_i, G]$, for $i = 0, 1, 2, \cdots$. Thus we get a sequence of normal subgroups of $G : G_0 = G \triangleright G_1 \triangleright G_2 \triangleright \cdots$. If there is some natural number n such that $G_n = \{e\}$, then G is called *nilpotent.* Also we can define another sequence of normal subgroups $G^0 = G \triangleright G^1 \triangleright G^2 \triangleright \cdots$ by letting $G^0 = G$ and $G^{i+1} = [G^i, G^i]$ for $i = 0, 1, 2, \cdots$. If there is some n such that $G^n = \{e\}$ then G is called *solvable.* The minimal n such that $G^n = \{e\}$ is called the *derived length* of G. A solvable group with derived length at most 2 is called *metabelian.* A solvable group G is called *polycyclic*, if for some k, G has a sequence of normal subgroups $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$ such that each N_i/N_{i+1} is cyclic. When each of the quotients N_i/N_{i+1} is infinite cyclic, G is said to be *poly-infinite-cyclic.*

Before stating our results for solvable groups, let us put them in a broader context. First note that topologically transitive actions of nonsolvable groups on the line are quite abundant. For example, the group generated by the two elements $f(x) = x+1$ and $g(x) = x^3$ is a free group [12, p. 37] and it is easy to verify that its action on the line is topologically transitive. The group of homeomorphisms of $\mathbb{R}$ that are piecewise-linear with respect to a finite subdivision of $\mathbb{R}$ is not solvable but doesn't contain a nonabelian free subgroup [2], and its action is clearly pseudo-k-transitive for all k.

In section 2, it is shown that minimal actions of commutative groups are building blocks of topologically transitive nilpotent group actions on $\mathbb{R}$. To illustrate this idea, for each finitely generated torsion free nilpotent group G, a topologically transitive $G \times \mathbb{Z}^2$ action on $\mathbb{R}$ is constructed. A more general result is also given, as we will now describe. First recall that a countable group G possesses a faithful orientation preserving action on the real line if and only if G is left orderable (see [10, Theorem 6.8]). Moreover, left orderable polycyclic groups are poly-infinite-cyclic [19]. Since the cyclic group $\mathbb{Z}$ obviously has no topologically transitive action on the line, it is therefore natural to look at noncyclic poly-infinite-cyclic groups. We have:

Theorem 1.1. *Every noncyclic poly-infinite-cyclic group possesses a faithful topologically transitive orientation preserving action on the real line.*

In particular, every finitely generated torsion free nilpotent group possesses a faithful topologically transitive orientation preserving action on the real line.

In section 3, two examples of pseudo-1-transitive but not of pseudo-2-transitive and of pseudo-2-transitive but not of pseudo-3-transitive metabelian group actions on $\mathbb{R}$ are given respectively. It is shown that each polycyclic solvable group action on $\mathbb{R}$ is at most pseudo-2-transitive, and if the derived length of a solvable group G is n, then the action of G is at most pseudo -$(4^n - 1)$-transitive. At the end of this section, it is shown that no nilpotent group action on $\mathbb{R}$ is pseudo-2-transitive.

In the following, all group actions on $\mathbb{R}$ refer to orientation preserving actions.

2 The construction of topologically transitive actions

In this section, we will construct some topologically transitive solvable group actions on $\mathbb{R}$. First we study minimal actions of nilpotent subgroups of $\mathrm{Homeo}_+(\mathbb{R})$. The following proposition is in fact a special case of Corollary 4.6 in [18], but for completeness we prove it again here.

Proposition 2.1. *Let G be a finitely generated nilpotent subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$. *If the action of G is minimal, then G must be commutative and be topologically conjugate to a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *consisting of translations on* $\mathbb{R}$.

Proof. From [17], we know that there is a G-invariant Borel measure μ on $\mathbb{R}$ which is finite on compact sets. Since G is minimal, the support of μ, $\mathrm{supp}(\mu) = \mathbb{R}$ and μ has no atoms. Let $h : \mathbb{R} \to \mathbb{R}$ be a map defined by

$$h(x) = \mu([0, x]) \text{ if } x \geq 0, \text{ and } h(x) = -\mu([x, 0]) \text{ if } x < 0.$$

Then it is easy to see that h is a homeomorphism. Now let $\tilde{G} = \{hgh^{-1} : \ g \in G\}$, then it is not difficult to check that each element of $\tilde{G}$ is an isometry. Since $\tilde{G}$ preserves the orientation of $\mathbb{R}$, $\tilde{G}$ consists of translations on $\mathbb{R}$. Thus $\tilde{G}$ is commutative and is conjugate to G by h. □

Remark 2.2. We can easily construct a minimal $\mathbb{Z}^2$ action on $\mathbb{R}$. Indeed, let L_a and L_b be two translations on $\mathbb{R}$ defined by $L_a(x) = x + a$ and $L_b(x) = x + b$ for all $x \in \mathbb{R}$, where a and b are rationally independent. Then the subgroup $< L_a, L_b >$ of $\mathrm{Homeo}_+(\mathbb{R})$ generated by L_a

and L_b is minimal.

Now we consider the structure of topologically transitive nilpotent group actions on $\mathbb{R}$. The following proposition indicates that, for nilpotent group actions on $\mathbb{R}$, minimal systems are building blocks of topologically transitive systems.

Proposition 2.3. *Let G be a finitely generated nilpotent subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is topologically transitive. Then there exists an open interval (α, β) (α may be $-\infty$ and β may be $+\infty$) such that the restriction of the action of the group $F = \{g \in G : g((\alpha, \beta)) = (\alpha, \beta)\}$ to (α, β) is minimal.*

Proof. If G is minimal then we need only let $\alpha = -\infty$ and $\beta = +\infty$. Otherwise, there is some $x_1 \in \mathbb{R}$ such that Gx_1 is not dense in $\mathbb{R}$. Let $a = \inf\{Gx_1\}$ and $b = \sup\{Gx_1\}$.

Claim 1: $a = -\infty$ and $b = +\infty$. Indeed, if $a \in \mathbb{R}$ then it is not difficult to see that a is a fixed point of G. Since G preserves the orientation of $\mathbb{R}$, we have $g((-\infty, a)) = (-\infty, a)$ and $g((a, +\infty)) = (a, +\infty)$ for each $g \in G$. This contradicts the topological transitivity of G. So $a = -\infty$. Similarly, we have $b = +\infty$.

Claim 2: $\overline{Gx_1}$ is nowhere dense in $\mathbb{R}$. Otherwise, there is some nonempty open set $U \subset \overline{Gx_1}$. For any nonempty open set $V \subset \mathbb{R}$, since G is topologically transitive, there is some $g \in G$ such that $g(U) \cap V \neq \emptyset$. Thus there is some $g' \in G$ such that $g'(x_1) \in U$ and $gg'(x_1) \in V$. By the arbitrariness of V, we get that x_1 is a transitive point. This contradicts our original assumption.

It follows from Claim 1 and Claim 2 that $\mathbb{R}\setminus\overline{Gx_1} = \bigcup_{i=-\infty}^{+\infty}(a_i, b_i)$, where $\{(a_i, b_i) : i \in \mathbb{Z}\}$ is a sequence of pairwise disjoint open intervals in $\mathbb{R}$. Let $F_1 = \{g \in G : g((a_0, b_0)) = (a_0, b_0)\}$ and let $G/F_1 = \bigcup_{i=-\infty}^{+\infty} g_i F_1$ be the coset decomposition of G with respect to F_1, where $g_i((a_0, b_0)) = (a_i, b_i)$ for all $i \in \mathbb{Z}$. This implies that $[G : F_1] = \infty$ and the restrictive action of F_1 on (a_0, b_0) is transitive. Let $(\alpha_1, \beta_1) = (a_0, b_0)$. If $F_1|_{(\alpha_1,\beta_1)}$ is not minimal, then, similar to the above discussions, we can get a subgroup F_2 of F_1 and an open interval $(\alpha_2, \beta_2) \subset (\alpha_1, \beta_1)$ such that $F_2|_{(\alpha_2,\beta_2)}$ is topologically transitive and $[F_1 : F_2] = \infty$. Going on in this way, if for every $i \geq 1$, $F_i|_{(\alpha_i,\beta_i)}$ is topologically transitive but is not minimal, then we obtain a sequence of open intervals $(\alpha_1, \beta_1) \supset (\alpha_2, \beta_2) \supset \cdots$ and a sequence of subgroups of G: $F_1 \supset F_2 \supset \cdots$

such that

(i) each (α_i, β_i) is F_i-invariant,

(ii) $F_i|_{(\alpha_i,\beta_i)}$ is topologically transitive but is not minimal, and

(iii) $[F_i : F_{i+1}] = \infty$, for $i = 1, 2, \cdots$.

But (iii) contradicts the fact that G is a finitely generated nilpotent group. So there must be some $n \in \mathbb{N}$ such that $F_n|_{(\alpha_n,\beta_n)}$ is minimal. This completes the proof. □

To illustrate the ideas in Proposition 2.3, we construct some topologically transitive actions of finitely generated nilpotent groups on $\mathbb{R}$. In the proof of the following proposition, we follow some ideas in [7].

Proposition 2.4. *Suppose G is a finitely generated torsion free nilpotent group. Then $G \times \mathbb{Z}^2$ acts on $\mathbb{R}$ faithfully and topologically transitively.*

Proof. We consider the group $\mathbb{Z}^n$ of n-tuples of integers and provide it with a linear order $\prec$ which is the lexicographic ordering, i.e. $(x_1, \cdots, x_n) \prec (y_1, \cdots, y_n)$ if and only if $x_i = y_i$ for $1 \le i < k$ and $x_k < y_k$ for some $0 \le k \le n$.

It is well known that each finitely generated torsion-free nilpotent group G admits a linear order $\prec$ which is invariant under left translations, and there exists an order preserving bijection $\jmath : G \to \mathbb{Z}^n$, i.e. $g_1 \prec g_2$ if and only if $\jmath(g_1) \prec \jmath(g_2)$. Thus $\jmath$ naturally induces an action of G on $\mathbb{Z}^n$ by letting $g(p_1, p_2, \cdots, p_n) = \jmath g \jmath^{-1}(p_1, p_2, \cdots, p_n)$ for all $(p_1, p_2, \cdots, p_n) \in \mathbb{Z}^n$.

Let $B : \mathbb{Z}^n \to \mathbb{R}$ be defined by

$$B(q_1, q_2, \cdots, q_n) = \sum_{j=1}^{n} q_j^{2n-2j+2},$$

and let

$$s = \sum_{(q_1,q_2,\cdots,q_n)\in\mathbb{Z}^n} \frac{1}{B(q_1, q_2, \cdots, q_n)}.$$

For $(p_1, p_2, \cdots, p_n)$ we define $\imath : \mathbb{Z}^n \to \mathbb{R}$ by

$$\imath(p_1, p_2, \cdots, p_n) = \sum_{(q_1,q_2,\cdots,q_n)\prec(p_1,p_2,\cdots,p_n)} \frac{1}{B(q_1, q_2, \cdots, q_n)}.$$

Then it is easy to see that $\imath$ is an order preserving injection from $\mathbb{Z}^n$ to $(0, s)$, i.e. $(p_1, p_2, \cdots, p_n) \prec (p'_1, p'_2, \cdots, p'_n)$ if and only if $\imath(p_1, p_2, \cdots, p_n) < \imath(p'_1, p'_2, \cdots, p'_n)$. Thus $\imath$ induced naturally an action of G on $\imath(\mathbb{Z}^n)$. Then we extend this action to the closure $\overline{\imath(\mathbb{Z}^n)}$ in $(0, s)$ by using continuity, and then extend it to $(0, s)$ by using affine extensions on the complementary intervals of this closure. Thus we get an orientation preserving action of G on $(0, s)$.

For each $(p_1, p_2, \cdots, p_n) \in \mathbb{Z}^n$, let $U_{(p_1,p_2,\cdots,p_n)} = (\imath(p_1, p_2, \cdots, p_n), \imath(p_1, p_2, \cdots, p_n + 1))$. Then it is easy to see that $(0, s) \setminus \overline{\imath(\mathbb{Z}^n)} = \cup_{(p_1,p_2,\cdots,p_n)\in\mathbb{Z}^n} U_{(p_1,p_2,\cdots,p_n)}$. Now let $f_{(0,0,\cdots,0)}$ and $h_{(0,0,\cdots,0)}$ be two homeomorphisms on $U_{(0,0,\cdots,0)}$ such that the action of $< f_{(0,0,\cdots,0)}, h_{(0,0,\cdots,0)} >$ on $U_{(0,0,\cdots,0)}$ is minimal (for the existence of such homeomorphisms, see Remark 2.2). Evidently, for each $(p_1, p_2, \cdots, p_n) \in \mathbb{Z}^n$, there is a unique $g \in G$ such that $g(U_{(0,0,\cdots,0)}) = U_{(p_1,p_2,\cdots,p_n)}$. Now define $f_{(p_1,p_2,\cdots,p_n)}$, $h_{(p_1,p_2,\cdots,p_n)} : U_{(p_1,p_2,\cdots,p_n)} \to U_{(p_1,p_2,\cdots,p_n)}$ by letting $f_{(p_1,p_2,\cdots,p_n)} = gf_{(0,0,\cdots,0)}g^{-1}$, $h_{(p_1,p_2,\cdots,p_n)} = gh_{(0,0,\cdots,0)}g^{-1}$. Then we define two homeomorphisms $f, h : (0, s) \to (0, s)$ by

$$f(x) = f_{(p_1,p_2,\cdots,p_n)}(x) \text{ for } x \in U_{(p_1,p_2,\cdots,p_n)}, \text{ and } f(x) = x \text{ for } x \in \overline{\imath(\mathbb{Z}^n)}, \text{ and}$$

$$h(x) = h_{(p_1,p_2,\cdots,p_n)}(x) \text{ for } x \in U_{(p_1,p_2,\cdots,p_n)}, \text{ and } h(x) = x \text{ for } x \in \overline{\imath(\mathbb{Z}^n)}.$$

From the above definitions, we see that $fh = hf$ and $fg = gf$, $hg = gh$ for all $g \in G$. Thus f, h and G generate a $G \times \mathbb{Z}^2$ action on $(0, s)$. It is not difficult to check that this action is topologically transitive. Since $(0, s)$ and $\mathbb{R}$ are homeomorphic, we also obtain a topologically transitive and faithful action of $G \times \mathbb{Z}^2$ on $\mathbb{R}$. □

Theorem 1.1, announced in Section 1 above, is more general than Proposition 2.4 and its proof relies on different ideas.

Proof of Theorem 1.1. Suppose that $G = N_0 \rhd N_1 \rhd \cdots \rhd N_k = \{e\}$ where $k > 1$ and each N_i/N_{i+1} is infinite cyclic. The proof is by induction on k. The theorem holds when $k = 2$, since the abelian case is given by Proposition 2.1, and in the nonabelian case, the group $< a, b|aba^{-1} = b^n >$ has a faithful minimal orientation preserving action on the line (see Proposition 3.1 below). It remains to deduce the result for G_k assuming it true for G_{k-1}. Let $a \in G_k \setminus G_{k-1}$. We let a act on the line by the unit translation $\phi(a) : x \mapsto x + 1$. By hypothesis, G_{k-1} has a faithful topologically transitive orientation preserving action on the open interval $(0,1)$, which we extend to an action ϕ on the closed interval $[0,1]$ by fixing the endpoints. Then extend this action of G_{k-1} to the line by setting, for each $i \in \mathbb{Z}$,

$$\phi(b)(x) = \phi(a^{-i}ba^{i})(x-i) + i$$

for all $x \in [i, i+1]$ and $b \in G_{k-1}$. We then define ϕ on G_k by setting $\phi(a^i b) = (\phi(a))^i \phi(b)$ for all $i \in \mathbb{Z}$ and $b \in G_{k-1}$. It is easy to verify that ϕ is a continuous group action and that it is faithful, topologically transitive and orientation preserving. Indeed, these claims are all obvious except possibly the fact that ϕ is an action, which is a calculation. Consider two arbitrary elements $g_1 = a^{l_1}b_1, g_2 = a^{l_2}b_2$ in G_k, where $b_1, b_2 \in G_{k-1}$. We must show that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$. For all $x \in [i, i+1]$ we have

$$\begin{aligned} \phi(g_1g_2)(x) &= \phi(a^{l_1}b_1a^{l_2}b_2)(x) = \phi(a^{l_1+l_2}a^{-l_2}b_1a^{l_2}b_2)(x) \\ &= \phi(a^{l_1+l_2})\phi(a^{-l_2}b_1a^{l_2}b_2)(x) = \phi(a^{-i}a^{-l_2}b_1a^{l_2}b_2a^{i})(x-i) + i + l_1 + l_2, \end{aligned}$$

while

$$\begin{aligned} \phi(g_1)\phi(g_2)(x) &= \phi(a^{l_1}b_1)\phi(a^{l_2}b_2)(x) = \phi(a^{l_1}b_1)(\phi(a^{-i}b_2a^{i})(x-i) + i + l_2) \\ &= \phi(a^{-i-l_2}b_1a^{i+l_2})\phi(a^{-i}b_2a^{i})(x-i) + i + l_1 + l_2 \\ &= \phi(a^{-i-l_2}b_1a^{l_2}b_2a^{i})(x-i) + i + l_1 + l_2, \end{aligned}$$

as required. □

3 Higher transitivity

In this section, we first give two examples of minimal but not of pseudo-2-transitive and of pseudo-2-transitive but not of pseudo-3-transitive solvable subgroups of $\mathrm{Homeo}_+(\mathbb{R})$. In the following, we use the symbols $\mathbb{R}_+$ and $\mathbb{R}_-$ to denote the set of positive numbers and the set of negative numbers respectively.

Proposition 3.1. *Let $T, S : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x + 1$ and $S(x) = \alpha x$ for some $\alpha > 1$ and for all $x \in \mathbb{R}$ respectively. Then the solvable group $G =< T, S >$ is minimal but is not pseudo-2-transitive.*

Proof. An easy computation shows that $S^{-n}TS^n(x) = x + \alpha^{-n}$ for all $n \in \mathbb{N}$. Define $L_{\alpha^{-n}} : \mathbb{R} \to \mathbb{R}$ by $L_{\alpha^{-n}}(x) = x+\alpha^{-n}$ for all $x \in \mathbb{R}$. Then these $L_{\alpha^{-n}}$ belong to G. For any nonempty open interval $U \subset \mathbb{R}$, choose an $n' \in \mathbb{N}$ such that $\alpha^{-n'} < \mathrm{diam}(U)$. So for any $x \in \mathbb{R}$, there must exist some $m \in \mathbb{N}$ such that $x + m\alpha^{-n'} \in U$, that is $L^m_{\alpha^{-n'}}(x) \in U$. Thus G is minimal.

Now we show that G is not pseudo-2-transitive. In fact, for any two different points $x, y \in \mathbb{R}$, let $d = |x - y|$. Then for any $g \in G$, from the definitions of T and S, we see that there are some $n \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ such that $g(x) = \alpha^n x + \beta$ and $g(y) = \alpha^n y + \beta$. Thus $|g(x) - g(y)| = |\alpha^n x - \alpha^n y| = \alpha^n d$. It follows that the set $\{|g(x) - g(y)| : g \in G\}$ is not dense in the set of positive real numbers $\mathbb{R}_+$. This implies that G is not pseudo-2-transitive. □

Lemma 3.2. *Given two numbers $\alpha > 0, \beta > 0$, let K denote $\mathbb{R}_+$ or $\mathbb{R}_-$ and let $M_\alpha, M_\beta : K \to K$ be defined by $M_\alpha(x) = \alpha x$ and $M_\beta(x) = \beta x$ for all $x \in K$. If $\log(\alpha)$ and $\log(\beta)$ are rationally independent, then the action of the group $G =< M_\alpha, M_\beta >$ on K is minimal.*

Proof. Suppose that $K = \mathbb{R}_+$. Let $a = \log(\alpha)$ and $b = \log(\beta)$. It is easy to see that the action of G on $\mathbb{R}_+$ is topologically conjugate to the action of $G' =< L_a, L_b >$ via the homeomorphism $h : \mathbb{R}_+ \to \mathbb{R},\ x \mapsto \log(x)$. Since a and b are rationally independent, from Remark 2.2, we get that G is minimal. Similarly, the conclusion also holds when $K = \mathbb{R}_-$.□

Proposition 3.3. *Let T, M_α, M_β be defined by $T(x) = x + 1, M_\alpha(x) = \alpha x$ and $M_\beta(x) = \beta x$ for all $x \in \mathbb{R}$ respectively. If $\alpha > 1, \beta > 1$, and, $\log(\alpha)$ and $\log(\beta)$ are rationally independent, then the solvable group $G =< T, M_\alpha, M_\beta >$ is pseudo-2-transitive but is not pseudo-3-transitive.*

Proof. For any two different points $x < y \in \mathbb{R}$ and any two nonempty open intervals $U \preceq V$ contained in $\mathbb{R}$, we will show that there is some $g \in G$ such that $g(x) \in U$ and $g(y) \in V$.

Since $U \preceq V$, there are two intervals $[a_1, b_1] \subseteq U$ and $[a_2, b_2] \subseteq V$ such that $b_1 < a_2$ and $|a_1 - b_1| = |a_2 - b_2|$. Let $c = |a_1 - b_1| = |a_2 - b_2|$ and $d = |x - y|$. From the proof of Proposition 3.1, we see that the set of translations $\{L_{\alpha^{-n}} : n \in \mathbb{N}\}$ lies in G. Since $\alpha > 1$, there is some $n' \in \mathbb{N}$ such that $\alpha^{-n'} < \min\{\frac{c}{6}, a_2 - b_1\}$. For the translation $L_{\alpha^{-n'}}$, there is some $p \in \mathbb{Z}$ such that

$$L^p_{\alpha^{-n'}}(b_1) < 0 < L^p_{\alpha^{-n'}}(a_2) \text{ and}$$

$$| \, |L^p_{\alpha^{-n'}}(b_1)| - |L^p_{\alpha^{-n'}}(a_2)| \, |=| \, |b_1 + p\alpha^{-n'}| - |a_2 + p\alpha^{-n'}| \, | \leqslant \alpha^{-n'}.$$

Denote $a'_i = L^p_{\alpha^{-n'}}(a_i)$ and $b'_i = L^p_{\alpha^{-n'}}(b_i)$ for $i = 1, 2$. From the inequalities above, we have that

$$|b'_2 + a'_1| = |b'_1 + a'_2| \leqslant \alpha^{-n'} < \frac{c}{6}. \tag{3.1}$$

Next choose $n'' \in \mathbb{N}$ such that $\alpha^{-n''} < \min\{\frac{d}{4}, \frac{c}{6}[\frac{4}{d}(\frac{c}{6} - a'_1)]^{-1}\}$. For $L_{\alpha^{-n''}}$, there exists some $q \in \mathbb{Z}$ such that

$$L^q_{\alpha^{-n''}}(x) < 0 < L^q_{\alpha^{-n''}}(y) \text{ and}$$

$$| \, |L^q_{\alpha^{-n''}}(x)| - |L^q_{\alpha^{-n''}}(y)| \, |=| \, |x + q\alpha^{-n''}| - |y + q\alpha^{-n''}| \, | \leqslant \alpha^{-n''}.$$

Let $x' = L^q_{\alpha^{-n''}}(x)$ and $y' = L^q_{\alpha^{-n''}}(y)$. Then it is not hard to see that

$$|x' + y'| \leqslant \alpha^{-n''} < \frac{c}{6}[\frac{4}{d}(\frac{c}{6} - a'_1)]^{-1} \text{ and } \; |x'| > \frac{d}{4}. \tag{3.2}$$

Since the action of $< M_\alpha, M_\beta >$ on $\mathbb{R}_-$ is minimal from Lemma 3.2, there are $s, t \in \mathbb{Z}$ such that

$$|M^s_\alpha M^t_\beta(x') - \frac{a'_1 + b'_1}{2}| = |\alpha^s \beta^t x' - \frac{a'_1 + b'_1}{2}| < \frac{c}{6}, \tag{3.3}$$

which implies that

$$\alpha^s \beta^t < \frac{1}{|x'|}(\frac{c}{6} + |\frac{a'_1 + b'_1}{2}|) < \frac{4}{d}(\frac{c}{6} - a'_1). \tag{3.4}$$

Then by the conditions (3.1)-(3.4) we have that

$$\begin{aligned}
& |M^s_\alpha M^t_\beta(y') - \frac{a'_2 + b'_2}{2}| \\
= \; & |\alpha^s \beta^t(-y') + \frac{a'_2 + b'_2}{2}| \\
\leqslant \; & |\alpha^s \beta^t(-y') - \alpha^s \beta^t x'| + |\alpha^s \beta^t x' - \frac{a'_1 + b'_1}{2}| + |\frac{a'_1 + b'_1}{2} + \frac{a'_2 + b'_2}{2}| \\
\leqslant \; & \alpha^s \beta^t |x' + y'| + |\alpha^s \beta^t x' - \frac{a'_1 + b'_1}{2}| + |\frac{a'_1 + b'_2}{2} + \frac{b'_1 + a'_2}{2}| \\
\leqslant \; & \frac{4}{d}(\frac{c}{6} - a'_1) \cdot \alpha^{-n''} + \frac{c}{6} + |b'_1 + a'_2| \\
< \; & \frac{4}{d}(\frac{c}{6} - a'_1) \cdot \frac{c}{6} \cdot [\frac{4}{d}(\frac{c}{6} - a'_1)]^{-1} + \frac{c}{6} + \frac{c}{6} \\
= \; & \frac{c}{2}
\end{aligned}$$

So $M^s_\alpha M^t_\beta(x') \in [a'_1, b'_1]$ and $M^s_\alpha M^t_\beta(y') \in [a'_2, b'_2]$, and thus $L^{-p}_{\alpha^{-n'}} M^s_\alpha M^t_\beta L^q_{\alpha^{-n''}}(x) \in [a_1, b_1] \subseteq U$ and $L^{-p}_{\alpha^{-n'}} M^s_\alpha M^t_\beta L^q_{\alpha^{-n''}}(y) \in [a_2, b_2] \subseteq V$. This implies that G is pseudo-2-transitive.

In the following, we show that G is not pseudo-3-transitive. In fact, for any $g \in G$, there are some $n, m \in \mathbb{Z}$ and $r \in \mathbb{R}$ such that $g(x) = \alpha^n \beta^m x + r$ for all $x \in \mathbb{R}$. Thus for any three points $x < y < z \in \mathbb{R}$, we have that

$$\frac{|g(x) - g(y)|}{|g(y) - g(z)|} = \frac{|\alpha^n \beta^m x - \alpha^n \beta^m y|}{|\alpha^n \beta^m y - \alpha^n \beta^m z|} = \frac{|x - y|}{|y - z|} \text{ for any } g \in G. \tag{3.5}$$

Now choose points $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ in $\mathbb{R}$ such that

$$\frac{a_2 - b_1}{b_3 - a_2} > 100 \frac{|x - y|}{|y - z|}.$$

Then from (3.5) we see that for all $g \in G$, $g(x) \in (a_1, b_1)$, $g(y) \in (a_2, b_2)$ and $g(z) \in (a_3, b_3)$ cannot occur simultaneously. This shows that the G-action is not pseudo-3-transitive. □

Recall that a measure μ on $\mathbb{R}$ is called *quasi-invariant* for a group $G \subset \text{Homeo}_+(\mathbb{R})$ if for each $g \in G$ there is a positive constant $\alpha(g)$ such that $g_* \mu = \alpha(g) \mu$ (see [18]).

Proposition 3.4. *Suppose that $G \subseteq$ Homeo$_+(\mathbb{R})$ is a group and has a quasi-invariant measure μ on $\mathbb{R}$ which is finite on compact sets. Then G is not pseudo-3-transitive.*

Proof. Assume to the contrary that G is pseudo-3-transitive and μ is a G-quasi-invariant measure on $\mathbb{R}$. First we claim that $\text{supp}(\mu) = \mathbb{R}$. In fact, fix an interval $[a, b] \subset \mathbb{R}$ such that $\mu([a, b]) > 0$. For any nonempty open interval $U \subset \mathbb{R}$, since G is pseudo-3-transitive, there is some $g \in G$ such that $g(U) \cap (-\infty, a - 1) \neq \emptyset$ and $g(U) \cap (b + 1, \infty) \neq \emptyset$. Thus $g(U) \supseteq [a, b]$. Since μ is quasi-invariant for G, there exists a number $\alpha(g) > 0$ such that

$$\mu(U) = \alpha(g)^{-1} \mu(g(U)) \geqslant \alpha(g)^{-1} \mu([a, b]) > 0.$$

By the arbitrariness of U, we see that $\text{supp}(\mu) = \mathbb{R}$.

Thus for any three points $x < y < z \in \mathbb{R}$, we have $\mu([x, y]) > 0$ and $\mu([y, z]) > 0$. Then for any $g \in G$,

$$\frac{\mu([gx, gy])}{\mu([gy, gz])} = \frac{\alpha(g) \mu([x, y])}{\alpha(g) \mu([y, z])} = \frac{\mu([x, y])}{\mu([y, z])},$$

which contradicts the assumption that G is pseudo-3-transitive. □

From Theorem 4.4 in [18], we see that each polycyclic solvable subgroup of $\text{Homeo}_+(\mathbb{R})$ must have a quasi-invariant measure μ on $\mathbb{R}$. Thus we get the following

Corollary 3.5. *Each polycyclic solvable subgroup of Homeo*$_+(\mathbb{R})$ *is not pseudo-3-transitive.*

Lemma 3.6. *Let G be a solvable subgroup of Homeo*$_+(\mathbb{R})$*. If G is pseudo-4k-transitive on $\mathbb{R}$ for some $k \in \mathbb{N}$, then the commutator subgroup $[G, G]$ is at least pseudo-k-transitive.*

Proof. Suppose $U_1 \preceq U_2 \preceq \cdots \preceq U_k$ and $V_1 \preceq V_2 \preceq \cdots \preceq V_k$ are arbitrary nonempty open intervals in $\mathbb{R}$. To see that $[G, G]$ is pseudo-k-transitive, we will show that there is some $f^{-1}g^{-1}fg \in [G, G]$ such that $f^{-1}g^{-1}fg(U_i) \cap V_i \neq \emptyset$ for all $i = 1, \cdots, k$. Without loss of generality, we can suppose that the interval length of each U_i or V_i is finite. Then there exists an $x \in \mathbb{R}$ such that $(\cup_{i=1}^{k}(U_i \cup V_i)) \cap [x, \infty) = \emptyset$. In the following, for an open interval $J = (a, b) \subset \mathbb{R}$, denote $J^+(\epsilon) = (b, b + \epsilon)$ and $J^-(\epsilon) = (a - \epsilon, a)$ for some small positive number ϵ. For brevity, we always use J^+ and J^- instead of $J^+(\epsilon)$ and $J^-(\epsilon)$ respectively.

Now take $3k$ pairwise disjoint open intervals of $(x + 1, \infty)$, $A_1 \preceq A_2 \preceq \cdots \preceq A_k \preceq B_1 \preceq B_2 \preceq \cdots \preceq B_k \preceq C_1 \preceq C_2 \preceq \cdots \preceq C_k$, such that the distance between any two adjacent intervals equals 1. Then there is some sufficiently small ϵ such that $A_i^+ \cap A_{i+1}^- = \emptyset$, $B_i^+ \cap B_{i+1}^- = \emptyset$ and $C_i^+ \cap C_{i+1}^- = \emptyset$ for all $i = 1, \cdots, k-1$, and, $A_k^+ \cap B_1^- = \emptyset$ and $B_k^+ \cap C_1^- = \emptyset$. For the following two sequences of open intervals:

$$U_1 \preceq U_1 \preceq U_2 \preceq U_2 \preceq \cdots \preceq U_k \preceq U_k \preceq B_1 \preceq B_1 \preceq B_2 \preceq B_2 \preceq \cdots \preceq B_k \preceq B_k \text{ and}$$

$$A_1^- \preceq A_1^+ \preceq A_2^- \preceq A_2^+ \preceq \cdots \preceq A_k^- \preceq A_k^+ \preceq C_1^- \preceq C_1^+ \preceq C_2^- \preceq C_2^+ \preceq \cdots \preceq C_k^- \preceq C_k^+,$$

since G is pseudo-4k-transitive, there is some $g \in G$ such that

$$g(U_i) \cap A_i^- \neq \emptyset,\ g(U_i) \cap A_i^+ \neq \emptyset \text{ for all } i = 1, \cdots, k, \tag{3.6}$$

$$g(B_i) \cap C_i^- \neq \emptyset,\ g(B_i) \cap C_i^+ \neq \emptyset \text{ for all } i = 1, \cdots, k. \tag{3.7}$$

From (3.6) and (3.7), it is not hard to see that

$$A_i \subseteq g(U_i) \text{ and } C_i \subseteq g(B_i) \text{ for each } i = 1, \cdots, k. \tag{3.8}$$

Similarly, for the two sequences of open intervals:

$$V_1 \preceq V_1 \preceq V_2 \preceq V_2 \preceq \cdots \preceq V_k \preceq V_k \preceq A_1 \preceq A_1 \preceq A_2 \preceq A_2 \preceq \cdots \preceq A_k \preceq A_k \text{ and}$$

$$B_1^- \preceq B_1^+ \preceq B_2^- \preceq B_2^+ \preceq \cdots \preceq B_k^- \preceq B_k^+ \preceq C_1^- \preceq C_1^+ \preceq C_2^- \preceq C_2^+ \preceq \cdots \preceq C_k^- \preceq C_k^+,$$

there is some $f \in G$ such that

$$f(V_i) \cap B_i^- \neq \emptyset,\ f(V_i) \cap B_i^+ \neq \emptyset \text{ for all } i = 1, \cdots, k,$$

$$f(A_i) \cap C_i^- \neq \emptyset,\ f(A_i) \cap C_i^+ \neq \emptyset \text{ for all } i = 1, \cdots, k.$$

Thus we have

$$B_i \subseteq f(V_i) \text{ and } C_i \subseteq f(A_i) \text{ for all } i = 1, \cdots, k. \tag{3.9}$$

From (3.8) and (3.9), we see that for all $i = 1, \cdots, k$,

$$fg(U_i) \supseteq f(A_i) \supseteq C_i \text{ and } gf(V_i) \supseteq g(B_i) \supseteq C_i.$$

Hence $fg(U_i) \cap gf(V_i) \neq \emptyset$, that is $f^{-1}g^{-1}fg(U_i) \cap V_i \neq \emptyset$ for each $i \in \{1, \cdots, k\}$. This completes the proof. □

Proposition 3.7. *Suppose G is a solvable subgroup of Homeo$_+(\mathbb{R})$ and its derived length is n. Then G is at most pseudo-$(4^n - 1)$-transitive.*

Proof. Assume to the contrary that G is pseudo-4^n-transitive. From Lemma 3.6, the commutator subgroup $G^1 = [G, G]$ is pseudo-4^{n-1}-transitive and $G^2 = [G^1, G^1]$ is pseudo-4^{n-2}-transitive. Continuing this process, we see that $G^n = [G^{n-1}, G^{n-1}]$ is topologically transitive on $\mathbb{R}$. However, as the derived length of G is n, $G^n = \{e\}$. This is a contradiction. □

Ping-pong game is an important technique in determining the existence of free subgroups or free sub-semigroups of some groups (see for example [13]). Let G be a group acting on $\mathbb{R}$. Suppose I, J and K are three closed intervals in $\mathbb{R}$ such that I and J are disjoint, $I \subset K$ and $J \subset K$. If there are two elements $g_1, g_2 \in G$ such that $g_1(K) \subset I$ and $g_2(K) \subset J$, then the pair $(g_1 : K \to I;\ \ g_2 : K \to J)$ is called a *ping-pong game*. It is not difficult to check that the semigroup S generated by such g_1 and g_2 is free. It is well known that a finitely generated nilpotent group does not contain a free sub-semigroup.

Proposition 3.8. *Each finitely generated nilpotent subgroup G of* Homeo$_+(\mathbb{R})$ *is not pseudo-2-transitive.*

Proof. Assume to the contrary that there is a finitely generated nilpotent subgroup G of Homeo$_+(\mathbb{R})$ which is pseudo-2-transitive. Let $I = [0, 1]$ and $J = [2, 3]$ be two disjoint closed intervals. Since G is pseudo-2-transitive, there are $g_1, g_2 \in G$ such that $g_1(I) \cap (-\infty, -1) \neq \emptyset$, $g_1(I) \cap (4, +\infty) \neq \emptyset$, $g_2(J) \cap (-\infty, -1) \neq \emptyset$ and $g_2(J) \cap (4, +\infty) \neq \emptyset$. Let $I' = g_1^{-1}([-1, 4])$ and $J' = g_2^{-1}([-1, 4])$. Then $(g_1^{-1} : [-1, 4] \to I'; \quad g_2^{-1} : [-1, 4] \to J')$ is a ping-pong game, and thus G contains a free semigroup S which is generated by g_1^{-1} and g_2^{-1}. This contradicts the fact that G is a finitely generated nilpotent group. $\square$

References

[1] L. A. Beklaryan, Groups of homeomorphisms of the line and the circle. Topological characteristics and metric invariants. (Russian) Uspekhi Mat. Nauk 59 (2004), no. 4(358), 3–68; translation in Russian Math. Surveys 59 (2004), no. 4, 599–660.

[2] Matthew G. Brin and Craig C. Squier, Groups of piecewise linear homeomorphisms of the real line. Invent. Math. 79 (1985), no. 3, 485–498.

[3] L. Burslem and A. Wilkinson, Global rigidity of solvable group actions on $\mathbb{S}^1$. Geom. Topol. 8 (2004), 877–924 (electronic).

[4] G. Cairns and A. Nielsen, On the dynamics of the linear action of $\mathrm{SL}(\mathrm{n}, \mathbb{Z})$. Bull. Austral. Math. Soc. 71 (2005), no. 3, 359–365.

[5] S.G. Dani and S. Raghavan, Orbits of Euclidean frames under discrete linear groups, Israel J. Math. 36 (1980), 300–320.

[6] B. Farb and J. Franks, Group actions on one-manifolds. II. Extensions of Holder's Theorem. Trans. Amer. Math. Soc. 355 (2003), 4385–4396.

[7] B. Farb and J. Franks, Groups of homeomorphism of one-manifolds. III. Nilpotent subgroups. Ergod. Th. Dynam. Sys. 23 (2003), 1467–1484.

[8] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Math. Systems Theory 1 (1967), 1–49.

[9] É. Ghys, Actions de réseaux sur le cercle. Invent. Math. 137 (1999), 199–231.

[10] É. Ghys, Groups acting on the circle. L'Enseign. Math. 47 (2001), 329–407.

[11] E. Glasner, Ergodic theory via joinings. Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[12] P. de la Harpe, Topics in geometric group theory, Chigago Lectures in Math., University of Chicago Press, Chicago, IL, 2000.

[13] S. Hurder, Dynamics of expansive group actions on the circle. (Available at http://www.math.uic.edu/hurder/).

[14] L. Lifschitz and D. Witte, Bounded generalization and lattices that cannot act on the line. arXiv:math/0604612v2 [math.GR] 19 Jun 2007.

[15] L. Lifschitz and D. Witte, Isotropic nonarchimedean S-arithmetic groups are not left orderable. C. R. Math. Acad. Sci. Paris 339 (2004), no. 6, 417–420.

[16] A. Navas, Groupes résolubles de difféomorphismes de l'intervalle, du cercle et de la droite. (French) [Solvable groups of diffeomorphisms of the interval, the circle and the real line] Bull. Braz. Math. Soc. (N.S.) 35 (2004), no. 1, 13–50.

[17] J. Plante, Foliations with measure preserving holonomy. Ann. of Math. (2) 102 (1975), 327–361.

[18] J. Plante, Solvable groups acting on the line. Trans. Amer. Math. Soc. 278 (1983), 401-414.

[19] A. H. Rhemtulla, Polycyclic right-ordered groups. Algebra, Carbondale 1980 (Proc. Conf., Southern Illinois Univ., Carbondale, Ill., 1980), pp. 230–234, Lecture Notes in Math., 848, Springer, Berlin, 1981.

[20] D.J.S. Robinson, A course in the theory of groups. Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.

[21] D. Witte, Amenable groups that act on the line. Algebr. Geom. Topol. 6 (2006), 2509–2518.

[22] D. Witte, Arithmetic groups of higher Q-rank cannot act on 1-manifolds. Proc. Am. Math. Soc. 122 (1994), 333–340.

Department of Mathematics, Suzhou University, Suzhou 215006, China
wangsuhuasz@yahoo.com.cn

Department of Mathematics, Suzhou University, Suzhou 215006, China
ehshi@yahoo.cn, ehshi@suda.edu.cn

Department of Mathematics, Suzhou University, Suzhou 215006, China
zhoulizhen@suda.edu.cn

Department of Mathematics, La Trobe University, Victoria 3086, Australia
G.Cairns@latrobe.edu.au