IMPARTIAL GAMES AS ACYCLIC DIGRAPHS

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ABSTRACT. This is a survey of impartial combinatorial games, seen from the perspective of finite acyclic directed graphs. This leads to the introduction of two notions that are natural in this setting: *simple* games and *bipartite* games. We give results on the number of these games. The main results on the paper concern subtraction games, where interesting families, called ultimately bipartite games, are examined. The basic theorems of the theory of nim values are established, emphasizing the order properties of the group. The paper offers an unconventional perspective on nim values, and it concludes with a review of several families of classical games.

1. INTRODUCTION

Combinatorial, two-player games are usually "defined" by an informal description, such as:

(a) The game has a finite number of possible *positions*, and a given *starting position*.
(b) The players take turns to *move*; i.e., choose one of the allowable *moves* from the current position to a new position, the set of allowable moves being given in the *rules of the game*.
(c) Each player has complete information about the rules of the game and the state of the game at their turn, and there is no element of luck (no rolling of dice, etc).
(d) The game is finite (i.e., the rules of the game are such that the same position can't be visited more than once).
(e) The game stops when a player can't play (i.e., has no possible move), and the winner is the last player to have played (in particular, these games have no draws or stalemates).

The word "combinatorial" is used to distinquish these games from games like poker, where there is an element of chance, and/or the players play simultaneously. There is a large literature on combinatorial games (see [19]), and an even larger one on the other sort of games. We will drop the adjective "combinatorial" in these notes. Furthermore, we restrict ourselves to games that are *impartial*; i.e., the rules are identical for each player. (For example, chess is not impartial; white isn't permitted to move any piece, but may only move white pieces). In Section 9 we will recall some standard impartial games.

Notice that the common use of the word "game" in English is readily prone to possible confusion. For example, we talk about "the game of chess", by which we mean the rules of chess, but we also talk about a particular "game" of chess played between two individuals, by which we mean the actual sequence of moves the players made when they played against each other. Also, when studying games, one is often led to argue by induction, so it is often

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useful to view each position as the start of a new game. So one might think of the positions as games themselves.

The approach we use in these notes is to adopt the following formal definition in terms of digraphs; recall that a digraph is a directed graph, without loops or multiple edges, and a digraph is acyclic if it contains no directed closed circuit.

Definition 1. An *impartial game* G is a finite acyclic digraph with precisely one source and one sink. The vertices of the digraph are the *positions* of the game, and the directed edges issuing from a position p are the possible *moves* from p.

The assumption that there is only one source (i.e., vertex with no incoming edges) means that this source is the (unique) starting position for the game. The assumption that there is only one sink (i.e., vertex with no outgoing edges) is just for convenience. The sink is the terminal, losing position; if a game were to have more than one sink, we could coalesce the sinks into a single vertex without effectively changing the game as far as the players would experience it. Nevertheless, our assumptions concerning the source and sink do have implications for the resulting theory; see Remark 5.

The notion of an impartial game as a digraph is an old idea, and would be of no surprise to anyone working in combinatorial game theory; see [5]. The restriction to finite acyclic graphs is one of convenience; most popular games are of this kind. It seems curious to us that this approach is not more widely pursued in the literature. Let us say straight away that this doesn't reduce the study of impartial games to the study of the category of finite acyclic digraphs! We'll explain why in the next section.

2. The Category of Impartial Games

The critical notion in combinatorial game theory is that of a *winning strategy.* At some position in the unfolding of a game, a player has a winning strategy if he or she has a plan for playing that is certain to win from the current position, regardless of what moves the other player makes on all subsequent moves. One says that a position is a *winning position* for a player if there is a winning strategy for that player, starting in that position. It is a basic result of game theory that if a position isn't a winning position then it is a *losing position*, in that there is no way the player can avoid defeat if the other player plays intelligently. This is sometimes expressed in the following terms: for every impartial game, either the first player has a winning strategy, or the second player has a winning strategy. In the literature, losing positions are often called $\mathcal{P}$ positions, and winning positions are called $\mathcal{N}$ positions. Here $\mathcal{N}$ stands for "player to move $\mathcal{N}$ext wins" and $\mathcal{P}$ stands for "$\mathcal{P}$revious player wins". It is a mystery to us why the more natural $\mathcal{W}$ for winning position and $\mathcal{L}$ for losing position was not adopted.

Winning and losing positions are more clearly defined, and very easy to understand, in terms of digraphs. For this we introduce the following central notions.

Definition 2. In an impartial game G, the *height* $h(p)$ of a position p is the length of the longest directed path from p to the sink. The *height* $h(G)$ of G is the height of its source.

Definition 3. In an impartial game G, the *value* of a position p is either 0 or 1 and is defined by induction on the height as follows:

(a) the sink has value 0,

(b) a position p of height $h > 0$ has value 1 if and only if there is a move from p to a position with value 0.

A position p is called a *winning* (resp. *losing*) *position* if it has value 1 (resp. 0). The *value* of G is the value of its source, and we say that G is a *first player win* (resp. *second player win*) if G has value 1 (resp. 0).

Remark 1. The value function can be interpreted in terms of Boolean algebra. Recall that in Boolean algebra one has the sums $0 \vee 0 = 0, 0 \vee 1 = 1, 1 \vee 1 = 1$, and the negation function $\tilde{0} = 1, \tilde{1} = 0$. Then the value of a position p is the Boolean sum of the negation of the values of the positions one can reach from p in a single move.

Remark 2. The above digraph formulation of impartial games should not be confused with the notion of a *rooted game tree*. The latter is the tree of all ways a given game may be played; i.e., it is the tree of all possible paths from the source to the sink.

When playing the game against an opponent, if it is your turn to play and happily you find yourself in a winning position (i.e., a position of value 1), then you should make a move that puts your opponent into a losing position (i.e., a position of value 0). Such a move exists by the definition of value; in the literature, this is sometimes called a *strategic move.* Then regardless of which move your opponent makes you will again be in a position of value 1 when it your next turn to play. Continuing in this way, eventually your opponent will find themselves in the sink and be unable to make a move.

If it is your turn to play and sadly you find yourself in a losing position (i.e., a position of value 0), then by definition, regardless of what move you make, your opponent will be put into a winning position.

Thus, from the perspective of the players, the *complete solution* of an impartial game G is just the determination of the value function of G. Indeed, at any given position, the value function tells the players what the strategic moves are, provided there are any. As we recall below in Section 7, game theorists often like to have even more than the determination of the value function.

The following reformulation is often useful. It may be regarded as the *fundamental lemma* of the theory of impartial games.

Lemma 1. *Suppose that the vertex set of an impartial game G is the disjoint union of two sets A, B such that*

(a) *for every vertex p in A, all the moves from p terminate in B,*
(b) *for every vertex p in B, there is a move from p that terminates in A.*

Then A is the set of losing positions, and B is the set of winning positions.

Proof. Suppose that A and B are as in the statement of the lemma. Since there are no moves issuing from the sink, the sink belongs to A. The proof can then be established by induction on the height of the vertices. ☐

The following definition distinguishes the study of impartial games from that of acylcic digraphs.

Definition 4. For impartial games G, H with respective value functions v_G, v_H, a *game homomorphism* is a digraph homomorphism $f : G \to H$ that sends sink to sink and respects the value functions; i.e., $v_H(f(p)) = v_G(p)$ for all $p \in G$.

Naturally, the *category of impartial games* is the category having impartial games as its objects and game homomorphisms as its morphisms.

Remark 3. For clarity we mention that in the above definition, a digraph homomorphism is a map on both the vertex and edge sets that respects adjacency. Injectivity and surjectivity of homomorphism are to be understood in the graph theoretic sense, i.e., respectively injective or surjective on both vertices and edges. Acyclic digraphs don't have loops, so a digraph homomorphism $f : G \to H$ between acyclic digraphs cannot have $f(p) = f(q)$ if there is a move between positions p and q. In particular, acyclic digraph homomorphisms preserve the length of paths. So if $f : G \to H$ is a digraph homomorphism that sends sink to sink, then f doesn't decrease height; i.e., $h(f(p)) \geq h(p)$ for all $p \in G$. If f is a surjective digraph homomorphism that sends sink to sink, then f does preserve height. However, in the nonsurjective case, f may not preserve height, even if f is a game homomorphism; see Figure 1. In this figure and all the diagrams in these notes, the moves are directed downwards, and the positions are drawn at their appropriate heights. Positions with value 0 (resp. 1) are represented by the circ symbol $\circ$ (resp. bullet symbol $\bullet$).

Figure 2 (resp. 3) gives an example of a height preserving, injective (resp. surjective) digraph homomorphism that isn't a game homomorphism. For impartial games G, H, a *game isomorphism* is a digraph isomorphism $f : G \to H$. Obviously a game isomorphism is a game homomorphism and its inverse is a game homomorphism.

Definition 5. A *subgame* of an impartial game H is a subdigraph G for which the natural inclusion $f : G \to H$ is a game homomorphism. If p is a position in an impartial game G, the *restriction* of G to p is the maximal subgame of G that has p as its source. So the restriction of G to p is the union of the paths from p to the sink.

3. Simple Impartial Games

In the same way that a group is simple if it has no nontrivial quotients, we pose:

Definition 6. An impartial game G is *simple* if every surjective game homomorphism $f : G \to H$ is an isomorphism.

This notion of a simple game is not to be confused with that of a *simple graph*, which is just a graph with no loops or multiple edges.

Notice that for every impartial game G, there is a simple impartial game H and a surjective game homomorphism $f : G \to H$. Indeed, this is clear by induction on the number of positions, since if G is not already simple there is a surjective game homomorphism $f : G \to H$ that isn't an isomorphism and hence H has fewer positions than G.

Hence the problem of determining the value function of impartial games is reduced to the determination of the value function of simple impartial games. Indeed, if $f : G \to H$ is a surjective game homomorphism, where H is simple, then the value function v_G is just the pull-back of the value function on H; i.e., $v_G(p) = v_H(f(p))$ for all $p \in G$.

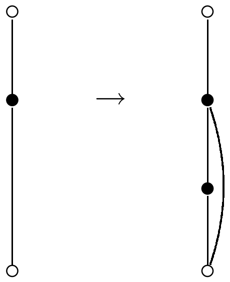


Figure 1. A game homomorphism that doesn't preserve height.

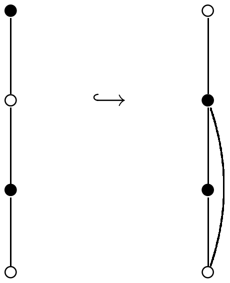


Figure 2. An injective digraph homomorphism that isn't a game homomorphism.

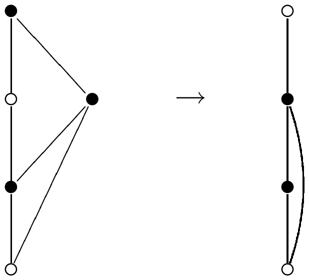
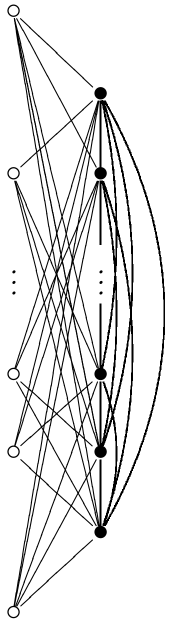
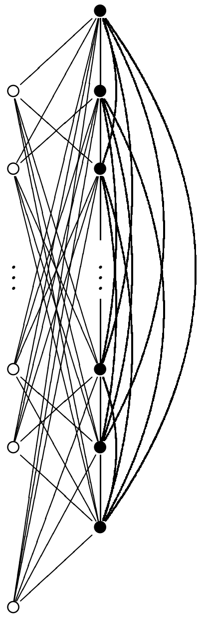


Figure 3. A surjective digraph homomorphism that isn't a game homomorphism.

Theorem 1. *Every simple impartial game G of height h is isomorphic to a subgame of one of the following two games U_0^h, U_1^h of the same height h:*


U_0^h


U_1^h

The digraphs U_0^h, U_1^h are explicitly described as follows: let $Z_m^0 = \{l_0, l_2, l_3, \dots, l_{m-2}, l_m\}$, $Z_m^1 = \{l_0, l_2, l_3, \dots, l_m\}$ and let K_n be the complete graph on the vertex set $\{w_1, w_2, \dots, w_n\}$. We make K_n a directed graph by orienting the edge from w_i to w_j whenever $i > j$. The digraph U_0^h (resp. U_1^h) is the digraph formed from the disjoint union of Z_h^0 and K_{h-1} (resp. Z_{h-1}^1 and K_h) by adding additional edges as follows: for all $i > j$ we add an edge from l_i to w_j and an edge from w_i to l_j, whenever these vertices exist.

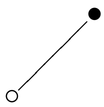
Proof. If G has value 0 (resp. 1), consider the unique map $f : G \to U_0^h$ (resp. $f : G \to U_1^h$) that preserves both height and value. This is clearly a game homomorphism. Let $\hat{f}$ denote the map defined by f from G onto the image of f. So $\hat{f}$ is a surjection and hence, as G is simple, $\hat{f}$ is an isomorphism; i.e., f is an injection. $\square$

Corollary 1. *An impartial game G of height h is a first player win if and only if there is a surjective game homomorphism from G to a subgame of U_1^h.*

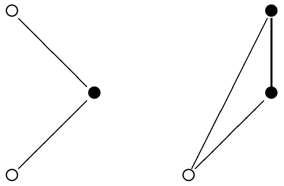
Corollary 2. *Here is the complete list of simple impartial games of height $h \leq 4$. In the diagrams for the different cases in height 3 and 4, we use the following convention:*

- *The edges of continuous lines $-$ are compulsory.*
- *The edges of dotted lines $\dots$ are optional.*
- *The edges of dashed lines $- - -$ are compulsory for at least one.*

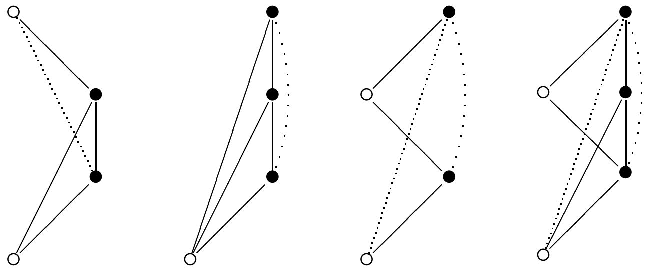
(a) *There is only one simple impartial game of height $h = 1$:*



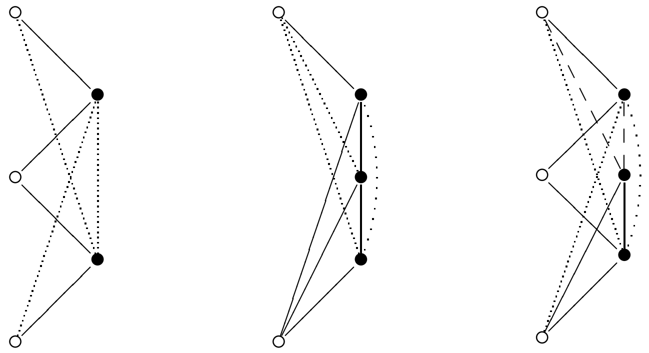
(b) *There are just two simple impartial games of height* $h = 2$*:*



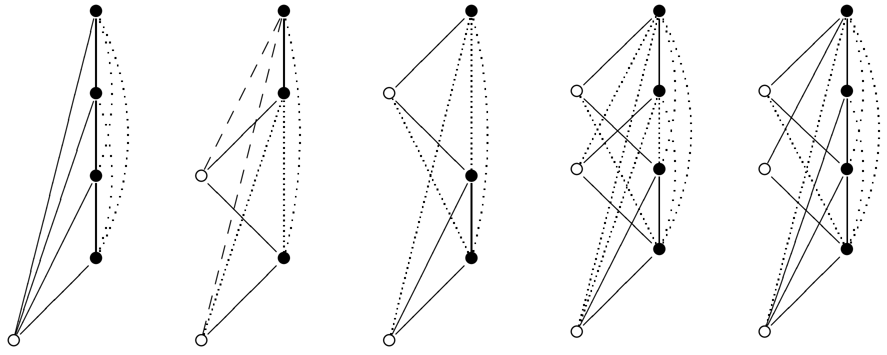
(c) *There are* $2 + 2 + 4 + 4 = 12$ *simple impartial games of height* $h = 3$*:*

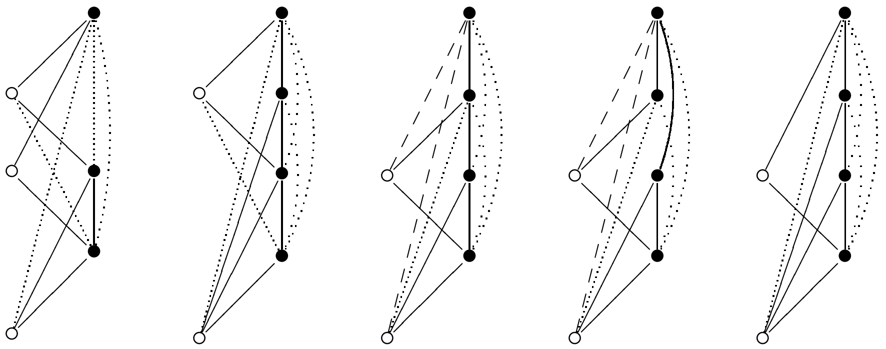


(d) *In total, there are* 512 *simple impartial games of height* $h = 4$*. There are* $2^3 + 2^3 + 3 \times 2^3 = 40$ *losing games:*



And there are $8 + 24 + 16 + 256 + 32 + 16 + 32 + 48 + 24 + 16 = 472$ *winning games:*





4. It really is an advantage to go first

It is well known that most impartial games are first player wins [26]. This is an old interest in game theory, dating back at least as far as Bouton's 1901/02 paper [9]. The purpose of this section is to show that most simple impartial games are first player wins. More precisely, we have:

Theorem 2. *Let* l_h *(resp.* w_h*) denote the number of losing (resp. winning) simple impartial games of height* h*. Then for* $h \geq 2$*,*

(a) $w_h \geq 2^{h-2} l_h$,
(b) $l_{h+1} \geq 2^{h/2-1} w_h$,
(c) $w_h \geq 2^{\frac{1}{4}(3h^2-11h+10)}$.

Before establishing this result we make some general observations. By Theorem 1, we need only count the simple subgames of the simple games U_0^h, U_1^h. Care needs to be taken however because not every subgame of a simple game is simple; Figure 4 shows a subgame of U_0^3 that isn't simple.

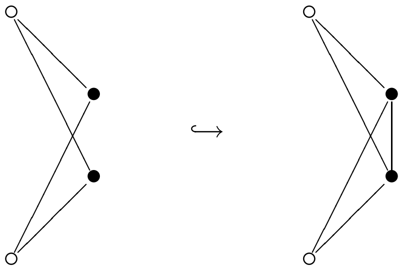


Figure 4. Not every subgame of a simple game is simple.

Notice that in all games, there are no losing positions of height 1. In a losing game of height $h > 1$, there are at least 2 losing positions, one of height 0 and one of height h, but there are no losing positions of height $h-1$, since otherwise they would be sources. So losing simple games have at most $h-1$ losing positions. In every game, from every losing position p of height $k > 0$, there is a move to a winning position of height $k-1$; indeed, otherwise, p wouldn't have height k. In particular, for simple games with i losing positions and j winning positions,

(a) in losing games, $j \geq i - 1$,

(b) in winning games, $j \geq i$.

In every game of height h, the source is connected to the sink by a path of length h. Thus, $i + j \geq h + 1$. So we have:

Lemma 2. *For simple games of height h with i losing positions and j winning positions,*

(a) *in losing games,* $j \geq \lfloor \frac{h}{2} \rfloor$,
(b) *in winning games,* $j \geq \lfloor \frac{h+1}{2} \rfloor$,

where $\lfloor x \rfloor$ *denotes the* integer part *of* x.

Proof of Theorem 2. To prove part (a), by Corollary 2, we may assume that $h \geq 3$. First notice that in a losing simple game G of height h with i losing positions and j winning positions, there is necessarily a winning position p_1 of height $h-1$, but there may or may not be a winning position of height $h-2$. One can turn the game G into a winning game by converting its source into a winning position. This can be achieved by performing the following:

(a) add a position p_2 and a move from the source to p_2, and make p_2 into a losing position of height $h-1$ as follows:
  (i) if there is a winning position p_3 of height $h-2$, add a move from p_2 to p_3, and add moves from p_2 to some subset of the other $j-2$ winning positions. This can be done in 2^{j-2} ways.
  (ii) if there isn't a winning position of height $h-2$, make one by adding a position p_3 and a move from p_3 to any position p_4 of height $h-3$, and if p_4 isn't a losing position, also add a move from p_3 to the sink. Now add a move from p_2 to p_3. Further, add moves from p_2 to some subset of the other $j-1$ winning positions. This can be done in 2^{j-1} ways.
(b) add moves from the source to some nonempty subset of the losing positions other than p_2; this can be done in 2^{i-1} ways.

From the losing simple game G, the above procedure produces at least $2^{j-2}2^{i-1} = 2^{i+j-3}$ distinct simple winning games. As we have already observed, $i + j \geq h + 1$. Hence G determines at least 2^{h-2} distinct simple winning games. Furthermore, no winning simple game can be contructed from 2 distinct losing games, using the above procedure. Hence $w_h \geq 2^{h-2} l_h$ as claimed.

Given a winning game G of height h, losing games of height $h+1$ can be constructed as follows: add a position p_1 and a move from p_1 to the source p_2 of G. Then add moves from p_1 to some subset of the other winning positions of G. From Lemma 2(b), there are at least $\lfloor \frac{h-1}{2} \rfloor$ ways of doing this. Simplifying gives (b).

The proof of (c) is by induction; from Corollary 2, it is true for $h = 2$. From (a) and (b) we have $w_{h+1} \geq 2^{3h/2-2} w_h$. So by the inductive hypothesis,

$$w_{h+1} \geq 2^{3h/2-2} 2^{\frac{1}{4}(3h^2-11h+10)} = 2^{\frac{1}{4}(3h^2-5h+2)} = 2^{\frac{1}{4}(3(h+1)^2-11(h+1)+10)}.$$

This establishes (c) and completes the proof of the theorem. □

Remark 4. The upshot of the above theorem is that the number of simple games, and the preponderance of winning simple games, both grow very rapidly with the height. At present, we do not know a recurrence relation for any of the sequences w_h, l_h or $w_h + l_h$.

5. Bipartite Games

Recall that a graph is bipartite if the vertex set can be written as a disjoint union of two subsets (the "parts") such that every edge connects vertices in different parts. It is well known and easy to see that for connected bipartite graphs, the parts are unique.

We introduce the following:

Definition 7. An impartial game G is *bipartite* if its underlying graph is bipartite.

Bipartite games are particularly easy to understand.

Proposition 1. *A bipartite game G is winning if and only if G has odd height.*

Proof. Note that if G has parts A, B, then the sets A, B satisfy the hypotheses of Lemma 1. So we have:

Lemma 3. *If an impartial game G is bipartite, then the two parts of G are the losings positions and the winning positions respectively.*

Thus there are no moves in G from a winning position to a winning position; every move is either from a winning position to a losing position, or from a losing position to a winning position. The path of maximal length from the source to the sink has length h, the height of G. So the source has value 1 if and only if h is odd. ☐

Example 1. In *Brussel sprouts*, one starts with an agreed number of crosses on the page. A move is played by connecting two arms of (possibly equal) crosses by a simple curve (avoiding all other crosses and previously drawn curves) and introducing a new cross by marking a bar across the middle of the curve. The players take turns to move and as usual, the loser is the first person who has no possible move. Figure 5 shows a game with one initial cross; it is a first player win. Invented by John H. Conway, Brussels Sprouts is something of a mathematical joke. A simple Euler characteristic calculation shows that for Brussels Sprouts played with n initial crosses, the first player wins if and only if n is odd, regardless of how the game is played [16, 4, 13]. Indeed, it is a simple exerice to show that Brussels Sprouts with n initial crosses is a bipartite game of height $5n - 2$.

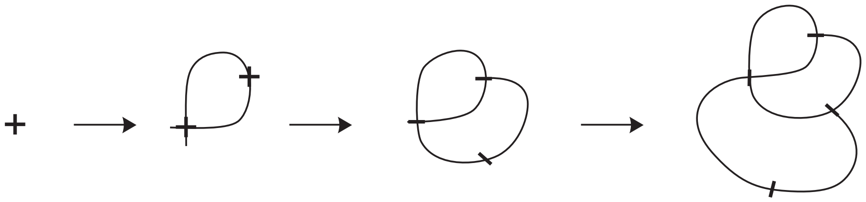


Figure 5. A game of Brussel sprouts

Bipartite games are essentially trivial from the player's perspective, as there is no strategy involved. Furthermore, it is easy to count the number of simple bipartite games.

Theorem 3. *There are precisely $2^{\lfloor \frac{h}{2} \rfloor \cdot \lfloor \frac{h-1}{2} \rfloor}$ simple bipartite games of height h.*

Proof. Let G be a simple bipartite game of height h and consider a path γ of length h from the source to the sink. By Lemma 3, the vertices on γ of even (resp. odd) height are losing (resp. winning) positions. Each position on γ of even (resp. odd) height is connected in G, to some subset of the positions of lower odd (resp. even) height. We claim that all the positions of G lie on γ; i.e., we claim that G has only one position of each height, from 0 to h (see Figure 6). Indeed, by Lemma 3, the source of G cannot be connected to both a winning and a losing position. So there can only be one position, p say, of height $h-1$. The restriction of G to p is a simple bipartite game of height $h-1$. Thus the claim follows by induction.

To count the number of simple bipartite games, let E_k (resp. O_k) denote the number of simple bipartite games of even (resp. odd) height $2k$ (resp. $2k+1$). One has $E_0 = O_0 = 1$ and from the above considerations, $O_k = 2^k E_k$ and $E_k = 2^{k-1} O_{k-1}$ for all $k \leq 1$. Hence $E_k = 2^{k^2-k}$ and $O_k = 2^{k^2}$. This establishes the required result.

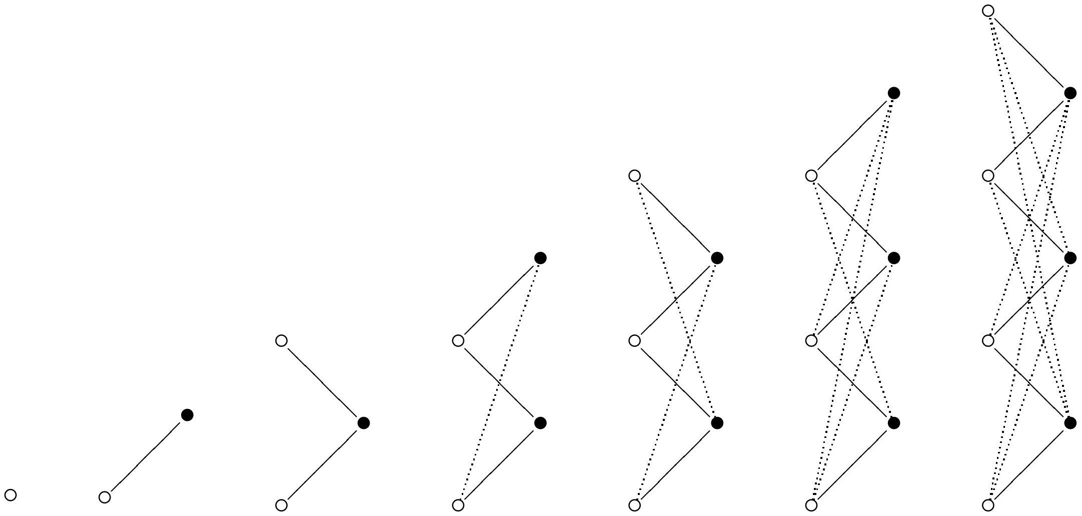


FIGURE 6. The simple bipartite games of height ≤ 6

□

Remark 5. It is well known that every graph G has a bipartite double cover $\hat{G}$, and that $\hat{G}$ is connected when G is not bipartite [3]. Note that the bipartite double cover of an impartial game is not an impartial game, since it has two sources and two sinks. Figure 7 shows an example of a nonbipartite game and its bipartite double cover.

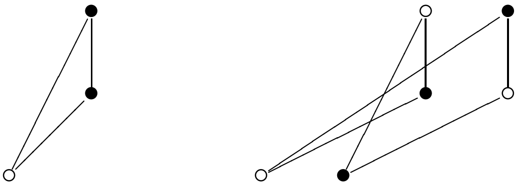


FIGURE 7. A game and its bipartite double cover

6. THE BASIC EXAMPLE: NIM

In a certain, somewhat restricted, sense to be explained below in Section 7, the game of Nim is the universal impartial game. Nim has a remarkable solution, due to Bouton [9], and the game is very widely known, perhaps due to its appearance in [25]. Nim is played from a start consisting of several piles (or heaps) of counters. On their turn, each player can remove as many counters as they like from a single pile. They can even remove a whole pile, but they can't take counters from two different piles in the same move. The winner is the player to remove the last counter.

We will denote by $N(i_1,\dots,i_n)$ the game of Nim that has n piles, with $i_1,\dots,i_n$ counters respectively. Clearly $N(i_1,\dots,i_n)$ has height $h = i_1 + \cdots + i_n$. In the game $N(h)$ with one pile, all the non-sink positions are winning positions. The graph of $N(h)$ is the complete graph K_{h+1} on $h+1$ vertices. The game $N(i_1,\dots,i_n)$ is the Cartesian product of the games $N(i_j)$ for $j = 1,\dots,n$. Figure 8 shows the simple game $N(6)$, and the game $N(3,3)$ and its surjection onto a simple game.

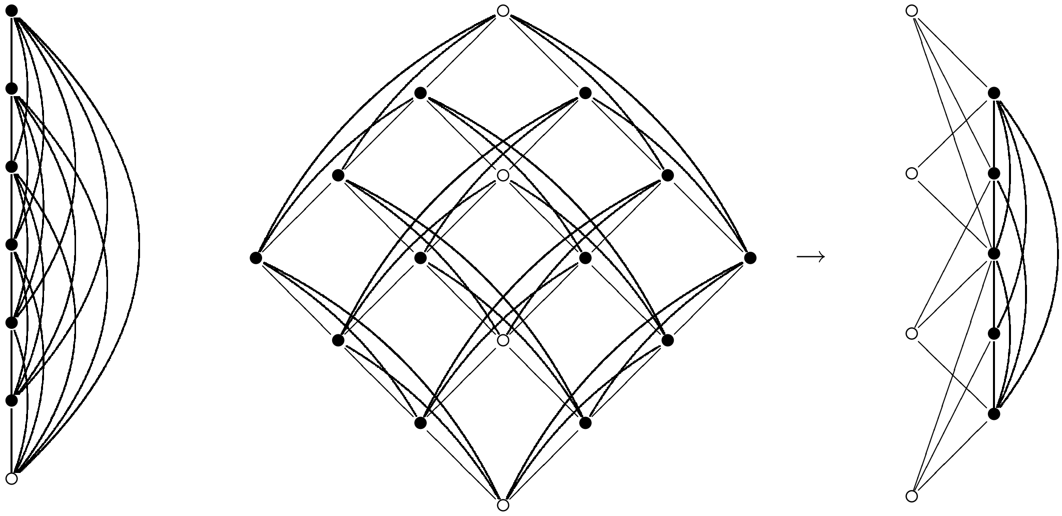


FIGURE 8. $N(6)$ and $N(3,3)$.

Consider the game $N(i,i)$. This is a second player win: the second player just copies the first player; whatever the first player does, the second player makes the same move in the other pile. Notice that after each move by the second player, the two piles have the same number of counters. In this way, regardless of how the first player plays, the second player is the last player to play, and so the second player wins. This is an example of the *Tweedledum-tweedledee strategy*, which is a basic idea in game theory.

7. DISJUNCTIVE SUMS AND THE NIM VALUE

The *disjunctive sum* $G_1 \oplus G_2$ of two impartial games G_1, G_2 is the Cartesian sum of the corresponding digraphs [14]; practically, $G_1 \oplus G_2$ can be played as follows: G_1 and G_2 are played side by side, and on each player's turn, a "move" is made by making a legal move in just one of the two component games G_1, G_2. This situation naturally arises in games where in the process of play, the game decomposes into distinct parts. Some games even begin in this form; notice that the Nim game $N(i_1,\dots,i_n)$ is the disjunctive sum of the n one-pile games $N(i_k)$. That is,

$$N(i_1,\dots,i_n) = N(i_1) \oplus \ldots \oplus N(i_n).$$

For impartial games G_1, G_2, it is not difficult to see that if G_1 is a losing game, then $G_1 \oplus G_2$ is a winning game if and only if G_2 is a winning game. However, if G_1 and G_2 are both winning games, one cannot say whether $G_1 \oplus G_2$ is a winning game without further information on G_1 and G_2.

Definition 8. Impartial games G_1, G_2 are said to be *equivalent* if $G_1 \oplus H$ is winning if and only if $G_2 \oplus H$ is winning, for all impartial games H.

There is a very satisfying general theory of disjunctive sums that extends Bouton's theory of Nim; this extension is due to Sprague [27] and Grundy [21]. First we recall a refinement of the notion of *value*:

Definition 9. In an impartial game G, the *nim value* of a position p, denoted $\mathcal{B}(p)$, is defined by induction on the height as follows:

(a) the sink has value 0
(b) a position p of height $h > 0$ has value i if and only if for each $0 \le j < i$ there is a move from p to a position with value j, but there is no move from p to a position with value i.

The nim value $\mathcal{B}(G)$ of G is the nim value of its source.

The following is a generalization of Lemma 1.

Lemma 4. *Suppose that the vertex set of an impartial game G is the disjoint union of sets $A_0, A_1, A_2, \ldots$ such that for each $i \ge 0$ and every vertex p in A_i,*

(a) *for each $j < i$ there is a move from p that terminates in A_j,*
(b) *no move from p terminates in A_i.*

Then A_i is the set of positions with nim value i.

Remark 6.

(a) The nim value is also known as the *Sprague-Grundy value.* However, it is arguably more appropriately termed the *Bouton value.* As this is not usual practice, we adopt the expression *nim value*, which is quite common.
(b) The nim value of the nim game $N(i)$ is i.
(c) The value of a position p in an impartial game is zero if and only if its nim value is zero.
(d) The nim value is often expressed in terms of the *minimal excluded value*, denoted mex; given a set A of non-negative integers, $\text{mex}(A)$ is the smallest non-negative integer that is not contained in A. The nim value of a position p is the mex of the nim values of the positions one can reach from p in a single move.
(e) The impartial games that have no positions with nim value greater than 1 are precisely the bipartite games.
(f) Obviously, game isomorphisms preserve nim values. However, surjective game homomorphisms don't necessarily preserve nim values; see Figure 9.

The set of all possible nim values is the set $\mathcal{N}$ of non-negative integers, equipped with its natural order. The key idea in the Sprague-Grundy theory is to use the group structure on $\mathcal{N}$ introduced for nim by Bouton: one uses the binary expansion of the elements and defines the group operation $\oplus$ to be binary addition without carries: so, for example $1 \oplus 1 = 0$, $1 \oplus 2 = 3$ and $1 \oplus 3 = 1 \oplus (1 + 2) = 2$. This makes $(\mathcal{N}, \oplus)$ an Abelian group in which every non-zero

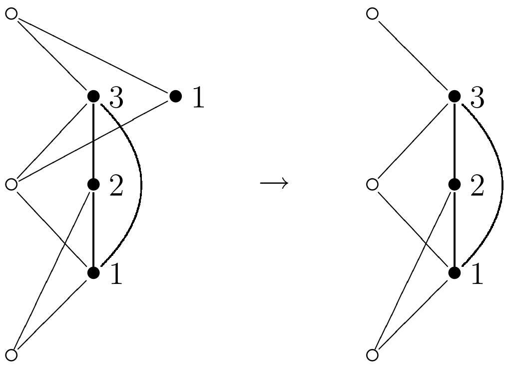


FIGURE 9. A surjective game homomorphism that doesn't preserve nim values.

element of $\mathcal{N}$ has order two; for each $i \geq 0$, the subgroup $\mathcal{N}_i = \{0, 1, \ldots, 2^i - 1\}$ is just the additive group of the Galois field of order 2^i. For information on the multiplicative structure, see [14, 22, 29]. The group operation $\oplus$ is variously designated in the literature as $+_2, \overset{*}{+}$ and in Grundy's paper, $\underset{N}{+}$. Of course, what is important is not just the group law, but its relation to the natural order on $\mathcal{N}$. The key property of $(\mathcal{N}, \oplus)$ is described in the following lemma.

Lemma 5. *For all $x, y, z \in \mathcal{N}$, if $z < x \oplus y$, then precisely one of the following conditions hold: either $x \oplus z < y$ or $y \oplus z < x$.*

Remark 7. The property in this lemma can be expressed in the following useful form: for all $x, y \in \mathcal{N}$ and all $z < x \oplus y$, precisely one of the following hold:

(a) either there exists $x' < x$ such that $z = x' \oplus y$,
(b) or there exists $y' < y$ such that $z = x \oplus y'$.

Indeed, if $x, y, z \in \mathcal{N}$ and $z < x \oplus y$, then from the property in this lemma, either $x \oplus z < y$ or $y \oplus z < x$. In the first case, let $y' = x \oplus z$. Then $y' < y$ and $x \oplus y' = x \oplus x \oplus z = z$. Conversely, if there exists $x' < x$ such that $z = x' \oplus y$, then $z \oplus y = x' < x$. The other case is similar.

Remark 8. Notice that the set $\mathcal{N}_k = \{0, 1, 2, \ldots, 2^k - 1\}$ is a subgroup of $\mathcal{N}$; its elements are the numbers whose binary expansions are of length $\leq k$. Obviously, the natural order on $\mathcal{N}_k$ still verifies the property of lemma 5.

Outline of the Proof of Lemma 5. The proof is attained by induction on i for $x, y, z \in \mathcal{N}_i$. One first verifies the claim for $\mathcal{N}_0$ and $\mathcal{N}_1 = \{0, 1\}$ and then uses the fact that the group $\mathcal{N}_i$ is an index 2 subgroup of $\mathcal{N}_{i+1}$ with quotient isomorphic to $\mathcal{N}_1$; the quotient map $\pi : \mathcal{N}_{i+1} \to \mathcal{N}_1$ being just the map that records the most significant digit in the binary expansion. $\square$

Theorem 4. *If a group G has a total order satisfying the property of Lemma 5, then G is Abelian, the identity element of G is a minimal element and all the nonidentity elements have order two. Furthermore, if G is finite, there is an order preserving group isomorphism between G and one of the groups $\mathcal{N}_k$.*

Proof. Let $(G, *)$ be a group possessing a total order satisfying the property of Lemma 5, and let e denote its identity element. Let z be any nonidentity element. Suppose that $z < e$.

Then $z < x \oplus y$ for $x = y = e$. But by hypothesis, only one of the conditions $x \oplus z < y$ or $y \oplus z < x$ can hold, and this is clearly impossible. Hence e is a minimal element. In the same way, $e < z \oplus z$ is impossible. So $z \oplus z = e$. Hence all nonidentity elements have order two, and consequently G is Abelian.

Now suppose that G is finite; so G is isomorphic to $\mathbb{Z}_2^k$ for some k; by relabelling the members of G if necessary, we may assume that $G = \{0, 1, 2, \dots, 2^k - 1\}$, with the natural order relation, so that 0 is the identity element. We will show that G is isomorphic to $\mathcal{N}_k$. The group $\mathcal{N}_k$ has generators 2^i, for $0 \leq i < k$. So it suffices to show that every word in these generators in $\mathcal{N}_k$ evaluates to the same element when computed in G. Since both G and $\mathcal{N}_k$ are Abelian and their nonidentity elements have order 2, it suffices to consider words with distinct letters. That is, it remains to show that for sums of distinct powers of 2 one has

$$2^{i_1} * \cdots * 2^{i_n} = 2^{i_1} \oplus \ldots \oplus 2^{i_n},$$

where $i_1 < \cdots < i_n$. Note that $2^{i_1} \oplus \ldots \oplus 2^{i_n} = 2^{i_1} + \cdots + 2^{i_n}$. So by induction, it suffices to show that $2^{i_1} + \cdots + 2^{i_n} = (2^{i_1} + \cdots + 2^{i_{n-1}}) * 2^{i_n}$. Thus it suffices to show that

$$x * 2^i = x + 2^i,$$

for all $0 \leq i < k$ and all $x < 2^i$. We establish this for each i by induction on x. The claim is true for $x = 0$, as 0 is the group identity.

We first show that for all $a, b \in G$, one has $a * b \leq a + b$, which we establish by induction on $a + b$. It is obvious for $a = b = 0$. Then in general, by Remark 7, for all $c < a * b$ either there exists $a' < a$ such that $c = a' * b$, or there exists $b' < b$ such that $c = a * b'$. By the inductive hypothesis, $a' * b \leq a' + b < a + b$ and $a * b' \leq a + b' < a + b$. So for all $c < a * b$ we have $c < a + b$. Hence $a * b \leq a + b$, as required.

We have just established that $x * 2^i \leq x + 2^i$. Our goal is to show that $x * 2^i \not< x + 2^i$. By the inductive hypothesis, $y * 2^i = y + 2^i$ for all $y < x$. Hence, because of the group structure, $x * 2^i \neq y + 2^i$ for any $y < x$. It remains to show that $x * 2^i \not< 2^i$.

Lemma 6. *For all $i \geq 0$, the set $G_i = \{0, 1, \dots, 2^i - 1\}$ is a subgroup of $(G, *)$.*

Proof. The proof is by induction on i and is obvious for $i = 0$. Suppose that G_i is a subgroup and let $y \in G_i$. Then as $2^i \notin G_i$, one has $y * 2^i \notin G_i$. Further, as established above, $y * 2^i \leq y + 2^i < 2^{i+1}$. So $y * 2^i \in G_{i+1} \backslash G_i$. As y varies across the 2^i elements of G_i, the element $y * 2^i$ varies across the 2^i elements of $G_{i+1} \backslash G_i$. Thus every element of $G_{i+1} \backslash G_i$ has the form $y * 2^i$ for some $y \in G_i$. Note that for all $y, y' \in G_i$,

$$(y * 2^i) * (y' * 2^i) = y * y' \in G_i \quad \text{and} \quad (y * 2^i) * y' = (y * y') * 2^i \in G_{i+1} \backslash G_i.$$

Thus G_{i+1} is closed under $*$ and is hence a subgroup. □

Returning to the proof of the theorem, note that if one had $x < 2^i$ and $x * 2^i < 2^i$, then one would have $x, x * 2^i \in G_i$ while $2^i \notin G_i$, contradicting the above lemma. This concludes the proof of the theorem. □

The following classical result is commonly regarded as the fundamental theorem of combinatorial game theory.

Theorem 5. *For all impartial games G_1, G_2, one has $\mathcal{B}(G_1 \oplus G_2) = \mathcal{B}(G_1) \oplus \mathcal{B}(G_2)$.*

Proof. Consider a position $(p_1, p_2) \in G_1 \oplus G_2$. For each $i \geq 0$, consider the set

$$A_i = \{(p_1, p_2) \in G_1 \oplus G_2 : \mathcal{B}(p_1) \oplus \mathcal{B}(p_2) = i\}.$$

Consider a position $(p_1, p_2) \in A_i$. There are two kinds of moves from (p_1, p_2); those of the form $(p_1, p_2) \to (p_1', p_2)$, where $p_1 \to p_1'$ is a move in G_1, and those of the form $(p_1, p_2) \to (p_1, p_2')$, where $p_2 \to p_2'$ is a move in G_2. By symmetry, it suffices to consider moves of the first kind. Suppose that $\mathcal{B}(p_1') \oplus \mathcal{B}(p_2) = i$; adding $\mathcal{B}(p_1)$ to both sides gives $\mathcal{B}(p_1') \oplus i = \mathcal{B}(p_1) \oplus i$ and so $\mathcal{B}(p_1') = \mathcal{B}(p_1)$, which is impossible. Thus the move $(p_1, p_2) \to (p_1', p_2)$ does not terminate in A_i.

Let $j < i$. By Lemma 4, it remains to show that there is a move from (p_1, p_2) that terminates in A_j. By Remark 7, either there exists $a < \mathcal{B}(p_1)$ such that $a \oplus \mathcal{B}(p_2) = j$, or there exists $b < \mathcal{B}(p_2)$ such that $\mathcal{B}(p_1) \oplus b = j$. In the first case there is a position p_1' in G with $\mathcal{B}(p_1') = a$ and in the second case there is a position p_2' in G with $\mathcal{B}(p_2') = b$. The move $(p_1, p_2) \to (p_1', p_2)$ or $(p_1, p_2) \to (p_1, p_2')$ has the required property. □

This clarifies the notion of equivalent games:

Corollary 3.

(a) *Impartial games G_1, G_2 are equivalent if and only if $\mathcal{B}(G_1 \oplus H) = \mathcal{B}(G_2 \oplus H)$ for all impartial games H.*

(b) *Impartial games G_1, G_2 are equivalent if and only if $\mathcal{B}(G_1) = \mathcal{B}(G_2)$.*

(c) *Every impartial game is equivalent to a game of nim.*

Proof. (b) First note that for all impartial games G, the game $G \oplus G$ is losing. So if games G_1, G_2 are equivalent, then $G_1 \oplus G_2$ is losing. Then $B(G_1 \oplus G_2) = 0$, and by Theorem 5, $B(G_1) \oplus B(G_2) = 0$. Hence $B(G_1) = B(G_2)$. Conversely, if $B(G_1) = B(G_2)$, then $B(G_1) = 0$ if and only if $B(G_2) = 0$, and so G_1 is losing if and only if G_2 is losing.

(a) follows from (b), by Theorem 5.

(c) As we have remarked above, $\mathcal{B}(N(i)) = i$. So for every impartial game G, one has $\mathcal{B}(N(\mathcal{B}(G))) = \mathcal{B}(G)$ and thus by (b), G and $N(\mathcal{B}(G))$ are equivalent. □

8. Subtraction games

A *subtraction game* is a game of nim in which there is an additional rule restricting how many counters one can remove from any pile; a *subtraction set* $S = \{s_1, s_2, s_3, \dots\}$ is given, and on each move the number of counters removed must belong to S. For example, when S is the set of odd positive integers, one can't remove an even number of counters, on any move. We will focus on the game with a single pile; let $\mathcal{S}(n, S)$ denote the subtraction game commencing with a single pile of n counters and subtraction set S. The game is said to be *finite* if S is finite. The game $\mathcal{S}(6, \{1, 2\})$ is shown in Figure 10; the labels indicate the number of counters for the corresponding position.

Before proceeding, we need to clarify what the positions in $\mathcal{S}(n, S)$ are. Consider the game $\mathcal{S}(7, \{2, 3\})$. The piles with 0 and 1 counters are both terminal situations; so we coalesce these into a single sink. Notice also that the heap with 6 counters can not be reached in this game; so this isn't a position. The game $\mathcal{S}(7, \{2, 3\})$ is shown in Figure 11. In general, if $s = \min(S)$, the piles with $0, 1, 2, \dots, s-1$ counters are all terminal situations; so they are coalesced into a single sink. The heaps with $n-s+1, n-s+2, \dots, n-1$ counters can not be reached in this game, and so they are not positions. There may be other heap sizes which are not positions. For example, if all the elements of S are multiples of some number

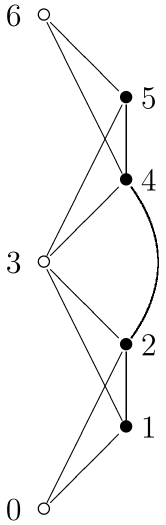


FIGURE 10. The game $\mathcal{S}(6, \{1, 2\})$.

k, then only the heap sizes that are congruent to n modulo k are positions, since these are the only positions that can be attained from the source. The following proposition shows that we may restrict our attention to games with $\gcd(S) = 1$.

Proposition 2. *The game* $\mathcal{S}(n, S)$ *is isomorphic to the game* $\mathcal{S}(\left\lfloor \frac{n}{\gcd(S)} \right\rfloor, S/\gcd(S))$.

Proof. Let $s = \gcd(S)$. The positions in $\mathcal{S}(n, S)$ all correspond to heap sizes that are congruent to n modulo s. Consider the map f from $\mathcal{S}(n, S)$ to $\mathcal{S}(\left\lfloor \frac{n}{s} \right\rfloor, S/s)$ which sends the heap of size m to the heap of size $\left\lfloor \frac{m}{s} \right\rfloor$. The map f is clearly a game isomomorphism. □

Notice that for a fixed subtraction set S, the game $\mathcal{S}(m, S)$ is the restriction of $\mathcal{S}(n, S)$ to the position with m counters, for all $n \geq m$ for which $\mathcal{S}(n, S)$ contains the position with m counters. Even when $\gcd(S) = 1$, there will usually be heap sizes that are not positions in $\mathcal{S}(n, S)$. Nevertheless, we do have:

Proposition 3. *Suppose that* $S = \{s_1, \ldots, s_k\}$ *where* $s_1 \leq \cdots \leq s_k$ *and* $\gcd(S) = 1$. *Then the pile with* k *counters is a position in the game* $\mathcal{S}(n, S)$ *for all* $n \geq k + (s_1 - 1)(s_k - 1)$.

This proposition is an immediate consequence of the following theorem of Brauer, which deals with the "coin exchange problem" of Frobenius. In the case $k = 2$, the bound is exact; this was due to Sylvester. For $k = 3$, see [24, 1, 2].

Theorem 6. [10]. *Given a finite set* $S = \{s_1, \ldots, s_k\}$ *of positive integers with* $s_1 \leq \cdots \leq s_k$ *and* $\gcd(S) = 1$, *every integer* $x \geq (s_1 - 1)(s_k - 1)$ *can be written as a linear combination* $x = a_1 s_1 + \cdots + a_k s_k$, *where the coefficients* $a_1, \ldots, a_k$ *are non-negative integers.*

Theorem 7.

(a) $\mathcal{S}(n, S)$ *is simple for all* n *if and only if* $1 \in S$,

(b) $\mathcal{S}(n, S)$ *is bipartite for all* n *if and only if the elements of* S *are all odd multiples of* $\min(S)$. *Moreover, in this case,* $\mathcal{S}(n, S)$ *is losing if and only if* $\left\lfloor \frac{n}{\min(S)} \right\rfloor$ *is even.*

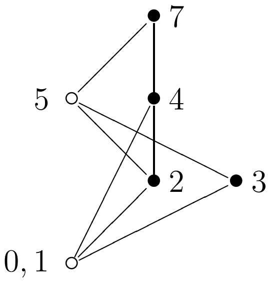


FIGURE 11. The game $S(7, \{2, 3\})$.

Proof. Let $s = \min(S)$. (a) If $1 \in S$, then the position with m counters has height m. Consequently, there is precisely one position at each height h for $0 \le h \le n$. So $\mathcal{S}(n, S)$ is simple. Conversely, if $1 \notin S$, the piles with $0, 1, 2, \dots, s-1$ counters are all terminal situations; so they are coalesced into a single sink. By Theorem 6, for sufficiently large n, the piles with s and $s+1$ counters are both positions in $\mathcal{S}(n, S)$. Moreover, they are both of value 1. So $\mathcal{S}(n, S)$ is not simple.

(b) First, we claim that when $1 \in S$ and the elements of S are all odd, $\mathcal{S}(n, S)$ is bipartite. Indeed, in this case, as we saw in (a), the game is simple and there is a position with m counters for all $0 \le m \le n$. Let A (resp. B) denote the set of positions having an even (resp. odd) number of counters. Each move starting in A ends in B, and visa-versa. So $\mathcal{S}(n, S)$ is bipartite. Moreover, $\mathcal{S}(n, S)$ is losing if and only if n is even.

Now suppose the elements of S are all odd multiples of $s = \min(S)$. Then only the heap sizes that are congruent to n modulo s are positions. Consider the map f from $\mathcal{S}(n, S)$ to $\mathcal{S}(\left\lfloor \frac{n}{s} \right\rfloor, S/s)$ which sends the heap of size m to the heap of size $\left\lfloor \frac{m}{s} \right\rfloor$. The map f is clearly a game isomomorphism. As we have just shown, $\mathcal{S}(\left\lfloor \frac{n}{s} \right\rfloor, S/s)$ is bipartite. Hence $\mathcal{S}(n, S)$ is bipartite. Moreover, $\mathcal{S}(n, S)$ is losing if and only if $\mathcal{S}(\left\lfloor \frac{n}{s} \right\rfloor, S/s)$ is losing and this occurs precisely when $\left\lfloor \frac{n}{s} \right\rfloor$ is even.

Conversely, suppose that $\mathcal{S}(n, S)$ is bipartite for all n. The piles with $0, 1, 2, \dots, s-1$ counters are coalesced into a single sink, of value 0. Then considering the removal of s counters, we have by induction that the pile with m counters has value 0 if and only if $\left\lfloor \frac{m}{s} \right\rfloor$ is even. The case with $s = 3$ is shown in Figure 12. Now suppose that S contains an element t and consider the piles with $t + i$ counters for $0 \le i < s$; these piles exist in $\mathcal{S}(n, S)$ for sufficiently large n, by Theorem 6. Removing t counters from these, we obtain positions with i counters, each of which has value 0. So the piles with $t + i$ counters all have value 1. Hence, from above, $\left\lfloor \frac{t+i}{s} \right\rfloor$ is odd for all $0 \le i < s$. Thus t is an odd multiple of s. ☐

Corollary 4. *The subtraction game $\mathcal{S}(n, \{s\})$ is losing if and only if $\left\lfloor \frac{n}{s} \right\rfloor$ is even.*

The following proposition is a reformulation of a description given without proof in [7, p. 530]. Here, $[x]_m$ denotes the reduction of x modulo m; i.e., $[x]_m = x - m. \lfloor x/m \rfloor$.

Proposition 4. *For $s_1 < s_2$, the subtraction game $\mathcal{S}(n, \{s_1, s_2\})$ is losing if and only if $\left\lfloor \frac{[n]_{s_1+s_2}}{s_1} \right\rfloor$ and $\left\lfloor \frac{[n]_{s_1+s_2}}{s_2} \right\rfloor$ are both even.*

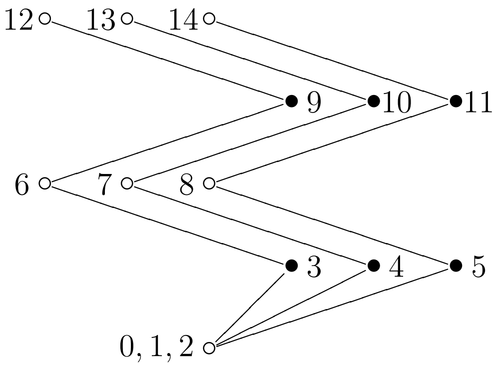


FIGURE 12. Part of $S(n, \{3\})$.

Proof. Let A be the set of n for which $\left\lfloor \frac{[n]_{s_1+s_2}}{s_1} \right\rfloor$ and $\left\lfloor \frac{[n]_{s_1+s_2}}{s_2} \right\rfloor$ are both even. The required result follows from Lemma 1, provided we show that:

(a) if $n \in A$, then $n - s_1$ and $n - s_2$ are both not in A,
(b) if $n \notin A$, then either $n - s_1$ or $n - s_2$ is in A.

For (a), if $n \in A$, then $\left\lfloor \frac{[n]_{s_1+s_2}}{s_1} \right\rfloor$ is even and so $\left\lfloor \frac{[n-s_1]_{s_1+s_2}}{s_1} \right\rfloor$ is odd, and similarly, $\left\lfloor \frac{[n]_{s_1+s_2}}{s_2} \right\rfloor$ is even and so $\left\lfloor \frac{[n-s_2]_{s_1+s_2}}{s_2} \right\rfloor$ is odd. Thus neither $n - s_1$ nor $n - s_2$ are in A.

For (b), for $n \notin A$, we consider two cases:

(i) If $\left\lfloor \frac{[n]_{s_1+s_2}}{s_1} \right\rfloor$ is odd and $\left\lfloor \frac{[n]_{s_1+s_2}}{s_2} \right\rfloor$ is even, then $s_1 \leq [n]_{s_1+s_2} < s_2$. So $0 \leq [n-s_1]_{s_1+s_2} < s_2$ and hence $\left\lfloor \frac{[n-s_1]_{s_1+s_2}}{s_2} \right\rfloor = 0$, which is even. Furthermore, as $\left\lfloor \frac{[n]_{s_1+s_2}}{s_1} \right\rfloor$ is odd, $\left\lfloor \frac{[n-s_1]_{s_1+s_2}}{s_1} \right\rfloor$ is even. So $n - s_1 \in A$.

(ii) If $\left\lfloor \frac{[n]_{s_1+s_2}}{s_2} \right\rfloor$ is odd, then $s_2 \leq [n]_{s_1+s_2} < s_1 + s_2$. So $0 \leq [n - s_2]_{s_1+s_2} < s_1$ and hence $\left\lfloor \frac{[n-s_2]_{s_1+s_2}}{s_2} \right\rfloor$ and $\left\lfloor \frac{[n-s_2]_{s_1+s_2}}{s_2} \right\rfloor$ are both even. So $n - s_2 \in A$.

In both cases, either $n - s_1$ or $n - s_2$ is in A. This completes the proof. □

One of the striking properties of the subtraction games $\mathcal{S}(n, \{s_1, s_2\})$ is that they are *periodic*. If we denote the value of $\mathcal{S}(n, \{s_1, s_2\})$ by $v(n)$, then by Proposition 4, $v(n)$ is periodic with period $s_1 + s_2$; i.e., $v(n + s_1 + s_2) = v(n)$ for all $n \geq 0$. At present, a complete solution is not known for subtraction games with 3 element subtraction sets $\{s_1, s_2, s_3\}$. Tables exhibiting some remarkable features are given in [6, Chap. 4]. As these authors remark: “There are obviously some new theorems waiting to be discovered” [6, p. 86]. One feature is that the games $\mathcal{S}(n, S)$ are not all periodic. Nevertheless, they are *ultimately periodic*; that is, there is an integer N and a *period* p such that $v(n+p) = v(n)$ for all $n \geq N$; see [7, p. 529]. Berlekamp–Conway–Guy observe that remarkably, in the tables provided in [6, p. 86], there is only one subtraction game that isn’t periodic. We exhibit infinite families of such games. In fact, the examples we give below in Theorem 8 are ultimately periodic in a very particular manner.

Definition 10. We say that a subtraction game with substraction set S is *ultimately bipartite* if there is an integer N such that for $n \geq N$, the sequence $v(n)$ is an alternating sequence of 0’s and 1’s.

Ultimately bipartite subtraction games are not uncommon. In the following result we give three infinite families of ultimately bipartite games, each family commencing with the game having subtraction set $\{3, 5, 9\}$.

Theorem 8. *The subtraction game* $\mathcal{S}(n, S_k)$ *is ultimately bipartite for each member of the following three families of subtractions sets:*

(a) $S_k = \{3, 5, 9, \dots, 2^k + 1\}$*, for all* $k \geq 3$*,*
(b) $S_k = \{3, 5, 2^k + 1\}$*, for all* $k \geq 3$*,*
(c) $S_k = \{k, k + 2, 2k + 3\}$*, for all odd* $k \geq 3$*.*

Proof. For each of the three given families of subtractions sets, and for k fixed, let $v(n)$ denote the value of the game $\mathcal{S}(n, S_k)$. In each case we will give a set A for which we claim that $v(n) = 0$ if and only if n belongs to A. This claim follows from Lemma 1, provided we show that:

(i) if $n \in A$, then $n - s \notin A$, for all $s \in S_k$,
(ii) if $n \notin A$, then $n - s_i \in A$, for some $s \in S_k$.

For the family (a), let $S_k = \{s_1, s_2, \dots, s_k\}$. Let $S_k + 1$ denote the set obtained by adding 1 to each of the members of S_k, and let $\mathbb{E}$ denote the set of non-negative even integers. We set:

$$A = \{1\} \cup \mathbb{E} \backslash (S_k + 1).$$

That is, $A = \{0, 1, 2, 8, 12, 14, 16, 20, \dots\}$. For condition (i), note that if n is odd and $n \in A$, then $n = 1$, and so for all i, we have $n - s_i < 0$ and so $n - s_i \notin A$. If n is even, then $n - s_i$ is odd and so $n - s_i \in A$ only if $n = s_i + 1$. But $s_i + 1 \notin A$. Thus $n \in A$ gives $n - s_i \notin A$ for all i.

For condition (ii), note that if $n \notin A$ and n is even and $n = s_i + 1$ for some i, so $n - s_i = 1 \in A$. On the other hand, if $n \notin A$ and n is odd, then $n \geq 3$. If $n - 3 \notin A$, then $n = s_i + 4$ for some i. But then $n - 5 = s_i - 1$. Thus $n - 5 \in A$ unless $s_i - 1 = s_j + 1$ for some j. Obviously this can only happen for $s_i = 5, s_j = 3$, and in this case $n = 9$. But then $n - 9 = 0 \in A$.

We have thus shown that A is the set of n for which $\mathcal{S}(n, S_k)$ has value zero. It remains to notice that for $n > 2^k + 2$, we have $v(n) = 0$ if and only if n is even, so the game $\mathcal{S}(n, S_k)$ is ultimately bipartite. This completes the proof for family (a).

For families (b) and (c) the proof is similar, but the details are more subtle. We content ourselves in exhibiting the sets A in each case, and we leave the verification of conditions (i) and (ii) to the reader. For the family (b), we work with congruences modulo 8. Notice that $2^k \equiv 0$ for all $k \geq 3$. Set

$$\begin{aligned}
D_1 &= \{i : i \equiv 4,\ 0 < i < 2^k\} \\
D_2 &= \{i : i \equiv 6,\ 0 < i < 2^k\} \\
C &= \{i : i \equiv 2,\ 2^k < i < 2^{k+1}\} \\
B &= \{i : i \equiv 1,\ 0 < i < 2^k\} \\
A &= B \cup \mathbb{E} \backslash (C \cup D_1 \cup D_2).
\end{aligned}$$

For the family (c), let $k = 2l + 1$ and set

$$\begin{aligned}
D_i &= \{j \in \mathbb{N} : j \text{ even}, k + 1 + i(3k + 5) \leq j \leq 2k + i(3k + 3)\}\\
C_i &= \{j \in \mathbb{N} : j \text{ odd}, 1 + i(3k + 5) \leq j \leq k - 2 + i(3k + 3)\}\\
D &= \bigcup_{i=0}^{l-1} (D_i \cup (D_{i+1} - (2k + 2)))\\
C &= \bigcup_{i=1}^{l-1} C_i\\
B &= \{j \in \mathbb{N} : 0 \leq j \leq k - 1\}\\
A &= B \cup C \cup \mathbb{E} \backslash D.
\end{aligned}$$

□

Notice that for the games in each of the families in the above theorem, for sufficiently large n, the game is winning if and only if n is odd. This is a rather curious situation: for n large, that is, for a large pile of counters, it is clear who has the winning position and furthermore, it doesn't matter how that player plays. However, as one plays the game and n eventually becomes small, it is no longer so easy to know what the strategic moves are. This is a general feature of all ultimately bipartite games: although it isn't obvious from the definition, there is no ultimately bipartite game such that, for sufficiently large n, the game is winning if and only if n is even.

Theorem 9. *If the subtraction game $\mathcal{S}(n, S)$ is ultimately bipartite, then for sufficiently large n, the game is winning if and only if n is odd.*

Proof. We employ the following:

Lemma 7. *Consider an arbitrary subtraction game $\mathcal{S}(n, S)$ for which the elements of the subtraction set S are all odd and greater than 2, and let $v(n)$ denote its value function. If n is odd and $v(n) = 0$, then $v(n - 1) = 0$.*

Proof. This result is best understood by imagining that one is playing the game. Suppose that one finds oneself in a losing position n where n is odd. Then, regardless of how one plays, the other player can always force a win, provided they play intelligently. In this case, the game terminates after an even number of moves; so, as the elements of S are all odd, the game terminates at some odd position n_0. Note that $0 < n_0 < \min(S)$. Now consider the situation where one finds oneself initially in position $n - 1$. The other player can use the same strategy they used when the game started at n. The result is that, regardless of how one plays, after an even number of moves, one finds oneself in position $n_0 - 1$, where $0 \leq n_0 - 1 < \min(S)$. So again one loses. □

Returning to the proof of Theorem 9, note that if $\mathcal{S}(n, S)$ is ultimately bipartite, the elements of S are all odd, because for sufficiently large n, the sequence $v(n)$ is an alternating sequence of 0's and 1's, and from any losing position n, the positions $n - s$ are necessarily winning, for all $s \in S$. Thus we may apply the above Lemma. We conclude that for sufficiently large n, there are no positions with n odd and $v(n) = 0$. □

Remark 9. Ultimate periodicity is such a ubiquitous concept in combinatorial game theory, some authors simply refer to it as periodicity, and refer to periodicity as "pure periodicicty".

So one does need to be careful when reading the literature as to the whether the term *periodic* is to be understood as ultimately periodic, or periodic in the strict sense. Also, the usual definition of "ultimately periodic" is that the *nim values* repeat beyond a certain point, and not just the values, as we have used. It is easy to see that for games with finite substraction set $S = \{s_1, \dots, s_k\}$ with $s_1 \leq \cdots \leq s_k$, the nim values are bounded above by k, and that any game which is ultimately periodic in the sense of this paper will also have ultimately repeating nim values. Furthermore, it is an immediate consequence of Ferguson's pairing property [6, p. 86] that if a subtraction game is ultimately bipartite, then beyond a certain point the sequence of nim values is also an alternating sequence of 0's and 1's.

9. More Examples

We conclude this paper with a brief look at some classical games. One of the themes of this paper has been that while the nim value is useful in some circumstances, it is the value (i.e., the 0–1 value) which is fundamental. Indeed, in our view, the preoccupation with nim values in combinatorial game theory is often misplaced.

9.1. **Octal games.** An *octal game* is similar to a subtraction game, but when removing counters one may also split a pile into two piles. To complicate matters, instead of a single subtraction set, one has three sets, S_0, S_1, S_2; on each turn one chooses one of the available piles, with k counters say, and performs one of the following operations on it, leaving all other piles untouched:

(a) if $k \in S_0$ one can remove the entire pile,
(b) if $s \in S_1$ with $s < k$ one can remove s counters from the pile, leaving a pile with $k - s$ counters,
(c) if $s \in S_2$ with $s < k$ one can remove s counters from the pile, leaving two non-empty piles with i and $k - s - i$ counters for some choice of i with $1 \leq i < k - s - i$.

Instead of stating the sets S_0, S_1, S_2, octal games are usually given as a sequence of numbers known as the *Guy-Smith code* or *octal code*: $.a_1a_2a_3\dots$, where $0 \leq a_i \leq 7$ for each i. The interpretation is that writing $a_i = a_{i,0} + 2a_{i,1} + 4a_{i,2}$ with $a_{i,j} \in \{0, 1\}$, we take $a_{i,j} = 1$ when $i \in S_j$. A subtraction game with subtraction set S is an octal game for which the only nonzero numbers in the code are 3's; $a_i = 3$ if and only if $i \in S$. For octal games with $a_i \geq 4$ for some i, play involves the splitting of piles, and so one is naturally lead to use nim values and employ Theorem 5.

An octal game is *finite* if its code has only finitely many nonzero numbers; in other words, the subtraction sets S_0, S_1, S_2 are all finite. Obviously, nim values are perfectly adapted to the study of non-subtraction, octal games. Considerable effort has been devoted to the study of finite octal games starting with a single pile with n counters, and the investigation of the ultimate periodicity of the nim value of these games. The main motivation for this seems to be the fact that it is still unknown whether all finite octal game have ultimately periodic nim values, and there are many specific games for which this question is still open. See Achim Flammenkamp's website [18].

9.2. **Moore's game.** Moore's game, nim_k, is a natural variation on nim, introduced and solved by Eliakim Hastings Moore in 1910 [23]. Here, for fixed natural number k, one plays with several piles of counters, as in nim, but on each turn, one may remove counters from one up to k piles. Thus nim_1 is the traditional game of nim. Moore's solution is that, like nim, one expresses the number c_i of counters in the i^{th} pile in binary:

$$c_i = c_{ip} + c_{i1}2^1 + c_{i2}2^2 + \cdots + c_{ij}2^j + \ldots$$

Then, the position is losing if and only if for each j, the sum $\sum_i c_{ij}$ is zero modulo $k+1$. So this theory coincides with Bouton's theory for $k = 1$, but for $k > 1$, the traditional nim values are quite inappropriate.

9.3. **Wythoff's game.** Wythoff's game is a famous game that was given a complete solution by Willem Abraham Wythoff in 1907 [30]. In this game, one plays with two piles of counters and on each move, one removes any number of counters from one heap, or the same number of counters from both piles. A pair of heaps (n, m), with $n < m$ is a losing position if and only there is a non-negative integer k such that $n = \lfloor k\phi \rfloor$ and $m = n + k$, where $\phi = (1 + \sqrt{5})/2$ is the Golden ratio. This remarkable solution is actually quite easy to establish; see Coxeter's entertaining paper [15]. The nim values play no part in this solution, but they have nevertheless been studied; see [8].

9.4. **Grundy's game.** Patrick Michael Grundy's game is a famous game that is still open. Here on each move, one splits a single heap into two heaps of different sizes. Here of course, the nim values are critical for the game. It is conjectured that the nim value of the heap with n counters is an ultimately periodic function of n [6, pp. 111-112].

9.5. **Welter's game.** C.P. Welter's game is usually played with coins on a strip, but let us explain it in terms of integers. The game begins with n distinct non-negative integers $a_1 < \cdots < a_n$. In deference to its usual form, we will say that n is the *number of coins*. Each move consists of replacing one of the integers by a smaller non-negative integer so that the resulting n integers are still distinct. We denote this game $W(a_1, \ldots, a_n)$ and following [14], we denote its nim value $[a_1|\ldots|a_n]$. The sink of $W(a_1, \ldots, a_n)$ is the position $W(0, \ldots, n-1)$.

Welter gave his game an intriguing complete solution, which we explain in the language of [14]. One examines the a_i to find a pair that are congruent modulo the largest possible power of 2. Suppose a_1, a_2 is such a pair of *mates* (of course, there may be more than one such pair). Then Welter's theory gives the nim value $[a_1|\ldots|a_n] = [a_1|a_2] \oplus [a_3|\ldots|a_n]$, where $\oplus$ denotes nim addition, as discussed above in Section 7. One continues recursively, thus reducing the problem to the cases $n = 1$ and $n = 2$. For $n = 1$, one obviously has $[a] = a$. For $n = 2$, Welter gave $[a|b] = (a \oplus b) - 1$. Note this last operation is rather curious, as it combines two group structures: nim arithmetic and usual arithmetic.

For example, $[3|5|7|9|11] = [3|11] \oplus [5|9] \oplus [7] = 7 \oplus 11 \oplus 7 = 11$.

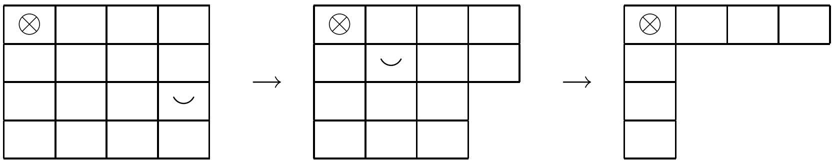
Let us say a few words about losing Welter games. Obviously $W(0)$ is the only losing Welter game with one coin. For the two coin game, one has $[a|b] = 0$ if and only if a is even and $b = a + 1$. For the four coin game, for mates a, b and c, d, one has

$$\begin{aligned} [a|b|c|d] = 0 &\Leftrightarrow ((a \oplus b) - 1) \oplus ((c \oplus d) - 1) = 0 \\ &\Leftrightarrow (a \oplus b) - 1 = (c \oplus d) - 1 \\ &\Leftrightarrow a \oplus b = c \oplus d \\ &\Leftrightarrow a \oplus b \oplus c \oplus d = 0. \end{aligned}$$

The three coin game $W(a, b, c)$ can be thought of as the four coin game $W(0, a+1, b+1, c+1)$. So $[a|b|c] = 0$ if and only if $(a + 1) \oplus (b + 1) \oplus (c + 1) = 0$. Conway remarks that "these observations... seem to have been made again and again by many people independently" [14, p. 164]. Welter's game is a good example of a game that is completely solved but still attracts interest. It is intriguing that nim values play a role, but in a rather mysterious manner.

9.6. **Chomp.** The game of *chomp* is played by two players with an $m \times n$ checker board made of chocolate, subject to the following rules:

(a) the two players take turns to choose a square and eat it and all the squares that are below and to the right of the chosen square, including the squares directly below and directly to the right of the chosen square,
(b) whoever eats the (poisoned!) square in the top left-hand corner loses. (In other words, the top left-hand corner square can't be eaten).



Chomp was invented by David Gale [20], though an equivalent game involving divisors of numbers had been proposed by Schuh in the early 1950's. There are many java implementations of Chomp. For example, see Thomas Ferguson's UCLA website [17]. Notice that every possible position in this game is a partition with no more than m parts, each no more than n. Explicitly, given distinct partitions $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_k)$, there is a move $a \mapsto b$ iff there exists $(i_0, j_0) \in \mathbb{N}^2$ such that $b_i = a_i$ for all $i < i_0$ and $b_i = \min\{a_i, j_0 - 1\}$ for all $i \geq i_0$. The Gaussian polynomial records how many positions there are; the i^{th} coefficient of the Gaussian polynomial is the number of positions of height i.

Chomp has a very nice feature; there is a winning strategy for the first player, but we don't yet know what this strategy is! To see that the first player can always win, consider the possible first move of eating the bottom right-hand square. Either this is the first move of a winning strategy, or the second play can win by chomping another square, but in the latter case, the first player could have eaten that square first, thus stealing the second player's winning strategy. Strategy stealing is not restricted to chomp:

Lemma 8. *Suppose that in an impartial game G there is move $p_1 \to p_2$ such that for every move $p_2 \to p_3$, there is a move $p_1 \to p_3$, then p_1 is a winning position.*

Proof. Suppose that p_1, p_2 are as in the statement of the lemma, and assume that p_1 is a losing position. Thus p_2 is necessarily a winning position, and so there is a move $p_2 \to p_3$ to some losing position p_3. Then by hypothesis, there is a move $p_1 \to p_3$, contradicting the assumption that p_1 is losing. ☐

Here are the losing positions with $m \leq n \leq 6$:

$(2,1)$	$(3,1,1)$	$(4,1,1,1)$	$(5,1,1,1,1)$	$(6,1,1,1,1,1)$
$(3,2)$	$(4,2,2)$	$(5,2,1,1)$	$(6,2,2,1,1)$	
$(4,3)$	$(5,3,2)$	$(5,3,3,2)$	$(6,3,1,1,1)$	
$(5,4)$	$(5,5,3)$	$(5,5,2,2)$	$(6,4,3,3,2)$	
$(6,5)$	$(6,3,3)$	$(6,2,2,2)$	$(6,4,4,3,3)$	
	$(6,4,2)$		$(6,6,3,3,3)$	
			$(6,6,4,3,2)$	
			$(6,6,5,4,2)$	
			$(6,6,6,5,2)$	

Here are some easily proved observations for partitions with $m \leq n$:

(a) The partitions $(i+1, i)$ are the only 2 part partitions with value zero,
(b) No partition with at least 3 parts, that commences $(i+1, i, ...)$, has value zero,
(c) If a partition has value zero and least part 1, then it has at least two 1's,
(d) The partition $(i, 1, 1, \ldots, 1)$ with i parts (i.e., $(i-1)$ 1's) has value zero,
(e) No other partition with i parts and largest part i has value zero,
(f) The fact that the first player has a winning strategy can be restated as follows: in each of the above columns, there is (at least) one partition that only has two numbers, i.e., it has the form $(a, \ldots, a, b, \ldots, b)$.

Here are some less obvious, but still elementary, observations for partitions with $m \leq n$ [31]:

(g) The only value zero 3 part partition with least part 1 is $(3, 1, 1)$,
(h) The only value zero 3 part partitions with least part 2 are the partitions $(i+2, i, 2)$, for $i \geq 2$.

To date, winning strategies are only known on chomp boards of very special shape (such as square boards and $2 \times n$ boards). For more on chomp, see [31, 32, 28, 12, 11]. Much of the literature has focused on chomp with 3 rows, which is a problem that is currently still open, and much of this work has concentrated on patterns of nim values of positions, though there is no obvious reason why nim values should have any particular relevance to Chomp.

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