OUTERPLANAR THRACKLES

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Abstract. We classify outerplanar thrackles and in particular, we establish Conway's thrackle conjecture for outerplanar thrackles. Furthermore, we show that no pair of vertices of an outerplanar thrackle can be joined by an edge in such a way that the resulting graph drawing is a thrackle.

1. Introduction

Recall that for a graph G, a planar graph drawing $\mathcal{D}(G)$ is *outerplanar* if, up to isotopy and inversion in the plane, the edges all lie on a disc with the vertices on the boundary. In this paper, the graphs are assumed to be finite and simple (i.e. no loops or multiple edges) and we only consider drawings in which the edges are represented by Jordan curves that meet each other only at common vertices or at normal crosssings.

Recall that a *thrackle* is a graph drawing in which, for every pair of distinct edges e_i, e_j, their intersection $e_i \cap e_j$ consists of a single point. Conway's thrackle conjecture states that for any thrackle in the plane, there are no more edges than vertices [12, 9, 3]. Obviously, it suffices to consider graphs whose vertices all have degree ≥ 2.

Lemma 1. *Let G be a finite simple (not necessarily connected) graph whose vertices all have degree ≥ 2. If G can be drawn as an outerplanar thrackle, then in fact G is a single cycle of odd length.*

Corollary 1. *Conway's thrackle conjecture holds for outerplanar thrackles.*

Further, one can be more explicit about the nature of thrackled cycles. The principle example of an outerplanar thrackled cycle is the odd musquash [4, 5]; for the n-cycle musquash, with n odd, the vertices are those of the regular n-gon, labelled cyclically, and for each i, vertex i is joined by straight edges to vertices $i + \frac{n\pm1}{2} \pmod{n}$. The 7-cycle musquash (or heptagram) is shown in Figure 1. Notice that one of the key features of musquashes is that they are *alternating*, in the following sense:

Definition 1. A thrackled cycle is *alternating* if for every edge e and every two-path fg vertex-disjoint from e, the crossings of f and g with e are opposite in orientation.

Theorem 1. *For a thrackled cycle in the plane the following are equivalent:*

(a) *the drawing is outerplanar,*
(b) *the drawing is alternating,*
(c) *up to an inversion in the plane, the drawing is Reidemeister equivalent to an odd musquash.*

We thank János Pach and Tom Zaslavsky for their valuable comments.

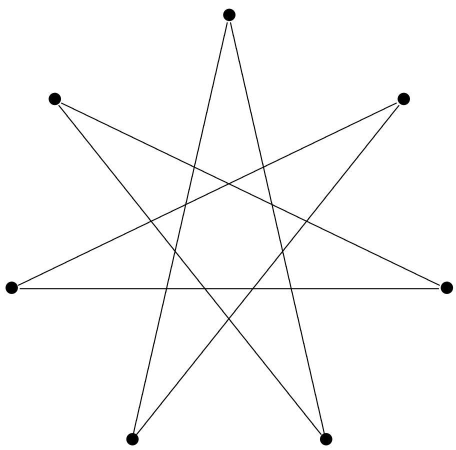

FIGURE 1. The 7-cycle musquash

The outerplanar hypothesis can be weakened somewhat. In the following result we consider drawings that can be obtained from outerplanar cycle drawings by adding an extra edge. In order for the resulting graph to be simple, we only consider new edges that connect vertices that are nonconsecutive in the original cycle. Notice that we are not assuming that the resulting drawing is outerplanar.

Theorem 2. *No pair of vertices of an outerplanar thrackled cycle can be joined by an edge in such a way that the resulting drawing is a thrackle.*

It is well known and not difficult to see that Conway's conjecture is equivalent to the fact that no pair of vertices of a thrackled cycle can be joined by a path without violating the thrackle condition. So people who are searching for a counterexample might naturally start by trying to add an edge to a known thrackled cycle. The above result can be rephrased as a caution: a counterexample to Conway's conjecture cannot be found by just adding a single edge to an odd musquash.

In order to announce our final result, we need a new definition.

Definition 2. For a drawing of a graph G in the plane, we say that the *crossing rank* is the smallest number r for which there are $r+1$ disjoint closed discs $D_0, \dots, D_r$ such that for $U = \mathbb{R}^2 \setminus \cup_i D_i$,

(a) no vertex lies in U,
(b) the edge crossings all lie in U,
(c) for each edge e, the intersection $e \cap U$ is connected (i.e., it is empty or consists of a single segment).

The rationale of the terminology "crossing rank" is that, if one regards the drawing as being on the sphere rather than the plane, the crossing rank is the rank of the first integral homology group of the closure $\bar{U}$ of U. Note that if G has n vertices, then every drawing of G has crossing rank $\leq n-1$, as one can choose the D_i to be small neighbourhoods of the vertices. Planar and outerplanar drawings have crossing rank 0. In general, crossing rank 0 drawings are not necessarily outerplanar; see Figure 2. However, we show that this does not occur for thrackles.

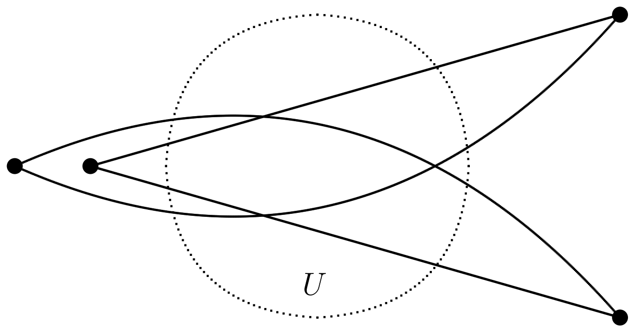

FIGURE 2. Crossing rank 0 but not outerplanar

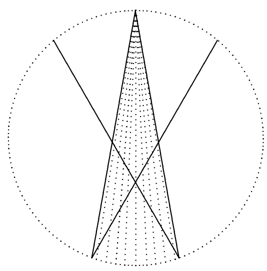

FIGURE 3. 4-path in a straight thrackled cycle

Theorem 3. *Consider a connected graph G with all vertices of degree ≥ 2. Every thrackle drawing of G of crossing rank 0 is outerplanar.*

2. Outerplanar Thrackled Graphs

In this Section, we prove Lemma 1. We first establish a lemma that we will use for both Lemma 1 and Theorem 1.

Lemma 2. *Every outerplanar thrackled cycle is of odd length and, up to an inversion in the plane, its drawing is Reidemeister equivalent to an odd musquash.*

Proof. Consider an outerplanar thrackled cycle $\mathcal{D}$. By an isotopy and an inversion of the plane if necessary, we can deform $\mathcal{D}$ to a thrackle lying within a regular polygon P whose vertices coincide with the vertices of $\mathcal{D}$. Replacing the edges of $\mathcal{D}$ by straight-line segments, we get a straight-line drawing $\mathcal{D}'$. By the Jordan curve theorem, $\mathcal{D}'$ is also a thrackle drawing; so it is a straight thrackle, in the sense of [12]. (Straight thrackles are also called geometric thrackles). In particular, by [12, Theorem 2], $\mathcal{D}'$ is an odd cycle. We claim that $\mathcal{D}'$ is a musquash. This is immediate for 3-cycles, so consider a straight thrackled n-cycle for odd $n \geq 5$. Then, as observed in the proof of [12, Theorem 2], every 4-path necessarily has the form shown in Figure 3 and furthermore, every other edge crosses the dotted cone. So there are the same number of vertices to the left of the dotted cone as there are to the right. Hence $\mathcal{D}'$ is an odd musquash.

It remains to show that $\mathcal{D}'$ is Reidemeister equivalent to the initial thrackle $\mathcal{D}$. In fact, it is possible to deform $\mathcal{D}$ to $\mathcal{D}'$ continuously, keeping the vertices fixed, in such a way that at every stage in the deformation, the drawing is a thrackle.

To perform the required deformation, we use the *curve-shortening flow* process (see [1, 2, 6] for details), which we apply simultaneously to every edge of $\mathcal{D}$. Given a smooth curve γ_0, consider the parabolic evolution equation $\gamma_u = k\nu$, with initial condition $\gamma(0) = \gamma_0$, where k is the (signed) curvature and ν is a unit normal. The set of curves $\{\gamma(u)\}$ is called the *curve-shortening flow*. It has the following properties:

(i) an embedded curve which initially lies in a strip between two parallel lines, with its endpoints on these lines, evolves within the strip and as $u \to \infty$, it tends to the straight line segment between the same endpoints [8, Theorem 2.6],
(ii) the number of self-intersections of a curve does not increase under the flow, and neither does the number of crossing points of two curves,
(iii) if two curves initially have no more than one crossing, then at no stage in the flow can they touch without crossing. (This property has a clear geometric demonstration: if two curves touch at some moment, then a short time before they must have had two crossing points.)

By property (i), a curve inside a strictly convex polygon, with fixed endpoints on the polygon, will evolve within the polygon. In particular, as the drawing $\mathcal{D}$ evolves, none of the edges pass through a vertex. Thus, in view of properties (ii) and (iii) above, the number of crossing points of any pair of edges is constant. So at each stage of the evolution of $\mathcal{D}$, the drawing is a thrackle. □

Remark 1. Although every outerplanar thrackled cycle is Reidemeister equivalent to an odd musquash, it need not be isotopic to one. In fact, an outerplanar thrackled cycle may have no isotopic straight-line realization at all. To see this, we start with a non-stretchable arrangement $\mathcal{C}$ of pseudolines [7, Section 3.1]. Such an arrangement $\mathcal{C}$ can be realized as a subset of an outerplanar thrackled cycle as follows. Embed $\mathcal{C}$ in a small disc inside a regular $2m$-gon P and join the endpoints of the segments of $\mathcal{C}$ with the vertices of P without creating any new crossings. Now let $\mathcal{D}$ be a straight-line thrackle drawing of a $2m+1$-cycle having a crossing point x of multiplicity m (to construct such a $\mathcal{D}$, choose $2m+1$ points v_i on the circle in such a way that every pair of points $v_i, v_{i+m},\ i = 0, \dots, m-1$, is antipodal, and then join every point v_i with $v_{i+m\,(\mathrm{mod}\ 2m+1)},\ i = 0, \dots, 2m$ by a straight line segment). Replacing a small disc centered at x by (a homothetic copy of) P with the arrangement $\mathcal{C}$ inside, we get an outerplanar thrackle containing $\mathcal{C}$ as a subset.

Proof of Lemma 1. Given Lemma 2, it remains to reduce the general case to that of a cycle. This is easily attained by repeating Perles' proof for geometric thrackles; see [10, Theorem 5.2] and [11, pp. 223–224]. Consider an outerplanar thrackle with its vertices on the boundary of a disc. Delete the left-most edge at each vertex. We claim that there is no remaining edge. Indeed, if an edge e remains between vertices v_1, v_2, then in the original drawing there were edges e_1, e_2 to the left of e at v_1, v_2 respectively. But then e_1, e_2 are disjoint, since neither can cross e. This contradicts the assumption that the drawing is a thrackle. □

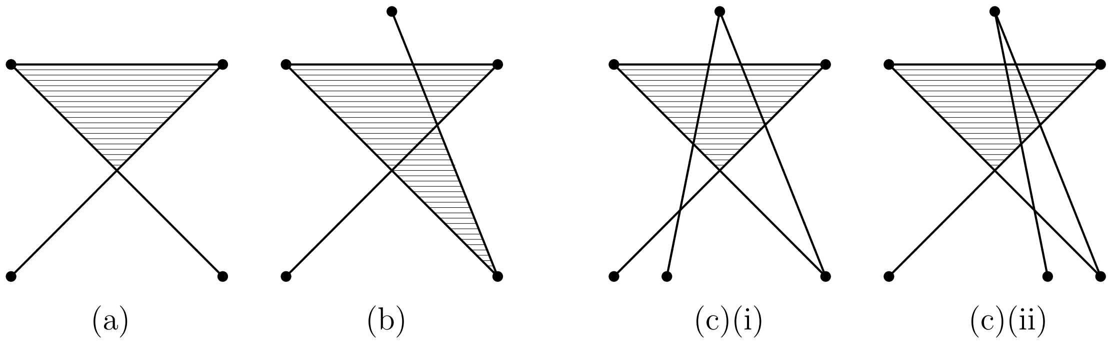

FIGURE 4. Alternating (a) three-, (b) four- and (c) five-paths

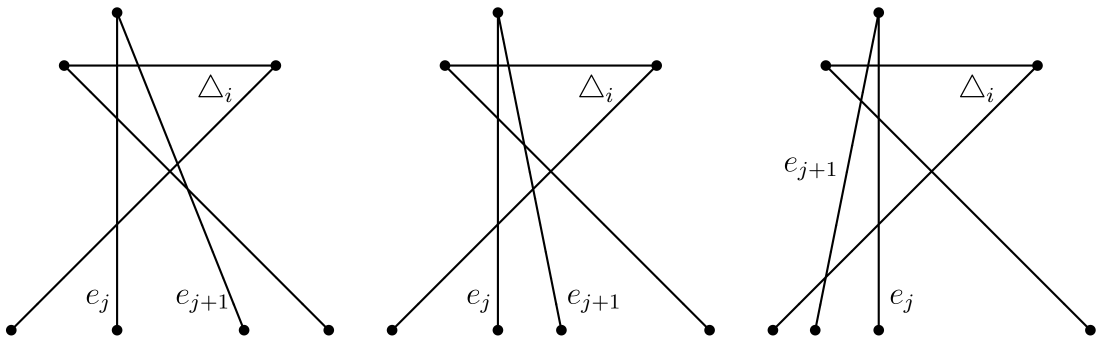

FIGURE 5

3. Outerplanar Thrackled Cycles

Proof of Theorem 1. Implication $a \Rightarrow c$ was established in Lemma 2. Implication $c \Rightarrow b$ follows immediately from the fact that Reidemeister moves do not violate the alternating condition. It remains to verify $b \Rightarrow a$.

It is easy to see that up to isotopy, change of orientation and an inversion of the plane, there is only one alternating path of length 3, one alternating path of length 4 and two alternating paths of length 5; see Figure 4.

Let c be an alternating thrackled cycle of length $n \geq 5$. Choose an orientation for c and label the edges of c in the cyclic order: $e_0, e_1, \dots, e_{n-1}$, with labels taken $\mod n$. Label the vertices $v_0, \dots, v_{n-1}$ so that e_i goes from v_i to v_{i+1}. Consider the drawings of any three consecutive edges e_{i-1}, e_i, e_{i+1}. The complement of the union of these edges consists of two open connected sets. Let $\triangle_i$ (the *i-th triangle*) be the connected component whose boundary contains only one endpoint of e_{i-1} (shaded in Figure 4a). Let $\mathcal{U} = \cup_{i=1}^{n} \triangle_i$. The set $\mathcal{U}$ is open and connected, as any two consecutive triangles intersect in an open connected set (see Figure 4b).

Note that as v_i is on the boundary of $\triangle_i$, the vertices $v_0, \dots, v_{n-1}$ are all contained in the closure of $\mathcal{U}$. We claim that the vertices $v_0, \dots, v_{n-1}$ all lie on the boundary of $\mathcal{U}$; in other words, for each i, none of the vertices are contained in the open set $\triangle_i$. Indeed, as one can see from Figure 4c, the endpoints of the edge e_{i+2} lie outside $\overline{\triangle_i}$. Furthermore, if the endpoints of an edge e_j lie outside $\overline{\triangle_i}$, then the same is true for the edge e_{j+1}, provided

$j+1 \neq i-1$ (for one possibility for e_j, the three Reidemeister equivalent cases are shown in Figure 5). So by induction, the vertices $v_0, \dots, v_{n-1}$ all lie on the boundary of $\mathcal{U}$, as claimed. Actually, this argument establishes something a little stronger: vertex v_i is only contained in the boundary of $\triangle_i$ and $\triangle_{i-1}$, and these triangles have the same intersection with a small neighbourhood of v_i. It follows that the vertices all lie on the boundary of the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$. Hence all the vertices lie on the boundary of the complement $\mathcal{O} = \mathbb{R}^2 \setminus \overline{\mathcal{U}}$, which is an open nonempty set in the plane. Moreover, by applying an inversion if necessary, one can make the domain $\mathcal{U}$ bounded, and hence $\mathcal{O}$ unbounded.

To prove that $\mathcal{D}(c)$ is outerplanar it remains to show that $\mathcal{O}$ is connected. To establish this we will show that $\overline{\mathcal{U}}$ is simply-connected and hence homeomorphic to a closed disc. Since the boundary of $\overline{\mathcal{U}}$ is a subset of $\mathcal{D}(c)$, it suffices to show that any closed loop in $\mathcal{D}(c)$ is contractible in $\overline{\mathcal{U}}$ which is equivalent to the fact that every cycle in $\mathcal{D}(c)$ is a boundary in the group of 1-chains of $\overline{\mathcal{U}}$ over $\mathbb{Z}_2$.

The group $H_1(\mathcal{D}(c), \mathbb{Z}_2)$ is generated by closed loops in $\mathcal{D}(c)$. If i, j, k are three distinct elements of $\{0, 1, \dots, n-1\}$, let $[ij]_k$ denote the segment of the drawing of the edge e_k between the crossing points with e_i and e_j, and call $T_{ijk} = [ij]_k + [jk]_i + [ki]_j$ a *triangular cycle*. By induction, one can show that any cycle in $\mathcal{D}(c)$ can be decomposed in a sum of triangular cycles, so the latter generate $H_1(\mathcal{D}(c), \mathbb{Z}_2)$. Moreover, $T_{ijk} + T_{ijl} = T_{kli} + T_{klj}$ for all i, j, k, l if we put $T_{iij} = 0$. Then for any i, j, s, one has

$$T_{i\,i+s\,k} = T_{i\,i+s-1\,k} + T_{i\,i+s\,i+s-1} + T_{k\,i+s-1\,i+s},$$

and by induction it follows that any triangular cycle is a sum of triangular cycles of the form T_{ii+1j}. Hence $H_1(\mathcal{D}(c), \mathbb{Z}_2)$ is generated by triangular cycles T_{ii+1j}. But any such cycle bounds in $\overline{\mathcal{U}}$: if $j = i+2$ or $j = i-1$, then T_{ii+1j} bounds $\triangle_{i+1}$ or $\triangle_i$, respectively; otherwise, both endpoints of $\mathcal{D}(e_j)$ are outside $\triangle_{i+1}$, as was shown above, and so T_{ii+1j} bounds a small subtriangle of $\triangle_{i+1}$.

□

4. Adding Edges to Outerplanar Cycles

Proof of the Theorem 2. The proof is based on the operation of *edge-removal* which is inverse to the Woodall's edge-insertion [12, Fig. 14]. Let $\mathcal{D}(G)$ be a thrackle drawing of a graph G and let $v_1v_2v_3v_4$ be a three-path in G such that $\deg v_2 = \deg v_3 = 2$. Let $A = \mathcal{D}(v_1v_2) \cap \mathcal{D}(v_3v_4)$. Removing the drawing of the edge v_2v_3, together with the portions of $\mathcal{D}(v_1v_2)$ and $\mathcal{D}(v_3v_4)$ from the point A to v_2 and v_3, respectively, we obtain a drawing of a graph with a single edge v_1v_4 in place of the three-path $v_1v_2v_3v_4$; see Figure 6.

Unlike edge-insertion, edge-removal does not always produce a thrackle drawing. Let $\triangle$ be the domain bounded by the arcs v_2v_3, Av_2, and v_3A, which does not contain v_1. We have the following Lemma:

Lemma 3. *Edge-removal results in a thrackle drawing if and only if $\triangle$ contains no vertices of $\mathcal{D}(G)$.*

Proof. The domain $\triangle$ contains no vertices of $\mathcal{D}(G)$ if and only if every edge e of the thrackle, apart from v_1v_2, v_2v_3, v_3v_4, crosses the boundary of $\triangle$ exactly twice, and this occurs precisely when e has only one point in common with the arc v_1Av_4. □

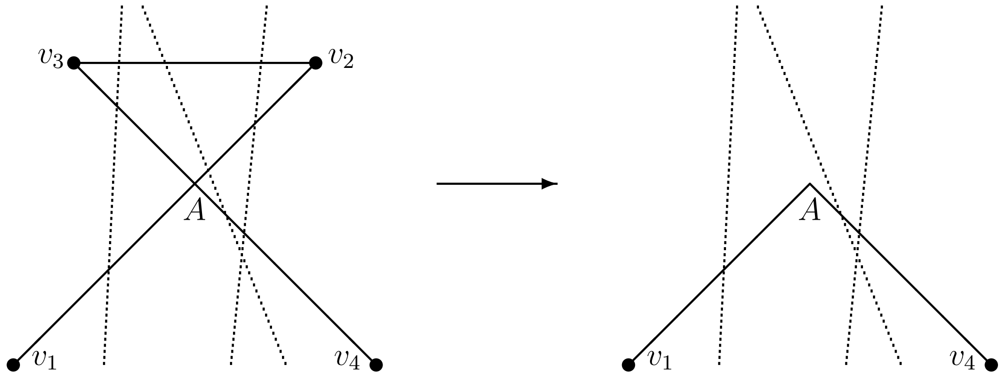

FIGURE 6. Edge-removal

To prove the theorem, assume that there exists a thrackle of a simple graph G obtained by joining two vertices of an outerplanar thrackled cycle c by an edge e. The graph G is therefore a theta graph, and the vertices of e separate the cycle c into two paths c_1, c_2. As c has odd length, by Lemma 1, we may assume that c_1 has odd length and c_2 has even length. Note that c_1 has length ≥ 3 as G is simple. Since c is outerplanar, using Lemma 3 we can perform edge-removal on any three-path $v_1v_2v_3v_4$ of c_1 with $\deg v_2 = \deg v_3 = 2$. The resulting drawing is again an outerplanar thrackled cycle with two vertices joined by an edge. The path c_2 is unaffected, and the length of c_1 is reduced by 2. Repeating the edge-removal process, we eventually reduce to the case where c_1 has length 3. But then the union of c_1 and e would be a thrackled 4-cycle, which is impossible. □

5. CROSSING RANK

Proof of the Theorem 3. Suppose a connected graph G, with all vertices of degree ≥ 2, has a crossing rank 0 drawing that is not outerplanar. Furthermore, suppose that G has the smallest number of edges of any such graph. We will use the fact that for the given drawing, every proper connected subgraph, with all vertices of degree ≥ 2, is outerplanar.

From the crossing rank 0 hypothesis, there is an open disc U such that all the vertices lie outside U and all the crossings lie inside U, and the intersection of each edge with U is connected.

Consider a vertex v and two adjacent edges e_1, e_2, as in Figure 7. Starting at v, the edges e_i cross the boundary ∂U of U at two points p_i respectively and subsequently exit U at two other points $q_i \in \partial U$, before terminating at vertices v_i, say. A priori, there are two possibilities, as in Figure 8: q_1, q_2 could lie in the same connected component of $\partial U \backslash \{p_1, p_2\}$, or they could lie in different components. We will refer to these cases as case A and case B respectively, and correspondingly we say that v is of type A or type B for the edges e_1, e_2.

Let α be the simple closed curve consisting of the segment of e_2 from p_2 to v, the segment of e_1 from v to p_1, and the segment of ∂U from p_1 to p_2 that does not contain q_1. Let V be the open domain with boundary α that does not contain v_1. So $v_2 \notin V$ in case A, while $v_2 \in V$ in case B.

Lemma 4.

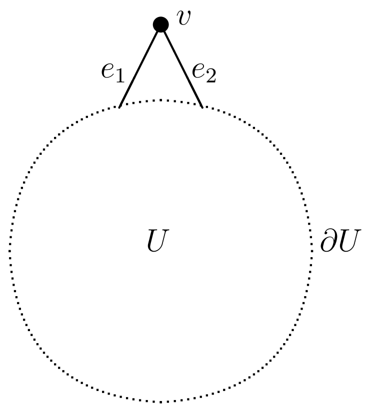

FIGURE 7. Two odd cycles produce one even cycle

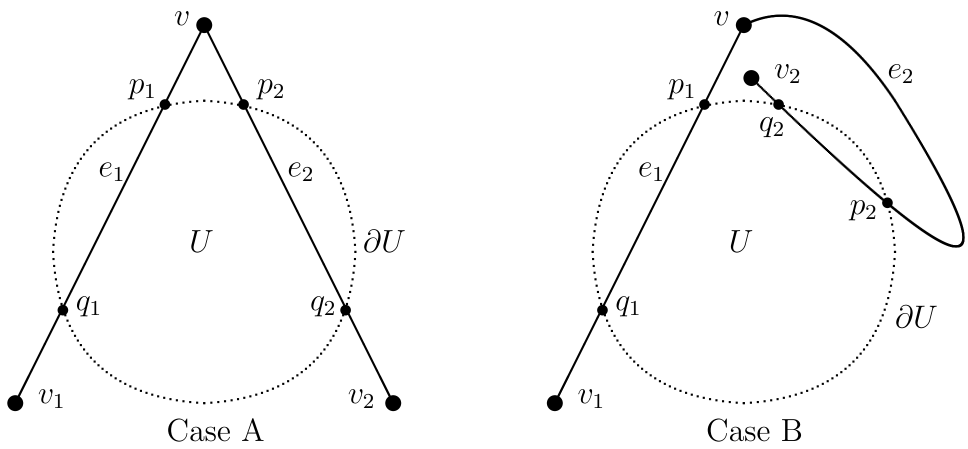

FIGURE 8. Two Cases

(a) *In case A, no vertex of G lies in V.*
(b) *In case B, the vertices v_1, v_2, v are the vertices of a 3-circuit.*

Proof. In case A, suppose that a vertex w lies in V. No edge of G incident with w can cross both edges e_1, e_2 while satisfying the rank 0 hypothesis. Thus, to verify the thrackle condition, each edge incident with w must either cross e_1 and terminate at v_2, or cross e_2 and terminate at v_1; see Figure 9. There are at least two edges e_3, e_4 incident with w and they must terminate at distinct vertices v_1, v_2. Thus, the edges e_1, e_2, e_3, e_4 form (in some order) a thrackled 4-circuit, contradicting [12].

In case B, let e_3 be an edge other than e_2 that is incident to v_2. If e_3 terminates at v_1, the edges e_1, e_2, e_3 form a 3-circuit. Otherwise e_3 crosses e_1; see Figure 10. In that case, if e_0 is an edge other than e_1 that is incident to v_1, then e_0 cannot cross both e_3 and e_2, as it cannot cross e_1. Thus e_0 must be incident to v_2. Now the edges e_1, e_2, e_0 form a 3-circuit.

□

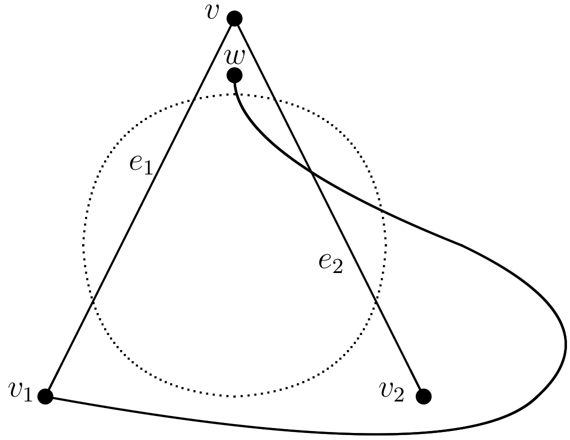

FIGURE 9

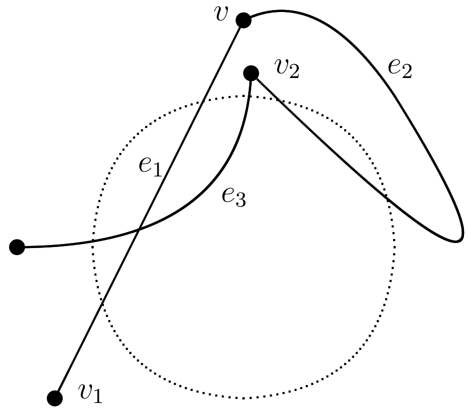

FIGURE 10

Returning to the proof of the theorem, notice that by Lemma 4, if the vertices of G are all of type A, the drawing is outerplanar and we are done. Suppose therefore that there is a vertex v of type B. So G contains a 3-circuit C, with vertices v, v_1, v_2. Note that G does not just consist of this circuit as the drawing has rank 0. Hence G must possess a subgraph of one of the kinds (i), (ii) or (iii) shown from left to right in Figure 11.

In case (iii), G contains a circuit K disjoint from the 3-circuit C. By hypothesis, K is outerplanar, and so by Lemma 1, K has odd length. But no graph with disjoint circuits of odd length can be drawn as a thrackle [12]. So case (iii) is impossible.

In case (ii), the vertices v, v_1, v_2 belong to a circuit K of length ≥ 5, and there is an edge e say, that does not belong to K which joins two of the vertices v, v_1, v_2. By hypothesis, K is outerplanar. But then the existence of the edge e violates Theorem 2. So case (ii) is impossible.

In case (i), there is a circuit K of length ≥ 3 that meets C at one of the vertices v, v_1, v_2. By hypothesis, K is outerplanar, and so by Lemma 1, K has odd length. Since C and K

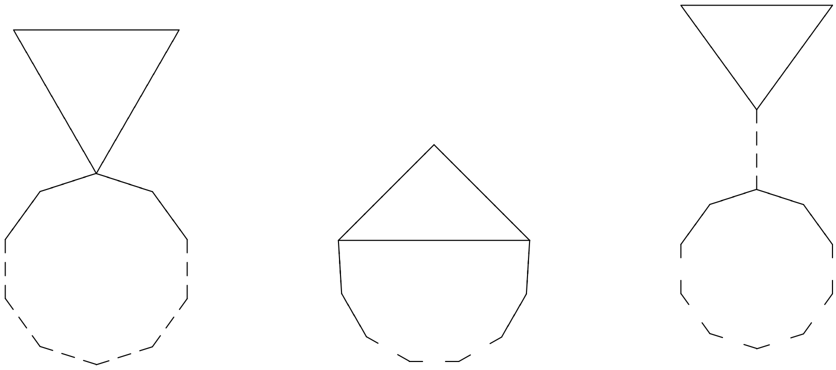

Figure 11. Possibilities (i), (ii) and (iii).

have odd length, they must cross each other at their common vertex; see [9] or [3]. It is easy to see that this cannot occur if the common vertex of C and K is v_1 or v_2. If the common vertex of C and K is v, then one finds that v is also a vertex of type B for the edges of K adjacent to v. Thus by Lemma 4, K also has length 3. But it is a simple exercise to show that the graph obtained by joining two triangles at a common vertex, cannot be drawn as a thrackle. So case (i) is impossible. This concludes the proof of the theorem. □

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