

Totally Goldbach Numbers and Related Conjectures

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Goldbach's famous conjecture is that every even integer n greater than 2 is the sum of two primes; to date it has been verified for n up to 10^{17} ; see [10, 13]. In order to establish the conjecture for a given even integer n , one optimistic approach is to simply choose a prime $p < n$, and check to see whether $n - p$ is prime. Of course, one has to make a sensible choice of p ; if $n - 1$ is prime, one shouldn't choose $p = n - 1$, and there is obviously no point choosing a prime p which is a factor of n . In this paper we examine the set of numbers n for which every "sensible choice" of p works:

Definition 1 A positive integer n is *totally Goldbach* if for all primes $p < n - 1$ with p not dividing n , we have that $n - p$ is prime. We denote by A the set of all totally Goldbach numbers.

It turns out that there are very few totally Goldbach numbers. We find:

$$A = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 18, 24, 30\}.$$

At first sight, it would seem very plausible that A is a small finite set. As everyone knows, the primes tend to become rarer as one proceeds along the real line; if $\pi(n)$ denotes the number of primes no greater than n , then one expects $\pi(n) \leq 2\pi(n/2)$ for all $n \geq 6$. Indeed, this was conjectured by Landau and proved by Rosser and Schoenfeld [16]. For n to be a member of A we require as many "sensible" primes p with $p < n/2$ as there are primes p with $n/2 < p < n - 1$. So we would have $n \notin A$ if we could show that $\pi(n)$ is less than $2\pi(n/2)$ minus the number of prime divisors of n . The Prime Number Theorem tells us that the density of the primes falls off on average with $1/\log(n)$. So for big n , there will tend to be considerably more primes between 1 and $n/2$ than there are between $n/2$ and n ; in fact, the difference is approximately $(2n \log 2)/(\log n)^2$. The number of prime divisors of n is more difficult to describe, but it grows much more slowly with n [14]. So we expect that large integers n will not belong to A . However, individual numbers seem to care little for expected "average" behaviour. Consistent with the falling frequency of prime numbers, Hardy and Littlewood conjectured (see for example [6]) that $\pi(x + y) \leq \pi(x) + \pi(y)$ for all sufficiently large x, y , but there are strongly held contrary views [19].

Before explaining how A can be determined, we first make some connections with three other closely related sets. Consider the set B of positive integers n such that every positive integer $r < n$ which is coprime to n is prime or 1. The members of B are called "very round numbers"; B appears as integer sequence A048597 in Sloane's Integer Sequences

web site [17]. Obviously $B \subseteq A$. Knowing A , one finds easily that

$$B = A \setminus \{5, 10\}.$$

According to [12, p. 281], the composition of B was first determined by Schatunowsky (1893) and independently by Wolfskehl (1901). Apparently, it was also obtained by Bonse (1907); see [11] for an account of this proof, which is elementary, and makes use of ‘‘Bonse’s inequality’’:

$$p_{n+1} < \sqrt{p_1 p_2 \cdots p_n},$$

where p_n denotes the n^{th} prime.

Another closely related set is $C = \{n \in \mathbb{N} : \varphi(n) \leq \tau(n)\}$, where φ is Euler’s Totient function and τ is the divisor counting function. Using the simple formulae for φ and τ (see for instance [15, p. 19]), one finds that

$$C = A \setminus \{5\}.$$

This set appears as integer sequence A020490 in Sloane’s Integer Sequences web site [17]. Despite the striking similarity between A and C , there is no obvious logical relation between the two sets; is their similarity merely a remarkable coincidence?

When examining Goldbach’s conjecture, for a given integer n , it is common to study the number $g(n)$ of ways of representing n as the sum of two primes. Obviously $g(n)$ is less than or equal to the number of primes p with $n/2 \leq p < n - 1$. Let the set D consist of those n for which $g(n)$ equals this maximum. Obviously $A \subseteq D$. In [4], Deshouillers, Granville, Narkiewicz and Pomerance showed that the maximum element of D is 210. It is easy to verify then that

$$D = A \cup \{7, 14, 16, 36, 42, 48, 60, 90, 210\}.$$

Of course, the determination of A is a simple consequence of the determination of D ; one just checks the elements of D to see which are totally Goldbach.

In their 1993 paper [4], two strategies are given for finding the maximal element n_0 of D . The first strategy relies on the following simple idea: if one can find primes p, q with $n/2 \leq p < n - q$ such that $p \equiv n \pmod{q}$, then $n - p$ would be a multiple of q and $n - p > q$; in this case, $n - p$ wouldn’t be prime and so n couldn’t belong to D . According to [4], using estimates for the number of primes $p \leq x$ with $p \equiv a \pmod{q}$, this strategy shows that D is finite and gives $n_0 \leq 10^{520}$. Unfortunately, this leaves too many cases to check, even by computer. Abandoning this approach, the authors of [4] then adopt a different strategy; using an argument involving sieve estimates, they obtain $n_0 \leq 2 \cdot 10^{24}$. Finally, using a computer to check the cases $n \leq 2 \cdot 10^{24}$, they arrive at $n_0 = 210$.

We will show that the first strategy of [4] is sufficient for the determination of our set A ; i.e., A can be determined once one has the bound 10^{520} . Our motivation for doing this is two-fold. Firstly, since A is a simpler set, it is only fitting that it have a simpler determination. (This was the original motivation for this work). Secondly, and perhaps more importantly, we will see that this leads us naturally to interesting questions concerning primes in a fixed residue class.

We proceed as follows. First show that A has no element n with $30 < n \leq 2 \cdot 10^6$ by directly applying the definition; this is easily accomplished by computer. Then suppose that $n \in A$ and $n > 2 \cdot 10^6$. Obviously n must be even. Assume first that $n \equiv 1 \pmod{3}$.

If q is prime, $q < n - 3$ and $q \equiv 1 \pmod{3}$, then $n - q$ is divisible by 3 and hence not prime; as $n \in A$, we conclude that q is a factor of n . Thus

$$n \geq 2 \prod_{\substack{q \text{ prime} \\ q < n-3 \\ q \equiv 1 \pmod{3}}} q \geq 2 \prod_{\substack{q \text{ prime} \\ q < 2 \cdot 10^6 - 3 \\ q \equiv 1 \pmod{3}}} q \geq 10^{1000},$$

where the last calculation is performed by computer. Similarly, if $n \equiv 2 \pmod{3}$, then n is at least twice the product of those primes $q < 2 \cdot 10^6 - 3$ for which $q \equiv 2 \pmod{3}$. Once again, one finds that $n \geq 10^{1000}$. So it remains to consider the case where n is divisible by 3 (and hence 6). Arguing as above, for each $a \in \{1, 2, 3, 4\}$,

$$n \geq 6 \prod_{\substack{q \text{ prime} \\ q < n-5 \\ q \equiv a \pmod{5}}} q \geq 6 \prod_{\substack{q \text{ prime} \\ q < 2 \cdot 10^6 - 5 \\ q \equiv a \pmod{5}}} q,$$

for $n \equiv a \pmod{5}$ and once again this gives $n \geq 10^{1000}$ in each case. So we may assume that n is divisible by 5 (and hence 30). Proceeding in this manner, we find that for each prime q up to the 351-st prime, 2371, and for each $a = 1, \dots, 2370$, one has $n \geq 10^{1000}$ for $n \equiv a \pmod{q}$. So we may assume that n is divisible by the product of the first 351 primes; but this also gives $n > 10^{1000}$, as claimed.

In all, the various calculations took less than 24 hours running Maple 9 on a Pentium IV 2.4GHz; the calculations were verified in a little over 2 days, running Mathematica 4 on a Macintosh G3.

Notice that the above argument used the assumption that $n > 2 \cdot 10^6$ to show that $n \geq 10^{1000}$. A complete determination of A , without recourse to [4], would be obtained if the above method could be extended indefinitely and thus turned into an induction argument; that is, assuming that n is greater than some sufficiently large number K , one could try to use the above method to show that n is greater than some larger number K' . Loosely speaking, this approach would work providing the primes q , in any given residue class, are not too sparse. What this asks for is effectively a modular version of Euclid's theorem; recall that Euclid's proof of the infinitude of primes can be rephrased as follows:

$$p_{n+1} < p_1 \cdot p_2 \cdot \dots \cdot p_n, \text{ for all } n \geq 2$$

where p_i is the i -th prime. This can be regarded as a weak version of Bonse's inequality [11], and a very weak version of Bertrand's postulate [1]. The simplest modular version of Euclid's theorem would be that for all primes q and for all $a = 1, 2, \dots, q - 1$,

$$r_{n+1} < r_1 \cdot r_2 \cdot \dots \cdot r_n, \text{ for all } n \geq 2 \tag{1}$$

where r_i is the i -th prime that is congruent to $a \pmod{q}$. Unfortunately, this doesn't hold in general. For example, the primes congruent to 3 $\pmod{13}$ are 3, 29, 107, \dots , but $107 \not\leq 3 \times 29$, and the primes congruent to 5 $\pmod{61}$ are 5, 127, 859, \dots , but $859 \not\leq 5 \times 127$, etc. In fact, if the twin prime conjecture is true, there are infinitely many counterexamples to (1); indeed, if $q, q + 2$ are twin primes, then the first two primes congruent to 2 \pmod{q} are $r_1 = 2, r_2 = q + 2$, and since $2q + 2$ is not prime, we must have $r_3 \geq 3q + 2$. Hence $r_3 \geq r_1 r_2$.

Nevertheless, computer calculations do seem to show that the following is true for small values of q and n .

Conjecture 1 *For all primes q and for all $a = 1, 2, \dots, q - 1$,*

$$r_{n+1} < r_1 \cdot r_2 \cdots r_n, \text{ for all } n \geq 3$$

where r_i is the i -th prime that is congruent to $a \pmod{q}$.

We have been unable to find a statement to this effect in the literature. There are known modular versions of Bertrand's postulate (see [7, 18, 9]), but these results are typically of the form: "for sufficiently large n , ...", and moreover, they are usually not uniform in q and a . The above conjecture is certainly consistent with the prime number theorem modulo q , which gives the asymptotic behaviour of the number $\pi(x; q, a)$ of primes at most x which are congruent to a modulo q :

"The expected asymptotic formula $\pi(x; q, a) \sim x/\varphi(q) \log x$ as $x \rightarrow \infty$ has long been known to hold but in all proofs given so far the dependence of the error term on the parameter q is rather poorly understood. For all we know it might even be the case that the asymptotic formula begins to represent the true state of affairs only after x is (almost) exponentially large compared to q ." from John Friedlander's MathSciNet review of [2].

Notice that Conjecture 1 would follow by induction if we could prove:

Conjecture 2 *For all primes q and for all $a = 1, 2, \dots, q - 1$,*

- (1) $r_4 < r_1 r_2 r_3$,
- (2) $r_{n+1} < r_n^2$, for all $n \geq 4$,

where r_i is the i -th prime that is congruent to $a \pmod{q}$.

Computer calculations appear to support part (1) of Conjecture 2, and in fact for (2), they seem to indicate that $r_{n+1} < r_n^2$, for all $n \geq 3$. Notice that $r_n \geq (2n - 3)q + a$. This is simply because the numbers congruent to $a \pmod{q}$ are:

$$a, q + a, 2q + a, 3q + a, \dots$$

so if a is odd, the smallest possible r_i would be

$$a, 2q + a, 4q + a, 6q + a, \dots$$

while if a is even, the smallest possible r_i would be

$$q + a, 3q + a, 5q + a, \dots$$

if $a \neq 2$ and

$$a, q + a, 3q + a, \dots$$

if $a = 2$. In each case, $r_n \geq (2n - 3)q + a$. So, to establish the second part of Conjecture 2, it suffices to show that $r_{n+1} < (2n - 3)^2 q^2$, or the somewhat stronger:

Conjecture 3 $r_n < 4(n - 3)^2 q^2$, for all $n \geq 4$, all primes q and all $a = 1, \dots, q - 1$.

In fact, it is easy to see, using the same kind of elementary arguments used above, that Conjecture 3 also implies the first part of Conjecture 2. So we have

$$\text{Conjecture 3} \Rightarrow \text{Conjecture 2} \Rightarrow \text{Conjecture 1.}$$

Moreover, it is not difficult to show that by the Bombieri–Friedlander–Iwaniec Theorem [3], Conjecture 3 holds “with few exceptions”. In fact, computer investigations indicate that the following may be true:

Conjecture 4 $r_n < (n + n \log n)q^2$, for all $n \geq 1$, all primes q and all $a = 1, \dots, q - 1$.

This conjecture is a generalization of an old conjecture of Schinzel and Sierpinski (see [12, p. 280 and p. 397]): $r_1 < q^2$ for all primes q and all $a = 1, \dots, q - 1$. At present the best result is Meng’s improvement of Heath-Brown’s version of Linnik’s theorem [8]: $r_1 < q^{4.5}$. So Conjecture 4 may be a long way away.

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References

- [1] M. Aigner and G. M. Ziegler, *Proofs from The Book*, Springer-Verlag, Berlin, 2004.
- [2] R. C. Baker and G. Harman, The Brun-Titchmarsh theorem on average, in *Analytic number theory, Vol. 1*, pp. 39–103, Birkhäuser Boston, Boston, MA, 1996.
- [3] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli. III, *J. Amer. Math. Soc.*, 2 (1989) 215–224.
- [4] J.-M. Deshouillers, A. Granville, W. Narkiewicz and C. Pomerance, An upper bound in Goldbach’s problem, *Math. Comp.* 61 (1993) 209–213.
- [5] P. Dusart, Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, *C. R. Math. Acad. Sci. Soc. R. Can.*, 21 (1999) 53–59.
- [6] ———, Sur la conjecture $\pi(x + y) \leq \pi(x) + \pi(y)$, *Acta Arith.*, 102 (2002) 295–308.
- [7] E. Fogels, On the existence of primes in short arithmetical progressions, *Acta Arith.* 6, (1960/1961) 295–311.
- [8] Z. Meng, The distribution of the zeros of L -functions and the least prime in some arithmetic progression, *Sci. China Ser. A*, 43 (2000) 937–944.
- [9] P. Moree, Bertrand’s postulate for primes in arithmetical progressions, *Comput. Math. Appl.* 26 (1993) 35–43.
- [10] T. Oliveira e Silva, “Goldbach conjecture verification” web page, <http://www.ieeta.pt/~tos/goldbach.html>
- [11] H. Rademacher and O. Toeplitz, *The enjoyment of mathematics*, Princeton University Press, 1957.
- [12] P. Ribenboim, *The new book of prime number records*, Springer-Verlag, New York, 1996.
- [13] J. Richstein, Verifying the Goldbach conjecture up to $4 \cdot 10^{14}$, *Math. Comp.* 70 (2001) 1745–1749.
- [14] G. Robin, Estimation de la fonction de Tchebychef θ sur le k -ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n , *Acta Arith.* 42 (1983), no. 4, 367–389.
- [15] H. E. Rose, *A course in number theory*, Clarendon Press, New York, 1994.
- [16] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, 6 (1962) 64–94.
- [17] N. J. A. Sloane, (2004), The On-Line Encyclopedia of Integer Sequences, published electronically at <http://www.research.att.com/~njas/sequences/>
- [18] T. Tatzawa, On Bertrand’s problem in an arithmetic progression, *Proc. Japan Acad.* 38 (1962) 293–294.

- [19] A. van der Poorten and T. Ah-Mah, On the twin prime conjecture, *Aust. Math. Soc. Gazette*, 31 (2004) 36–39.

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