Brussels Sprouts and Cloves

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Introduction

Sprouts is a well-known 2-person pen-and-paper game; see [**20**], [**18**], [**9**], [**15**], [**12**, Chap. 37], and the *World Game of Sprouts Association* Web site [**19**]. One starts with an agreed number of dots on the page. A *move* in this game is played by connecting two (possibly equal) dots by a simple curve (carefully avoiding any other dot) and introducing a new dot on the middle of the curve. There are only two rules:

1. no two curves can cross,
2. no more than 3 curves can issue from any dot.

The players take turns to move; the loser is the first person who has no possible move. FIGURE 1 shows a game with one initial dot; here the second player has won.

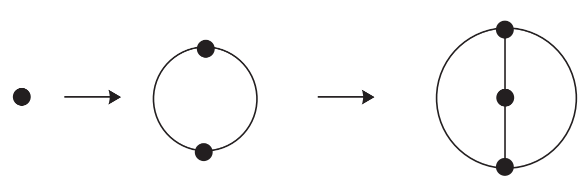

Figure 1

Introduced by Michael S. Paterson and John H. Conway in 1967, Sprouts has interesting combinatorial and topological aspects [**7, 17**]. In particular, there is the daunting *Sprouts conjecture*: for a game with n initial dots, there is a winning strategy for the first player if and only if n is congruent to 3, 4 or 5 modulo 6. This conjecture has been verified only for small values of n, by (computer) exhaustion [**1**].

There is a variation of Sprouts called Brussels Sprouts; here one begins with crosses rather than dots, and with each move one introduces a new cross by marking a bar across the middle of the curve. FIGURE 2 shows a game with one initial cross.

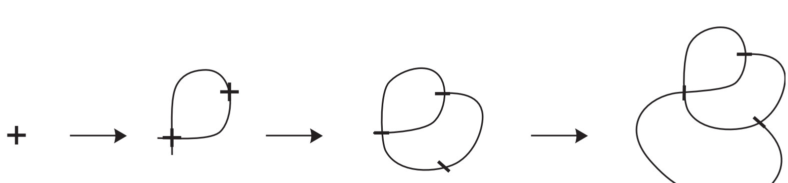

Figure 2

Due to Conway, Brussels Sprouts is something of a mathematical joke. A simple Euler characteristic calculation shows that there is no strategy involved in the game; for Brussels Sprouts played with n initial crosses, the first player wins if and only if n is odd, regardless of how the game is played [**9, 3**]. In fact, the same is true if one plays the game on any orientable surface, rather than the plane. It is perhaps less well known that on non-orientable surfaces, Brussels Sprouts is more interesting; here it is a game of strategy. Nevertheless, the game is amenable to analysis. One has:

THEOREM 1. *For a game of Brussels Sprouts on a compact surface M, without boundary, of Euler characteristic χ, played with m initial crosses, there is a winning strategy for the second player if and only if m and χ are both even.*

Theorem 1 is stated and key elements of a proof are outlined in [**13**]. It is also stated in [**10**]. As far as we are aware, a detailed proof of the theorem hasn't appeared in the literature.

The purpose of this paper is to introduce a generalization of Brussels Sprouts, which we call Cloves, and to state and prove Theorem 1 in this more general context. Before defining Cloves, let us describe the sort of game that we will be able to treat with Cloves; in doing so, the choice of name "Cloves" will become apparent. Consider a number of dots on the plane with a non-zero number of "free arms" connected to each dot, as in FIGURE 3.

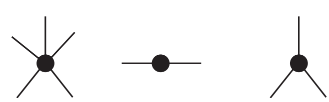

Figure 3

As in Brussels Sprouts, a move is made by connecting two free arms with a curve and marking a bar across the middle of the curve, thus adding two new free arms, one on each side of the curve. Notice that like Brussels Sprouts, such a game has the key feature that as the game is played, the number of free arms remains constant; in each move one uses two arms and adds two new free arms. The game of Cloves that we will now define is equivalent to the game we just described, but its presentation is quite different.

DEFINITION 1. A game of *Cloves* begins with a compact surface M with b boundary curves and on each boundary curve there is a non-zero number of dots. A *move* is played as follows:

(a) connect two distinct dots by a simple curve γ and remove the two dots,

(b) cut M along γ,

(c) introduce two new dots, one on each of the curves resulting from the splitting of γ.

The game finishes, as in Sprouts, when a player can't make a move.

FIGURE 4 shows a game of Cloves, starting with a single disc with three dots. The surfaces in FIGURE 4 have been drawn to emphasize the cuts that have been made, and the dotted lines shows the moves that are about to be made. The same game is more conveniently drawn as in FIGURE 5. FIGURE 6 shows a game of Cloves starting with an annulus with two dots.

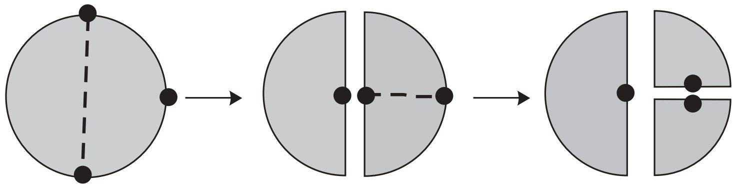

Figure 4

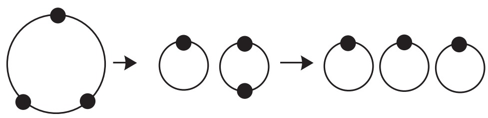

Figure 5

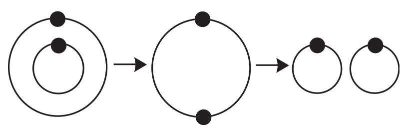

Figure 6

In what way is Cloves a generalization of Brussels Sprouts? Consider a game of Brussels Sprouts, starting with a collection of crosses on some surface M. For each cross, we do the following: rub out the cross, remove a small disc from the surface where the cross was located, and put 4 dots around the boundary curve. In Brussels Sprouts, we draw a curve γ between two crosses; since the subsequent curves cannot cross γ, the curve γ is effectively part of the boundary of the playing surface. In Cloves, this is formalized by splitting the surface along γ. For example, FIGURE 7 shows a game of Brussels Sprouts (thought of as being played on a large sphere), while FIGURE 8 shows the corresponding game of Cloves.

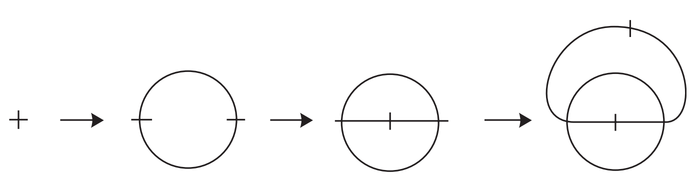

Figure 7

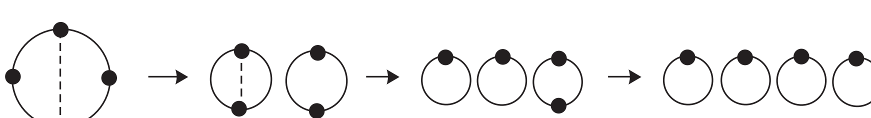

Figure 8

Fundamental theory of surfaces

Before continuing with Cloves, we need to recall some of the fundamental results of surface theory. The two basic constructions on a surface are the *handle* and the *cross-*

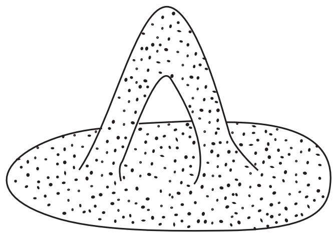

Figure 9

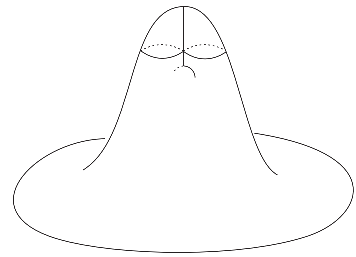

Figure 10

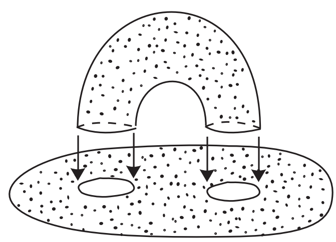

Figure 11

cap which are shown in Figure 9 and Figure 10, respectively. Both surfaces have a single boundary curve. The handle is easy to understand; it is obviously orientable and can be constructed by removing two small discs from a bigger disc, and attaching a cylinder to the small holes, as in Figure 11. In order to avoid 3 dimensional drawings, we will depict handles as in Figure 12. For example, Figure 13 shows a closed curve that passes under and over the handle. Notice that if you attach a disc to the boundary of a handle, you get a torus.

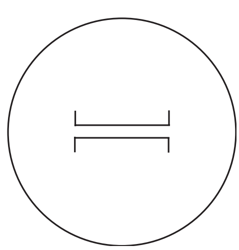

Figure 12

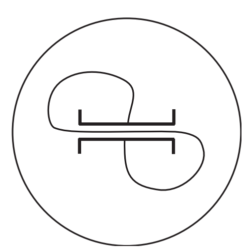

Figure 13

Cross-caps are non-orientable and are harder to visualize; see [**14, 2**]. Nevertheless, they are easy to understand since they are equivalent topologically to Möbius strips. Recall that a Möbius strip is obtained from an annulus, by cutting and reglueing it with a twist, as shown in Figures 14 and 15. If you attach a disc to the boundary of a cross-cap (or Möbius strip), you get a projective plane [**5**]. We represent cross-caps as in Figure 16. The convention here is that as a curve passes across the cross-cap symbol, the orientation is reversed, as shown in Figure 17.

Recall that the Euler characteristic χ of a triangulated surface M is the alternating sum $V - E + F$ of the number of its vertices, edges and faces, respectively. (See [**8**, Ch. 5] and [**16**, Part 1, Ch.1] for information about the Euler characteristic and Euler's formula.) The following theorem summarizes some of the fundamental results of surface theory; see [**11, 5, 10**].

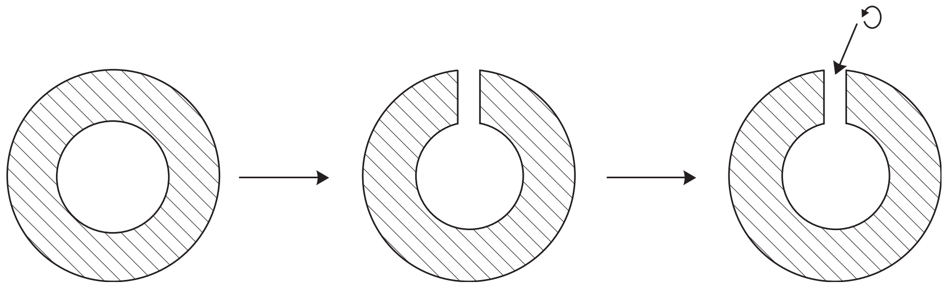

Figure 14

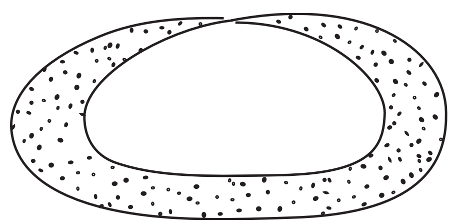

Figure 15

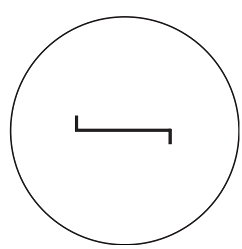

Figure 16

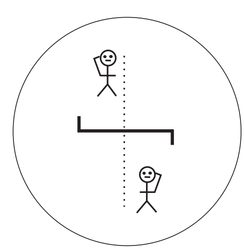

Figure 17

THEOREM 2.

(a) *Every compact connected orientable surface M with boundary is homeomorphic to a disc with a number, h say, of holes and a number, g say, of handles. The Euler characteristic of such a surface is $\chi = 1 - h - 2g$ and M has $h + 1$ boundary curves.*

(b) *Every compact connected non-orientable surface M with boundary is homeomorphic to a disc with h holes and k cross-caps. The Euler characteristic of such a surface is $\chi = 1 - h - k$ and M has $h + 1$ boundary curves.*

(c) *Any two compact connected orientable (resp. non-orientable) surfaces that have the same number of boundary curves and the same Euler characteristic are homeomorphic.*

Remark 1. Even the first part of this theorem is non trivial; FIGURE 18 shows one of the consequences. Perhaps the most striking and non-intuitive part of the above theorem is its corollary: a disc with three cross-caps is homeomorphic to a disc with one cross-cap and one handle; see [**11**].

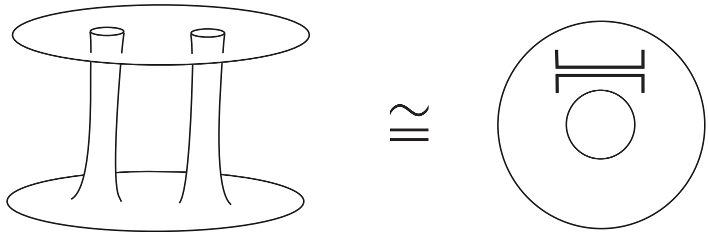

Figure 18

Remark 2. Notice that part (b) of Theorem 2 fails unless M is connected. The problem here is that in general M may have some connected components that are orientable and some components that aren't. Only on the non-orientable components can each handle be replaced by a pair of cross-caps. Notice that if M has c connected components and b boundary curves, then by applying the above theorem to each component separately, one obtains $\chi(M) \leq 2c - b$.

Notation. In what follows, we will write $\chi(M), b(M), c(M)$ etc., if there is any possible confusion as to the surface M.

Cutting up surfaces

As Cloves are played by cutting surfaces along curves, let us briefly recall the effect of this construction on the topology of the surface. The general situation is that we have a connected compact surface M with b boundary components, and Euler characteristic $\chi(M)$, and we cut M along a simple curve γ that joins distinct points on the boundary of M. Consider the resulting surface M'.

LEMMA 1. $\chi(M') = \chi(M) + 1$.

Idea of proof. It suffices to notice that each move adds one to the alternating sum $V - E + F$ of vertices, edges and faces in any triangulation; see FIGURE 19, where E has been reduced by one, while V and F are unchanged. ■

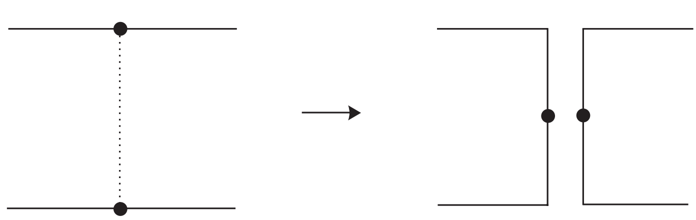

Figure 19

If the end points of γ lie on distinct boundary components, M' has one fewer boundary curves than M, and is still connected; see FIGURE 20.

If the end points of γ lie on the same boundary component, there are two possibilities. Let $\hat{\gamma}$ denote the loop obtained by composing γ with a path back around the boundary of M (there are two choices for $\hat{\gamma}$ depending on which return path one takes along the boundary). If $\hat{\gamma}$ is an orientation preserving loop, the resulting surface M'

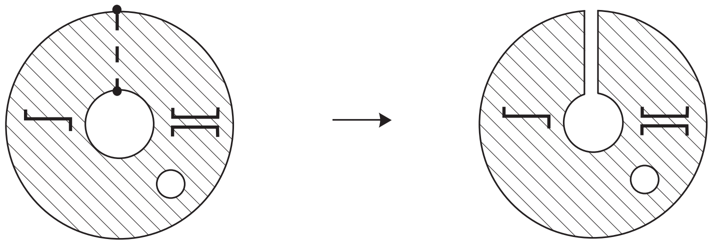

Figure 20

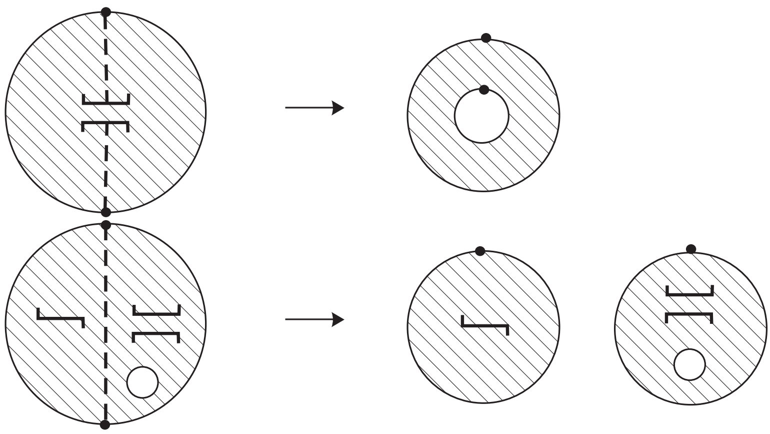

Figure 21

has one more boundary curve than M and may have 1 or 2 connected components; see the two examples in FIGURE 21.

If $\hat{\gamma}$ is orientation reversing, then M' is connected, and has the same number of boundary curves as M; see FIGURE 22. Notice that the condition that $\hat{\gamma}$ is orientation reversing depends only on γ.

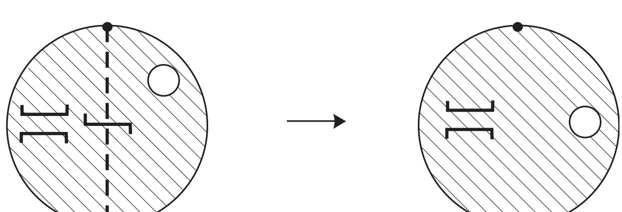

Figure 22

Preparatory results

Consider a game of Cloves with n moves and let M_i denote the surface that results from the ith move. Thus $M_0 = M$ and M_n is the surface at the end of the game. First notice that from the definition of Cloves, one immediately has:

LEMMA 2.

(a) *At each stage of the game, every boundary curve has at least one dot.*

(b) *At the end of the game, each connected component has precisely one boundary curve and it has precisely one dot.*

For Brussels Sprouts, Theorem 1 connects the existence of a winning strategy to the Euler characteristic and the number of crosses. For Cloves, one would want to establish a connection between the existence of a winning strategy and the topological invariants:

- the Euler characteristic $\chi(M)$,
- the number b of boundary curves,
- the number c of connected components,
- the number d of dots.

Unfortunately, one can find games for which the given topological invariants are the same but for which the games have quite different outcomes. For example, the two games in FIGURE 23 both have $b = 2$, $c = 2$, $d = 4$, $\chi = 1$, but the second player is the only possible winner of the game at the top of FIGURE 23, while the first player wins the other game.

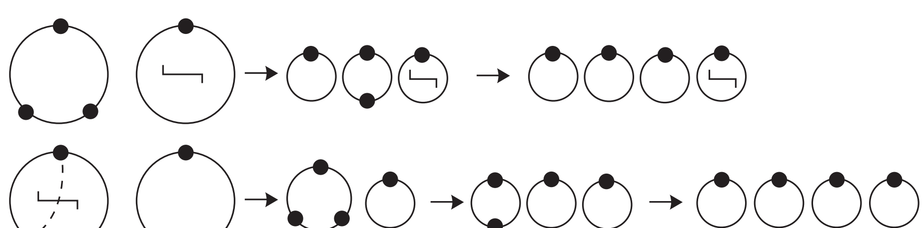

Figure 23

It might seem that in generalizing from Brussels Sprouts to Cloves, we have made the situation harder. Nevertheless, there is a topological description of the game, as we will show. The first important observation is:

LEMMA 3. *If $b = d$, the second player has a winning strategy.*

Proof. The assumption that $b = d$ implies that there is exactly one dot on each boundary curve. The proof is by induction on the integer $j(M) = b - c$. When $j(M) = 0$, each connected component has exactly one boundary curve, and then the hypothesis $b = d$ implies that each connected component has exactly one dot. There is thus no possible move and so the second player wins (without playing a move).

If $j(M) > 0$, the first player does have a possible move, which consists of drawing a curve from a dot on one boundary curve to a dot on another boundary curve lying on the same connected component of M. This is possible since, as $b > c$, there is a connected component with more than one boundary curve. The surface M' that results from this move has $b - 1$ boundary curves, while the number c of connected components is unchanged. Moreover, M' has a (unique) boundary curve with exactly two dots; see FIGURE 24. The second player can now choose a path γ between these two dots that remains very close to the boundary of M'. Cutting along γ gives a disc, with a single dot on its boundary, and a surface M'' homeomorphic to M, with $d - 1$

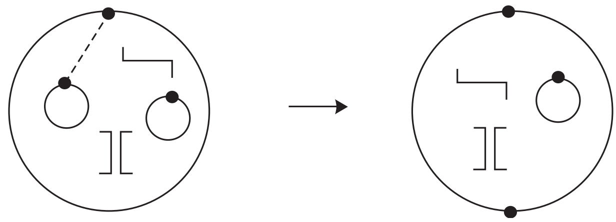

Figure 24

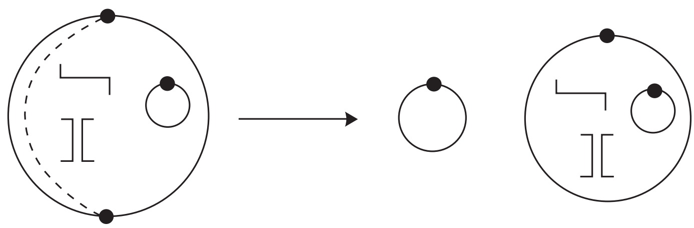

Figure 25

dots; see FIGURE 25. Applying the inductive hypothesis to M'', we obtain the required result. ■

Game theory generalities

At this point, it is useful to pause to consider some general aspects of the game of Cloves. First, Cloves is a *finite game*; that is, given an initial surface M, there is a finite bound on the possible length of the game. In fact, by Lemma 2(b), each game terminates at a surface, M' say, whose connected components have precisely one boundary curve and precisely one dot. So, if M has d dots, $\chi(M') \leq d$. By Lemma 1, if the game lasted for n moves, then $\chi(M') = \chi(M) + n$. Thus $n \leq d - \chi(M)$. This gives an upper bound on the length of a game in terms of the number of dots and the topology of the initial surface M. The following lemma is a key feature of finite games and can be easily established by induction.

LEMMA 4. *For any finite game, either the first player has a winning strategy, or the second player has a winning strategy.*

Another important aspect of Cloves is that it is a game in which *the last player to move is the winner*. Consider the context of all possible abstract two-player games in which the last player to move is the winner. Recall that the *disjunctive sum* of two such games G_1, G_2, is the game $G_1 + G_2$ in which G_1 and G_2 are played side by side, so to speak, and on each player's turn, a "move" is made by making a legal move in just one of the two component games G_1, G_2; see [**6**]. One has the following general fact:

LEMMA 5. *Consider arbitrary finite games G_1, G_2 in which the last player to move is the winner. Assume that the second player has a winning strategy in G_1. Then the first player has a winning strategy in $G_1 + G_2$ if and only if the first player has a winning strategy in G_2.*

Idea of proof. For example, if the second player has a winning strategy in both G_1 and G_2, then the second player can win $G_1 + G_2$ by adopting the following strategy: at each turn, play (sensibly) in whichever game the first player just played in. ■

Remark 3. When the first player has a winning strategy in both G_1 and G_2, one cannot say who will win $G_1 + G_2$ without further information. For some games the first player will have a winning strategy in $G_1 + G_2$, while in other games the second player will have the advantage.

Notice that for two games G_1, G_2 of Cloves, on initial surfaces M_1, M_2 respectively, the game G of Cloves that starts with the disjoint union $M_1 \cup M_2$ is the disjunctive sum $G_1 + G_2$. Thus Lemmas 3 and 5 show that, for a game of Cloves on a surface M, if one of the connected components C of M has the same number of dots as boundary curves, then C plays no role in the game; that is, as far as the outcome of the game is concerned, one may just ignore C. Consequently, for all intents and purposes, we may assume that in every connected component of M, there are more dots than boundary curves. We are now ready to move on to our main result.

Main result

THEOREM 3. *Consider a game of Cloves played on an initial surface M with b boundary curves, d dots and Euler characteristic χ. Assume that in every connected component of M, there are more dots than boundary curves. Then the second player has a winning strategy if and only if b, d, χ are either all odd or all even.*

Proof. The proof is by induction on the integer $j = d - \chi$. Notice that by hypothesis, each connected component has more dots than boundary curves; thus $d \geq b + c$. Hence, as $c \leq b$, we have $d \geq 2c$. By Remark 2 following Theorem 2, $\chi \leq 2c - b$. Thus

$$j = d - \chi \geq d - 2c + b \geq b.$$

Hence the smallest possible value of j is $j = 1$, which occurs when $b = 1$, $c = 1$, $d = 2$, $\chi = 1$; in this case, M is a single disc with two dots on its boundary. The winner is obviously the first player. So Theorem 3 holds for $j = 1$.

Now suppose that $j = d - \chi > 1$ and assume that Theorem 3 holds for all surfaces with smaller j. First suppose that M is orientable. We have:

LEMMA 6. *If M is orientable, the first player wins if and only if $b + d$ is odd (regardless of how the game is played).*

Proof. For an orientable surface M, one has $\chi(M) \equiv b \pmod 2$ by Theorem 2(a). By Lemma 2(b), every connected component of the surface M_n, at the end of the game, has exactly one boundary curve and exactly one dot. Since the number of dots is constant throughout the game, M_n has d connected components and thus $\chi(M_n) \equiv d \pmod 2$. Hence, by Lemma 1,

$$n = \chi(M_n) - \chi(M) \equiv b + d \pmod 2.$$

As the first player wins if and only if n is odd, the proof is complete. ∎

Notice that Lemma 6 proves Theorem 3 in the orientable case. Indeed, one has $\chi \equiv b \pmod 2$ for orientable surfaces, so the condition "b, d, χ all odd or all even" is the same as "b, d both odd or both even," which is the same as "$b + d$ is even."

Now suppose that M is non-orientable. Notice that the condition "b, d, χ are all odd or all even" is equivalent to the condition: $b + d$ and $b + \chi$ are both even. We consider four cases; we show that the first player has a winning strategy in the first three cases, while the second player has a winning strategy in the fourth case.

Case 1. $b + d$ odd, $b + \chi$ even. The first player must first choose a good connected component to play in. He/she should choose a connected component C with $d(C) > b(C) + 1$, if one exists. If there isn't one, then $d(C) = b(C) + 1$ for each connected component C. In this case, since $b(M) + d(M)$ is odd, there is an odd number of connected components. Hence, since $b(M) + \chi(M)$ is even, there is a connected component C for which $b(C) + \chi(C)$ is even. The first player should choose such a connected component.

Having chosen the connected component C, the first player cuts C along a curve joining two dots on the same boundary curve of C, in such a way that the resulting surface C' is the disjoint union of a homeomorphic copy C' of C, with one fewer dot than C, and a disc with a single dot on its boundary; see FIGURE 25. Eliminating this disc, we see that the effect of this move is to replace M by a surface M' for which $b(M') + d(M')$ is even, while $b(M') + \chi(M')$, which has remained unchanged, is still even. Moreover, the integer j has decreased by one.

It remains to apply the inductive hypothesis. To do so, we must ensure that no connected component of our surface has the same number of boundary curves as dots. This is certainly the case for M' if $d(C) > b(C) + 1$ for the connected component C in which the first move was played. In this case, $d(C') > b(C')$ and we can apply the inductive hypothesis to M'. In the other case, from the choice of C, we have $d(C') = b(C')$ and $b(C') + \chi(C') = b(C) + \chi(C)$ is even. By Lemma 3 the second player has a winning stratgey on C', and so by Lemma 5 we can eliminate that component. The elimination of C' doesn't change the parity of $b(M') + d(M')$ or $b(M') + \chi(M')$ and so we can apply the inductive hypothesis to the resulting surface M''. Since the second player has a winning strategy on M' or M'', the first player can adopt that stratgegy to win the game from this point.

Case 2. $b + d$ odd, $b + \chi$ odd. Since $b(M) + \chi(M)$ is odd, M has a connected component C for which $b(C) + \chi(C)$ is odd. In particular, C is non-orientable. The first player's strategy is as follows: choose a boundary curve γ of C with two dots, x and y. Draw a curve from x out to a cross-cap, around the cross-cap and back near x, and hug the boundary of C around to y; see FIGURE 26. Splitting C along γ, we obtain a cross-cap with a single dot on its boundary, and a connected component C' with one fewer dots and one fewer cross-caps than C. Eliminating the cross-cap, the surface M' we obtain has $b(M') + d(M')$ even and $b(M') + \chi(M')$ even. This time the integer j has decreased by 2. If $d(C) > b(C) + 1$, then $d(C') > b(C')$ and we can apply the inductive hypothesis to M'. If $d(C) = b(C) + 1$, then $d(C') = b(C')$. Thus, as $b(C') + \chi(C')$ is even, we can eliminate C' from M' without changing the parity of $b(M') + d(M')$ or $b(M') + \chi(M')$. As in case 1, applying the inductive hypothesis, we conclude that the first player has a winning strategy.

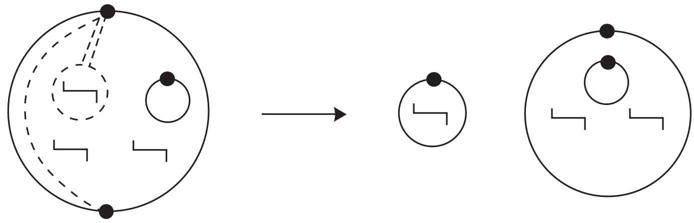

Figure 26

Case 3. $b + d$ even, $b + \chi$ odd. Once again, as $b(M) + \chi(M)$ is odd, M has a non-orientable connected component C for which $b(C) + \chi(C)$ is odd. We choose a boundary curve of C with 2 dots x and y, but this time we join x to y by a curve γ such that splitting along γ gives a connected surface, as in FIGURE 22. For the resulting surface M', j has been reduced by one, b and d are unchanged, and χ has increased by one so $b + \chi$ is now even. As b and d are unchanged, we haven't introduced a connected component with the same number of boundary curves as dots. Hence we may apply the inductive hypothesis, thus giving the first player a winning strategy.

Case 4. $b + d$ even, $b + \chi$ even. We must show that regardless of the move the first player chooses, the second player will find himself in a winning position. There are three possibilities:

Subcase 4a. The first player draws a curve that connects dots on distinct boundary curves, as in FIGURE 24. The resulting surface M' has one fewer boundary curves, and the same number of dots as M. The integer j has decreased by one and $b(M') + d(M')$ is odd. Clearly, we have not created a connected component with the same number of boundary curves as dots; so we can apply the inductive hypothesis (case 1).

Subcase 4b. The first player draws a curve γ that connects dots on the same boundary curve in such a way that splitting along γ produces an extra boundary curve; see the two examples in FIGURE 27.

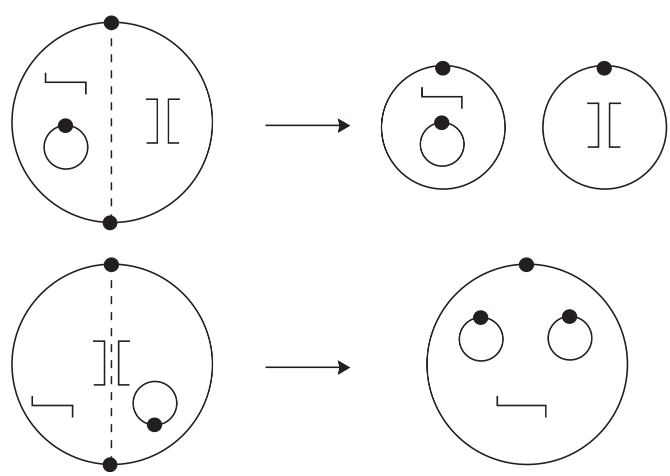

Figure 27

As the number of boundary curves has increased by one, $b(M') + d(M')$ is odd. It is possible that one or both of the newly created connected components has the same number of boundary curves as dots, but eliminating these will not change the parity of $b(M') + d(M')$ nor change the winner of the game, in view of Lemmas 3 and 5. In either case, the integer j has decreased by at least one. So we may apply the inductive hypothesis (case 1 or 2).

Subcase 4c. The first player draws a curve γ that connects dots on the same boundary curve in such a way that splitting along γ doesn't produce an extra boundary curve; see FIGURE 28. In this case, χ decreases by one. Thus $b(M') + d(M')$ is even and $b(M') + \chi(M')$ is odd. As the number of dots and boundary curves is unchanged, we have not created a connected component with the same number of boundary curves as dots, and we can apply the inductive hypothesis (case 3).

This completes the inductive step, and finishes the proof of Theorem 3. ■

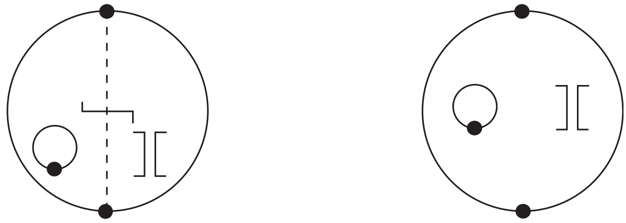

Figure 28

Remark 4. Notice that Theorem 1 is a corollary of Theorem 3. Indeed, for a game of Brussels Sprouts with m crosses on a surface N, the corresponding game of Cloves has $b = m$, $d = 4m$ and $\chi = \chi(N) - m$. Thus m and $\chi(N)$ are both even if and only if b, d and χ are all even or all odd.

Remark 5. Cloves is closely related to a game called Stars and Stripes [**4**]. In this game, one begins with a general collection of dots with free arms, as in FIGURE 3, and one plays the game as in Brussels Sprouts, except that for each move, when the player draws a new curve γ, the player can choose whether or not to draw a cross on γ. Thus the games of Cloves may be thought of as being between Brussels Sprouts and Stars and Stripes.

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