CSE21DMO Solutions to Discrete Mathematics Exam, 2004

1. (a) (i) $\sum_{k=8}^{32} 2k$ (ii) $\sum_{i=1}^{n} \frac{1}{3i-1}$

(b)
$$S_3 = 2\times S_2 + \tfrac{3}{2}\times S_1 = 2\times 3 + \tfrac{3}{2}\times 2 = 6+3 = \mathbf{9}$$
$$S_4 = 2\times S_3 + \tfrac{4}{3}\times S_2 = 2\times 9 + \tfrac{4}{3}\times 3 = 18+4 = \mathbf{22}$$
$$S_5 = 2\times S_4 + \tfrac{5}{4}\times S_3 = 2\times 22 + \tfrac{5}{4}\times 9 = 44+11\tfrac{1}{4} = \mathbf{55\tfrac{1}{4}}$$

2.

For n=1: $\text{LHS} = \frac{1}{1\times 2} = \frac{1}{2}$ $\text{RHS} = \frac{1}{1+1} = \frac{1}{2}$ ✓

For n=k: Assume, $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k\times(k+1)} = \frac{k}{k+1}$

For n=k+1:
$$\text{LHS} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k\times(k+1)} + \frac{1}{(k+1)\times(k+2)}$$
$$= \left[\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k\times(k+1)}\right] + \frac{1}{(k+1)\times(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)\times(k+2)} \qquad \text{by the Inductive Assumption}$$
$$= \frac{k\times(k+2)}{(k+1)\times(k+2)} + \frac{1}{(k+1)\times(k+2)}$$
$$= \frac{k^2+2k+1}{(k+1)\times(k+2)} = \frac{(k+1)^2}{(k+1)\times(k+2)} = \frac{(k+1)\times(k+1)}{(k+1)\times(k+2)}$$
$$= \frac{k+1}{k+2}$$
$$= \text{RHS}$$

By Induction, $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n\times(n+1)} = \frac{n}{n+1}$ for $n \geq 1$

3. (a)

Function	$\sqrt{n}$	n	$\log_2(n)$	$\log_2(n!)$	$n^{1/3}$
Number	3	4	1	5	2

(b)
$$\left|0.6n^2\log_2(n) + \tfrac{1}{2}n^2 + 12n\log_2(n) + n + 5\log_2(n)\right|$$
$$= 0.6n^2\log_2(n) + \tfrac{1}{2}n^2 + 12n\log_2(n) + n + 5\log_2(n) \qquad \text{as all terms are +ve}$$
$$\leqslant 0.6n^2\log_2(n) + 0.1n^2\log_2(n) + 0.1n^2\log_2(n) + 0.1n^2\log_2(n) + 0.1n^2\log_2(n)$$
$$= 1.0n^2\log_2(n)$$

Provided:

$\frac{1}{2}n^2 \leqslant 0.1n^2\log_2(n)$	$12n\log_2(n) \leqslant 0.1n^2\log_2(n)$	$n \leqslant 0.1n^2\log_2(n)$	$5\log_2(n) \leqslant 0.1n^2\log_2(n)$
$5n^2 \leqslant n^2\log_2(n)$	$120n\log_2(n) \leqslant n^2\log_2(n)$	$10n \leqslant n^2\log_2(n)$	$50\log_2(n) \leqslant n^2\log_2(n)$
$5 \leqslant \log_2(n)$	$120 \leqslant n$	$10 \leqslant n\log_2(n)$	$50 \leqslant n^2$
$2^5 \leqslant n$	$120 \leqslant n$	$10 \leqslant n$	$\sqrt{50} \leqslant n$

Choosing: $c = 1$ and $M = \max\left\{32, 120, 10, \sqrt{50}\right\}$

we have $0.6n^2\log_2(n) + \frac{1}{2}n^2 + 12n\log_2(n) + n + 5\log_2(n) \in O\left(n^2\log_2(n)\right)$

4. (a) Homogeneous version: $I_n^H - I_{n-1}^H = 0$

Characteristic equation: $a - 1 = 0$

$\implies a = 1$

General Homogeneous solution: $I_n^H = A\times 1^n = A$

Particular solution, try: $I_n^P = B\times n^2 + C\times n$

$$\begin{aligned}
\text{LHS} &= B\times n^2 + C\times n - \left[B\times(n-1)^2 + C\times(n-1)\right]\\
&= B\times n^2 + C\times n - \left[B\times(n^2-2n+1) + C\times n - C\right]\\
&= B\times n^2 + C\times n - \left[B\times n^2 - 2Bn + B + C\times n - C\right]\\
&= B\times n^2 + C\times n - B\times n^2 + 2Bn - B - C\times n + C\\
&= 2Bn - B + C\\
\text{RHS} &= 2n+1
\end{aligned}$$

So $2B = 2$ and $-B + C = 1$

$\implies B = 1$ and $C = 2$

Particular Solution is: $I_n^P = 2n^2 + 2n$

General Solution: $I_n = I_n^P + I_n^H = n^2 + 2n + A$

Using initial condition: $I_1 = 1 = 1^2 + 2\times 1 + A$

$\implies 1 + 2 + A = 1 \implies A = -2$

Solution is: $\boxed{I_n = n^2 + 2n - 2}$

(b) $I_{30} = 30^2 + 2\times 30 - 2 = 900 + 60 - 2 = 958$

5. (a) f(A) = **J** f(B) = **G** f(C) = **I** f(D) = **F** f(E) = **H**

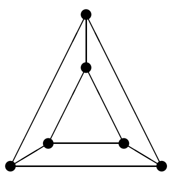
OR f(A) = **J** f(B) = **G** f(C) = **I** f(D) = **H** f(E) = **F**

(b) Eulerian Path: **ab bc ca ae ed db bf fa ag gc cd df fe eg gb** (Edge List)

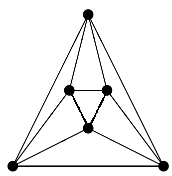
Eulerian Path: **a b c a e d b f a g c d f e g b** (Vertex List)

6. (a) In Graph 4 the vertices of degree three are not adjacent. (*Other reasons are possible*)

(b) (i) Plane Graph for **A**



(ii) Plane Graph for **B**



7. (a)

List to be Sorted	After Pass 1	After Pass 2	After Pass 3	After Pass 4	After Pass 5
Map	Ker	Basis	Basis	Basis	Basis
Ker	Map	Ker	Ker	Ker	Dim
Basis	Basis	Map	Map	Map	Ker
Vector	Vector	Vector	Vector	Linear	Map
Linear	Linear	Linear	Linear	Vector	Linear
Dim	Dim	Dim	Dim	Dim	Vector

(b)

	Number of Comparisons	Number of Exchanges
Pass 1	1	1
Pass 2	2	2
Pass 3	1	0
Pass 4	3	2
Pass 5	5	4

8. $e_1 = v_2v_6$ $e_2 = v_1v_2$ $e_3 = v_3v_5$ $e_4 = v_2v_4$ $e_5 = v_1v_6$

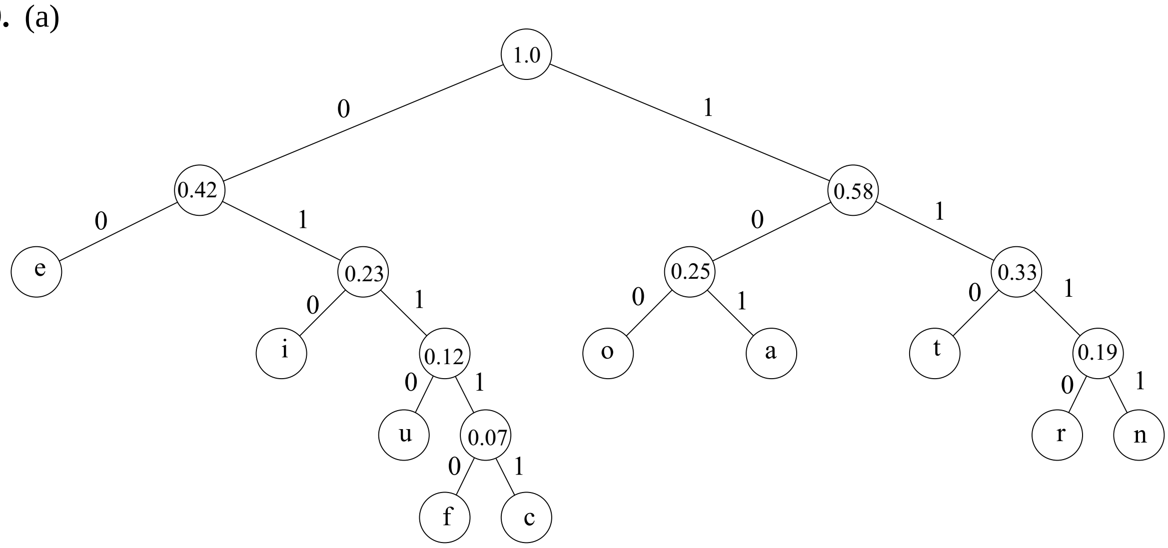
$e_6 = v_2v_5$ $e_7 = v_2v_3$ $e_8 = v_5v_6$ $e_9 = v_1v_4$ $e_{10} = v_3v_4$

	N(1)	N(2)	N(3)	N(4)	N(5)	N(6)	Edges	Weight
Initially	1	2	3	4	5	6	$\varnothing$	0
1.	1	2	3	4	5	2	e_1	1
2.	1	1	3	4	5	1	e_1, e_2	3
3.	1	1	3	4	3	1	e_1, e_2, e_3	6
4.	1	1	3	1	3	1	e_1, e_2, e_3, e_4	10
5.	1	1	3	1	3	1	e_1, e_2, e_3, e_4	10
6.	1	1	1	1	1	1	e_1, e_2, e_3, e_4, e_6	16

The edge set for the minimal spanning tree consists of edges: $\boxed{e_1, e_2, e_3, e_4, e_6}$

The weight of the minimal spanning tree is: **16**

9. (a)



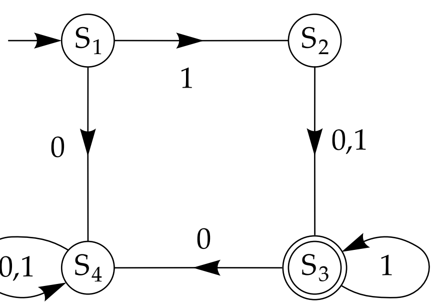
Character	Code	Character	Code
a	101	**n**	1111
c	01111	**o**	100
e	00	**r**	1110
f	01110	**t**	110
i	010	**u**	0110

(b) **acute** encodes as: 101 01111 0110 110 00

(c) 0111011101010111111001010011111

decodes as: **fraction**

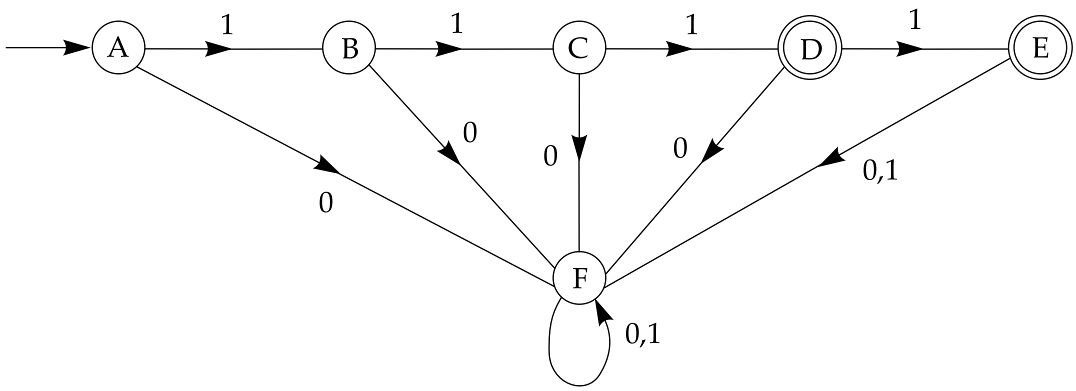
10. (a) (i)



(ii) In symbolic notation: $1(0|1)1^*$

OR $1(0|1)1^n \quad n \geqslant 0$

(b)



11. (a) There are 11 letters: 2 C's 2 A's 2 L's 1 U 1 T 1 I 1 O and 1 N.

So there are: $\binom{11}{2,2,2,1,1,1,1,1} = \frac{11!}{2!\times 2!\times 2!\times 1!\times 1!\times 1!\times 1!\times 1!} = \mathbf{4989600}$ arrangements.

(b) **GLENELG**

Case 1. Delete **G** So for **GLENEL**, $\binom{6}{1,2,2,1} = \frac{6!}{1!\times 2!\times 2!\times 1!} = 180$ arrangements

Case 2. Delete **L** So for **GLENEG**, $\binom{6}{2,1,2,1} = \frac{6!}{2!\times 1!\times 2!\times 1!} = 180$ arrangements

Case 3. Delete **E** So for **GLENLG**, $\binom{6}{2,2,1,1} = \frac{6!}{2!\times 2!\times 1!\times 1!} = 180$ arrangements

Case 4. Delete **N** So for **GLEELG**, $\binom{6}{2,2,2} = \frac{6!}{2!\times 2!\times 2!} = 90$ arrangements

Total of: **630** arrangements.

(c)

	Length: **n=2**	**n=5**	**n**
How many bit strings of length … ?	$\mathbf{2^2 = 4}$	$\mathbf{2^5 = 32}$	$\mathbf{2^n}$
How many contain no 1's ?	$\mathbf{{}^2C_0 = 1}$	$\mathbf{{}^5C_0 = 1}$	**1**
How many contain two 1's ?	$\mathbf{{}^2C_2 = 2}$	$\mathbf{{}^5C_2 = 10}$	$\mathbf{{}^nC_2 = n}$
How many contain three 1's ?	**0**	$\mathbf{{}^5C_3 = 10}$	$\mathbf{{}^nC_3}$
How many are palindromes ?	$\mathbf{2^1 = 2}$	$\mathbf{2^3 = 8}$	For **n** even: $\mathbf{2^{(n/2)}}$ For **n** odd: $\mathbf{2^{\frac{n+1}{2}}}$

12.