1. (a) $1+2+3+\cdots+n=\sum_{i=1}^{n} i$

   (b) $1^2+2^2+3^2+\cdots+n^2=\sum_{i=1}^{n} i^2$ (c) $1+3+5+\cdots+2n-1=\sum_{i=1}^{n}(2i-1)$

   (d) $\sum_{i=1}^{n} i^5+\sum_{i=1}^{n} i^7=2\left(\sum_{i=1}^{n} i\right)^4$ (e) $\sum_{i=1}^{n} i^3=\left(\sum_{i=1}^{n} i\right)^2$

2. (a)
$$1\times 2+3\times 4+\cdots+(2n-1)\times 2n=\sum_{r=1}^{n}(2r-1)2r=\sum_{r=1}^{n}(4r^2-2r)$$
$$=4\sum_{r=1}^{n} r^2-2\sum_{r=1}^{n} r \qquad \text{by Basic Rules (i) and (ii)}$$

   (b)
$$1^2+3^2+\cdots+(2n-1)^2=\sum_{r=1}^{n}(2r-1)^2$$
$$=\sum_{r=1}^{n}(4r^2-4r+1)$$
$$=4\sum_{r=1}^{n} r^2-4\sum_{r=1}^{n} r+\sum_{r=1}^{n} 1 \qquad \text{by Basic Rules (i) and (ii)}$$

3. (a) $2n^2+6\in\mathbf{O}(n^2)$
   if there are positive real numbers c and M, such that $|2n^2+6|\le c|n^2|$ for all $n\ge M$.

   (b) $n^2+2n\log n+\frac{1}{2}\in\mathbf{O}(n^3)$
   if there are positive real numbers c and M, such that $|n^2+2n\log n+\frac{1}{2}|\le c|n^3|$ for all $n\ge M$.

   (c) $\frac{1}{2}n^3-6n^2+\frac{5}{3}\in\mathbf{O}(n^2)$
   if there are positive real numbers c and M, such that $|\frac{1}{2}n^3-6n^2+\frac{5}{3}|\le c|n^2|$ for all $n\ge M$.
   The modulus signs are not really necessary on the RHS in (a) and (c) above as $n^2=|n^2|$.

4. Many different answers are possible.
$$|f(n)|=|2n^2+6|\leqslant|2n^2|+|6| \quad \text{by the triangle inequality}$$
$$=2n^2+6 \qquad \text{for } n\geqslant 1$$
$$\leqslant 2n^2+6n^2=8n^2 \qquad \text{since } 6\leqslant 6n^2$$
   choosing $c=8$ and $M=1$ we have: $2n^2+6\in\mathbf{O}(n^2)$

5. $|3n^3-2n^2+\frac{1}{2}|\leqslant|3n^3|+\left|-2n^2\right|+|\frac{1}{2}|$ by the triangle inequality
$$=3n^3+2n^2+\tfrac{1}{2} \qquad \text{for } n\geqslant 1$$
$$\leqslant 3n^3+2n^3+\tfrac{1}{2}n^3 \qquad \text{as } 2n^2\leqslant 2n^3 \quad \& \quad \tfrac{1}{2}\leqslant\tfrac{1}{2}n^3$$
$$=5\tfrac{1}{2}n^3$$
   Choosing $c=5\frac{1}{2}$ & $M=1$ we have: $3n^3-2n^2+\frac{1}{2}\in\mathbf{O}(n^3)$

6. $|\frac{1}{3}n^2-2n\log n+\frac{1}{6}n-2|\leqslant|\frac{1}{3}n^2|+|-2n\log n|+|\frac{1}{6}n|+|-2|$ by the $\triangle$ inequality
$$=\tfrac{1}{3}n^2+2n\log n+\tfrac{1}{6}n+2 \qquad \text{for } n\geqslant 1$$
$$\leqslant\tfrac{1}{3}n^2+2n^2+\tfrac{1}{6}n^2+2n^2 \qquad \text{as } \log n\leqslant n \;\&\; \tfrac{1}{6}n\leqslant\tfrac{1}{6}n^2 \;\&\; 2\leqslant 2n^2$$
$$=4\tfrac{1}{2}n^2$$
   Choosing $c=4\frac{1}{2}$ & $M=1$ we have $\frac{1}{3}n^2-2n\log n+\frac{1}{6}n-2\in\mathbf{O}(n^2)$

7. (a) $|f(n)|=\alpha n^2+\beta n+\gamma$ since $\alpha,\beta,\gamma\geqslant 0$ and $n\geqslant 1$
$$\leqslant\alpha n^2+\beta n^2+\gamma n^2 \quad \text{since } \gamma\leqslant\gamma n^2 \quad \& \quad \beta n\leqslant\beta n^2 \text{ for } n\geqslant 1$$
$$=(\alpha+\beta+\gamma)n^2$$
   $\therefore$ putting $c=\alpha+\beta+\gamma$, $M=1$ we have
   $|f(n)|\leqslant cn^2$ for all $n\geqslant M$

   (b) $|\alpha n^2+\beta n+\gamma|\leqslant|\alpha|n^2+|\beta|n+|\gamma|$ $\forall\alpha,\beta,\gamma\in\mathbb{R}$
   Now use $|\alpha|$ $|\beta|$ $|\gamma|$ in the argument for part (i).