1. (a) (i) $\sum_{k=0}^{n} 2^k$ **2 MARKS** (ii) $\sum_{k=2}^{20} \frac{1}{5k}$ **2 MARKS**

(b) $S_2 = 2 \times S_1 + 2^2 = 2 + 4 = \mathbf{6}$ $S_3 = 3 \times S_2 + 2^3 = 18 + 8 = \mathbf{26}$ **2 MARKS**

2. For n=1: LHS = 2

RHS $= 1 \times (1+1) = 1 \times 2 = 2$ ✓

For n=k: Assume $2 + 4 + 6 + \cdots + 2k = k(k+1)$.

For n=k+1: LHS $= 2 + 4 + 6 + \cdots + 2k + 2(k+1)$

$= [2 + 4 + 6 + \cdots + 2k] + 2(k+1)$

$= k \times (k+1) + 2(k+1)$ by Inductive Assumption

$= (k+1) \times (k+2)$ ✓

By Induction, $2 + 4 + 6 + \cdots + 2n = n \times (n+1)$ **5 MARKS**

3. (a) **2 MARKS**

Function	2^n	$n!$	3^n	n^3	$n^2 2^n$
Number	2	5	4	1	3

(b)

$$\begin{aligned} |2.2n^2 + 1.1n\log_2(n) - 20n + 15| &\leq |2.2n^2| + |1.1n\log_2(n)| + |-20n| + |15| \quad \text{by } \triangle\text{-inequality} \\ &= 2.2n^2 + 1.1n\log_2(n) + 20n + 15 \\ &\leq 2.2n^2 + 1.1n^2 + 20n^2 + 15n^2 \\ &= 38.3n^2 \end{aligned}$$

Choosing $c = 38.3, M = 1$; we have $2.2n^2 + 1.1n\log_2(n) - 20n + 15 \in O(n^2)$ **5 MARKS**

(c) $\sum_{k=1}^{n-1} k \in O(n^2)$ **2 MARKS**

4. (a) (i) $a + b + c + d = \mathbf{10}$ (ii) $\mathbf{0} \leqslant a \leqslant \mathbf{10}$ **2 MARKS**

(b) $\sum_{r=1}^{n} \binom{n}{r} = \mathbf{2^n - 1}$ **2 MARKS**

(c) $\binom{7}{2,2,2,1} = \frac{7!}{2! \times 2! \times 2! \times 1!} = \frac{5040}{8} = 630$ There are 2 G's, 2 L's, 2 E's and 1 N. **2 MARKS**

(d) **630** (i) delete G : $\binom{6}{2,2,1,1} = \frac{6!}{2! \times 2! \times 1! \times 1!} = \frac{720}{4} = 180$ arrangements of LENELG

(ii) delete L : $\binom{6}{2,2,1,1} = \frac{6!}{2! \times 2! \times 1! \times 1!} = \frac{720}{4} = 180$ arrangements of GENELG

(iii) delete E : $\binom{6}{2,2,1,1} = \frac{6!}{2! \times 2! \times 1! \times 1!} = \frac{720}{4} = 180$ arrangements of GLNELG

(iv) delete N : $\binom{6}{2,2,2} = \frac{6!}{2! \times 2! \times 2!} = \frac{720}{8} = 90$ arrangements of GLEELG

TOTAL: 630 arrangements. **2 MARKS**

(e) **360**

Can delete 2 G's, 2 E's, 2 L's or GN, LN or EN $6 \times \binom{5}{2,2,1} = 6 \times 30 = 180$ arrangements

Can delete GL, GE or LE $3 \times \binom{5}{2,1,1,1} = 3 \times 60 = 180$ arrangements.

Total: 360 arrangements. **4 MARKS**

5.

6 MARKS – $\frac{1}{3}$ each	**n = 3**	**n = 8**	**n**
How many strings of length ... ?	$3^3 = 27$	$3^8 = 6561$	3^n
How many contain no $\boldsymbol{a}$'s?	$2^3 = 8$	$2^8 = 256$	2^n
How many contain one $\boldsymbol{a}$?	${}^3C_1 \times 2^2$	${}^8C_1 \times 2^7$	${}^nC_1 \times 2^{n-1}$
How many contain two $\boldsymbol{a}$'s?	${}^3C_2 \times 2^1$	${}^8C_2 \times 2^6$	${}^nC_2 \times 2^{n-2}$
How many contain three $\boldsymbol{a}$'s?	${}^3C_3 = 1$	${}^8C_3 \times 2^5$	${}^nC_3 \times 2^{n-3}$
How many are palindromes?	$3 \times 3 = 9$	3^4	$\begin{cases} 3^{\frac{n}{2}} & \text{if } \mathbf{n} \text{ is even} \\ 3^{\frac{n+1}{2}} & \text{if } \mathbf{n} \text{ is odd} \end{cases}$

6. (a)

(1) Using a characteristic equation

Characteristic equation: $a - 1 = 0$

$\implies a = 1$

General solution: $I_n = A \times 1^n = A$

Using initial condition: $I_1 = A = \frac{1}{2}$

Solution: $\boxed{I_n = \frac{1}{2}}$

(2) Using iteration

$I_n = I_{n-1}$

$I_2 = I_1 = \frac{1}{2}$

$I_3 = I_2 = \frac{1}{2}$

Guess: $\boldsymbol{I_n = \frac{1}{2}}$

2 MARKS

(b)

(1) Solution:

Characteristic equation: $a^2 - 1 = 0$

$\implies (a-1)(a+1) = 0 \implies a = 1$ or $a = -1$

General solution: $J_n = A \times 1^n + B \times (-1)^n$

or $J_n = A + B \times (-1)^n$

Using initial conditions: $J_0 = A + B = 0$

$J_1 = A - B = \frac{1}{2}$

$\implies A = \frac{1}{4} \quad B = -\frac{1}{4}$

Solution: $\boxed{J_n = \frac{1}{4} - \frac{1}{4} \times (-1)^n}$

(2) Check:

Check initial conditions:

$J_0 = \frac{1}{4} - \frac{1}{4} = 0$

$J_1 = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$ ✓

For the recurrence:

LHS$= J_n - J_{n-2}$

$= \frac{1}{4} - \frac{1}{4} \times (-1)^n - \left[\frac{1}{4} - \frac{1}{4} \times (-1)^{n-2}\right]$

$= \frac{1}{4} - \frac{1}{4} \times (-1)^2(-1)^{n-2} - \frac{1}{4} + \frac{1}{4}(-1)^{n-2}$

$= \left[\frac{1}{4} - \frac{1}{4}\right] + \left[-\frac{1}{4} + \frac{1}{4}\right] \times (-1)^{n-2}$

$= 0 + 0 = 0$ ✓

5 + 3 MARKS

7. (a)

List to be sorted	After Pass 1	After Pass 2	After Pass 3	After Pass 4	After Pass 5
Hash	Bubble*	Bubble*	Bubble*	Bubble*	Bubble*
Insert	Insert	Hash*	Hash*	Hash*	Hash*
Bubble	Hash	Insert	Insert*	Insert*	Insert*
Quick	Quick	Quick	Quick	Merge*	Merge*
Select	Select	Select	Select	Select	Quick*
Merge	Merge	Merge	Merge	Quick	Select

(b)

	Pass 1	Pass 2	Pass 3	Pass 4	Pass 5
Number of Comparisons	5	4	3	2	1
Number of Exchanges	1	1	0	1	1

(c) $\mathbf{n - 1}$ passes

(a) **4 MARKS** (b) **2 MARKS** (c) **1 MARK**