Lexicographic Ordering, Recurrence Reluations & A Special Formula

Lexicographic Ordering:

For the next definition we consider a "*word*" to be any finite string of symbols out of an ordered list of symbols. The ordered list is referred to as the "*alphabet*".

Definition A word $s_1s_2\ldots s_n$ precedes a word $t_1t_2\ldots t_m$ in *lexicographic order* if there is an integer K such that
(a) $s_i = t_i$ for $i < K$ and s_K precedes t_K in the alphabet,
(b) $s_i = t_i$ for $i \leq n$ and $n < m$.

A systematic way of enumerating permutations is given by the following algorithm. Here we use $x < y$ to mean x is before y (in the alphabet).

Algorithm *(Given a permutation $a_1a_2\ldots a_n$, the algorithm will produce the next permutation in lexicographic order.)*
(a) Find K, such that $a_K < a_{K+1} > a_{K+2} > \cdots > a_n$.
(b) Interchange a_K with the smallest of $a_{K+1},\ldots,a_n$ that *exceeds* a_K in the alphabet.
(c) Put the *new* $a_{K+1},\ldots,a_n$ in increasing order.

Algorithm (Sensible Version)
(a) Starting from the right, move left until you find the first " $<$ " sign.
(b) Interchange the element to the left of this sign with the next biggest element (in the ordering) which occurs to the right of the $<$ sign.
(c) Reverse the order of the new set of elements to the right of the $<$ sign.

Example 1 Find the next word, in lexicographic order, after the permutation $gedfcba$.
We have $g > e > d < f > c > b > a$, so Step (a) identifies $a_K = d$.
The smallest of $fcba$ which exceeds d is f so, by Step (b),
we interchange f and d giving $g\ e\ f\underline{d\ c\ b\ a}$.
By Step (c), we now rewrite $dcba$ in increasing order giving $gefabcd$.

Question 1: For the characters a, b, c, d, e, f, g:
(a) How many permutations are there? (b) Which is the first permutation?
(c) Which is the last permutation? (d) How many permutations begin with d?
(e) Which permutation succeeds $acgfedb$

An Example of Setting up a Recurrence Relation (Difference Equation)

$\alpha =$	1	2	3	4	5	$\longrightarrow$	$n-1$	n	$n+1$	$\longrightarrow$
$2^\alpha =$						$\longrightarrow$				$\longrightarrow$
S_α	S_1	S_2	S_3	S_4	S_5	$\longrightarrow$				$\longrightarrow$

Firstly, fill in the blank cells in the table above.

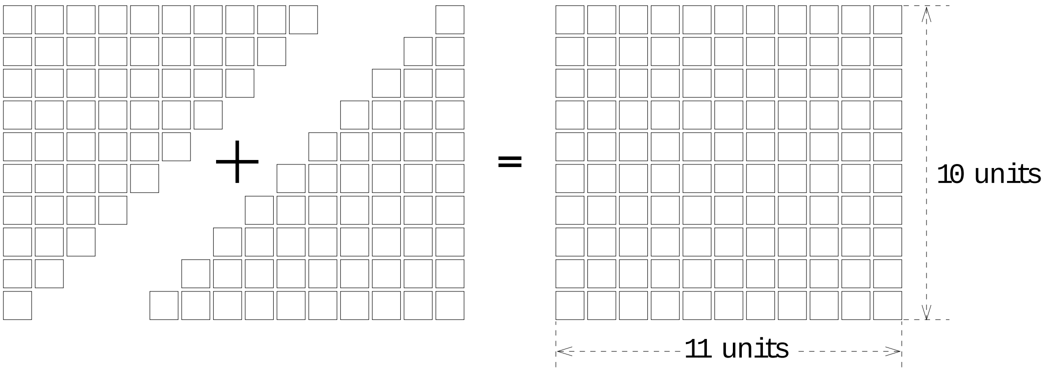
$S_5 - S_4 = \ldots\ldots$ $S_4 - S_3 = \ldots\ldots$ $S_3 - S_2 = \ldots\ldots$

Then calculate: $S_n - S_{n-1} = \ldots\ldots\ldots\ldots$ A suitable Initial Condition is: $S_1 = \ldots\ldots$
Comment: This is why these relationships are called <u>Difference Equations</u>.

Another version of this Difference Equation is: $S_n = \ldots\ldots \times S_{n-1}$, again with $S_1 = 2$.

To set up a Recurrence Relation, we need to compare Stage "$\boldsymbol{n}$" of a process and Stage "$\boldsymbol{n-1}$" of the process.

A Fundamental Result: The Sum of the First "k" +ve integers



Area on Left $= 1+2+3+4+5+6+7+8+9+10$
Area in Middle $= 1+2+3+4+5+6+7+8+9+10$
Area on Right $= 10 \times 11$
Area on Left + Area in Middle = Area on Right
So, $2 \times (1+2+3+4+5+6+7+8+9+10) = 10 \times 11$

Dividing both sides by 2 $\rightarrow$ $1+2+3+4+5+6+7+8+9+10 =$ ....

Generalising this for the sum $1+2+3+\cdots+k$
$2 \times (1+2+3+\cdots+k) =$ ....

Again, dividing both sides by 2 $\rightarrow$ $1+2+3+\cdots+k =$ ....