1. (a),(b),(c) In each case List 3 is: $\begin{array}{c}1\\2\\\underline{3}\\4\end{array}$ and one comparison is required to move from List 3 to List 4.

(d) 3! comparisons are required.

(e),(f) List 3 is: $\begin{array}{c}1\\2\\\underline{4}\\3\end{array}$ and in each case 2 comparisons are required to move from List 3 to List 4.

(g) The total number required is $(2 \times 3!)$

(h) List 3 is: $\begin{array}{c}1\\3\\\underline{4}\\2\end{array}$

(i) In each case 3 comparisons are required.

(j) The total number required is $T_3 + (3 \times 3!)$

(k) $\begin{array}{c}2\\3\\\underline{4}\\1\end{array}$

(l) $T_3 + (3 \times 3!)$

2. (a)

$$\begin{aligned} A_n &= \tfrac{n}{n!}T_{n-1} + \tfrac{(n-1)!}{n!} \times \Big(1 + 2 + \cdots + (n-1) + (n-1)\Big) \\ &= \tfrac{T_{n-1}}{(n-1)!} + \tfrac{1}{n} \times \Big(1 + 2 + \cdots + (n-1) + (n-1)\Big) \\ &= A_{n-1} + \tfrac{1}{n} \times \Big(1 + 2 + \cdots + (n-1) + (n-1)\Big) \end{aligned}$$

(b) $A_n - A_{n-1} = \frac{1}{n} \times \Big(1 + 2 + \cdots + (n-1) + (n-1)\Big)$

(c) $A_n - A_{n-1} = \frac{1}{n}\Big(\frac{1}{2}n^2 - \frac{1}{2}n + n - 1\Big) = \frac{1}{n}\Big(\frac{1}{2}n^2 + \frac{1}{2}n - 1\Big) = \frac{1}{2}n + \frac{1}{2} - \frac{1}{n}.$

3. (a) $U_n - U_{n-1} = A_n - A_{n-1} + \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n-1} \frac{1}{r} = A_n - A_{n-1} + \frac{1}{n} = \frac{n}{2} + \frac{1}{2}.$

(b) $U_n - U_{n-1} = \frac{n}{2} + \frac{1}{2}.$

Homogeneous version: $U_n^H - U_{n-1}^H = 0$

Characteristic equation: $a - 1 = 0 \implies a = 1$

General Homogeneous solution: $U_n^H = B \times 1^n = B.$

Since one term of the RHS is a solution of the homogeneous equation, we use the multiplication by n rule:

Particular solution – try $U_n^P = n \times (Cn + D) = Cn^2 + Dn.$

$$\begin{aligned} \text{LHS} \;=\; U_n^P - U_{n-1}^P &= Cn^2 + Dn - C(n-1)^2 - D(n-1) \\ &= Cn^2 + Dn - C(n^2 - 2n + 1) - (Dn - D) \\ &= Cn^2 + Dn - Cn^2 + 2Cn - C - Dn + D \\ &= 2Cn - C + D \\ \text{RHS} &= \tfrac{n}{2} + \tfrac{1}{2} \end{aligned}$$

so $2C = \frac{1}{2}$ and $-C + D = \frac{1}{2}$, giving $C = \frac{1}{4}$ and $D = \frac{3}{4}$.

$U_n^P = \frac{1}{4}n^2 + \frac{3}{4}n$ is a particular solution and the general solution is: $U_n = \frac{1}{4}n^2 + \frac{3}{4}n + B$ for some $B \in \mathbb{R}$.

(c) $A_2 = 1.$ **(d)** $\frac{5}{2} = U_2 = \frac{4}{4} + \frac{6}{4} + B = \frac{10}{4} + B$, so $B = 0.$

(e) $|A_n| = A_n \leqslant \frac{1}{4}n^2 + \frac{3}{4}n \leqslant \frac{1}{4}n^2 + \frac{3}{4}n^2 = n^2$ for all n.

Choosing $c = 1$ and $M = 1$, $A_n \in O(n^2)$.