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Endoprimal implication algebras

J. G. Pitkethly

Abstract. An algebra **A** is endoprimal if, for all $n \in \mathbb{N}$, the only maps $f: A^n \to A$ which preserve the endomorphisms of **A** are the n-ary term functions of **A**. The theory of natural dualities has been a very effective tool for finding finite endoprimal algebras. We study endoprimality within the variety of implication algebras, which does not contain any non-trivial dualisable algebras. We show that there are no non-trivial finite endoprimal implication algebras. We also give some examples of infinite implication algebras which are endoprimal.

Implication algebras model the implication operation of Boolean algebras. The class of Boolean algebras is the quasivariety generated by the two-element Boolean algebra $\underline{\mathbf{B}} = \langle \{0, 1\}; \vee, \wedge, ', 0, 1 \rangle$. Similarly, we define the class $\mathfrak{I}$ of implication algebras to be the quasivariety generated by the two-element implication algebra $\underline{\mathbf{I}} = \langle \{0, 1\}; \to, \vee, 1 \rangle$, where $a \to b := a' \vee b$ for all $a, b \in \{0, 1\}$. It was shown by Abbott [1] that $\mathfrak{I}$ is the variety determined by the equations

$$(x \to y) \to x \approx x,$$

$$(x \to y) \to y \approx (y \to x) \to x,$$

$$x \to (y \to z) \approx y \to (x \to z)$$

involving $\to$ and the equations $1 \approx x \to x$ and $x \vee y \approx (x \to y) \to y$. In particular, the fundamental operation $\to$ incorporates both $\vee$ and 1.

Abbott [1] established a bijective correspondence between the variety of implication algebras and the class of all upper-bounded join-semilattices for which every principal filter is a Boolean lattice. To show how this correspondence is set up, we firstly consider an implication algebra **A**. The algebra **A** has an upper-bounded

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join-semilattice as a term-reduct where, for each $a \in A$ and $b, c \in \uparrow a$, the greatest lower bound of b and c is given by $b \wedge c = ((b \to a) \vee (c \to a)) \to a$, and the complement of b in $\uparrow a$ is given by $b \to a$. On the other hand, if $\langle A; \vee, 1 \rangle$ is an upper-bounded semilattice such that $\uparrow a$ is a Boolean lattice for all $a \in A$, then we can construct an implication algebra on A by defining $a \to b$ to be the complement of $a \vee b$ in $\uparrow b$.

Let $\mathbf{A}$ be any algebra. Then $\mathbf{A}$ is n-**endoprimal** if every map $f: A^n \to A$ that preserves the endomorphisms of $\mathbf{A}$ is an n-ary term function of $\mathbf{A}$. An algebra is **endoprimal** if it is n-endoprimal for all $n \in \mathbb{N}$.

Endoprimal algebras have been characterised within several quasivarieties of logic-based algebras. Endoprimal distributive lattices were classified by Márki and Pöschel [9]. A non-trivial distributive lattice is endoprimal if and only if it is not relatively complemented. Davey and Pitkethly [7] showed, for example, that every Boolean algebra is endoprimal, every Heyting algebra in the quasivariety generated by a finite chain or by $\mathbf{2}^2 \oplus \mathbf{1}$ is endoprimal, and that a Stone algebra is endoprimal if and only if it is Boolean or its dense set is not relatively complemented.

The results in [7] were obtained using the theory of natural dualities. Instead of proving directly that a finite algebra is endoprimal, we may seek to show that the algebra is endodualisable. Although endodualisability is a stronger and more technical condition than endoprimality, the accumulated machinery of duality theory can make endodualisable algebras easier to identify. Our study of implication algebras will not make direct use of duality theory, so we omit a detailed definition of endodualisability. The interested reader may consult the introduction of [7] or the monograph Clark and Davey [2]. A finite algebra $\mathbf{M}$ is said to be endodualisable provided the endomorphism monoid End $\mathbf{M}$ yields a duality on $\mathbb{ISP}\,\mathbf{M}$. If $\mathbf{M}$ is endodualisable, then End $\mathbf{M}$ yields a duality on the free algebras in $\mathbb{ISP}\,\mathbf{M}$, which implies that $\mathbf{M}$ is endoprimal (First Duality Theorem [2]).

Márki and Pöschel obtained their description of endoprimal distributive lattices without the aid of natural dualities. However, their result was re-proved in the finite case using duality theory by Davey et al. [5]. The complete characterisation follows easily from Priestley duality for distributive lattices and the general results of Davey [4] and Davey and Pitkethly [7].

In this paper we consider endoprimality in a variety to which we cannot apply the tools of natural dualities. Davey and Werner [8] gave the two-element implication algebra as the first example of a non-dualisable algebra. It is shown in Davey et al. [6] that no non-trivial implication algebra is dualisable. So there are no non-trivial endodualisable implication algebras.

We show that there are no non-trivial finite endoprimal implication algebras. Infinite endoprimal implication algebras can be built up from free implication algebras. We give a description of the n-endoprimal implication algebras for each

$n \in \mathbb{N}$, which we use to give an example of an endoprimal implication algebra which does not have a retraction onto any non-trivial free algebra. If **B** is an infinite Boolean lattice, then the join-semilattice on the set $B \backslash \{0\}$ determines an endoprimal implication algebra.

1. Finite implication algebras

In order to show that certain implication algebras are not endoprimal, we use the fact that implication algebras do not have any near-unanimity term functions. This was proven by Mitschke [10]. The result also follows indirectly from the theory of natural dualities: the NU Duality Theorem [2] tells us that every finite algebra with a near-unanimity term is dualisable, and [8, 6] show that the two-element implication algebra is not dualisable.

We will prove that implication algebras do not have near-unanimity terms by giving a complete description of the term functions of the two-element implication algebra. Our characterisation will also be used to find endoprimal implication algebras. Note that $I := \{0, 1\}$ and $\pi_k : I^n \to I$ denotes the kth projection function, for $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$.

LEMMA 1.1. *Let $n \in \mathbb{N}$ and let $f: I^n \to I$. Then f is a term function of $\underline{\mathbf{I}}$ if and only if $f^{-1}(0) \subseteq \pi_k^{-1}(0)$ for some $k \in \{1, \ldots, n\}$.*

Proof. Assume that $f^{-1}(0) \subseteq \pi_k^{-1}(0)$ for some $k \in \{1, \ldots, n\}$. For each $a \in \{0, \ldots, 2^n - 2\}$, let $\bar{a}$ denote the element of I^n corresponding to the binary expression for a. Then, for each $a \in \{0, \ldots, 2^n - 2\}$, there is a term function $t_a : I^n \to I$, given by

$$t_a = \bigvee \{\pi_i \mid \bar{a}(i) = 0\} \vee \bigvee \{\pi_i \to \pi_j \mid \bar{a}(i) = 1 \quad \text{and} \quad \bar{a}(j) = 0\},$$

such that $t_a^{-1}(0) = \{\bar{a}\}$.

Define a sequence of n-ary term functions of $\underline{\mathbf{I}}$ by

$$s_{-1} := \pi_k$$

and

$$s_a := \begin{cases} s_{a-1} & \text{if } f(\bar{a}) = 0, \\ t_a \to s_{a-1} & \text{otherwise,} \end{cases}$$

for all $a \in \{0, \ldots, 2^n - 2\}$. We will show by induction that f is equal to the term function s_{2^n-2}. Assume that $f\restriction_{\{\bar{0}, \ldots, \overline{a-1}\}} = s_{a-1}\restriction_{\{\bar{0}, \ldots, \overline{a-1}\}}$ and $f^{-1}(0) \subseteq s_{a-1}^{-1}(0)$.

Case (i). $f(\bar{a}) = 0$. We have $s_a(\bar{a}) = s_{a-1}(\bar{a}) = 0$, since $f^{-1}(0) \subseteq s_{a-1}^{-1}(0)$. Therefore $f\restriction_{\{\bar{0}, \ldots, \bar{a}\}} = s_a\restriction_{\{\bar{0}, \ldots, \bar{a}\}}$ and $f^{-1}(0) \subseteq s_a^{-1}(0)$.

Case (ii). $f(\bar{a}) = 1$. Since $t_a\restriction_{I^n\backslash\{\bar{a}\}} = \underline{1}\restriction_{I^n\backslash\{\bar{a}\}}$, we know that

$$s_a\restriction_{I^n\backslash\{\bar{a}\}} = (t_a \to s_{a-1})\restriction_{I^n\backslash\{\bar{a}\}} = (\underline{1} \to s_{a-1})\restriction_{I^n\backslash\{\bar{a}\}} = s_{a-1}\restriction_{I^n\backslash\{\bar{a}\}}.$$

We also have

$$s_a(\bar{a}) = (t_a \to s_{a-1})(\bar{a}) = t_a(\bar{a}) \to s_{a-1}(\bar{a}) = 0 \to s_{a-1}(\bar{a}) = 1.$$

So $f\restriction_{\{\bar{0}, \ldots, \bar{a}\}} = s_a\restriction_{\{\bar{0}, \ldots, \bar{a}\}}$ and $f^{-1}(0) \subseteq s_a^{-1}(0)$.

It now follows that $f\restriction_{I^n\backslash\{(1, \ldots, 1)\}} = s_{2^n-2}\restriction_{I^n\backslash\{(1, \ldots, 1)\}}$. Since $f^{-1}(0) \subseteq \pi_k^{-1}(0)$, we have $f(1, \ldots, 1) = 1 = s_{2^n-2}(1, \ldots, 1)$, whence $f = s_{2^n-2}$.

It remains to establish that, for every term function $t: I^n \to I$, there exists $k \in \{1, \ldots, n\}$ such that $t^{-1}(0) \subseteq \pi_k^{-1}(0)$. Assume that the term functions $t_1: I^n \to I$ and $t_2: I^n \to I$ satisfy $t_1^{-1}(0) \subseteq \pi_{k_1}^{-1}(0)$ and $t_2^{-1}(0) \subseteq \pi_{k_2}^{-1}(0)$ for some $k_1, k_2 \in \{1, \ldots, n\}$. Then $(t_1 \to t_2)^{-1}(0) \subseteq t_2^{-1}(0) \subseteq \pi_{k_2}^{-1}(0)$. The required result now follows by induction. □

Each implication algebra $\mathbf{A}$ has a join-semilattice term-reduct. If a greatest lower bound for elements $a, b \in A$ exists in $\mathbf{A}$, it will be denoted by $a \wedge b$.

LEMMA 1.2. *Let* $\mathbf{A}$ *be a non-trivial implication algebra and let* $n \in \mathbb{N}$ *with* $n \geq 2$. *If the near-unanimity function* $m: A^n \to A$, *given by*

$$m(a_1, \ldots, a_n) = \bigwedge_i \bigvee_{j \neq i} a_j,$$

is well defined on $\mathbf{A}$, *then* $\mathbf{A}$ *is not* n*-endoprimal.*

Proof. Every principal filter of $\mathbf{A}$ is a Boolean lattice where, for each $a \in A$ and $b \in \uparrow a$, the complement of b in $\uparrow a$ is given by $b \to a$. Let $a \in A$ and let B be a finite + subset of $\uparrow a$. Since $\uparrow a$ satisfies the de Morgan laws, we have $\bigwedge B = \bigvee (B \to a) \to a$.

Assume that $m: A^n \to A$ is well defined. Let $e \in \operatorname{End} \mathbf{A}$, let $a_1, \ldots, a_n \in A$ and define $a := m(a_1, \ldots, a_n)$. Then $\bigvee_{j \neq i} a_j \in \uparrow a$ and $e(\bigvee_{j \neq i} a_j) \in \uparrow e(a)$ for all $i \in \{1, \ldots, n\}$. This gives us

$$\begin{aligned} m(e(a_1), \ldots, e(a_n)) &= \bigwedge_i \bigvee_{j \neq i} e(a_j) = \bigwedge_i e\left(\bigvee_{j \neq i} a_j\right) \\ &= \bigvee_i \left(e\left(\bigvee_{j \neq i} a_j\right) \to e(a)\right) \to e(a) \\ &= e\left(\bigvee_i \left(\left(\bigvee_{j \neq i} a_j\right) \to a\right) \to a\right) \\ &= e\left(\bigwedge_i \bigvee_{j \neq i} a_j\right) = e(m(a_1, \ldots, a_n)). \end{aligned}$$

Thus m preserves the endomorphisms of **A**.

Since **A** is non-trivial, we can assume that $\underline{\mathbf{I}}$ is a subalgebra of **A**. Suppose that m is a term function of **A**. Then $m\restriction_{I^n}: I^n \to I$ is a term function of $\underline{\mathbf{I}}$. Let $k \in \{1, \ldots, n\}$ and define $x_k \in I^n$ by

$$x_k(i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $m\restriction_{I^n}(x_k) = 0$ and $\pi_k(x_k) = 1$, which implies $m\restriction_{I^n}^{-1}(0) \nsubseteq \pi_k^{-1}(0)$. This contradicts the characterisation of the term functions of $\underline{\mathbf{I}}$ given in Lemma 1.1. Thus m is not a term function of **A**, whence **A** is not n-endoprimal. □

Our first main result follows easily from the previous lemma.

THEOREM 1.3. *There are no non-trivial finite endoprimal implication algebras.*

Proof. Let **A** be a non-trivial finite implication algebra. Then we can form the set $\{c_1, \ldots, c_{n-1}\}$ of all minimal elements of **A**. We will prove that **A** is not n-endoprimal. Using Lemma 1.2, it suffices to show that the near-unanimity function $m: A^n \to A$, given by

$$m(a_1, \ldots, a_n) = \bigwedge_i \bigvee_{j \neq i} a_j,$$

is well defined on **A**.

Let $a_1, \ldots, a_n \in A$. We have $A = {\uparrow}c_1 \cup \cdots \cup {\uparrow}c_{n-1}$, so there must be some $m \in \{1, \ldots, n-1\}$ with distinct $k, l \in \{1, \ldots, n\}$ such that $a_k, a_l \in {\uparrow}c_m$. This implies that $\bigvee_{j \neq i} a_j \in {\uparrow}c_m$ for all $i \in \{1, \ldots, n\}$. Since ${\uparrow}c_m$ is a lattice, it follows that $m(a_1, \ldots, a_n) = \bigwedge_i \bigvee_{j \neq i} a_j$ exists. □

2. Infinite implication algebras

The free algebras of a quasivariety, having a plentiful supply of endomorphisms, seem to be the most likely candidates for endoprimality. All infinitely generated free algebras must be endoprimal. We will show that finitely generated free implication algebras can also be used to construct endoprimal algebras. The main result of this section is an example of an endoprimal implication algebra which does not have a retraction onto any non-trivial free algebra.

We say that $\mathbf{B}$ is a **subretract** of $\mathbf{A}$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and there is a homomorphism $\alpha: \mathbf{A} \to \mathbf{B}$ such that $\alpha\restriction_B = \mathrm{id}_{\mathbf{B}}$. The map α is called a **subretraction**. An algebra $\mathbf{C}$ is a **retract** of $\mathbf{A}$ if $\mathbf{C}$ is isomorphic to a subretract of $\mathbf{A}$. We will use the following general result from Davey and Pitkethly [7].

THEOREM 2.1. *Let* $\mathbf{A}$ *be an algebra and let* $\mathbf{B}$ *be a subretract of* $\mathbf{A}$. *Assume that* $f: A^k \to A$ *and* $g: A^k \to A$ *preserve the endomorphisms of* $\mathbf{A}$. *Then* $f(B^k) \subseteq B$. *If* $\mathbf{A} \in \mathbb{ISP}\,\mathbf{B}$ *and* $g\restriction_{B^k} = f\restriction_{B^k}$, *it follows that* $g = f$.

The next theorem follows directly from the definition of endoprimality.

THEOREM 2.2. *Let* $\mathscr{A}$ *be a quasivariety, let* $\mathbf{A} \in \mathscr{A}$ *and let* $n \in \mathbb{N}$. *If* $\mathbf{F}_{\mathscr{A}}(n)$ *is a retract of* $\mathbf{A}$, *then* $\mathbf{A}$ *is* n*-endoprimal. In particular, every infinitely generated free algebra is endoprimal.*

Proof. Assume that $\alpha: \mathbf{A} \to \mathbf{B}$ is a subretraction and that $\mathbf{B}$ is freely generated by $\{b_1, \ldots, b_n\}$ in $\mathscr{A}$. Let $f: A^n \to A$ preserve the endomorphisms of $\mathbf{A}$. Then $f(B^k) \subseteq B$ by Theorem 2.1. This implies that $f(b_1, \ldots, b_n) = t(b_1, \ldots, b_n)$ for some n-ary term function t of $\mathbf{A}$.

Let $a_1, \ldots, a_n \in A$ and define the homomorphism $\beta: \mathbf{B} \to \mathbf{A}$ so that $b_i \mapsto a_i$ for all $i \in \{1, \ldots, n\}$. The map $e := \beta \circ \alpha$ is an endomorphism of $\mathbf{A}$ with $e(b_i) = \beta \circ \alpha(b_i) = \beta(b_i) = a_i$ for all $i \in \{1, \ldots, n\}$. We have

$$\begin{aligned} f(a_1, \ldots, a_n) &= f(e(b_1), \ldots, e(b_n)) = e(f(b_1, \ldots, b_n)) \\ &= e(t(b_1, \ldots, b_n)) = t(e(b_1), \ldots, e(b_n)) \\ &= t(a_1, \ldots, a_n), \end{aligned}$$

whence $f = t$. Thus $\mathbf{A}$ is n-endoprimal. $\square$

Let $\mathbf{A}$ and $\mathbf{B}$ be implication algebras. We can construct the implication algebra $\mathbf{A} \oplus \mathbf{B}$, with underlying set $(A\backslash\{1\}) \dot\cup (B\backslash\{1\}) \dot\cup \{1\}$, by defining

$$a \to b = \begin{cases} a \to^{\mathbf{A}} b & \text{if } a, b \in A, \\ a \to^{\mathbf{B}} b & \text{if } a, b \in B, \\ b & \text{otherwise.} \end{cases}$$

The algebras $\mathbf{A}$ and $\mathbf{B}$ are both retracts of $\mathbf{A} \oplus \mathbf{B}$. This construction can be used to create endoprimal implication algebras from free implication algebras. The algebra $\bigoplus \{\mathbf{F}_{\mathfrak{I}}(n) \mid n \in \mathbb{N}\}$ is endoprimal, and $\mathbf{A} \oplus \mathbf{F}_{\mathfrak{I}}(\mathbb{N})$ is endoprimal for all $\mathbf{A} \in \mathfrak{I}$.

The remainder of this section is dedicated to producing a general description of endoprimal implication algebras. We will be able to give examples of endoprimal implication algebras which do not have retractions onto any non-trivial free algebras.

Let $\mathbf{A}$ be an implication algebra. A non-empty set $F \subseteq A$ is a **filter** of $\mathbf{A}$ if F is an increasing subset of $\mathbf{A}$ and $a \wedge b \in F$ for all $a, b \in F$ which have a lower bound in $\mathbf{A}$. Abbott [1] shows that a set $F \subseteq A$ is a filter of $\mathbf{A}$ if and only if $1 \in F$ and $b \in F$ for all $a, b \in A$ such that $a, a \to b \in F$. It is noted in Davey and Werner [8] that a proper filter F of $\mathbf{A}$ is maximal if and only if $a \to b \in F$ for all $a, b \in A \backslash F$. Each maximal filter F of $\mathbf{A}$ induces a non-constant homomorphism $\chi_F \colon \mathbf{A} \to \underline{\mathbf{I}}$. Conversely, every non-constant homomorphism $x \colon \mathbf{A} \to \underline{\mathbf{I}}$ determines a maximal filter $x^{-1}(1)$ of $\mathbf{A}$.

Let $a_1, \ldots, a_n \in A$ and let $\mathscr{F}$ be a set of maximal filters of $\mathbf{A}$. We will say that $\{a_1, \ldots, a_n\}$ is an n-**element partition of A under** $\mathscr{F}$ if $\bigcup \mathscr{F} = A$ and, for all $F \in \mathscr{F}$, there is a unique $i \in \{1, \ldots, n\}$ such that $a_i \in F$.

LEMMA 2.3. *Let* $\mathbf{A}$ *be an implication algebra and let* $n \in \mathbb{N}$ *with* $n \geq 2$. *If* $\mathbf{A}$ *has an* n*-element partition, then* $\mathbf{A}$ *is* n*-endoprimal.*

Proof. Assume that $\{a_1, \ldots, a_n\}$ is an n-element partition of $\mathbf{A}$ under $\mathscr{F}$. Let $f \colon A^n \to A$ preserve the endomorphisms of $\mathbf{A}$ and suppose that f is not a term function of $\mathbf{A}$. We can assume that $\mathbf{A}$ is non-trivial and that $\underline{\mathbf{I}}$ is a subretract of $\mathbf{A}$. Using Theorem 2.1 and Lemma 1.1, we must have $f\restriction_{I^n}^{-1}(0) \nsubseteq \pi_k\restriction_{I^n}^{-1}(0)$ for all $k \in \{1, \ldots, n\}$. It follows that, for each $k \in \{1, \ldots, n\}$, there exists $x_k \in I^n$ such that $f(x_k) = 0$ and $x_k(k) = 1$.

Define $x \in A^n$ by

$$x(i) = \bigvee \{a_k \mid x_k(i) = 1\}$$

and let $F \in \mathscr{F}$. By definition, there exists a unique $l \in \{1, \ldots, n\}$ such that $a_l \in F$. Since $\{a_k \mid k \neq l\} \subseteq A \backslash F$ and F is a maximal filter, we must have $\bigvee \{a_k \mid k \neq l\} \notin F$. So $x(i) \in F$ if and only if $x_l(i) = 1$. This gives us

$$\chi_F(f(x)) = f(\chi_F(x)) = f(x_l) = 0,$$

which implies $f(x) \notin F$. This contradicts our assumption that $\bigcup \mathscr{F} = A$. □

If an implication algebra $\mathbf{A}$ has a minimum element 0, then $\mathbf{A}$ forms a Boolean lattice with complementation given by $a' = a \to 0$ for all $a \in A$. We will refer to bounded implication algebras as **Boolean implication algebras**. We will embed each implication algebra into a Boolean implication algebra so that we can make use of the extra available structure.

LEMMA 2.4. *Every implication algebra has an embedding into a Boolean implication algebra.*

Proof. Let $\underset{\sim}{\mathbf{I}}$ denote the discrete topological space on $\{0, 1\}$ and define $\mathfrak{X}$ to be the class of all compact totally-disconnected topological spaces. For all $\mathbf{A} \in \mathfrak{I}$, the set $\mathfrak{I}(\mathbf{A}, \underline{\mathbf{I}})$, of all homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{I}}$, forms a closed subspace of the product space $\underset{\sim}{\mathbf{I}}^A$. So we can define the map $D: \mathfrak{I} \to \mathfrak{X}$ by $D(\mathbf{A}) = \mathfrak{I}(\mathbf{A}, \underline{\mathbf{I}})$ for all $\mathbf{A} \in \mathfrak{I}$. We can define $E: \mathfrak{X} \to \mathfrak{I}$ by assigning as $E(\mathbf{X})$ the implication algebra $\mathfrak{X}(\mathbf{X}, \underset{\sim}{\mathbf{I}}) \leq \underline{\mathbf{I}}^X$ of all continuous maps from $\mathbf{X}$ to $\underset{\sim}{\mathbf{I}}$.

Let $\mathbf{A} \in \mathfrak{I}$. Define $e_{\mathbf{A}}: \mathbf{A} \to ED(\mathbf{A})$ by $e_{\mathbf{A}}(a)(x) = x(a)$ for all $a \in A$ and $x \in D(\mathbf{A})$. It is straightforward to show that $e_{\mathbf{A}}$ is an embedding from $\mathbf{A}$ into the Boolean implication algebra of all continuous maps from $D(\mathbf{A})$ into $\underset{\sim}{\mathbf{I}}$. (The construction we are using is known as a preduality – see Clark and Davey [2] or Davey [3].) □

In the proof of the previous lemma, we actually embed each implication algebra $\mathbf{A}$ into the implication algebra term-reduct of the free Boolean algebra generated by $\mathbf{A}$ (see [11]).

We are now able to give a characterisation of the n-endoprimal implication algebras for each $n \in \mathbb{N}$. Every non-trivial implication algebra has a retraction onto $\underline{\mathbf{I}} \cong \mathbf{F}_{\mathfrak{I}}(1)$ and is therefore 1-endoprimal by Theorem 2.2.

THEOREM 2.5. *Let* $\mathbf{A}$ *be a non-trivial implication algebra and let* $n \in \mathbb{N}$ *with* $n \geq 2$. *The following are equivalent*:

(i) $\mathbf{A}$ *is* n-*endoprimal*;

(ii) $\mathbf{A}$ *has an* n-*element partition*;

(iii) *there exist* $a_1, \ldots, a_n \in A$ *such that* $\{\bigvee_{j \neq i} a_j \mid i \in \{1, \ldots, n\}\}$ *does not have a lower bound in* $\mathbf{A}$;

(iv) *the* n-*ary near-unanimity function* $m: A^n \to A$, *given by*

$$m(a_1, \ldots, a_n) = \bigwedge_i \bigvee_{j \neq i} a_j,$$

is not well defined on $\mathbf{A}$.

Proof. Since every principal filter of an implication algebra is a Boolean lattice, the conditions (iii) and (iv) are equivalent. We have already shown ¬(iv) ⇒ ¬(i) and (ii) ⇒ (i) in Lemmas 1.2 and 2.3. It remains to prove that (iii) ⇒ (ii).

Lemma 2.4 tells us that $\mathbf{A}$ is a subalgebra of a Boolean implication algebra $\mathbf{A}^+$. Let $\mathscr{U}(\mathbf{A}^+)$ denote the set of all maximal filters of $\mathbf{A}^+$. Then, for all $F \in \mathscr{U}(\mathbf{A}^+)$ with $A \nsubseteq F$, the set $F\restriction_A := F \cap A$ is a maximal filter of $\mathbf{A}$.

Assume that $a_1, \ldots, a_n \in A$ such that $\{\bigvee_{j \neq i} a_j \mid i \in \{1, \ldots, n\}\}$ does not have a lower bound in $\mathbf{A}$. Define $b_i := a_i$, for all $i \in \{1, \ldots, n-1\}$, and define $b_n := (a_1 \vee \cdots \vee a_{n-1}) \to a_n$. Then the set $B := \{b_1, \ldots, b_n\}$ has n elements. We want to show that B is an n-element partition of $\mathbf{A}$ under

$$\mathscr{F} := \{F\restriction_A \mid F \in \mathscr{U}(\mathbf{A}^+) \text{ with } |B \cap F| \leq 1\}.$$

Since

$$b_1 \vee \cdots \vee b_n = (a_1 \vee \cdots \vee a_{n-1}) \vee ((a_1 \vee \cdots \vee a_{n-1}) \to a_n) = 1,$$

we must have $|B \cap F| \geq 1$ for all $F \in \mathscr{U}(\mathbf{A}^+)$. So, for each $F\restriction_A \in \mathscr{F}$, there is a unique $i \in \{1, \ldots, n\}$ such that $b_i \in F\restriction_A$. We will prove that $\bigcup \mathscr{F} = A$.

Suppose there is some $a \in A \setminus \bigcup \mathscr{F}$. Let $F \in \mathscr{U}(\mathbf{A}^+)$ with $a \in F$. We must have $|B \cap F| \geq 2$, which implies that $b_k, b_l \in F$ for distinct $k, l \in \{1, \ldots, n\}$. So $\bigvee_{i \neq j} (b_i \wedge b_j) \in F$. For all $i, j \in \{1, \ldots, n-1\}$ we have $b_i \wedge b_j = a_i \wedge a_j$ and

$$\begin{aligned}
b_i \wedge b_n &= a_i \wedge ((a_1 \vee \cdots \vee a_{n-1}) \to a_n)\\
&= a_i \wedge ((a_1 \vee \cdots \vee a_{n-1})' \vee a_n)\\
&= a_i \wedge ((a_1' \wedge \cdots \wedge a_{n-1}') \vee a_n)\\
&= a_i \wedge (a_1' \vee a_n) \wedge \cdots \wedge (a_{n-1}' \vee a_n)\\
&= (a_i \wedge (a_i' \vee a_n)) \wedge (a_1' \vee a_n) \wedge \cdots \wedge (a_{n-1}' \vee a_n)\\
&= (a_i \wedge a_n) \wedge (a_1' \vee a_n) \wedge \cdots \wedge (a_{n-1}' \vee a_n)\\
&\leq a_i \wedge a_n.
\end{aligned}$$

This gives us $\bigvee_{i \neq j} (a_i \wedge a_j) \geq \bigvee_{i \neq j} (b_i \wedge b_j) \in F$, and therefore $\bigwedge_i \bigvee_{j \neq i} a_j = \bigvee_{i \neq j} (a_i \wedge a_j) \in F$.

We have shown that $\bigwedge_i \bigvee_{j \neq i} a_j \in F$ for all $F \in \mathscr{U}(\mathbf{A}^+)$ with $a \in F$. This implies that $a \leq \bigwedge_i \bigvee_{j \neq i} a_j$. So a is a lower bound for $\{\bigvee_{j \neq i} a_j \mid i \in \{1, \ldots, n\}\}$ in $\mathbf{A}^+$. Since the inclusion map from $\mathbf{A}$ into $\mathbf{A}^+$ is one-to-one and join-preserving, it is automatically an order-embedding. So a is also a lower bound for $\{\bigvee_{j \neq i} a_j \mid i \in \{1, \ldots, n\}\}$ in $\mathbf{A}$. This contradicts our original assumption and therefore $\bigcup \mathscr{F} = A$. Thus B is an n-element partition of $\mathbf{A}$ under $\mathscr{F}$. □

Unfortunately, this characterisation of n-endoprimality does not always make it easy to decide whether a given implication algebra is endoprimal. We will use our result to give some examples of endoprimal implication algebras which do not have any retractions onto non-trivial free algebras.

For each implication algebra $\mathbf{A} = \langle A; \rightarrow, \vee, 1 \rangle$, define the bounded semilattice $\mathbf{A}_\perp = \langle A \dot{\cup} \{0\}; \vee, 0, 1 \rangle$ with minimum element 0.

EXAMPLE 2.6. Let $\mathbf{A}$ be an implication algebra such that $\mathbf{A}_\perp$ is an infinite Boolean lattice. Then $\mathbf{A}$ is endoprimal.

Proof. Let $n \in \mathbb{N}$ with $n \geq 2$. Since $\mathbf{A}_\perp$ is an infinite Boolean lattice, there exist $a_1, \ldots, a_n \in A$ such that $a_i \wedge a_j = 0$ for distinct $i, j \in \{1, \ldots, n\}$ and $a_1 \vee \cdots \vee a_n = 1$. Let $\mathscr{U}(\mathbf{A}_\perp)$ denote the set of ultrafilters of $\mathbf{A}_\perp$. For every $F \in \mathscr{U}(\mathbf{A}_\perp)$ there is a unique $i \in \{1, \ldots, n\}$ such that $a_i \in F$. Each member of $\mathscr{U}(\mathbf{A}_\perp)$ is also a maximal filter of $\mathbf{A}$. So $\{a_1, \ldots, a_n\}$ forms an n-element partition of $\mathbf{A}$ under $\mathscr{U}(\mathbf{A}_\perp)$. □

In [7], endoprimality is characterised within some dualisable quasivarieties. For each of these quasivarieties there is some $k \in \mathbb{N}$ for which endoprimality is equivalent to k-endoprimality. There is no similar finiteness condition for endoprimality within the variety of implication algebras. For each $k \in \mathbb{N}$ with $k \geq 2$, the implication algebra $\mathbf{B}_k$, where $\mathbf{B}_{k\perp}$ is a Boolean lattice with k atoms, is k-endoprimal but not $(k+1)$-endoprimal.

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La Trobe University
Bundoora
Victoria 3083
Australia
e-mail: *J.Pitkethly@latrobe.edu.au*