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Modal restriction semigroups: towards an algebra of functions

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Restriction semigroups model algebras of partial maps under composition and domain. Here we consider restriction semigroups for which the usual Boolean operations on domains are modelled. Such algebras are capable of modelling the usual modal operators considered in dynamic logic. Indeed adding a natural functional variant of union to the signature gives a deterministic version of the modal semirings of Möller and Struth, but also a monoidal version of the classical restriction categories of Cockett and Manes. Other operations modelled are intersection and (in the finite case) functional iteration. In each case, axiomatizations of the concrete functional examples are given, leading to algebraic models of partial maps incorporating all the domain-related and set-theoretic operations previously considered. Our algebras furnish natural algebraic semantics for the logics of deterministic computer programs, leading to new results for some variants of Propositional Dynamic Logic.

Keywords: Restriction semigroups, partial functions, relation algebra, Eq-monoid, dynamic algebra, Kleene algebra with domain, propositional dynamic logic, modal semirings.

1. Introduction

The class of restriction semigroups models algebras of partial functions on a set under the binary operation of composition plus the unary operation of restriction of the identity map to the domain of a function. A finitely based variety with its own "Cayley theorem", this class has independently arisen in numerous different contexts and has accumulated equally many different names.

The set of partial functions $\mathcal{P}(X)$ on a set X is a restriction semigroup under composition and the operation of the restriction of the identity map to the domain of a function, the latter a unary operation we henceforth refer to as *domain*. Conversely, every restriction semigroup is embeddable in such a functional example. (If 0 and/or 1 are included in the signature, then these can be represented as the empty function and the identity function respectively if one adds in the usual multiplicative laws and the properties $D(0) = 0$ and/or $D(1) = 1$.)

This representation theorem for restriction semigroups was apparently first explicitly proved by Trokhimenko [64] (who in fact went further and considered a multiplace function version). The characterization can also be obtained by adapting the work of Schweizer and Sklar [60], and has been rediscovered in various guises by subsequent authors, including the present authors [31] (where they arise as *twisted left closure semigroups*) and Manes [45] (where they arise as *guarded semigroups*).

Restriction semigroups can also be well motivated as generalisations of inverse semigroups, along the lines of left ample semigroups. Indeed left ample semigroups constitute a (proper) subquasivariety of weakly left ample semigroups (studied in Gomes and Gould [20] for example), in turn a proper subquasivariety of the class of weakly left E-ample semigroups, which is itself just the class of restriction semigroups (see [21,22] for example).

A category-theoretic formulation of the domain operation was independently obtained and characterized by Cockett and Lack [8] in the setting of "restriction categories". A monoidal restriction category is nothing but a restriction monoid (restriction semigroup which is a monoid). More recently, Gould and Hollings have presented a variant of the ESN theorem of inverse semigroup theory [44] applying to restriction semigroups, by establishing an equivalence between the class of restriction semigroups and the class of inductive constellations (generalisations of categories in which domain but not range information is present).

The operations of composition and domain for binary relations on a set are considered by Möller and Struth in [47], in the setting of *modal semirings*, where an operation modelling relational union is also present. Modal semirings admit the usual modalities of dynamic logic, a form of program logic significant in the theory of program verification.

A feature of modal semirings is that the set of domain elements forms a Boolean algebra, modelling all the usual Boolean operations on sets; moreover there is a join operation, modelling relational union. These provide an elegant algebraic approach to various program logics. In [47], modal semirings are enriched to give Kleene algebras with domain, which model looping as well as branching.

Recent work of Cockett and Manes [9] has enriched the notion of a restriction category to allow modelling of loop-free deterministic propositional dynamic logic. Their concept of a "classical restriction category" models categories of partial functions equipped with domain and domain complement, along with a partial operation of domain-disjoint union of partial functions of the same type.

In this article we consider restriction semigroups for which the domain elements form a Boolean algebra, modelling the usual set operations. Using the method of determinative pairs developed by Boris Schein and others (see [56]), we obtain a finite quasi-equational axiomatization of the concrete algebras, which provides a representation theorem for the algebras considered in [33]. We call this quasivariety the class of *modal restriction semigroups.*

The signature is readily enlarged to model (compatible) functional union, yielding a structure that is both a deterministic analog of modal semirings, and a monoidal version of classical restriction categories. Other natural operations definable on partial functions may be added to the modal restriction semigroup signature, and are readily axiomatized based on the determinative pair method used here. This leads to to some of the richest function algebras so far axiomatized, incorporating all known set-theoretic and domain-related operations previously considered in the literature, along with some novel ones related to program algebra operations. The additional operations we consider are partial map intersection and a functional analog of the reflexive transitive closure (or Kleene closure) of a binary relation.

The structures considered here provide natural algebraic models of composition-closed systems of modal operators, in precisely the way appropriate for the modelling of deterministic computer programs. With the additional operations added, the algebras model computer progam constructions such as "`if-then-else`" branching and "`while`" loops, so our results can be interpreted as axiomatizations of the function algebras associated with deterministic program logics such as *strict deterministic propositional dynamic logic* (SDPDL) as studied by Halpern and Reif [24]. The results are also related to earlier work [36] in which semigroups of piecewise defined transformations were investigated in the context of the `if-then-else` construction.

Importantly, our results cover the operation of intersection. Recent work of the first author with R. Hirsch [28] has revealed that there can be no fully nondeterministic (that is, relational) analogue of the results in the present article because the class of finite algebraic structures isomorphic to relational (that is, nondeterministic) models of dynamic logics with program intersection is not recursive. In contrast, the axiomatization given here yields a polynomial time algorithm for the corresponding problem for functional (deterministic) models.

We make a contribution to dynamic logic by providing a complete axiomatization for the loop-free fragment of strict deterministic propositional dynamic logic (PDL) with intersection. This turns out to be substantially simpler than the known axiomatizations for loop-free PDL with intersection. We also show that the deterministic elements of a modal semiring (in a natural sense) form a modal restriction semigroup with functional union, a fact that easily extends to Kleene algebras with domain.

2. Algebras of functions

2.1. *Background on Relation Algebra*

All of the operations considered here make sense for binary relations on a set as well as for (partial) functions. Indeed the relational setting has seen much work over many years, and is known to a far greater number of researchers. So to establish context, we begin by briefly reviewing the relational case.

First, some standard terminology. A binary relation on a set X is any subset of $X \times X$, and the set of all binary relations on X will be denoted by $\mathcal{B}(X)$. This set is a semigroup when equipped with the binary operation of composition.

The modern theory of relation algebras was begun by Tarski in the 1940's; see [63]. The notion was introduced in order to model as many of the natural definable operations on $\mathcal{B}(X)$ as possible. Of course this includes composition, but also all the usual set-theoretic (Boolean) operations (since binary relations are just sets of ordered pairs), that is: union $\cup$, intersection $\cap$, and set-theoretic complement, along with two nullary operations corresponding to the empty relation 0 and the full relation on X. Also modelled were the unary operation of converse and a nullary operation modelling the diagonal relation 1. Although Tarski's axiomatization was not complete for the intended models (algebras of actual binary relations under these operations), his definition has been influential throughout mathematics, logic and computer science.

A further (non-Boolean) operation of importance, especially in computer science applications, is reflexive transitive closure or *Kleene closure*. Given $\rho \in \mathcal{B}(X)$, define ρ^* to be the smallest reflexive transitive relation containing ρ; hence

$$\rho^* = \bigcup_{n \geq 0} \rho^n,$$

where $\rho^0 = 1$. The signature of the Kleene algebras of Kozen (see [40]) includes this operation, together with composition and union.

In modal semirings (see [47]), the elements model binary relations in $\mathcal{B}(X)$, and the operations modelled are composition and union, plus a unary operation D modelling domain ($D(a)$ is the restriction of the identity function to the domain of $a \in \mathcal{B}(X)$). The domain elements of the form $D(a)$, or "tests", are assumed to form a Boolean algebra (in which meet is product), modelling all the usual set operations on subsets of X.

The domain operation D arises naturally in the setting of PDL, itself a strong motivation for domain semirings, as we discuss further below. Modal semirings provide algebraic models of loop-free PDL, whereas KAD (Kleene algebra with domain) as in [13], an enrichment of domain semirings in which looping is modelled by the Kleene asterate operator * (modelling reflexive transitive closure), provides models for full PDL.

2.2. *Function algebras*

The study of abstract algebraic properties of functions goes back at least as far as Menger [46], and the 1970 survey by Schein [52] already contains dozens of different operations and relations on functions. It is not possible here to give a full survey of the literature concerning algebraic structures of functions; instead we cite relevant articles as we introduce the operations. We also direct the reader to the survey article by Schein [52] or that of the authors [37] (see also the survey by Schein and Trokhimenko [59] for algebras of multiplace functions, and the survey article on reducts of relation algebras by Schein [58], which also has some references on functions).

Following Schein [57], or the book Harel, Kozen and Tiuryn [27, p. 9] for example, we use the term *function on* X to mean a functional binary relation on X. The binary relation ρ on X is *functional* if $(a, b) \in \rho$ and $(a, c) \in \rho$ imply $b = c$; in this case we write $b = \rho(a)$ whenever $(a, b) \in \rho$.

Let $\mathcal{P}(X)$ denote the set of all functions on X. Let $\Delta(X)$ denote the set of restrictions of the identity function (diagonal element) in $\mathcal{P}(X)$ or $\mathcal{B}(X)$. $\Delta(X)$ is obviously a subsemilattice of both semigroups, isomorphic to the semilattice of subsets of X under intersection, and arises as the image of the domain operator D in both cases.

2.3. *The operations considered here.*

As discussed, we are modelling restriction semigroups of partial functions equipped with additional operations. In fact all the enrichments of restriction semigroups we consider at least have the domain complement operation, which we describe first.

Domain complement P

For any function $f \in \mathcal{P}(X)$ or $\mathcal{B}(X)$, we let $P(f)$ denote the identity map restricted to the complement of the domain of f, and we call P *domain complement*. On $\mathcal{P}(X)$, the operation P is equivalent to D together with the partial operation of complementation defined only on domain elements (which is what is needed to give $\Delta(X)$ Boolean algebra structure), in the following sense: $P(a) = D(a)'$, where α' is the restriction of the identity to the complement of the domain of $\alpha \in \Delta(X)$, while conversely we have $D = P^2$ and $\alpha' = P(\alpha)$. Both D and $'$ are part of the signature of modal semirings, which can therefore be defined in terms of P instead.

Semigroups of functions (and relations) under composition equipped with domain complement were considered by the authors in [33] as special cases of semigroups equipped with a generalized "pseudocomplement" operation, but no abstract characterizations were given. Our first task will be to give such an axiomatization and show its correctness. The resulting abstract class of algebras is a finitely-based (proper) quasivariety, a class we call the class of *modal restriction semigroups.*

Functional analogs of union

The set of binary relations $\mathcal{B}(X)$ on the set X is closed under all the familar Boolean set operations. Union in particular has proved to be a very important operation, for example in the setting of models of program algebras and logics having relational semantics. By contrast, $\mathcal{P}(X)$ is evidently not closed under union in general (but see Schein [54,55] for a characterisation of semigroups of functions that are closed under union).

Natural functional analogs of union on $\mathcal{P}(X)$ exist. For example, in [9], operations of disjoint and compatible union are modelled for functions having the same source and target. For partial functions on a set X, we can define $f \vee g$ as the union of f, g providing f, g agree where both are defined: this is *compatible union.*

Compatible union is easily seen to be equivalent to a total operation, in the presence of composition and domain complement. Thus, for $f, g \in \mathcal{P}(X)$, we define the *preferential union* $f \sqcup g$ of f with g to be the union of f with the restriction of g to the places where f is undefined: $f \sqcup g = f \cup (P(f)g)$. It is easy to see that $f \sqcup g = f \vee (P(f)g)$, and conversely that $f \vee g = f \sqcup g$ if $D(f)g = D(g)f$ (that is if they agree on their common domain). Hence if a semigroup of functions closed under P is closed under one of $\vee$ and $\sqcup$, it is closed under the other. Indeed the partial operation of union of functions having no common domains ("domain-disjoint union") is easily seen to be equivalent (in the presence of P) to both $\vee$ and $\sqcup$.

In [9], restriction categories on which domain complement and domain-disjoint unions are modelled are called *classical restriction categories.* So $\mathcal{P}(X)$ equipped with its modal restriction monoid operations and domain-disjoint union is a monoidal classical restriction category.

From the perspective of program algebra, the operation "`if` α `then` f `else` g" is important. Whether f, g are functions or just binary relations, this operation can be expressed in terms of composition, P and $\sqcup$ as $(\alpha f) \sqcup (P(\alpha)g)$, where α, β are domain elements, or *tests.* This is identical in form to the formulation of `if-then-else` in terms of union used in relational models. However, in the presence of domain complement, $\sqcup$ is polynomially equivalent to `if-then-else`, because $f \sqcup g =$ `if` $D(f)$ `then` f `else` g. By contrast, relational union is strictly richer than `if-then-else`.

Our axiomatization of restriction semigroups of functions equipped with domain complement easily extends to cover preferential union. The resulting class is a finitely based variety (the one quasi-equation in the definition of modal restriction semigroups being replaced by an equation).

Intersection and related operations

Aside from composition, together with domain and all Boolean operations on domains (equivalently, domain complement), $\mathcal{P}(X)$ is closed under the following operations.

- *Intersection,* $\cap$. Clearly $\mathcal{P}(X)$ is closed under set-theoretic intersection. Algebras of functions under composition and intersection were axiomatized by Garvac'kii [18]; they form a finitely based variety.
- *Set-theoretic difference,* $\setminus$. $\mathcal{P}(X)$ is also closed under set-theoretic difference, and indeed intersection can be expressed in terms of this. Algebras of

functions under composition and set difference were axiomatized by Schein in [57]; again, the class is a finitely based variety.
- *The tie operation,* $\bowtie$. Given two functions f, g, the function $f \bowtie g$, pronounced "f tie g", is the identity map restricted to the points at which f and g do not disagree—that is, for which either f and g are defined and act identically or are both undefined. Again, intersection can be expressed in terms of this operation together with composition. Algebras of function and relations under composition and $\bowtie$ were considered by the second author [62], but no axiomatizations were given.

There are many other domain- and Boolean-type operations and relations considered in the literature. However, as far as the authors are aware, all can be defined on $\mathcal{P}(X)$ in terms of some combination of D, P, $\bowtie$, $\cap$, $\setminus$, 0, 1 (in the presence of composition, which we assume throughout).

In [37] it is observed that, in the presence of composition, *any one of the following combinations of operations is sufficiently strong to define all of the operations just described* (whence, all domain- or Boolean-type operations on semigroups of functions previously considered in the literature):

- $\{\cap, P\}$;
- $\{\setminus, D, 1\}$;
- $\{\bowtie, 0\}$.

In particular then, each of these combinations yields a polynomially equivalent class of enriched semigroups, all equivalent to that obtained by adding intersection to composition and domain complement.

Again building on our methods for modal restriction semigroups, we give characterizations of these structures (and their injective variants): again, the resulting class is a finitely based variety. In view of known equivalences, this completes the characterization of such enriched semigroups of functions (and injective functions), where the additional operations considered can be chosen to be any possible subset of $\{D, P, \bowtie, \cap, \setminus, 0, 1\}$ that includes P as a term; this is all subsets aside from $\{\bowtie\}$ and the two equivalent cases $\{\setminus, D\}$ and $\{\setminus, D, 0\}$.

Further equivalences

Other notions of functional union can be formulated. The *restricted union* $f \curlyvee g$ of f and g is the function given by the (necessarily disjoint) union of $f \cap g$ with f restricted to the places where g is undefined and with g restricted to the places where f is undefined. The restricted union of f and g is easily seen to be the largest functional domain restriction of the relational union $f \cup g$.

Easy examples show that neither restricted union nor preferential union is definable in terms of composition, P and intersection. But if we adjoin either one of these to that signature, the other is definable, indeed both agree with the usual union on pairs of functions whose union is still a function, and both reduce to unions of functions with disjoint domains – thus

$$f \sqcup g = f \cup (P(f)g) = f \curlyvee (P(f)g),$$

while

$$f \curlyvee g = (f \cap g) \cup (P(g)f) \cup (P(f)g) = ((f \cap g) \sqcup (P(g)f)) \sqcup (P(f)g).$$

Hence $\curlyvee$ and $\sqcup$ are interdefinable in the presence of P and intersection.

In fact the operation P itself (whence D) is definable in terms of $\backslash$ when either of $\curlyvee$ or $\sqcup$ is present. Indeed

$$P(f) = (1 \curlyvee f)\backslash(f \cap 1) = (f \sqcup 1)\backslash f,$$

where $f \cap 1$ is obtained via $f\backslash(f\backslash 1)$. A further equivalence comes from considering $P, \curlyvee$: using these operations we may define $0, 1$ using just P, as well as intersection via $f \cap g := P^2(f)P^2(g)(f \curlyvee g)$. Hence all four of the following combinations are equivalent in expressive strength on $\mathcal{P}(X)$:

- $\{P, \cap, \sqcup\}$;
- $\{P, \curlyvee\}$;
- $\{\bowtie, \sqcup, 0\}$;
- $\{\backslash, \sqcup, 1\}$.

We characterize the abstract class of semigroups of functions equipped with the operations P, $\sqcup$ and $\cap$: again, a finitely based variety is obtained.

Maximal iterate

Finally, we consider a novel functional analog of the operation of reflexive transitive closure r^* of a binary relation r. Now r^* has the infinitary definition $r^* := \bigcup_{i \in \omega} r^i$ (where $r^0 = 1$), but again, $\mathcal{P}(X)$ is not closed under *. Note that, if we consider the binary relation r as a "multivalued function" then the property $(a, b) \in r^*$ can be expressed as "there exists an iterate of a under r equal to b". We propose the following functional (or deterministic) analog. Continuing with the notation $f^0 = 1$, we let the *maximal iterate* $f^\uparrow$ of f be the function

$$\{(a, b) \mid (\exists n \geq 0)\ f^n(a) = b\ \ \&\ \ f^{n+1} \text{ is undefined at } a\}.$$

The reader should have no trouble verifying that $f^\uparrow$ can equivalently be defined by

$$f^\uparrow := \bigcup_{i \geq 0} \left(f^i P(f)\right) = f^* P(f).$$

The maximal iterate of f acts as the identity where f is undefined, and otherwise returns the last well-defined iterate under f. It is easy to see that $P(f^\uparrow)$ is the largest invariant set under f that lies wholly within its domain.

Again, the link with program algebra is very tight. The usual program construct "`while` α `do` f" can be expressed as $(\alpha f)^\uparrow P(\alpha)$, again identical in form to a familiar formulation for relational models of programs in which the Kleene closure operator is used rather than maximal iteration. However, $f^\uparrow = \texttt{while}\ D(f)\ \texttt{do}\ f$, showing that $^\uparrow$ is precisely rich enough in the presence of domain complement to capture `while-do`, in contrast to $*$ in the relational case, which is strictly richer. We return to these links with program algebras and logics in more detail in the final section.

Maximal iteration has an inherently infinitary definition, and we speculate that there is no recursively enumerable axiomatization for quasi-equations if the finiteness restriction is dropped. Supporting this, we note that there is no recursively enumerable axiomatization for the valid quasiequations holding in relational Kleene algebras (meaning abstract algebras of binary relations under composition plus $\cup, ^*, 0, 1$); for example, see [25]. If this is true, then the best one can hope for is a simple set of axioms that is complete for functional models restricted in various ways (such as being defined on a finite set).

We provide a finite set of axioms sufficient to characterize modal restriction semigroups of functions equipped with the operation of $^\uparrow$ or indeed the operations $\cap, ^\uparrow$ under the assumption of finiteness of the underlying set X. The resulting axioms prove sufficient to interpret important program logics discussed in the final section.

3. The characterizations

We now systematically work through the characterizations of the algebras of functions under the various operations just discussed. Characterizations will be obtained for algebras of functions equipped with composition and domain complement plus any subset of $\{\sqcup, \cap, ^\uparrow\}$. In every case, a set of laws is presented which is obviously satisfied by the algebra of functions in question. Completeness follows once it is shown that any algebra satisfying the laws may be faithfully represented as an algebra of functions.

The same basic representation approach works in all cases: it is an application of the method of determinative pairs, developed and used extensively by Boris Schein and his colleagues. We turn to this next.

3.1. *Generalities: representations and determinative pairs*

Throughout what follows, a *relation semigroup* (resp. *function semigroup*) is any algebra whose elements are binary relations (resp. functions) on some set X and whose operations include the operation of composition. In general, when $\mathcal{F}$ is some family of operations (aside from composition) defined on $\mathcal{B}(X)$ or $\mathcal{P}(X)$, a *relation* $\mathcal{F}$*-semigroup* is a relation semigroup whose operations are those in $\mathcal{F}$, together with composition; likewise for function $\mathcal{F}$-semigroups. The set X is the *universe* of a relation semigroup in $\mathcal{B}(X)$ (resp. function semigroup in $\mathcal{P}(X)$).

By the *abstract class* of a class K of relation semigroups or function semigroups, we mean the isomorphism closure $\mathbb{I}(K)$ of K. Thus, the abstract class of function $\{\}$-semigroups is just the class of all semigroups, and the abstract class of function $\{D\}$-semigroups is the class of restriction semigroups. Our goal is to first extend this result to provide axioms for function $\{P\}$-semigroups, and then to extend that to the various enrichments described above.

When considering elements of $\mathcal{B}(X)$, we usually write 0 for the empty relation $\varnothing$, 1 for the diagonal relation Δ, and $\Delta(X)$ for the subset consisting of all restrictions of the identity map (as previously). The operation of composition will usually be written as concatenation.

In accordance with standard convention, our functions act on the right of a set X: the function ϕ acts as $x \mapsto x\phi$. (This is the reverse of the convention used by the authors in previous related articles and so occasional translations are required when consulting some references.)

By a *functional representation* of a semigroup with extra operations $\mathbf{S}$ we mean a homomorphism from $\mathbf{S}$ into a function semigroup (in which the semigroup multiplication is preserved as composition).

It is straightforward to check that for any choice of subset $\mathcal{F}$ of the operations described above, the abstract class of function $\mathcal{F}$-semigroups is closed under the taking of both subalgebras and direct products. Hence the existence of a faithful functional representation is equivalent to there being, for each distinct pair of elements a, b a functional representation separating a from b. This last fact is well known, and considered at some length in [37] (for example), but the argument is easy and we give it.

Thus consider the abstract class of function $\mathcal{F}$-semigroups, where $\mathcal{F}$ is some subset of $\{D, P, \bowtie, \cap, \setminus, 0, 1\}$. Closure under subalgebras is trivial. For closure under direct products, observe that if $\{\mathbf{S}_i \mid i \in I\}$ is a family of function $\mathcal{F}$-semigroups on the (disjoint) sets $\{X_i \mid i \in I\}$, then the map from the direct product $\prod_{i\in I} \mathbf{S}_i$ to the union $\bigcup_{i\in I} \mathcal{P}(X_i)$ given by $(f_i)_{i\in I} \mapsto \bigcup_{i\in I} f_i$ preserves all of the operations in $\mathcal{F}$, and is easily seen to be an isomorphism.

Schein's Fundamental Theorem of Relation Algebras as in [52] (whose finer details are too technical to recall here) ensures that, except when † is present, the abstract classes of function semigroups will be axiomatizable in universal first order logic. Moreover, since they are closed under taking direct products and subalgebras, it follows that they will necessarily be axiomatizable using quasi-equations.

The representation of restriction semigroups as function $\{D\}$-semigroups is based on a straightforward regular representation, similar to that used to prove the Vagner-Preston theorem for inverse semigroups. However, for the case of function $\{P\}$-semigroups, a more complicated representation is needed: it must both represent the restriction semigroup of partial functions, and the various Boolean operations on domain elements. The following lemma summarizes the general approach taken.

Lemma 1. *Let* $\mathcal{F} \subseteq \{D, P, \bowtie, \cap, \setminus, 0, 1\}$*, and let* $\mathcal{A}_{\mathcal{F}}$ *be the class of function* $\mathcal{F}$*-semigroups. Let* Σ *be a set of laws satisfied by* $\mathcal{A}_{\mathcal{F}}$*. If for every model* $\mathbf{M}$ *of* Σ *and every* $a \neq b$ *in* M*, there is a (not necessarily faithful) representation* $\phi_{a,b}$ *of* $\mathbf{M}$ *into a member* $\mathbf{A}_{a,b}$ *of* $\mathcal{A}_{\mathcal{F}}$ *that separates* a *from* b*, then* $\mathbf{M}$ *is isomorphic to a function* $\mathcal{F}$*-*

semigroup and the class of models of Σ *is precisely the isomorphism closure of* $\mathcal{A}_{\mathcal{F}}$*. Furthermore, if* $\mathbf{M}$ *and each* $\mathbf{A}_{a,b}$ *is a function* $\mathcal{F}$*-semigroup on a finite universe, then* $\mathbf{M}$ *is isomorphic to a function* $\mathcal{F}$*-semigroup on a finite universe.*

Proof. The membership of an algebra $\mathbf{A}$ in a class $\mathcal{K}$ of algebras closed under taking isomorphic copies of subalgebras of direct products is equivalent to the property that for every $a \neq b$ in A there is a homomorphism ϕ from $\mathbf{A}$ into a member $\mathbf{B}$ of $\mathcal{K}$ for which $\phi(a) \neq \phi(b)$. The finiteness claim follows from the fact that direct products can be represented using unions of universes, as above. □

The means of defining the representations $\phi_{a,b}$ will be a special case of an approach developed by Schein; see [56] for a detailed exposition. The method is closely related to the notion of a coset representation of a group, except that instead of a group acting on the cosets of a subgroup, we have a semigroup acting on the equivalence classes of a right congruence (that is, an equivalence relation stable under right translations: $x \mathrel{\epsilon} y \Rightarrow xs \mathrel{\epsilon} ys$). The action is made partial by removing one of the equivalence classes. Since all of our semigroups have a multiplicative 0 and 1, we present the method under this assumption since it yields some simplifications.

Let $\mathbf{M}$ be a monoid with 0. If ϵ is a right congruence on $\mathbf{M}$ and W is the ϵ-class of 0, then the pair (ϵ, W) is called a *determinative pair* for $\mathbf{M}$. Let N denote the set of all ϵ-classes *except* W. Then every determinative pair gives rise to a partial action ψ of the monoid $\mathbf{M}$ on N. For any $x \in M$ we define the (partial) function ψ_x on N by $(a/\epsilon)\psi_x = ax/\epsilon$, provided that $ax \notin W$; otherwise, ψ_x is undefined at a/ϵ. The right congruence property ensures that this is well-defined.

The following proposition has a routine proof, which can be found in [56] for example.

Proposition 2. *Let* $\mathbf{M}$ *be a monoid with* 0*, let* (ϵ, W) *be a determinative pair, and assume that* $1 \notin W$ *(equivalently,* $\epsilon \neq M^2$ *where* M *is the universe of* $\mathbf{M}$*). The morphism* $\psi : \mathbf{M} \to \mathcal{P}(N)$ *associated with* (ϵ, W)*, given by* $a \mapsto \psi_a$*, preserves multiplication and represents* 1 *as the identity map and* 0 *as the empty map. Also, if* $a \notin W$ *and* $a/\epsilon \neq b/\epsilon$*, then* $\psi_a \not\subseteq \psi_b$*. If* $\mathbf{M}$ *is finite, then* ψ *is a monoid representation into a function semigroup with a finite universe.*

Thus let $\mathcal{F} \subseteq \{P, \bowtie, \cap, \setminus, 0, 1\}$, with $0, 1 \in \mathcal{F}$. In order to axiomatize function $\mathcal{F}$-semigroups, our task is to identify an abstract class of algebras defined by laws holding for all function $\mathcal{F}$-semigroups, and for which, for any algebra A in this class and any unequal $a, b \in A$, there is a determinative pair (ϵ, W) for which $(a, b) \notin \epsilon$ and the mapping $x \mapsto \psi_x$ respects all operations in $\mathcal{F}$.

This determinative pairs approach is used by Schein in [57] to establish an axiomatization for the case of composition and set difference, and by Dudek and Trokhimenko in [14] for the case of domain plus intersection; however, their results cannot be used here since the construction of ϵ in [57] does not represent the D operation, while that in [14] does not represent P (or the $\setminus$ operation) correctly.

3.2. *Function $\{P\}$-semigroups*

Following [31] and [22], we say that a *(left) restriction semigroup* is a semigroup equipped with a unary operation D satisfying the following laws:

- $D(x)x = x$ (6)
- $D(xy) = D(x)D(xy)$ (7)
- $D(x)D(y) = D(y)D(x)$ (8)
- $xD(y) = D(xy)x$ (the *twisted* law for D) (9)

In [31], the law $D(D(x)) = D(x)$ was included, but can readily be shown to follow from those listed above.

While we use the symbol D for the unary "domain" operation, many other notations have appeared in the literature, including each of $C, R, g, {}^{+}, {}^{\dagger}$ and *. As mentioned, the class of restriction semigroups is just the class of function $\{D\}$-semigroups. This extends to function $\{D, 0, 1\}$-semigroups when the the obvious axioms involving $0, 1$ are added.

Several properties of restriction semigroups will be very useful in what follows. We present these laws (and their names) as in [31].

- $D(D(x)) = D(x)$ (10)
- $D(xD(y)) = D(xy)$ (the *left congruence* law) (11)
- $D(D(x)y) = D(x)D(y)$ (the *normality* law) (12)

Very easy equational deductions reveal that within the variety of unary semigroups defined by (6)–(8) plus (10), the twisted law implies the left congruence law ([31, Proposition 3.2] for example) and the left congruence law implies the normality law (see [32, p. 400] for example). Semigroups of relations with D (such as relational Kleene algebras with domain [13]) satisfy (11), whence (12), but generally do not satisfy (9).

In any restriction semigroup S, the subset

$$D(S) = \{D(a) \mid a \in S\} = \{e \in S \mid e = D(e)\}$$

of *domain elements* is a subsemigroup of S which is a semilattice (modelling restrictions of the identity function). We will now consider S such that $D(S)$ is a Boolean algebra, with complement $'$ say.

As discussed above, adding the partial operation $'$ is equivalent to replacing both D and $'$ by the single operation of domain complement P. Hence for reasons of mathematical parsimony, the operation D will be supplanted by the domain complement operation P.

Note that any function $\{P\}$-semigroup will contain 0 (because $P(f)f = 0$ for all $f \in \mathcal{P}(X)$) and 1 (because $P(0) = 1$). Hence the function $\{P, 0, 1\}$-semigroups are polynomially equivalent to the function $\{P\}$-semigroups. However, we explicitly include the nullary operations 0 and 1 in the signature for ease of presentation (but with no loss of generality). We also make use of the derived domain operation $D = P^2$.

Definition 3. *The class of modal restriction semigroups is the quasivariety of algebras* $\langle S; \cdot, P, 0, 1\rangle$ *in which* $\langle S; \cdot, 0, 1\rangle$ *is a monoid with* 0 *and*

- $P(x)x = 0$ (13)
- $P(0) = 1$ (14)
- $xP(y) = P(xy)x$ (*the* twisted *law for* P) (15)
- $\alpha x = \alpha y \ \ \& \ \ P(\alpha)x = P(\alpha)y \ \Rightarrow \ x = y$ (16)

where Greek letters are shorthand for terms of the form $P(z)$ *for some variable* z (*distinct for distinct Greek letters*).

It is not hard to check that $\mathcal{P}(X)$ is a modal restriction semigroup when equipped with composition $\cdot$ and the operation P of restriction of the identity function to the complement of the domain of a function. Indeed, these simple and elegant axioms turn out to abstractly characterize function $\{P\}$-semigroups. The proof of the following theorem covers the remainder of this section.

Theorem 4. *The abstract class of function* $\{P, 0, 1\}$*-semigroups is the class of modal restriction semigroups.*

First, some notation: throughout the remainder of the article, Greek letters will represent elements of $P(S) = \{P(s) \mid s \in S\} = D(S)$, we define the relation $\leq$ by $x \leq y$ if $x = \alpha y$ for some α, and we let α' abbreviate $P(\alpha)$.

Consider a semigroup S with a nullary element 0 in which there is a subsemilattice L (ordered, as usual, by $e \leq f$ if $e = ef$) with $\max\{e \in L \mid es = 0\}$ existing for every $s \in S$. Such systems were studied by the authors in [33], where they were called *semilattice pseudocomplemented semigroups* (or SP-semigroups). It was shown that the class of SP-semigroups can be captured equationally as unary semigroups by defining a unary operation P by $P(s) = \max\{e \in L \mid es = 0\}$; the needed laws are the following:

- $P(x)x = 0$ (17)
- $P(x)P(0) = P(x)$ (18)
- $P(x)P(y) = P(y)P(x)$ (19)
- $P(x)P(y) = P(x)P(P(x)y)$ (20)

With these axioms given, L can be chosen to be the subsemigroup generated by the set $P(S) := \{P(s) \mid s \in S\}$ (see Propositions 1.1 and 1.2 of [33]).

It was observed in [33] that the modal restriction semigroup $\mathcal{P}(X)$ is an example of an SP-semigroup satisfying the twisted law (see [33, p. 2897]); hence any function $\{P\}$-semigroup is a twisted SP-semigroup. Theorem 4 therefore characterizes those SP-semigroups embedding into some $\mathcal{P}(X)$.

Proposition 5. *The following hold in any modal restriction semigroup* $\langle S; \cdot, P, 0, 1\rangle$.

- $\langle P(S); \cdot, \vee, ', 0, 1\rangle$ *is a Boolean algebra, where* $\alpha \vee \beta := (\alpha'\beta')'$ (21)
- $\langle S; \cdot, D, 0, 1\rangle$ *is a restriction semigroup* (22)
- $\leq$ *is a partial order, stable under left and right multiplication* (23)
- $P(x) = \max\{\alpha \in P(S) \mid \alpha x = 0\}$ (24)
- $P(x)P(y) = P(x)P(P(x)y)$ (25)
- $P(\alpha)x = P(\alpha x)x$ (26)
- $\alpha x = \alpha y \ \& \ \beta x = \beta y \Rightarrow (\alpha \vee \beta)x = (\alpha \vee \beta)y$ (27)
- $x\alpha = \beta x \Rightarrow x\alpha' = \beta' x$ (28)
- $x\alpha = \beta x \ \& \ x\gamma = \delta x \Rightarrow x(\alpha \vee \gamma) = (\beta \vee \delta)x$ (29)

where Greek letters are shorthand for terms of the form $P(z)$*, for some variable* z *(distinct for distinct variables),* α' *is shorthand for* $P(\alpha)$*, and* $D = P^2$.

In the proof we indicate application of laws (13), (15) and (16) of Definition 3 and of laws (25)–(29) by placing the corresponding equation number above the relevant equality symbols. The other properties are used without reference. We adopt a similar convention throughout, although we omit reference to law (13) after the following proof.

Proof. First observe that

$$P(x)P(x) \overset{15}{=} P(P(x)x)P(x) \overset{13}{=} P(0)P(x) \overset{14}{=} P(x). \qquad (30)$$

So $P(S) := \{t \in S \mid \exists s \in S : \ P(s) = t\}$ consists of idempotent elements. Now we use this and (16) to prove that $P(S)$ is a semilattice. First note that $P(x)P(x)P(y) \overset{30}{=} P(x)P(y)$, while

$$P(x)P(y)P(x) \overset{15}{=} P(P(x)y)P(x)P(x) \overset{30}{=} P(P(x)y)P(x) \overset{15}{=} P(x)P(y).$$

Also, we have $P(P(x))P(x)P(y) \overset{13}{=} 0P(y) = 0$, while

$$P(P(x))P(y)P(x) \overset{15}{=} P(P(P(x))y)P(P(x))P(x) \overset{13}{=} P(P(P(x))y)0 = 0.$$

Hence, by (16) (with $\alpha = P(x)$) we have

$$P(x)P(y) = P(y)P(x). \qquad (33)$$

We use the idempotency and commutativity of $\cdot$ on $P(S)$ freely throughout the remainder of the proof.

Now we prove (25): $P(x)P(P(x)y) = P(P(x)y)P(x) \overset{15}{=} P(x)P(y)$. Next we show that $P^2(x)x = x$, which is part of (22). We have $P(x)P^2(x)x \overset{13}{=} 0 \overset{13}{=} P(x)x$ and $P^2(x)P^2(x)x = P^2(x)x$, so that (16) (with $\alpha := P(x)$) shows that $P^2(x)x = x$.

We now make use of some facts in [33] (though each claim is a relatively straightforward equational deduction). Unary semigroups satisfying (13), (14), (25), (33), and $P^2(x)x = x$ were studied by the authors in [33, §3], where they are called *closable SP-semigroups*. These are examples of SP-semigroups, all of which satisfy property (24) as is shown in [33]. Moreover it is shown in [33, Corollary 3.2] that property (21) holds (property $0 = P^2(0)$ comes from [33, Proposition 1.1]) and that each of the laws (6)–(8) as well as (10) hold for the derived operation $D := P^2$. Furthermore, (9) (the twisted law for D) follows from (15) in any closable SP-semigroup; see base of page 2903 in [33]. Thus (22) is true. Property (23) holds on any restriction semigroup; see [31] for example.

For (26) we have

$$\alpha P(\alpha)x \overset{13}{=} 0 \overset{13}{=} \alpha P(x)x \overset{25}{=} \alpha P(\alpha x)x$$

and also that

$$P(\alpha)P(\alpha)x = P(\alpha)x = P(0)P(\alpha)x = P(P(\alpha)\alpha x)P(\alpha)x \overset{15}{=} P(\alpha)P(\alpha x)x,$$

so $P(\alpha)x = P(\alpha x)x$ by (16).

For (27), assume that $\alpha x = \alpha y$ and $\beta x = \beta y$ hold. Then

$$\alpha(\alpha \vee \beta)x = \alpha x = \alpha y = \alpha(\alpha \vee \beta)y$$

while

$$P(\alpha)(\alpha \vee \beta)x = P(\alpha)\beta x = P(\alpha)\beta y = P(\alpha)(\alpha \vee \beta)y,$$

so that $(\alpha \vee \beta)x = (\alpha \vee \beta)y$ by (16).

For (28), suppose $x\alpha = \beta x$. Then

$$xP(\alpha) \overset{15}{=} P(x\alpha)x = P(\beta x)x \overset{26}{=} P(\beta)x = P(\beta)x.$$

Finally, for (29), suppose $x\alpha = \beta x$ and $x\gamma = \delta x$. Then $xP(\alpha) = P(\beta)x$ and $xP(\gamma) = P(\delta)x$ by (28), so

$$\beta x(\alpha \vee \gamma) = x\alpha(\alpha \vee \gamma) = x\alpha = \beta x = \beta(\beta \vee \delta)x,$$

and similarly,

$$P(\beta)x(\alpha\vee\gamma) = xP(\alpha)(\alpha\vee\gamma) = xP(\alpha)\gamma = x\gamma P(\alpha) = \delta xP(\alpha) = \delta P(\beta)x = P(\beta)(\beta\vee\delta)x$$

and so we must have $x(\alpha \vee \gamma) = (\beta \vee \delta)x$ by (16), as required. □

We now proceed to the representation of modal restriction semigroups, which will complete the proof of Theorem 4.

Let S be a modal restriction semigroup, F a filter of $P(S)$. Define

$$\epsilon_F = \{(s,t) \in S \times S \mid \exists\, \alpha \in F : \alpha s = \alpha t\},$$

$$W_F = \{s \in S \mid P(s) \in F\}.$$

Noting that $W_F = \{s \in S \mid \exists \alpha \in F : \alpha s = 0\}$, the following is immediate.

Proposition 6. *Let* $\langle S; \cdot, P, 0, 1\rangle$ *be a modal restriction semigroup,* F *a filter of* $P(S)$*. Then* (ϵ_F, W_F) *is a determinative pair:* ϵ_F *is a right congruence and* W_F *is the* ϵ_F*-class containing* 0*.*

A key result is the following, which has parallels in earlier work (see for example [57]). A simplifying feature here is that ultrafilters can be used.

Lemma 7. *Let* S *be a modal restriction semigroup. Suppose* $a, b \in S$ *are such that* $a \not\leq b$*. Then there is an ultrafilter* F *of* $P(S)$ *for which* $a \notin W_F$ *and* $(a, b) \notin \epsilon_F$*.*

Proof. Assuming $b \not\geq a$, we first consider the filter F_0 in $P(S)$ generated by the subset $\{\alpha \in P(S) \mid \alpha a = a\} \cup \{\beta' \in P(S) \mid \beta a = \beta b\}$. For such a filter to be proper, we must ensure that $0 \notin F_0$. But if in fact $0 \in F_0$, then we would have $\alpha, \alpha_1', \ldots, \alpha_n' \in F$ for which $\alpha\alpha_1'\alpha_2' \cdots \alpha_n' = 0$, where $\alpha a = a$ and $\alpha_i a = \alpha_i b$ for each i, and so by iterated application of quasiequation (27), we obtain $\beta a = \beta b$ where $\beta = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$. Hence

$$0 = \alpha\alpha_1'\alpha_2' \cdots \alpha_n' = \alpha(\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n)' = \alpha\beta',$$

so $\alpha \leq \beta$, and post-multiplying by a gives $a = \alpha a \leq \beta a = \beta b$, so $a \leq b$, a contradiction. So $0 \notin F_0$.

As every proper filter extends to an ultrafilter, F_0 can be embedded in an ultrafilter F, in particular containing all α for which $\alpha a = a$. Assume (for a contradiction) that $a \in W_F$. Then $\beta a = 0 = a0$ for some $\beta \in F$, and so $\beta' a = a1$ by (28), so $\beta' \in F$, and so $0 = \beta\beta' \in F$, which is false. So $a \notin W_F$. Lastly, F also contains all α' for which $\alpha a = \alpha b$, hence no α for which $\alpha a = \alpha b$, and so $(a, b) \notin \epsilon_F$. □

For each $a, b \in S$ with $a \not\leq b$, let $F_{a,b}$ be a fixed ultrafilter satisfying Lemma 7.

Now we are ready to prove the sufficiency of the modal restriction semigroup axioms (with respect to functional representability).

Lemma 8. *Suppose* S *is a modal restriction semigroup. For any* $a, b \in S$ *with* $a \not\leq b$*, the semigroup representation* ψ *associated with the determinative pair* $(\epsilon_{F_{a,b}}, W_{F_{a,b}})$ *is a modal restriction semigroup homomorphism separating* a *from* b*.*

Proof. Let F, ϵ and W stand for $F_{a,b}$, $\epsilon_{F_{a,b}}$ and $W_{F_{a,b}}$ respectively.

By Proposition 2, ψ correctly represents 0 and 1, as well as multiplication (as composition). Also, as a is not ϵ-related to b, Proposition 2 shows that $\psi_a \neq \psi_b$. It remains to verify that P is correctly represented; that is, that $\psi_{P(s)} = P(\psi_s)$.

Let $s, t \in S$ and suppose that $t \in \mathrm{dom}(\psi_{P(s)})$. So $t, tP(s) \notin W$. In particular then, $P(tP(s)) \notin F$, so that $P^2(tP(s)) \in F$. We now use this to show that $t \,\epsilon\, tP(s)$, so that $\psi_{P(s)}$ is a restriction of the identity map.

Let $\alpha = P^2(tP(s)) \in F$. Since $P^3 = P$, we have

$$\alpha t = \alpha P^2(tP(s))t = \alpha P(P(tP(s))t)t \overset{15}{=} \alpha P(tP(P(s)))t \overset{15}{=} \alpha tP(P(P(s))) = \alpha tP(s).$$

So $t \mathrel{\epsilon} tP(s)$ as required.

Now as $\psi_{P(s)}\psi_s = \psi_{P(s)s} = \psi_0 = 0$, it remains to show that $\psi_{P(s)}$ is defined wherever ψ_s is undefined. Suppose ψ_s is undefined at t/ϵ for some $t \notin W$; that is, $P(ts) \in F$. Then $P(ts)t \overset{15}{=} tP(s) = tP(s)P(s) \overset{15}{=} P(ts)tP(s)$, so by definition, $t/\epsilon\, \psi_{P(s)} = t/\epsilon$. □

Theorem 4 follows immediately from Lemmas 1 and 8, and from the fact that whenever $\mathbf{S}$ is a modal restriction semigroup and $a \neq b$ then either $a \not\lesssim b$ or $b \not\lesssim a$.

Next we turn to the characterization of function $\{P\}$-semigroups of injective functions. Let an element s of a modal restriction semigroup S be called *injective* if it satisfies

$$\forall x \forall y \ \left(xs = ys \to xD(s) = yD(s)\right). \tag{44}$$

If S is a function $\{P, 0, 1\}$-monoid, and $s \in S$ is an injective function, it is easy to see that s satisfies (44). The following shows that (44) is also sufficient under our representation.

Proposition 9. *Let S be a modal restriction semigroup. An element $s \in S$ is represented as an injective function by ψ if and only if s is an injective element.*

Proof. Assume that s is injective and consider $a, b \in S$ with $as/\epsilon_F = bs/\epsilon_F \notin W_F$. So there is $\alpha \in F$ with $\alpha as = \alpha bs$. Then we have $\alpha aD(s) = \alpha bD(s)$, and by twistedness of D (equation (9)) we have $\alpha D(as)a = \alpha D(bs)b$. Hence $\alpha D(as)D(bs)a = \alpha D(as)D(bs)b$, and $\alpha D(as)D(bs)$ is in F because each of $\alpha, D(as), D(bs)$ are in F. So by definition, a and b are ϵ_F-related, as required. □

As an immediate corollary we have a complete quasi-equational characterisation of the abstract class of injective function $\{P, 0, 1\}$-semigroups.

Corollary 10. *An algebra $\langle S, \cdot, P, 0, 1\rangle$ is isomorphic to an injective function $\{P, 0, 1\}$-semigroup if and only if it is a modal restriction semigroup satisfying the quasi-identity*

$$xs = ys \Rightarrow xD(s) = yD(s).$$

To complete the section we show that the quasi-identity (16) in the definition of a modal restriction semigroup cannot be replaced by any set of equations; hence the class of function $\{P\}$-semigroups is a proper quasivariety.

Proposition 11. *The class of modal restriction semigroups is a proper quasivariety.*

Proof. This will follow immediately from Example 12 below. □

Let $\mathbf{Q}$ be the function $\{P, 0, 1\}$-semigroup on the domain $\{1, 2, 3\}$ consisting of the identity function, the zero function and the following functions:

$$a := \begin{pmatrix} 1\,2\,3 \\ 1\,3\,3 \end{pmatrix}, b := \begin{pmatrix} 1\,2\,3 \\ 1\,2\,2 \end{pmatrix}, \qquad \gamma := \begin{pmatrix} 2\,3 \\ 2\,3 \end{pmatrix},$$
$$\gamma' := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \gamma a = a\gamma = \begin{pmatrix} 2\,3 \\ 3\,3 \end{pmatrix}, \gamma b = b\gamma = \begin{pmatrix} 2\,3 \\ 2\,2 \end{pmatrix}.$$

Example 12. The algebra $\mathbf{Q}$ is isomorphic to a function $\{P, 0, 1\}$-semigroup, but has a quotient that is not.

Proof. The first statement is trivial as $\mathbf{Q}$ is a function $\{P, 0, 1\}$-semigroup. Now let θ be the equivalence relation on Q identifying all three of γa, γb and γ. Note that the operation P on $\mathbf{Q}$ agrees on each of these three elements, so that P is preserved on $\mathbf{Q}/\theta$. That multiplication is preserved is routine and left to the reader. Thus $\mathbf{Q}/\theta$ is a well defined quotient algebra of $\mathbf{Q}$. However in $\mathbf{Q}/\theta$ we have $\gamma a/\theta = \gamma b/\theta$ (which both equal γ/θ) and $\gamma' a/\theta = \gamma' b/\theta$ (both equal γ'/θ). Thus $\mathbf{Q}$ fails the law (16), hence is not isomorphic to a function $\{P, 0, 1\}$-semigroup. $\square$

Example 12 can also be used to establish that the laws (13)–(16) provide an irredundant axiomatization for modal restriction semigroups. Indeed the non-representability of $\mathbf{Q}/\theta$ demonstrates that (16) cannot be removed. Law (15) cannot be removed, because when X is a set of size more than 1, the relation $\{P, 0, 1\}$-semigroup of all relations on a set X fails (15) but satisfies the remaining laws. For law (13), take any nontrivial monoid M with 0 and set $P(x) = 1$ for every $x \in M$ — all of the axioms except for (13) hold. Finally, to show the necessity of law (14), consider the three element semigroup on the set $\{0, 1, a\}$ in which 0 and 1 have the usual multiplicative properties, and $aa = 1$, define P by letting P map $1 \mapsto 0 \mapsto a \mapsto 0$, and note that all of the axioms hold except (14).

Note that the variety of modal restriction semigroups is subtractive in the sense of Ursini [65], as the term $t(x, y) := P(y)x$ satisfies $t(x, x) = 0$ and $t(x, 0) = x$. Subtractive varieties are congruence permutable at 0 (meaning that for any two congruences θ, ρ on an algebra A in the variety, $(x, 0) \in \theta \circ \rho$ if and only if $(x, 0) \in \rho \circ \theta$) and admit a working commutator theory.

3.3. *Function* $\{P, \sqcup\}$*-semigroups*

We now turn to the operation of preferential union, as defined in Section 2. It turns out that with Theorem 4 in hand, it is straightforward to characterize $\sqcup$.

In functional models, we have $f \sqcup g = h$ if and only if $D(f)h = f$ and $P(f)h = P(f)g$. So the extra axioms (beyond the axioms for modal restriction semigroups) required to characterize the abstract class of function $\{P, \sqcup\}$-semigroups are just the two laws

- $D(x)(x \sqcup y) = x$ and (46)
- $P(x)(x \sqcup y) = P(x)y$. (47)

This gives us the main theorem of this section.

Theorem 13. *The abstract class of function $\{P, \sqcup, 0, 1\}$-semigroups consists of the modal restriction semigroups equipped with binary operation $\sqcup$ satisfying* (46) *and* (47)*.*

Modal restriction semigroups with a binary operation called "preferential join" $\sqcup$ satisfying the laws of Theorem 13 will be called *modal restriction semigroups with preferential join.* These are our deterministic analogs of modal semirings.

Modal restriction semigroups with preferential join are monoidal classical restriction categories as in [9], since by Lemma 7.10 there, the latter are restriction semigroups with zero that have "stable" joins of domain-disjoint elements (meaning elements x, y for which $D(x)D(y) = 0$). Here, stability is left-distributivity (in our terms) of multiplication over such joins, which is easily seen to hold (see the next proposition).

Conversely, it may be shown that any monoidal classical restriction category is equivalent to a modal restriction semigroup with preferential join, if one defines $x \sqcup y := x \vee P(x)y$ (where $\vee$ is stable join of domain-disjoint elements). The proof is routine and is mostly omitted, although we show law (15). Note that by Lemma 6.10 (2) of [9], multiplication distributes on both sides of $\vee$ in a monoidal classical restriction category S, and $D(S)$ is a Boolean algebra, so for $x, y \in S$, we have, on writing $P(s)$ for the complement of $D(s)$, and exploiting the fact that S is a restriction semigroup,

$$\begin{aligned}
P(xy)x &= P(xy)x(D(y) \vee P(y)) \\
&= P(xy)xD(y) \vee P(xy)xP(y) \\
&= P(xD(y))xD(y) \vee P(xy)xP(y) \\
&= 0 \vee P(xy)xP(y) \\
&= xD(y)P(y) \vee P(xy)xP(y) \\
&= D(xy)xP(y) \vee P(xy)xP(y) \\
&= (D(xy) \vee P(xy))xP(y) \\
&= xP(y).
\end{aligned}$$

So monoidal classical restriction categories are really the same objects as modal restriction semigroups with $\sqcup$. Thus, one way of viewing Theorem 13 is as a functional representation theorem for the monoidal versions of classical restriction categories as in [9]. We conjecture that the representation can be extended to arbitrary classical restriction categories.

We next present some further useful laws for modal restriction semigroups with preferential join (which, in light of the comments just made, might just as well be called "classical restriction monoids").

Proposition 14. *The following laws hold on a modal restriction semigroup with* *preferential join* $\langle A; \cdot, \sqcup, P, 0, 1\rangle$:

- $\langle A; \cdot, 0, 1\rangle$ *is a monoid with* 0
- $\langle A; \sqcup, 0\rangle$ *is an idempotent monoid* (*a* band *monoid*) *additionally satisfying* $x \sqcup y \sqcup x = x \sqcup y$ (48)
- $\langle P(A); \cdot, \sqcup, P, 0, 1\rangle$ *is a Boolean algebra, where* $\cdot$ *is meet,* $\sqcup$ *is join and* P *is complement* (49)
- $P(x \sqcup y) = P(x)P(y)$ (*an extension of De Morgan's law*) (50)
- $D(x \sqcup y) = D(x) \vee D(y)$ (*the join law*) (51)
- $P(P(y)x) = P(P(y)D(x)) = D(y) \vee P(x)$ (*P-normality law*) (52)
- $x \sqcup y = y \sqcup x \Rightarrow (x \sqcup y)z = xz \sqcup yz$ (53)
- $x \sqcup y = x \sqcup P(x)y = P(x)y \sqcup x$ (54)
- $x(y \sqcup z) = xy \sqcup xz$ (*left distributivity law*) (55)
- $(x \sqcup y)z = xz \sqcup P(x)yz$ (*right distributivity law*) (56)
- $(\alpha \sqcup \beta)x = \alpha x \sqcup \beta x$ (57)
- $x \leq y \Leftrightarrow y = x \sqcup y.$ (58)

Proof. We leave it to the reader to verify that these laws are actually satisfied by function semigroups (that is, are necessary): this can be done either by direct argument on $\mathcal{P}(X)$, or (by completeness of our axioms) by equational deductions from our existing axioms. □

Quasi-equation (53) is used frequently below. The premise condition is equivalent to the preferential union coinciding with the conventional union, so it is satisfied exactly by union-compatible functions (see Section 2). In particular we always have $\alpha f \sqcup \alpha' g = \alpha' g \sqcup \alpha f$.

The following result contrasts with Proposition 11.

Proposition 15. *An SP-semigroup* (*see laws* (17) *to* (20)) *which is also a monoid satisfying the twisted law* (15), *and with binary operation* $\sqcup$ *satisfying the equational properties* (48), (49), (54), (55) *and* (57) *of Proposition 14, is a modal restriction semigroup with preferential join. Conversely, any modal restriction semigroup with preferential join satisfies these laws. Hence the abstract class of function* $\{\sqcup, P, 0, 1\}$*-semigroups is an equational class.*

Proof. Most of the axioms for modal restriction semigroups with preferential join (see Definition 3) are being assumed here or follow immediately. For example $(\alpha')' = \alpha$ follows because of (49). Only the implication (16) and the preferential join laws (46) and (47) remain.

For (16), assume $\alpha x = \alpha y$ and $\alpha' x = \alpha' y$ and then observe that

$$x = 1x = (\alpha \sqcup \alpha')x \overset{57}{=} \alpha x \sqcup \alpha' x = \alpha y \sqcup \alpha' y \overset{57}{=} y.$$

For (46), we have

$$D(x)(x \sqcup y) \overset{54}{=} D(x)(x \sqcup P(x)y) \overset{55}{=} D(x)x \sqcup D(x)P(x)y \overset{49}{=} x \sqcup 0 \overset{48}{=} x.$$

Finally, for (47) we have $P(x)(x \sqcup y) \overset{55}{=} P(x)x \sqcup P(x)y \overset{13}{=} 0 \sqcup P(x)y \overset{48}{=} P(x)y$.

The converse direction is immediate. □

If w is a semigroup word (involving only multiplication), we say that $D(w)$ is a *closed word*. An SP-monoid term is said to be in *reduced form* if it is a product αw, where α is a Boolean combination of closed words and w is a semigroup word. An SP-monoid with preferential join term is said to be in *reduced form* if it is a preferential join of reduced form SP-monoid terms.

Lemma 16. *In the variety of modal restriction semigroups with preferential join, every term is equivalent to a reduced form term.*

Proof. This is an easy consequence of the laws so far, in particular, left-to-right applications of twistedness (15), the normal laws (12) and (52) for D and P, and the left and right distributivity properties (55) and (56). As an example,

$$\begin{aligned}
(x \sqcup y)P((xP(y)) \sqcup y) &\overset{15}{=} (x \sqcup y)P((P(xy)x) \sqcup y)\\
&\overset{50}{=} (x \sqcup y)P(P(xy)x)P(y)\\
&\overset{56}{=} [xP(P(xy)x)P(y)] \sqcup [P(x)yP(P(xy)x)P(y)]\\
&\overset{15}{=} [P(xP(xy)x)xP(y)] \sqcup [P(x)P(yP(xy)x)yP(y)]\\
&\overset{15}{=} [P(P(xxy)xx)P(xy)x] \sqcup [P(x)P(P(yxy)yx)P(yy)y]\\
&\overset{52}{=} [P(P(xxy)D(xx))P(xy)x]\\
&\qquad\qquad \sqcup [P(x)P(P(yxy)D(yx))P(yy)y]\\
&\equiv [(D(xxy) \vee P(xx))P(xy)x]\\
&\qquad\qquad \sqcup [P(x)(D(yxy) \vee P(yx))P(yy)y]. \quad \square
\end{aligned}$$

We conclude this subsection with a universal algebraic observation. We first recall that the *syntactic congruence* $\mathrm{Syn}(\theta)$ of an equivalence relation θ on an algebra $\mathbf{A}$ is the largest congruence contained within θ. If there is a finite set of terms $F = \{t_i(x, z_1, z_2, \ldots) \mid i = 1, \ldots, n\}$ such that

$$\mathrm{Syn}(\theta) = \{(a, b) \in A^2 \mid (\forall c_1, c_1, \ldots \text{ in } A)(\forall t \in F) \quad t(a, c_1, \ldots)\, \theta\, t(b, c_1, \ldots)\}$$

then F is said to determine $\mathrm{Syn}(\theta)$. The set of terms F determines syntactic congruences in a variety $\mathcal{V}$ if it determines every syntactic congruence in every algebra in $\mathcal{V}$; in this case $\mathcal{V}$ is said to have *finitely determined syntactic congruences* (FDSC), and F is said to *determine syntactic congruences* in $\mathcal{V}$.

Proposition 17. *The variety of modal restriction semigroups with preferential join has FDSC.*

Proof. A complete deduction procedure—known as *shadowing*—for establishing that F determines syntactic congruences in a variety is established by Clark, Davey, Freese and Jackson [7, §3] (although the property itself is much older; see discussion in [7]). Lemma 16 shows that every term may be reduced to a preferential join of $\sqcup$-free terms in which every variable is under at most two applications of P. While we omit the details, this makes it easy to identify a finite set of terms F in $\{\cdot, P, \sqcup\}$ from which the shadowing procedure of [7] may be applied, thus showing that FDSC is a property of the variety of modal restriction semigroups with preferential join $\square$

One of the key uses of the FDSC property is that it guarantees the equivalence of some important topological conditions. Recall that a topological space is *Boolean* if it is compact and totally disconnected; see [6, IV.4] for example.[a] A *Boolean topological algebra* is an algebra whose universe is a Boolean topological space which is *compatible*, in the sense that all operations of the algebra are continuous. Finite algebras admit the discrete topology which is compatible and Boolean. Profinite algebras—algebras arising as inverse limits of finite algebras—also inherit a natural compatible Boolean topology that makes them topologically residually finite. Indeed, it is well known that a Boolean topological algebra is topologically residually finite if and only if it is an inverse limit of finite structures (for example, see Almeida and Weil [2] or [7, Theorem 8.1] and associated discussion). The FDSC property is the only general algebraic property that guarantees topological residual finiteness, whence profiniteness, of a Boolean topological algebra. This fact has been independently observed by a number of authors; probably the first completely general version of it is given in Day [11] (see [7] for further discussion of the history).

The following fact now follows immediately from Proposition 17 and is useful in Section 3.6.

Corollary 18. *A modal restriction semigroup with preferential join admitting a compatible Boolean topology is topologically residually finite, whence is an inverse limit of finite modal restriction semigroups with preferential join.*

3.4. *Function* $\{P, \cap\}$*-semigroups*

We now enlarge the signature of modal restriction semigroups by including intersection. Function $\{D, \cap\}$-semigroups were characterized independently in both [14] and [32]; in the latter case, an elementary Cayley-style regular representation was shown to be sufficient, rather than the more general approach of determinative pairs.

As explained in Section 2, function $\{P, \cap\}$-semigroups are polynomially equivalent to function $\{\bowtie, 0\}$-semigroups. So although it would suffice to find a system of extra laws relating to intersection and then show that the representation of the

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[a] Sometimes Boolean spaces are called Stone spaces; see Johnstone [38] for example.

previous section extends, various simplifications facilitate a direct proof involving function $\{\bowtie, 0\}$-semigroups.

Definition 19. *The class of modal Eq-monoids is the variety of algebras of the form* $\langle S; \cdot, \bowtie, 0\rangle$ *in which* $\langle S; \cdot, 0, 1\rangle$ *is a monoid with* 0 *and*

- $\alpha^2 = \alpha$
- $\alpha\beta = \beta\alpha$
- $x \bowtie x = 1$ (62)
- $(\alpha\beta) \bowtie 1 = \alpha\beta$ (63)
- $(x \bowtie y)x = (x \bowtie y)y$ (64)
- $x(y \bowtie z) = (xy \bowtie xz)x$ (*the twisted law*) (65)
- $(\alpha')' = \alpha$

where Greek letters are shorthand for terms of the form $(r \bowtie s)$ *for variables* r, s (*distinct for distinct Greek letters*), *and* α' *is shorthand for* $(\alpha \bowtie 0)$.

As in the previous section, unnumbered equations are used without reference.

An obvious consequence of (65) and the second commutativity law above is $\alpha(x \bowtie y) = \alpha(\alpha x \bowtie \alpha y)$. Algebras satisfying this identity along with the laws of Definition 19 up to (64) have been called (left) *Eq-monoids* by the second author [62] (where law (65) is called the *E-deterministic law*). Eq-monoids are in turn the simplest cases of the E-structures considered in [16], where the operation $\bowtie$ acts as a kind of operational version of equality within an algebra. It follows (see [62]) that the elements of the form $r \bowtie s$ form a multiplicative subsemilattice L and that $s \bowtie t$ is the maximal left "equalizer" in L of s and t:

$$s \bowtie t = \max\{\alpha \in L \mid \alpha s = \alpha t\}; \tag{66}$$

indeed an equivalent way to define Eq-monoids is in terms of the existence of such maximal left equalizers from some fixed subsemilattice L for all elements a, b.

A familiar example of an Eq-monoid is any Boolean algebra L, with the monoid multiplication chosen to be the Boolean meet and $\bowtie$ defined to be the usual biconditional, or "if-and-only-if" operation; certainly (66) holds, but in fact this example is a modal Eq-monoid. Similarly, both $\langle \mathcal{P}(X); \cdot, \bowtie, 0, 1\rangle$ and $\langle \mathcal{B}(X); \cdot, \bowtie, 0, 1\rangle$ satisfy (66) and so are Eq-monoids; moreover, the reader can easily verify that $\mathcal{P}(X)$ is a modal Eq-monoid, whereas $\langle \mathcal{B}(X); \cdot, \bowtie, 0, 1\rangle$ fails the twisted law (65).

We shall prove the following result.

Theorem 20. *The abstract class of function* $\{\bowtie, 0, 1\}$*-semigroups is the variety of modal Eq-monoids.*

The proof of Theorem 20 covers the remainder of this section.

As in the previous section, to prove sufficiency it is easier to work with an expanded list of laws.

Proposition 21. *The following properties hold in any modal Eq-monoid* $\langle S; \cdot, \bowtie, 0, 1\rangle$ *(here* $P(x) := x \bowtie 0$*).*

- $\langle S; \cdot, P, 0, 1\rangle$ *is a modal restriction semigroup*
- $(x \bowtie y) = (y \bowtie x)$ (67)
- $(x \bowtie sy)(s \bowtie t) = (x \bowtie ty)(s \bowtie t)$ (68)
- $xs \bowtie xt = x \bowtie x(s \bowtie t)$ (69)

Proof. For the first property, all of the modal restriction semigroup laws follow easily except for (16). But suppose $\alpha x = \alpha y$ and $\alpha' x = \alpha' y$. Then $x \bowtie y \geq \alpha$ and $x \bowtie y \geq \alpha'$, so $x \bowtie y \geq \alpha \vee \alpha' = 1$, so $x \bowtie y = 1$ and so $x = (x \bowtie y)x = (x \bowtie y)y = y$.

Laws (67) and (68) hold in any Eq-monoid, as is shown in [62]. For (69), let $\alpha = x \bowtie x(s \bowtie t)$. Now

$$(xs \bowtie xt)x = (xs \bowtie xt)(xs \bowtie xt)x \overset{65}{=} (xs \bowtie xt)x(s \bowtie t)$$

so that $xs \bowtie xt$ is a left equalizer of x and $x(s \bowtie t)$. By (66) we have $xs \bowtie xt \leq \alpha$, or equivalently, $xs \bowtie xt = \alpha(xs \bowtie xt)$. But then we also have

$$\begin{aligned} xs \bowtie xt &= \alpha(xs \bowtie xt) \\ &\overset{65}{=} (\alpha xs \bowtie \alpha xt)\alpha \\ &\overset{64}{=} (\alpha x(s \bowtie t)s \bowtie \alpha xt)\alpha \\ &\overset{64}{=} (\alpha x(s \bowtie t)t \bowtie \alpha xt)\alpha \\ &\overset{64}{=} (\alpha xt \bowtie \alpha xt)\alpha \\ &\overset{62}{=} \alpha. \end{aligned}$$

So (69) holds. □

We now return to representations.

Let S be a modal Eq-monoid. Since S is a modal restriction semigroup (if we set $P(x)$ to be the term operation $x \bowtie 0$), the determinative pair construction of the previous section is well-defined for S, and we use it here. Notice that property (66) shows that when F is a filter, ϵ_F can be written as $\{(s,t) \mid s \bowtie t \in F\}$.

Lemma 22. *Let* S *be a modal Eq-monoid. For any ultrafilter* F *of* $P(S)$*, the representation* ψ *corresponding to the determinative pair* (ϵ_F, W_F) *preserves* $\bowtie$.

Proof. (We follow the notation as in the proof of Lemma 8.) For $s, t \in S$, $\psi_{(s \bowtie t)}$ is a restriction of the identity having domain at least a subset of the set on which ψ_s and ψ_t do not disagree (since $(s \bowtie t)s \overset{64}{=} (s \bowtie t)t$ for all $s, t \in S$). To show that it is the whole of this subset, suppose that $u/\epsilon\, \psi_s = u/\epsilon\, \psi_t$ for some $u \in S$, so

$us/\epsilon = ut/\epsilon$, and so $(u \bowtie u(s \bowtie t)) \overset{69}{=} us \bowtie ut \in F$. Then $u(s \bowtie t)/\epsilon = u/\epsilon$ and so $u/\epsilon\, \psi_{(s \bowtie t)} = u/\epsilon$ as required. $\square$

Theorem 20 follows immediately from this lemma and Lemma 1.

Proposition 9 carries over to the modal Eq-monoid case too (as the representation, ψ, is the same), but here we can replace the implication (44) by an equational property. One can prove directly from the Eq-monoid axioms that the injective property (44) for an element s is equivalent to the following property:

$$\forall x \forall y\ (x \bowtie y)ys = (xs \bowtie ys)ys. \tag{71}$$

However this claim also follows from the next proposition, which parallels Proposition 9 and its proof.

Proposition 23. *Let S be a modal Eq-monoid. The element $s \in S$ is represented as an injective function by ψ if and only if s is an injective element of S.*

Proof. Suppose $x/\epsilon, y/\epsilon \neq W_F$, with $xs/\epsilon = ys/\epsilon \neq W_F$. Then $xs \bowtie ys \in F$, so because $(x \bowtie y)xs \bowtie xs = (xs \bowtie ys)xs \bowtie xs \geq xs \bowtie ys \in F$ on using (71), we have that $(x \bowtie y)xs \bowtie xs \in F$.

Suppose $x \bowtie y \notin F$. Then since F is an ultrafilter, $(x \bowtie y) \bowtie 0 \in F$, and so $F \ni ((x \bowtie y) \bowtie 0)((x \bowtie y)xs \bowtie xs) \leq 0 \bowtie xs$, so $P(xs) \in F$, contradicting $xs \notin W_F$. Hence $x \bowtie y \in F$, and so $x/\epsilon = y/\epsilon \neq W_F$, as required. $\square$

Corollary 24. *An algebra $S = \langle S; \cdot, \bowtie, 0\rangle$ is isomorphic to an injective function $\{\bowtie, 0\}$-semigroup if and only if it is a modal Eq-monoid satisfying the identity $(x \bowtie y)yz = (xz \bowtie yz)yz$.*

Notice that Equation (71) can be written as $(x \wedge y)z = xz \wedge yz$, where $\wedge$ is the term function given by $x \wedge y := (x \bowtie y)x$ and agreeing with intersection in functional models. Of course left distributivity of multiplication over meet is also a consequence of the modal Eq-monoid axioms (since it holds in functional models and our axioms are complete for function models).

As noted in [62], any variety of Eq-monoids is 0-regular, in the sense that all congruences are determined by the class containing 1; this follows because the term $d(x, y) = x \bowtie y$ satisfies $d(x, y) = 1$ if and only if $x = y$. (See Corollary 1.7 in [23].)

3.5. *Function $\{P, \cap, \sqcup\}$-semigroups*

It is straightforward to introduce $\cap$ to the signature of modal restriction semigroups with preferential join, or equivalently, $\sqcup$ to the signature of modal Eq-monoids, and the representation theorems immediately carry over.

Theorem 25. *The abstract class of function $\{\bowtie, \sqcup, 0, 1\}$-semigroups consists of the modal Eq-monoids equipped with a binary operation $\sqcup$ satisfying* (46) *and* (47).

We call such an enriched modal Eq-monoid a *modal Eq-monoid with* $\sqcup$. This class is our closest functional analog to Tarski's relation algebras (possibly including the partial inverse operation, which can be covered using the method of [37]), and is easily reformulated in terms of either $\{\backslash, \sqcup, 1\}$ or $\{P, \curlyvee\}$ (using the equivalences described in Section 2).

In contrast to Tarski's representable relation algebras (which have undecidable finite membership problem, [29]), the finite equational axiomatization given by Theorem 25 shows that the finite membership problem in this functional case is solvable in polynomial time: to establish membership, just check for satisfaction of the axioms.

We showed in Section 2 that P is a term function in $\cdot, \backslash, \sqcup$, or equivalently, $\cdot, \backslash, \curlyvee$. So because Schein's representation in [57] for function $\{\backslash\}$-semigroups does not preserve P, neither $\sqcup$ nor $\curlyvee$ can be defined equationally in terms of $\backslash$.

The class of modal Eq-monoids with $\sqcup$ has two strong algebraic properties: congruence distributivity (the lattice of congruences is distributive) and congruence permutability (for any two congruences θ, ρ on a modal Eq-monoid, $\theta \circ \rho = \rho \circ \theta$). Hence they form an arithmetical variety (see [6, §II.12]).

Proposition 26. *The variety of modal Eq-monoids is congruence distributive, whence so is the variety of modal Eq-monoids with* $\sqcup$; *the latter is also congruence permutable and congruence regular.*

Proof. The congruence distributivity of the functional Eq-monoids was proved by a structural argument in [37], however we observe here that congruence distributivity follows using only the term operation of restricted union: $[x, y, z] := (x \cap y) \cup (x \cap z) = x \backslash ((x \backslash y) \backslash z)$; the argument can be found in Delić [12] for example.

Congruence permutability follows because the term

$$\rho(x, y, z) = (y \bowtie z)x \sqcup (x \bowtie y)z$$

is a Malcev term, satisfying $\rho(x, y, y) = x$ and $\rho(x, x, y) = y$ (see [6, II.12.2]).

Finally we prove congruence regularity. Consider a modal Eq-monoid with $\sqcup$, M, a congruence θ on M and an element $a \in M$. It is already known that congruences on Eq-monoids are determined by the set $1/\theta \cap P(M)$, consisting of closed elements congruent to 1; see [62] or [30, Example 5] (this simply uses the fact that $x = y$ if and only if $x \bowtie y = 1$). So it suffices to show that $\{b \bowtie c \mid b, c \in a/\theta\} = 1/\theta \cap P(M)$. The containment $\{b \bowtie c \mid b, c \in a/\theta\} \subseteq 1/\theta$ follows from law (62). Now observe that for all $\alpha \in 1/\theta \cap P(M)$ we have that $(\alpha a \sqcup \alpha')\ \theta\ (1a \sqcup 0) = a$ and that $(\alpha a \sqcup \alpha') \bowtie a = \alpha$ (as can be checked by arguing in terms of functions), showing that $\{b \bowtie c \mid b, c \in a/\theta\} \supseteq 1/\theta$ holds. □

By contrast, the variety of modal restriction semigroups with preferential join seems to have no obvious congruence regularity conditions. Consider the example obtained from the algebra of all functions on a set X that are either totally defined constant maps or are equal to the empty map or the identity; then any equivalence relation whose blocks include $\{0\}$ and $\{1\}$ is a congruence.

3.6. *Function $\{P, {}^{\uparrow}\}$-semigroups*

We next present axioms for maximal iteration that are sound, and indeed complete for cases in which the underlying set X is finite. (As explained earlier, there is no reason to expect that maximal iteration can be effectively axiomatized in general.) The axioms we give also model many properties of maximal iteration relevant in program algebra, as we show in the final section.

We consider the following system of laws, easily seen to hold on $\mathcal{P}(X)$. (We use the same abbreviations as in previous sections.)

- $D(x)x^{\uparrow} = xx^{\uparrow}$ (73)
- $P(x)x^{\uparrow} = P(x)$ (74)

We say that a modal restriction semigroup S equipped with a unary operation ${}^{\uparrow}$ called maximal iteration is a *modal restriction semigroup with* ${}^{\uparrow}$ if laws (73) and (74) hold.

Lemma 27. *In a modal restriction semigroup, if $ab = D(a)b$ and $P(a)b = P(a)$, then*

$$a^n P(a) \leq b \text{ for all } n \geq 0.$$

Proof. We use induction on n. Now $P(a)b = P(a)$ implies that $P(a) \leq b$, giving the $n = 0$ case. Assuming the $n = k$ case, we have $a^k P(a) = \beta b$ for some β, and so

$$a^{k+1}P(a) = aa^kP(a) = a\beta b \overset{15}{=} P(a\beta')ab = P(a\beta')D(a)b \leq b,$$

and the result follows. □

Corollary 28. *Let S be a modal restriction semigroup with ${}^{\uparrow}$. Under any functional representation $\phi : S \to \mathcal{P}(X)$ of the modal restriction semigroup S, the operation ${}^{\uparrow}$ is represented as a partial map at least as big as $(a\phi)^{\uparrow}$.*

Proof. This is immediate since $s^{\uparrow} \geq s^n P(s)$ for all $n \geq 0$, because these latter terms are all represented correctly by the previous results. □

In particular then, the above corollary applies to our determinative pair representation. To do better than this, we need an additional property. We say a modal restriction semigroup with ${}^{\uparrow}$ is *dynamic* if the following law holds.

- $\alpha x = \alpha x \alpha \Rightarrow \alpha x^{\uparrow} = \alpha x^{\uparrow} \alpha$ (77)

Note that law (77) arises very naturally in the setting of program logics: it is nothing but an algebraic encoding of the "while-rule" in Hoare logic, and is equivalent to a key inference rule in all forms of propositional dynamic logic (see below) — hence the use of the word "dynamic". In fact this law is equivalent to an equation.

Proposition 29. *A modal restriction semigroup with* $^\uparrow$ *is dynamic if and only if the following equation holds*:

$$\alpha P(x\alpha')(xP(\alpha x\alpha'))^\uparrow\alpha' = 0. \tag{78}$$

Proof. First note that (78) is equivalent under an application of (15) to the law

$$\alpha P(\alpha x\alpha')(xP(\alpha x\alpha'))^\uparrow\alpha' = 0. \tag{79}$$

Assume Equation (79) holds globally, and that α, x are such that $\alpha x\alpha' = 0$. Then

$$\alpha x^\uparrow\alpha' = \alpha P(\alpha x\alpha')(xP(\alpha x\alpha'))^\uparrow\alpha' \overset{79}{=} 0,$$

showing (77).

Conversely, suppose (77) holds. For fixed α, x, let $\beta = P(\alpha x\alpha')$, so that $\beta\alpha x\alpha' \overset{13}{=} 0$. But then $0 = \beta\alpha x\beta\alpha' = \beta\alpha(x\beta)(\beta\alpha)'$, and so by applying (77), we have $\beta\alpha(x\beta)^\uparrow(\beta\alpha)' = 0$. Hence $\alpha\beta(x\beta)^\uparrow\alpha' = 0$, which is (79). $\square$

Lemma 30. *Let S be any dynamic modal restriction semigroup with* $^\uparrow$. *Then for all $x \in S$,*

$$P(x^\uparrow) = \max\{\beta \in P(S) \mid \beta \leq D(x), \beta x\beta = \beta x\}.$$

Proof. Now $x^\uparrow \geq P(x)$ by (74), so $P(x^\uparrow) = D(x^\uparrow)' \leq D(x)$.

Moreover, $xx^\uparrow = D(x)xx^\uparrow$, and so $P(x^\uparrow)xx^\uparrow = 0$. Thus $P(xx^\uparrow) \geq P(x^\uparrow)$ by (24). Then $P(x^\uparrow)xP(x^\uparrow) \overset{15}{=} P(x^\uparrow)P(xx^\uparrow)x = P(x^\uparrow)x$.

Finally, suppose $\beta x\beta = \beta x$ and $\beta \leq D(x)$. Then $\beta x^\uparrow = \beta x^\uparrow\beta$ by (77) and $\beta P(x) = 0$. So

$$\beta P(x^\uparrow) \overset{15}{=} P(\beta x^\uparrow)\beta = P(\beta x^\uparrow\beta)\beta = P(\beta x^\uparrow P(x)\beta)\beta = P(0)\beta = \beta,$$

so that $\beta \leq P(x^\uparrow)$ as required. $\square$

Note that in $\mathcal{P}(X)$ (and using "$y \in \alpha$" as an obvious abbreviation),

$$P(x^\uparrow) = \{a \mid (\forall n \geq 0)\ x^n(a) \in D(x)\}.$$

There is evidently a connection with dynamical systems: given the system determined by $f \in \mathcal{P}(X)$, $P(f^\uparrow)$ is the largest invariant subset of the domain of f. The above result gives a description of this set in terms of $\Delta(X)$.

The dynamic property of modal restriction semigroups with $^\uparrow$ is one convenient way to try to axiomatize $^\uparrow$, but there are others. In the Kleene algebras with tests (KAT) of Kozen, the relational analog of the dynamic property is actually a corollary of other quasi-equations holding in Kleene algebra generally (see Theorem 1, Section 3 of [41]), laws for which we can give deterministic analogs. All of the following hold in $\mathcal{P}(X)$, for example.

- $x^{\uparrow}P(x) = x^{\uparrow}$, (84)
- $xy \leq y \Rightarrow x^{\uparrow}y \leq y$, (85)
- $xy \leq x \Rightarrow xy^{\uparrow} \leq x$. (86)

The first two of these give rise to a pleasing description of $a^{\uparrow}$.

Proposition 31. *Let S be a modal restriction semigroup with preferential join and $^{\uparrow}$ satisfying* (84) *and* (85)*. Then for all $a \in S$, $a^{\uparrow}$ is the smallest* (*under the natural P order*) *solution x to*

$$D(a)x = ax \;\;\&\;\; P(a)x = P(a).$$

Proof. If b is one such solution, then $ab = D(a)b \leq b$ and so $a^{\uparrow}b \leq b$ by (85), so that

$$a^{\uparrow} = a^{\uparrow}P(a) = a^{\uparrow}P(a)b = a^{\uparrow}b \leq b.$$

Conversely, clearly $a^{\uparrow}$ is one such b. □

Something analogous happens with the operation * in a Kleene algebra with tests: r^* is the least "reflexive transitive element" in the algebra, and any representation in terms of relations will represent it as the least reflexive transitive relation containing the image of r, but only amongst those relations that are images of elements in the algebra [42].

3.7. *Function $\{P, \sqcup, {}^{\uparrow}\}$-semigroups*

In the presence of $\sqcup$, we can go further. We define (dynamic) modal restriction semigroups with preferential join and $^{\uparrow}$ in the obvious manner. The next fact follows immediately from Propositions 15 and 29.

Corollary 32. *The class of dynamic modal restriction semigroups with preferential join and $^{\uparrow}$ is a variety.*

Since the operation $\sqcup$ in a modal restriction semigroup with preferential join is associative (48), we can unambiguously adopt the notation $\bigsqcup_{i=0}^{m} p_i$ for $p_0 \sqcup p_1 \sqcup \cdots \sqcup p_m$. For each $i \in \omega$, let $W_i(x)$ denote the term $\bigsqcup_{i=0}^{n-1} x^i P(x)$.

Lemma 33. *Let $\mathbf{M}$ be a modal restriction semigroup with preferential join and $^{\uparrow}$. For $a \in M$, $a^{\uparrow} \geq W_n(a)$ for all $n \geq 0$.*

Proof. Each $a^iP(a) \leq a^{\uparrow}$ by Lemma 27, or equivalently, $a^iP(a) = D(a^iP(a))a^{\uparrow}$. Using right distributivity over preferential joins of tests, we have

$$W_n(a) = \bigsqcup_{i=0}^{n-1} a^iP(a) = \bigsqcup_{i=0}^{n-1} \left(D(a^iP(a))a^{\uparrow}\right) \overset{57}{=} \left(\bigsqcup_{i=0}^{n-1} D(a^iP(a))\right) a^{\uparrow} \leq a^{\uparrow}.$$ □

Recall that an element a of a semigroup is *periodic* if there are $n, m \in \mathbb{N}$ such that $a^n = a^{n+m}$; the smallest n for which this holds is called the *index* of a.

Lemma 34. *Let* $\mathbf{M}$ *be a modal restriction semigroup with* $^\uparrow$. *If* $a \in M$ *is periodic with index* n, *then for all* $k \geq n$ *we have* $a^k P(a) = 0$.

Proof. Suppose $a^n = a^{n+m}$ for some $n, m \in \mathbb{N}$. For any $i \geq 1$, $P(a)a^i = P(a)aa^{i-1} = 0$, so $P(a) \leq P(a^i)$. Now suppose $a^n = a^{n+m}$ holds. So there is $\ell \geq n$ such that $a^\ell a^\ell = a^\ell$ and $a^k a^\ell = a^k$ for any $k \geq n$. Then $a^k P(a) = a^k a^\ell P(a) = a^k a^\ell P(a^\ell) P(a) \overset{15}{=} a^k P(a^\ell a^\ell) a^\ell P(a) = a^k P(a^\ell) a^\ell P(a) = 0$, as required. □

Lemma 34 shows that for a periodic element a in a modal restriction semigroup with preferential join and $^\uparrow$, the sequence $W_1(a)$, $W_2(a)$, … is eventually constant. In functional models, this implies that $a^\uparrow = W_i(a)$ for sufficiently large i. We now prove that this is also a consequence of our laws for $^\uparrow$ in the dynamic case. (In contrast to the previous cases, we cannot deduce such facts from their truth on functional models because we have not proved that our axioms for $^\uparrow$ are complete.)

Lemma 35. *Let* $\mathbf{M}$ *be a dynamic modal restriction semigroup with preferential join and* $^\uparrow$. *If* $a \in M$ *is periodic, with* $a^n = a^{n+m}$ *for some* $n, m \in \mathbb{N}$, *then* $a^\uparrow = W_n(a)$.

Proof. From repeated application of rule (50) in Proposition 14, we have that $P(W_n(a)) = \Pi_{i=0}^{n-1} P(a^i P(a))$, and as $a^n P(a) = 0$ by Lemma 34, we have, on applying (15) n times,

$$aP(W_n(a)) \overset{15}{=} \left(\Pi_{i=0}^{n-1} P(a^{i+1} P(a))\right) a \geq P(W_n(a))a,$$

so that $P(W_n(a))aP(W_n(a)) = P(W_n(a))a$. Hence by Lemma 30, $P(W_n(a)) \leq P(a^\uparrow)$, so $D(a^\uparrow) \leq D(W_n(a))$. But from Lemma 33, $W_n(a) \leq a^\uparrow$, so that $W_n(a) = D(W_n(a))a^\uparrow = D(W_n(a))D(a^\uparrow)a^\uparrow = D(a^\uparrow)a^\uparrow = a^\uparrow$. □

Hence in dynamic modal restriction semigroups with preferential join and $^\uparrow$ in which every element is periodic (for example, the finite ones), the $^\uparrow$ operation is an *implicit operation* in the operations $\cdot, P, \sqcup, 0, 1$. Implicit operations feature in the application of finite semigroups to finite state automata and the theory of regular languages; see [1] for example.

Corollary 36. *The class of finite dynamic modal restriction semigroups with preferential join and* $^\uparrow$ *coincides with the class of functional* $\{\sqcup, P, ^\uparrow, 0, 1\}$*-semigroups on finite domains, and is a pseudovariety: closed under the taking of subalgebras, homomorphic images and finitary direct products.*

Proof. Certainly all functional $\{\sqcup, P, ^\uparrow, 0, 1\}$-semigroups are dynamic modal restriction semigroups with preferential join and $^\uparrow$, and if they have a finite domain, then they are necessarily finite.

Conversely, let **M** be a finite dynamic modal restriction semigroup with preferential join and $^\uparrow$ with n elements. Then it is also a finite modal restriction semigroup with preferential join and $^\uparrow$, and hence is isomorphic as a modal restriction semigroup with preferential join and $^\uparrow$ (under ι, say) to a functional one $\mathbf{M}'$ on a finite domain. In both **M** and $\mathbf{M}'$ we have $a^\uparrow = W_n(a)$, and since the isomorphism preserves the term operation W_n, it also preserves $^\uparrow$.

The last claim follows because of Corollary 32. □

These facts also demonstrate that maximal iterate can be captured as a limit on not-necessarily periodic models, provided that the right kind of topology is present.

Corollary 37. *Let* $\mathbf{S} = \langle S; \cdot, P, \sqcup, 0, 1\rangle$ *be a modal restriction semigroup with preferential join admitting a compatible Boolean topology.*

(1) For any element $s \in S$*, the sequence* $(W_n(s))_{n\in\omega}$ *has a unique limit point, denoted by* $s^\triangle$*.*
(2) $\langle S; \cdot, P, \sqcup, ^\triangle, 0, 1\rangle$ *is isomorphic to a function* $\{P, \sqcup, ^\uparrow, 0, 1\}$*-semigroup (and hence satisfies all of* (73)–(77)*).*
(3) If $^\$: S \to S$ *is a function making* $\langle S; \cdot, P, \sqcup, ^\$, 0, 1\rangle$ *a modal restriction semigroup with preferential join and* $^\uparrow$ *satisfying the dynamic law* (77)*, then* $^\$$ *coincides with* $^\triangle$*.*

Proof. Claim (1) follows immediately from the fact that **S** is an inverse limit of finite modal restriction semigroups with preferential join (Corollary 18) and that in a finite modal restriction semigroup with preferential join, the sequence $(W_n(s))_{n\in\omega}$ is eventually constant. Now the value of $\triangle$ is correctly represented as maximal iterate in any finite modal restriction semigroup with preferential join, and because continuous functions preserve limits, and **S** is topologically residually finite, it follows that $\langle S; \cdot, P, \sqcup, ^\triangle, 0, 1\rangle$ is residually representable as a modal restriction semigroup with preferential join and $^\uparrow$, with $^\triangle$ is preserved as maximal iterate. But the class of representable function $\{P, \sqcup, ^\uparrow\}$-semigroups is closed under arbitrary direct products and subalgebras, so in fact $\langle S; \cdot, P, \sqcup, ^\triangle, 0, 1\rangle$ is faithfully representable as a function $\{P, \sqcup, ^\uparrow\}$-semigroup, proving (2).

For (3), note that in a finite modal restriction semigroup with preferential join and $^\uparrow$, the limit of the sequence $(W_n(s))_{n\in\omega}$ necessarily satisfies the equations (73)–(77). Hence this is true also of the function $^\triangle$ in **S**. So both \$ and $\triangle$ make **S** into a dynamic modal restriction semigroup with preferential join and $^\uparrow$ and then Lemma 30 shows that $D(s^\$) = D(s^\triangle)$. Now $s^\triangle$ is the limit of the sequence $(W_n(s))_{n\in\omega}$ that is nondecreasing in the $\leq$ order (which is a closed subset of the space $S \times S$). Hence it is the least upper bound of this sequence. However, $s^\$$ is also an upper bound of this sequence because repeated applications of $s^\$ = P(s) \sqcup ss^\$$ leads to $s^\$ = W_n(s) \sqcup s^n s^\$$. Hence $s^\triangle \leq s^\$$, showing that $s^\triangle = D(s^\triangle)s^\$ = D(s^\$)s^\$ = s^\$$, as required. □

One can prove analogs of Lemma 35 and Corollary 36 using laws (84) and (85) rather than (77), but we do not pursue this here.

3.8. *All other cases*

Because the same representation technique is used throughout, $\cap$ may be combined with any of the cases already considered with no further laws needed, thereby fulfilling the promise made at the beginning of this section that all subsets of $\{\sqcup, \cap, {}^{\uparrow}\}$ may be combined with composition and P to give finite axiomatizations.

4. Links with Propositional Dynamic Logic

4.1. *Intoduction*

The algebras discussed above have non-trivial connections to deterministic versions of propositional dynamic logic (PDL), in much the same way that Kleene algebras with domain relate to the non-deterministic version. We can use results of PDL to easily solve the equational problem for function $\{P, \sqcup, {}^{\uparrow}\}$-semigroups in the general (possibly infinite) case; conversely our results shine light on deterministic variants of PDL involving intersection.

First, some background. *Dynamic logic* is an extension of modal logic originally intended for reasoning about computer programs and later applied to other fields, such as linguistics, philosophy and artificial intelligence. It originated with the work of Pratt; see [49] and before long it gave birth to a propositional form due to Fischer and Ladner; see [17]. We now give a brief introduction to the language and relational semantics of Propositional Dynamic Logic (PDL), but for more extensive discussion and references we refer the reader to a book such as Harel, Kozen and Tiuryn [27] or the modal logic book by Blackburn, de Rijke and Venema [5].

In PDL there are two kinds of variables: atomic programs and atomic propositions. We will use $p, q, \ldots$ for programs and Greek letters $\alpha, \beta, \ldots$ for propositions. Each program p determines a unary operator of "modal necessity" on the propositions, denoted by $[p]$; it also determines a "modal possibility" operator $\langle p\rangle$ which is the operator $\neg[p]\neg$. Each proposition α produces a "test program" $\alpha?$.

Programs can be built from other programs using the binary operations of regular expressions, namely $\cup$, $;$, *, while propositions are built using the usual Boolean connectives $\wedge, \vee, \neg, 0, 1$ and by applying the modal operators $[p]$, $\langle p\rangle$ (for each program p). In this language the programming constructions of `if-then-else` and `while-do` are expressed as follows:

$$\texttt{if } \alpha \texttt{ then } p \texttt{ else } q \;=\; \alpha?; p \cup \neg\alpha?; q \tag{90}$$

$$\texttt{while } \alpha \texttt{ do } p \;=\; (\alpha?; p)^*; (\neg\alpha?). \tag{91}$$

We now briefly review the relational semantics for PDL. Programs are viewed as binary relations on some set X, and the propositions are viewed as subsets of X. The connectives $\cup$, $;$, * are interpreted as union, composition and reflexive transitive closure respectively, while the Boolean connectives for propositions are just intersection, union and complement (with $0 = \varnothing$ and $1 = X$). Deterministic programs are interpreted as functions. The test $\alpha?$ is the restriction of the identity relation to the subset α, while the proposition $[p]\alpha$ is given as $\{x \in X \mid (x, y) \in p \Rightarrow y \in \alpha\}$.

Remark 38. When the relations p and q are functions then so are `if` α `then` p `else` q and `while` α `do` p.

Indeed the definitions of `if-then-else` and `while` as in (90) and (91) agree with those given earlier when the relations are assumed to be functional. Generally, a pair $(x, y) \in X^2$ is contained in `if` α `then` p `else` q if and only if either both $x \in \alpha$ and $(x, y) \in p$ or both $x \notin \alpha$ and $(x, y) \in q$. Similarly, (x, y) is contained in the relation `while` α `do` p if and only if there is a non-negative integer n and points $x = x_0, \ldots, x_n = y$, with (x_i, x_{i+1}) in p for $i = 0, \ldots, n-1$ and with each of $x_0, \ldots, x_{n-1}$ in α, but $x_n \notin \alpha$.

The interpretation of the modal operators is of particular interest: a point $x \in X$ is contained in $[p]\alpha$, provided that applying p to x cannot lead to α being false. So $[p]\alpha$ is the "weakest precondition" (meaning "largest subset of X"), under which executing p cannot lead to α being false. Similarly, $\langle p\rangle\alpha$ corresponds to the subset of X for which it is possible that running p will lead to α being true. When p is deterministic—meaning corresponds to a function—then $\langle p\rangle\alpha$ is nothing other than the preimage of the truth set of α under p. The importance of the modal operators lies in the kind of question that program verification attempts to address. A piece of code p is desired to produce the specification α, when executed under the input conditions β. The truth of this is precisely the statement $\beta \le [p]\alpha$.

The basic PDL has many variants, some of which we now list.

- *Strict PDL* (SPDL) is a weakening of PDL in which the program connectives $\cup$ and * are replaced by the weaker programming constructions `if-then-else` and `while`. (See Goldblatt [19] or Passy [48].)
- *PDL with intersection* (IPDL) is PDL enriched with $\cap$ (intersection) adjoined as an extra binary connective for programs. (See Danecki [10], and Balbiarni and Vakarelov [3].)
- *Deterministic PDL* (DPDL) is obtained by requiring that the atomic programs in PDL be deterministic. In the relational semantics, this corresponds to atomic programs corresponding to functions. (See Ben-Ari, Halpern and Pnueli [4].)
- *Strict deterministic PDL* (SDPDL) is obtained by requiring the atomic programs in SPDL to be deterministic. (See Halpern and Reif [24].)
- *Deterministic PDL with intersection* (DIPDL) is just DPDL with $\cap$. (See Harel [26].)
- *SDPDL with intersection* (SDIPDL). This variant does not seem to have been studied in its own right. The decidability of SDIPDL is mentioned as being open in the Stanford Encyclopedia of Philosophy [61].

- *Loopless PDL, DPDL*, etc. These are restrictions of the above, where * (or `while`) is omitted.

The strict deterministic variants of PDL (that is, SDPDL and SDIPDL) have purely functional semantics, with no non-functional relations generated, since the program variables are functions and $\mathcal{P}(X)$ is closed under both `if-then-else` and `while-do` (as in Remark 38). (Further variations of PDL, as well as discussion of some of the variants above, may be found in [27, Chapter 16].)

4.2. *Algebraic models*

The relational semantics for PDL provides a family of operations on binary relations and subsets of the underlying domain. The corresponding algebra will be two-sorted (a program sort consisting of relations and a propositional sort consisting of subsets of the underlying domain): in the language of the present article it would produce relation $\{\cup, {}^*\}$-semigroup on one sort, with a Boolean algebra of subsets of the underlying domain and with the modal and query operations connecting the two sorts. A well-known axiomatic system modelling precisely this is *dynamic algebra*; see [50]. While we do not recall the precise laws defining dynamic algebra, we mention that the relational models share the same operations and relational semantics as PDL, except that the query operation $\alpha \mapsto \alpha?$ is not typically included in the signature. When the signature is expanded to include query (thus enabling the expression of `if-then-else` and `while-do` via the definitions in Equation (90) and (91)), the system is usually known as *dynamic algebra with tests*, or sometimes just *test algebra.*

More recent approaches to algebraically modelling PDL have internalised the propositional sort (adjusting other operations accordingly): each proposition α can be identified with its query program $\alpha?$. This simplifies the language by eliminating the need for two sorts. The axiomatic system *Kleene algebra with domain* (KAD) is a recent model of this kind, first introduced in [13].

Here we focus on the class of relation $\{\cup, P, {}^*, 0, 1\}$-semigroups, which are the natural relational models of KAD. We detail how relation $\{\cup, P, {}^*, 0, 1\}$-semigroups interpret all of the relational operations arising from the relational semantics of PDL, and moreover that they are essentially equivalent to dynamic algebras with tests.

Let $\mathbf{S}$ be a relation $\{\cup, P, {}^*, 0, 1\}$-semigroup on the domain X. The elements of S are binary relations on X. The program composition, union and iteration * of PDL already share the same semantics as composition, union and * in $\mathbf{S}$. Propositions are modelled by elements of $P(S)$, which are precisely those elements of S that are restrictions of the identity. Thus if α and β are elements of $P(S)$, then proposition conjunction $\alpha \wedge \beta$ is modelled by the composite $\alpha\beta$, whose domain is precisely the intersection of the domains of α and β. Proposition negation is modelled by $\neg\alpha := P(\alpha)$. Internalisation of tests, makes the query operation redundant; essentially, $\alpha?$ is the same as α in $\mathbf{S}$.

Finally, the modal operator of necessity is modelled by

$$[r]\alpha := P(r\alpha'). \tag{92}$$

The reader will easily verify that $P(r\alpha')$ is precisely the restriction of the identity to the places where $[r]\alpha$ is true under the relational semantics for PDL. As $\langle r\rangle\alpha = \neg[r]\neg\alpha$, we have $\langle r\rangle\alpha = P(P(r\alpha'')) = D(r\alpha)$ (where D here corresponds to PP, the derived domain operation); again this of course is the identity map restricted to precisely those points at which $\langle r\rangle\alpha$ is true. (This correspondence between P and the modal operators makes sense for functions as well as for relations: if the relation r is functional, then $[r]\alpha$ is modelled by $P(r\alpha')$.)

In this way the relation $\{\cup, P, {}^*, 0, 1\}$-semigroup $\mathbf{S}$ gives rise to a dynamic algebra with tests on X. Conversely, any dynamic algebra with tests on X gives rise to a relation $\{\cup, P, {}^*, 0, 1\}$-semigroup if one sets $P(r) := [r]0$ and then drops the Boolean sort. So there is a kind of definitional equivalence between relational dynamic algebras with tests (two-sorted) and relation $\{\cup, P, {}^*, 0, 1\}$-semigroups (one-sorted).

One can apply a very similar analysis to the other variants of PDL described above. In particular, we again note that in the presence of P (and again using D for PP),

- preferential union and `if-then-else` are interdefinable:

$$p \sqcup q = \texttt{if } D(p) \texttt{ then } p \texttt{ else } q \quad \text{and} \quad \texttt{if } \alpha \texttt{ then } p \texttt{ else } q = \alpha p \sqcup \alpha' q \tag{93}$$

- but also, maximal iterate and `while-do` are interdefinable:

$$p^{\uparrow} = \texttt{while } D(p) \texttt{ do } p \qquad \text{and} \qquad \texttt{while } \alpha \texttt{ do } p = (\alpha p)^{\uparrow}\alpha'. \tag{94}$$

Thus function/relation $\{P, \sqcup, 0, 1\}$-semigroups are equivalent to function/relation $\{P, \texttt{if-then-else}, 0, 1\}$-semigroups,
and function/relation $\{P, \sqcup, {}^{\uparrow}, 0, 1\}$-semigroups are definitionally equivalent to function/relation $\{P, \texttt{if-then-else}, \texttt{while-do}, 0, 1\}$-semigroups. We remark that without the operation P, the authors have shown [36] that in both the functional and relational cases, the class of representable $\{\texttt{if-then-else}\}$-semigroups is not finitely axiomatised (regardless of the presence of 0, 1 or `while`).

Table 1 lists some variants of PDL including several simplified forms in which looping and possibly also branching are not modelled. Listed next to each is the associated class of relation or function semigroup, based on the equivalences (92), (93) and (94). Row 8 consists of the fragment of PDL involving program composition, modal operators, query and the usual operations of propositional logic on propositions. Row 9 is the same but with intersection included; this system is given the name KR in Section 6.5 of Blackburn, de Rijke and Venema [5, p. 367], where it is observed that the deterministic version is decidable.

In each case in Table 1, the algebraic models interpret the logic in the sense that a proposition α in the language of the logic is relationally valid if and only if $\alpha = 1$ is an identity satisfied by the corresponding family of algebras of relations.

Variant of PDL	Associated kind of relation or function semigroup
1. PDL	Relation $\{\cup, P, {}^*, 0, 1\}$-semigroups ($\equiv$ test algebras)
2. DPDL	Relation $\{\cup, P, {}^*, 0, 1\}$-semigroups generated by functional elements
3. SPDL	Relation $\{\sqcup, {}^{\uparrow}, P, 0, 1\}$-semigroups
4. IPDL	Relation $\{\cap, \cup, P, {}^*, 0, 1\}$-semigroups
5. DIPDL	Relation $\{\cap, \cup, P, {}^*, 0, 1\}$-semigroups generated by functional elements
6. SDPDL	Function $\{\sqcup, {}^{\uparrow}, P, 0, 1\}$-semigroups
7. SDIPDL	Function $\{\cap, \sqcup, {}^{\uparrow}, P, 0, 1\}$-semigroups
8. $\{\cup, {}^*\}$-free DPDL	Function $\{P, 0, 1\}$-semigroups
9. $\{\cup, {}^*\}$-free DIPDL	Function $\{P, \cap, 0, 1\}$-semigroups
10. loop-free SDPDL	Function $\{P, \sqcup, 0, 1\}$-semigroups
11. loop-free SDIPDL	Function $\{P, \cap, \sqcup, 0, 1\}$-semigroups

Table 1. Some variants of PDL and associated classes of algebras of relations.

Quasi-equations holding in the family of algebras of relations cannot in general be expressed as propositions in the logical language. We also observe that the following conditions are easily seen to be equivalent on relational models:

$$\alpha p = \alpha p \beta \Leftrightarrow \alpha p \beta' = 0 \Leftrightarrow \alpha \leq [p]\beta \Leftrightarrow \{\alpha\}p\{\beta\} \tag{95}$$

The last expression here is a Hoare triple, a proposition that is true provided that whenever α is true at the execution of p, then β is true if and when p halts. The first three of these equalities are algebraic expressions in any of the algebraic languages involved in Table 1 and it is easy to prove their equivalence using suitable axioms from earlier sections of this article.

There is a well known relational representation for dynamic algebras (definitionally equivalent to the algebras in row 1) due to Dexter Kozen [39]. This representation makes assumptions of halting in loops (similar to the periodicity assumption made in Lemma 35), but also additional "separation" assumptions that do not hold in all representable models (even in the loop-free case). Hence the representation does not axiomatize the row 1 relation semigroups, even in the loop-free case in which Kleene closure is omitted.

The authors are not aware of any representation results for the classes described in rows 2–5; moreover submitted work of the first author and Robin Hirsch [28] will show that there is no algorithm to decide (faithful) representability of an arbitrary finite structure as an algebra of the kind described in row 4.

In the present article, we have provided representations for the algebras described in rows 6–11; where looping is allowed (rows 6 and 7), we require similar assumptions on the halting of `while` loops as for the dynamic algebra case considered by Kozen, but *no separation conditions.*

4.3. *The equational problem for function $\{P, \sqcup, {}^\uparrow\}$-semigroups*

Results for SDPDL may be deployed to establish the following result.

Theorem 39. *Let u and v be terms in the operations $\cdot, P, \sqcup, {}^\uparrow, 0, 1$.*

(1) An equation of the form $D(u) = D(v)$ holds for all function $\{P, \sqcup, {}^\uparrow\}$-semigroups if and only if it is a consequence of the dynamic modal restriction semigroup with preferential join and ${}^\uparrow$ axioms.

(2) A general equation of the form $u = v$ holds for all function $\{P, \sqcup, {}^\uparrow\}$-semigroups if and only if the equation $D(ux) = D(vx)$ is an equational consequence of the dynamic modal restriction semigroup with preferential join and ${}^\uparrow$ axioms.

The argument is similar to those using the notion of an "intensional" dynamic algebra (see Pratt [51]), and relies on the fact that the axioms for dynamic modal restriction semigroups with preferential join and ${}^\uparrow$ subsume the axioms of SDPDL (that is, when translated into our algebraic language according to the recipes established earlier, our axioms imply all the axioms and rules of SDPDL). Such proofs of subsumption are common: for example KAD subsumes PDL in exactly the same sense, as shown in [15].

Corollary 40. *The equational theory of function $\{P, \sqcup, {}^\uparrow\}$-semigroups is PSPACE-complete.*

Proof. Theorem 39 shows that to decide the validity of $u = v$ it suffices to decide validity of the test equation $D(ux) = D(vx)$. But this is equivalent to an SDPDL formula and the validity problem for SDPDL formulas is known to be PSPACE-complete [24]. □

The results run in the other direction also. It is a standard exercise of PDL to show that the rule of inference

$$\text{from } \alpha \to [p]\alpha \text{ infer } \alpha \to [p^*]\alpha$$

may be replaced by the axiom

$$[p^*](\alpha \to [p]\alpha) \to (\alpha \to [p^*]\alpha).$$

(See [61] for example.) It is perhaps of some interest to note that something similar can be done with SDPDL. When interpreted in the language of function $\{P, \sqcup, {}^\uparrow\}$-semigroups, the above rule of inference (which also occurs in SDPDL) is easily seen to be equivalent to the dynamic law (77). Proposition 29 may therefore be deployed in order to replace this rule of inference of SDPDL by an axiom, namely the one obtained by interpreting Equation (78) within SDPDL. Hence SDPDL can also be axiomatized without the need for any inference rules.

4.4. *Embedding into* KAD

As mentioned above, *Kleene algebra with domain* (KAD) is an axiomatic system modelling relation $\{\cup, P, {}^*\}$-semigroups; indeed every such relation semigroup is a model of KAD, although the converse is false. The reduct of KAD obtained by dropping * is the theory of *modal semirings*, which again is insufficient to axiomatize relation $\{\cup, P\}$-semigroups; compare this with Theorem 13.

KAD provides an algebraic semantics for PDL, meaning that it subsumes the KAD axioms (proves them when they are interpreted as statements in KAD as for $\{P, \cup, {}^*\}$-semigroups). But the move to the deterministic variant DPDL of PDL can be made by the addition of a single law, which can readily be shown to be equivalent to the twisted law when interpreted in KAD. However, this DPDL law is only assumed for atomic programs. It follows that strict programs built from the atomic programs in DPDL must also satisfy this same law.

Because the proof of this last fact can be translated into the language of KAD (or modal semirings in the loop-free case) to give a valid proof there, it follows that the set $W(S)$ of "deterministic elements" in a model S of KAD (or in any modal semiring S), namely

$$W(S) = \{a \in S \mid a\alpha = D(a\alpha)a \text{ for all } \alpha \in D(S)\},$$

is closed under the program operations of concatenation, `if-then-else` and `while-do`. If $S = \mathcal{B}(X)$, then it is not hard to see that $W(S) = \mathcal{P}(X)$, Generally, $W(S)$ is closed under D (and indeed P), so it follows that $W(S)$ constitutes a subset closed under $\{\cdot, P, \sqcup, {}^\uparrow\}$, moreover one satisfying the laws of modal restriction semigroups with preferential join and $^\uparrow$ (omit $^\uparrow$ in the modal semiring case).

Thus if S is a modal semiring or a finite model of KAD, its "deterministic subreduct" $W(S)$ is functionally representable even if S is not itself relationally representable. It follows easily that *every* modal restriction semigroup with preferential join, hence every function $\{P, \sqcup\}$-semigroup, is embeddable in $W(S)$ for some modal semiring S; this carries over to finite modal restriction semigroups with preferential join and $^\uparrow$ and finite models of KAD.

4.5. *Axiomatizing* SDIPDL

The decidability of full SDIPDL is currently unknown and so it would be very interesting to know the complexity of the equational theory of the function $\{P, \cap, \sqcup, {}^\uparrow\}$-semigroups: any solution to the equational problem for these would imply a solution to the validity problem for SDIPDL. For comparison, we comment that the validity problem for SDPDL is PSPACE-complete [24], while PDL is EXPTIME-complete [17]. However IPDL is decidable [10] (although 2-EXPTIME-hard [43]), while valid propositions of DIPDL are not even recursively enumerable (the validity problem is Π^1_1-hard; see [26]).

We can already give axioms for "loop-free" SDIPDL (that is, SDIPDL without *while*), based on our equational characterization of function $(P, \sqcup, \cap)$-semigroups as modal Eq-monoids with $\sqcup$. This is because it is possible to translate identities for modal Eq-monoids with $\sqcup$ into tautologies of SDIPDL, as we now show.

First note that any equation $p = q$ in the language of modal Eq-monoids with $\sqcup$ can equivalently be expressed in the form $(p \bowtie q) = 1$. Moreover we can always write

$$(p \bowtie q) = D(p \cap q) \vee P(p)P(q) = \langle p \cap q\rangle 1 \vee [p]0 \vee [q]0$$

which immediately gives an axiomatization of loop-free SDIPDL: simply rewrite the non-Boolean laws in Definition 19 plus the two laws obtained from Equations (46) and (47) according to the above recipe (recalling also that $D(f) = \langle f\rangle 1$ and $P(f) = [f]0$).

Defining the class of dynamic modal Eq-monoids with $\sqcup$ and $^\uparrow$ in the obvious way, we conjecture that their axioms are at least complete for functionally valid test equations (as in Theorem 39). If so, these would immediately convert into axioms for full SDIPDL, in the same way as for the loop-free case.

5. Conclusion and open problems

Building on the notion of a restriction semigroup by first modelling Boolean operations on domains, we have developed an algebraic theory of functions encompassing all set-theoretic or domain-related operations previously considered, together with deterministic analogs of union and Kleene closure. The resulting theory is rich enough to provide some kind of match to the most popular algebras of relations, along with their applications to some of the most important logics of computer science. Many questions remain to be explored.

Completeness. Are the axioms for modal restriction semigroups with preferential join and $^\uparrow$, either satisfying the dynamic property or perhaps (84) and (85), complete for equations? Similar questions arise for modal Eq-monoids with $\sqcup$ and $^\uparrow$ (with immediate ramifications for axiomatizing SDIPDL, as discussed above).

Decidability and complexity. What is the precise complexity of the universal Horn theory of functional modal restriction semigroups with $\sqcup$ and $^\uparrow$ (or modal Eq-monoids with $\sqcup$ and $^\uparrow$)? Of course, these theories are undecidable, since that is already the case for semgroups. For modal Eq-monoids with $\sqcup$ and $^\uparrow$ we also have the unresolved question of the decidability of the equational theory (equivalently, the decidability of SDIPDL). We mention that the authors have shown in [34] that the equational theory of function $\{D, \cap, 0, 1\}$-semigroups is decidable, which lacks only test complementation in comparison to modal Eq-monoids.

Residual finiteness. It is known that free dynamic algebras are residually finite and relationally representable (Pratt [50]). This means that any *equation* holding in all finite relational models also holds in arbitrary relational models, which resolves any issues concerning the distinction between finite models and infinite models for dynamic algebra equations. Is the same true for equations over function $\{P, \sqcup, {}^{\uparrow}\}$-semigroups? What about for function $\{\bowtie, \sqcup, {}^{\uparrow}\}$-semigroups?

Range-related operations. There is also the issue of extending the representations presented here to incorporate range-related operations. For example, is there a way to represent the operation R (restriction of the identity map to the range of a function) simultaneously with the domain-related operations considered here? Schein [53] showed that D and R can be captured simultaneously (as a finitely axiomatized (proper) quasivariety of representable algebras), while the authors have extended this to include intersection, giving a finitely axiomatized variety of functionally representable algebras, as in [35]. But these methods do not appear to extend to complement-related operations. On the other hand, determinative pair methods do not seem able to cope with range-related operations such as R (or a range version of $\bowtie$). New methods appear to be needed for these two-sided cases.

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