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Semilattices with closure

Marcel Jackson

Abstract. We examine the varieties of semilattices (called CSL's) with an additional closure operator treated as a unary operation. Topics investigated include the word problem in free CSL's, the lattice of CSL varieties, the finite basis problem for the identities of finite CSL's and a representation as intersection closed subsets of topological spaces.

1. Introduction

The standard definition (see [5]) of a closure operator on a set A of subsets of a set S is a mapping $C\colon A \to A$ satisfying

$$X \subseteq C(X), \quad X \subseteq Y \to C(X) \subseteq C(Y) \quad \text{and} \quad C(C(X)) = C(X)$$

(for X, $Y \in A$). This idea is easily extended to semilattices as follows: let $\mathbf{A} := \langle A, \cdot \rangle$ be a meet semilattice (written multiplicatively) and $C\colon A \to A$ be a unary operation on A such that $\langle A, \cdot, C\rangle$ satisfies

$$x \leq C(x), \quad x \leq y \to C(x) \leq C(y) \quad \text{and} \quad C(C(x)) \approx C(x).$$

Such a structure will be called a *closure semilattice* or a CSL. These structures have been briefly studied in a more general setting in [11]. There the idea of a closure semilattice is extended further to semigroups $\mathbf{S}$ containing a subsemilattice E such that for any element $x \in S$, the unique element $\min\{e \in E : xe = x\}$ exists and is denoted by $C(x)$. The corresponding structures are called *right closure semigroups* (or *RC-semigroups*) and turn out to share a number of structural similarities with the class of inverse semigroups (indeed every inverse semigroup with inverse $^{-1}$ can be made into a RC-semigroup by letting $C(x)$ equal $x^{-1}x$). The class of RC-semigroups is also very closely related to a number of other semigroup-related structures such as those investigated in [1], [2], [7], [8], [14] and [10]. Closure semilattices arise naturally in the theory of RC-semigroups: for example, Proposition 2.2 of [11] states that each possible right closure operation on an inverse semigroup is determined by its restriction to the idempotent elements (which form a sub-semilattice).

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It is shown in [11] that the class of RC-semigroups forms a variety of unary semigroups given by the identities

(1) $x(yz) \approx (xy)z$;
(2) $xC(x) \approx x$;
(3) $C(x)C(y) \approx C(y)C(x)$;
(4) $C(C(x)) \approx C(x)$; and
(5) $C(xy)C(y) \approx C(xy)$.

It is also shown in [11] that the class of CSL's is exactly the class of RC-semigroups whose multiplicative semigroup is a semilattice and so the class of CSL's form a variety given by the identities 1–5 above along with $xx \approx x$ and $xy \approx yx$ (obviously identity 3 above follows immediately from $xy \approx yx$). We will denote this variety by $\mathcal{CSL}$. Note that since CSL's are commutative, the right handed nature of RC-semigroups disappears. Note also, that (as follows from the first definition of an RC-semigroup above) the law $C(x)C(y) \approx C(C(x)C(y))$ is a consequence of identities 1–5.

As well as these semigroup connections, CSL's are also loosely connected to the *closure algebras* introduced in [17]. These are algebras $\langle B, \wedge, \vee, ', C, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that $\langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra and C is a unary operation satisfying $C(1) \approx 1$, $C(0) \approx 0$, $C(C(x)) \approx C(x)$, $x \wedge C(x) \approx x$, $C(x \vee y) \approx C(x) \vee C(y)$. It easily follows that closure algebras also satisfy $C(x \wedge y) \approx C(x \wedge y) \wedge C(y)$ and therefore the reduct of a closure algebra to the signature $\{\wedge, C\}$ is a CSL. We will return to this connection in Section 3 where we will show that the variety of CSL's is exactly the class of subreducts of closure algebras to the signature $\{\wedge, C\}$.

Another way of motivating the idea of a RC-semigroup is by first translating the definition of a closure algebra on a Boolean algebra to the equivalent notion on a Boolean Ring and then generalising this to the class of all rings to give *closure rings* (see [9]). The variety of RC semigroups is then the variety generated by the reduct of closure rings to the multiplicative and closure part. Thus CSL's are connected to closure algebras in two (albeit similar) ways.

We begin with a description of the free CSL's and then (in Section 3) give a representation of CSL's as intersection and closure closed subsets of topological spaces. We then (in Section 4 and Section 5) investigate the base of the lattice of CSL varieties. In Section 6 we give a description of the simple (congruence free) CSL's and examine the subvariety they generate. We finish with an examination of the finite basis problem for the identities of finite CSL's; this and related problems are completely solved for CSL's with identity element.

Unless otherwise defined, the notation used below is standard; the reader is directed to [5] for details.

2. Free CSL's

While it is well known (see [6] for example) that the semigroup variety given by $\{(xy)z \approx x(yz),\ xx \approx x,\ xy \approx yx\}$ is the only nontrivial variety of semilattices, it is easily seen that there are at least two nontrivial varieties of CSL's: that given (within $\mathcal{CSL}$) by $x \approx C(x)$ and that given by $C(x) \approx C(y)$. A model of the first of these varieties can be found by adjoining the identity map as a closure operation to the two element semilattice. For the second variety, one can take the same semilattice and define the closure operation by letting $C(x)$ be the top element for any x. In fact these two varieties are exactly the atoms in the lattice of subvarieties of $\mathcal{CSL}$ [11] and thus are generated by the respective two element models described above. The CSL variety defined by $x \approx C(x)$ will be called the *variety of identity CSL's* or simply $\mathcal{I}$ (since the closure operation is the identity map) and the CSL variety defined by $C(x) \approx C(y)$ will be called the *variety of monoidal CSL's* or simply $\mathcal{M}$ (since these are exactly the semilattices with identity element 1 such that $C(x) = 1$ for every x). The two element CSL's generating these varieties will be denoted $\mathbf{I}$ and $\mathbf{M}$ respectively.

While the CSL $\mathbf{I}$ is of no further importance in this section, it is easily seen that $\mathbf{M}$ is in fact the free CSL on one generator. We shall now show that for $n > 1$, the n-generated free CSL is infinite. This is a distinct departure from the usual semilattice case (the n-generated free semilattice is easily seen to have exactly $2^n - 1$ elements) and is more similar in flavour to the situation for the closure algebras of [17]. For these algebras, even the 1-generated free algebra is infinite and structurally very complicated: for any $n \in \mathbb{N}$, it contains an interval isomorphic to the n-generated free Heyting algebra [4]. Here however, we will give a complete (and quite workable) solution to the word problem for free CSL's that yields a reasonably simple Hasse diagram representation in the 2-generator case.

Our method for describing the word problem in the free CSL's is to construct a CSL whose elements are equivalence classes of finite rooted trees whose vertices are labeled by semilattice terms (called *term trees*). Every term tree will have an associated CSL term, with equivalent trees having terms that are equivalent modulo the CSL laws. Recall that a rooted tree is a finite meet semilattice where no pair of incomparable elements have an upper bound. The unique minimal element is called the *base* and is denoted by $\mathrm{b}(\mathcal{T})$.

Definition 2.1. A *term tree* is a finite rooted tree $\mathcal{T}$ and a function (called a *labelling*) $\ell_{\mathcal{T}} \colon \mathcal{T} \to \mathbf{F}_{\mathrm{SL}}$, where $\mathbf{F}_{\mathrm{SL}}$ is a free semilattice with identity (denoted 1) on some countable set of free generators and such that no maximal element is labelled 1.

A *subtree* of a term tree $\mathcal{T}$ will be a principal order filter, that is, a subset of $\mathcal{T}$ (viewed as a semilattice) of the form $\{x : x \geq \alpha\}$ for some $\alpha \in \mathcal{T}$. We denote this subtree by $\mathcal{T}_\alpha$ and give it the labelling $\ell_\mathcal{T}\restriction_{\mathcal{T}_\alpha}$. If $\alpha > \mathrm{b}(\mathcal{T})$, then we let the unique element of $\mathcal{T}$ covered by $\mathrm{b}(\mathcal{T}_\alpha)$ be called the *root* of $\mathcal{T}_\alpha$ and denote it by $\mathrm{r}(\alpha)$ (note that the root of a subtree is not its base). If $\mathcal{T}$ is a term tree then we will say that the *height* of an element $t \in \mathcal{T}$ is the number of elements of $\mathcal{T}$ ordered strictly beneath it. The height of $\mathcal{T}$ will be the maximum of all the heights of its elements.

Definition 2.2. A *closed term* is a term equal to a product of terms of the form $C(s)$ for some CSL-term s. A *closure free term* (or *semilattice term*) is a CSL term that contains no applications of the unary operation C.

If s and t are two semilattice terms we will use the notation $s \leqslant t$ to denote the fact that s is equivalent modulo the semilattice laws to a subterm of t. In other words $s \leqslant t$ if and only if $t \leq s$ under the usual order on the free semilattice.

An order preserving map $\mu\colon \mathcal{S} \to \mathcal{T}$ between term trees that satisfies (for each s in $\mathcal{S}$) the rule $\ell_\mathcal{S}(s) \leqslant \ell_\mathcal{T}(\mu(s))$ is said to be a *label-morphism.* Two term trees are said to be *label-isomorphic* if there is a bijective label-morphism between them whose inverse is also a label-morphism (in this case it is easily established that the two trees are isomorphic as semilattices and that the image of each element has the same label).

With every term tree $\mathcal{T}$ we may associate a CSL term $\Xi(\mathcal{T})$ as follows. If $\mathcal{T}$ is a term tree of height 0 whose base is labelled by w then the associated CSL term $\Xi(\mathcal{T})$ is the (closure free) term w. Note that $w \neq 1$ in this case. Now let $\mathcal{T}$ be a height $k+1$ term tree with base label w (now possibly equal to 1) and with maximal proper subtrees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ (whose height is necessarily at most k) whose corresponding CSL terms are $t_1, t_2, \ldots, t_n$. Then $\Xi(\mathcal{T})$ is the term $wC(t_1)C(t_2)\cdots C(t_n)$ if $w \neq 1$, and $C(t_1)C(t_2)\cdots C(t_n)$ otherwise. It is easy to see how this association can be reversed to go from a given CSL term t to a term tree. We will use $\Xi^{-1}(t)$ to denote the term tree $\mathcal{T}$ for which $\Xi(\mathcal{T}) = t$.

It should be clear to the reader that under an obvious definition of product and closure, the set of term trees over some countable set of variables will become a countably generated free semilattice with unary operation (satisfying no laws other than those that are consequences of the semilattice laws). Therefore there is a (fully invariant) congruence on this structure that yields the free CSL on countably many generators. We aim to find a nice algorithm for deciding the corresponding equivalence of term trees.

The construction of term trees is not essential to the description of the free CSL that we give below, however it simplifies many of the arguments. Also the introduction of term trees provides an interesting two pronged approach to the problem: while our characterisation is in essence a syntactic one (it relies on an analysis of deductions of identities), we are able to use arguments of semantic character since term trees are effectively algebraic structures themselves. The fundamental tool in examining term trees will be the following construction.

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two term trees. A *fragmentation map* from $\mathcal{A}$ into $\mathcal{B}$ is a label-morphism $\mu\colon \mathcal{A} \to \mathcal{B}$ such that $\mu(\mathrm{b}(\mathcal{A})) = \mathrm{b}(\mathcal{B})$. We say that $\mathcal{A}$ is a fragment of $\mathcal{B}$ if such a fragmentation map exists. If $\mathcal{A}$ and $\mathcal{B}$ are both fragments of each other then we will say that they are *fragment equivalent*. When $\mathcal{A}$ and $\mathcal{B}$ are fragment equivalent we will write $\mathcal{A} \sim \mathcal{B}$ and we write $K(\mathcal{A})$ to denote the set of all term trees fragment equivalent to $\mathcal{A}$.

Note that $\sim$ is indeed an equivalence on the set of all term trees over the free semilattice monoid $\mathbf{F}_{\mathrm{SL}}$. Also note that a fragmentation map goes from a fragment to its parent term tree and that the fragment may appear "larger" than the parent term tree. For example, every term tree over a finite alphabet $x_1, x_2, \ldots, x_n$ is a fragment of the one element term tree with label $x_1 \cdots x_n$.

Two elementary but useful properties of fragmentation maps (which we give without proof) are the following.

Lemma 2.4. (i) *If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are term trees and $\mu_1\colon \mathcal{A} \to \mathcal{B}$ and $\mu_2\colon \mathcal{B} \to \mathcal{C}$ are fragmentation maps then $\mu_2 \circ \mu_1\colon \mathcal{A} \to \mathcal{C}$ is a fragmentation map.*

(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are term trees and $\mu\colon \mathcal{A} \to \mathcal{B}$ is a fragmentation map, then for all $\alpha \in \mathcal{A}$, $\mu\restriction_{\mathcal{A}_\alpha}\colon \mathcal{A}_\alpha \to \mathcal{B}_{\mu(\alpha)}$ is a fragmentation map (where $\mu\restriction_{\mathcal{A}_\alpha}$ is the restriction of μ to the subtree $\mathcal{A}_\alpha$).*

The crucial connection between fragments of term trees and CSL identities is the following theorem describing the equational theory of CSL's. The proof of this theorem constitutes most of the remainder of this section.

Theorem 2.5. *If $\mathcal{A}$ and $\mathcal{B}$ are term trees then $\Xi(\mathcal{B}) \approx \Xi(\mathcal{A})$ is a CSL law if and only if $\mathcal{A}$ and $\mathcal{B}$ are fragment equivalent.*

One half of Theorem 2.5 is easy.

Lemma 2.6. *If $\mathcal{A}$ and $\mathcal{B}$ are term trees and $\mathcal{A}$ is a fragment of $\mathcal{B}$ then $\Xi(\mathcal{B}) \approx \Xi(\mathcal{B})\Xi(\mathcal{A})$ is an identity satisfied by all CSL's (that is, $\Xi(\mathcal{B}) \leq \Xi(\mathcal{A})$).*

Proof. The proof is by induction on the height of $\mathcal{A}$. Firstly, if $\mathcal{A}$ is of height 0 then $\mathcal{A}$ is a single element term tree with label w say. In this case, $\Xi(\mathcal{B})$ is of the form wv where v is a (possibly empty) CSL term and so $\Xi(\mathcal{B}) \approx vw \approx vww \approx \Xi(\mathcal{B})\Xi(\mathcal{A})$.

Now fix some integer $k \geq 0$ and assume that whenever $\mathcal{C}$ is a term tree of height at most k and $\mathcal{C}$ is a fragment of a term tree $\mathcal{D}$ then $\Xi(\mathcal{D}) \leq \Xi(\mathcal{C})$ holds in every CSL. Consider the case when $\mathcal{A}$ is a height $k+1$ term tree that is a fragment of a term tree $\mathcal{B}$ with fragmentation map μ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the list of atoms (height 1 elements) of $\mathcal{A}$. Now for each $i \leq n$, $\mathcal{A}_{\alpha_i}$ is of height at most k and since (by Lemma 2.4 (ii)) $\mathcal{A}_{\alpha_i}$ is a fragment of $\mathcal{B}_{\mu(\alpha_i)}$ the induction hypothesis implies that $\Xi(\mathcal{B}_{\mu(\alpha_i)}) \approx \Xi(\mathcal{B}_{\mu(\alpha_i)})\Xi(\mathcal{A}_{\alpha_i})$ is a law of every CSL. For each $i \leq n$ we now consider two cases. Firstly, if $\mu(\alpha_i) = \mathrm{b}(\mathcal{B})$, then $\mathcal{B} = \mathcal{B}_{\mu(\alpha_i)}$. In this case $\Xi(\mathcal{B}) \approx \Xi(\mathcal{B}_{\mu(\alpha_i)}) \approx \Xi(\mathcal{B}_{\mu(\alpha_i)})\Xi(\mathcal{A}_{\alpha_i}) \approx \Xi(\mathcal{B}_{\mu(\alpha_i)})\Xi(\mathcal{A}_{\alpha_i})C(\Xi(\mathcal{A}_{\alpha_i})) \approx \Xi(\mathcal{B}_{\mu(\alpha_i)})C(\Xi(\mathcal{A}_{\alpha_i})) \approx \Xi(\mathcal{B})C(\Xi(\mathcal{A}_{\alpha_i}))$ by the induction assumption. In the second case, $\mu(\alpha_i) > \mathrm{b}(\mathcal{B})$ and we can write $\Xi(\mathcal{B})$ in the form

$$\ell_{\mathcal{B}}(\mu(\mathrm{b}(\mathcal{A})))t_0C(t_1C(t_2C(\cdots t_nC(\Xi(\mathcal{B}_{\mu(\alpha_i)})C(t_{n+1}\cdots))\cdots)))$$

where $t_0, t_1, \ldots, t_n, \ldots$ are (possibly empty) CSL terms. Denote $\Xi(\mathcal{B}_{\mu(\alpha_i)})$ by v_i, $\Xi(\mathcal{A}_{\alpha_i})$ by w_i and $\ell_{\mathcal{B}}(\mu(\mathrm{b}(\mathcal{A})))$ by w.

Now

$$\begin{aligned}
&wt_0C(t_1C(t_2C(\cdots t_nC(v_iC(t_{n+1}\cdots))\cdots)))\\
&\approx wt_0C(t_1C(t_2C(\cdots t_nC(v_iw_iC(t_{n+1}\cdots))\cdots))) \quad \text{(by induction hypothesis)}\\
&\approx wt_0C(t_1C(t_2C(\cdots t_nC(v_iC(t_{n+1}\cdots)w_i)\cdots)))\\
&\approx wt_0C(t_1C(t_2C(\cdots t_nC(v_iC(t_{n+1}\cdots)w_i)C(w_i)\cdots))) \quad \text{(by law 5)}\\
&\quad\cdots \quad \text{(repeated applications of law 5 from right to left)}\\
&\approx wt_0C(t_1C(t_2C(\cdots t_nC(v_iC(t_{n+1}\cdots)w_i)C(w_i)\cdots)C(w_i))C(w_i))C(w_i)\\
&\approx wt_0C(t_1C(t_2C(\cdots t_nC(v_iC(t_{n+1}\cdots)w_i)\cdots)))C(w_i)\\
&\qquad \text{(by repeated applications of law 5 from left to right)}\\
&\approx \Xi(\mathcal{B})C(\Xi(\mathcal{A}_{\alpha_i})).
\end{aligned}$$

By performing this procedure for each $i \leq n$, a derivation of the law $\Xi(\mathcal{B}) \approx \Xi(\mathcal{B})C(\Xi(\mathcal{A}_{\alpha_1}))\cdots C(\Xi(\mathcal{A}_{\alpha_n}))$ is obtained. Writing $\Xi(\mathcal{B})$ as wv for some CSL term v, we have that

$$\begin{aligned}
\Xi(\mathcal{B}) &\approx wv\\
&\approx vwC(\Xi(\mathcal{A}_{\alpha_1}))\cdots C(\Xi(\mathcal{A}_{\alpha_n}))\\
&\approx vwwC(\Xi(\mathcal{A}_{\alpha_1}))\cdots C(\Xi(\mathcal{A}_{\alpha_n}))\\
&\approx \Xi(\mathcal{B})\Xi(\mathcal{A})
\end{aligned}$$

as required. Hence by induction, the claim is true for term trees of any height. □

Note that if $\mathcal{A}$ is a fragment of a subtree of $\mathcal{B}$ with fragmentation map μ then $\Xi(\mathcal{B}_{\mu(\mathrm{b}(\mathcal{A}))}) \approx \Xi(\mathcal{B}_{\mu(\mathrm{b}(\mathcal{A}))})\Xi(\mathcal{A})$ is a CSL law by the previous lemma. Using repeated applications of CSL law 5 (as in the proof of the previous lemma) we obtain the following.

Corollary 2.7. *If $\mathcal{A}$ and $\mathcal{B}$ are term trees and $\mathcal{A}$ is a fragment of a subtree of $\mathcal{B}$ then the following are satisfied by all CSL's:*
(i) $\Xi(\mathcal{B}) \approx \Xi(\mathcal{B})C(\Xi(\mathcal{A}))$;
(ii) $C(\Xi(\mathcal{B})) \approx C(\Xi(\mathcal{B}))C(\Xi(\mathcal{A}))$.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are the term trees for $xC(xy)$ and $yC(xy)$ then $\mathcal{A}$ is a fragment of a subtree of $\mathcal{B}$ and vice versa and yet it is not hard to find examples of CSL's not satisfying $xC(xy) \approx yC(xy)$. Thus it is not sufficient to test for the equivalence of two term trees by verifying that each is a fragment of a subtree of the other.

The next step in the proof of Theorem 2.5 is to show that each term tree (and each corresponding CSL term) can be reduced to a (unique) normal form term tree. The idea will be based on the notion of fragment equivalence (see Definition 2.3).

If $\mathcal{T}$ is a term tree and $\mathcal{T}'$ is a subposet of $\mathcal{T}$ that forms a meet semilattice (but not necessarily the same meet as in $\mathcal{T}$) given the labelling $\ell_{\mathcal{T}'} := \ell_{\mathcal{T}}\restriction_{\mathcal{T}'}$ then we say that $\mathcal{T}'$ is a *restriction* of $\mathcal{T}$. Note that $\mathcal{T}'$ is a term tree, even though it may not be a subtree of $\mathcal{T}$ or even a subsemilattice of $\mathcal{T}$. The following proposition contains three ways of finding a fragment equivalent restriction of a given term tree. (Recall that $\mathrm{r}(\alpha)$ denotes the root of $\mathcal{T}_\alpha$.)

Proposition 2.8. *Let $\mathcal{T}$ be a term tree.*

(Reduction 1) *Let $\alpha > \mathrm{b}(\mathcal{T})$. If $\ell_{\mathcal{T}}(\alpha) \leqslant \ell_{\mathcal{T}}(\mathrm{r}(\alpha))$, then $\mathcal{T}$ is fragment equivalent to the restriction $\mathcal{T}'$ obtained by removing α.*

(Reduction 2) *Let $\alpha > \mathrm{b}(\mathcal{T})$ be such that there is a unique β with $\mathrm{r}(\beta) = \alpha$. If $\ell(\alpha) \leqslant \ell(\beta)$ then $\mathcal{T}$ is fragment equivalent to the restriction $\mathcal{T}'$ obtained by removing α.*

(Reduction 3) *Let α and β be incomparable elements of $\mathcal{T}$ such that $\mathrm{r}(\alpha) \leq \beta$. If $\mathcal{T}_\alpha$ is a fragment of $\mathcal{T}_\beta$, then $\mathcal{T}$ is fragment equivalent to the restriction $\mathcal{T}'$ obtained by removing $\mathcal{T}_\alpha$.*

In each case the inclusion map from $\mathcal{T}'$ into $\mathcal{T}$ is injective but not surjective.

Proof. We will only prove the last statement; the other proofs are similar and are left to the reader. Let $\mathcal{T}$ and $\mathcal{T}'$ be as in Reduction 3. Because $\mathcal{T}'$ is a labelled subsemilattice of $\mathcal{T}$, the map $\mu_1 \colon \mathcal{T}' \to \mathcal{T}$ defined by $\mu_1(\alpha) = \alpha$ is an injective but not surjective fragmentation map. Now let $\phi \colon \mathcal{T}_\alpha \to \mathcal{T}_\beta$ be a fragmentation map associated with $\mathcal{T}_\alpha$ being a fragment of $\mathcal{T}_\beta$. Define a map $\mu_2 \colon \mathcal{T} \to \mathcal{T}'$ as follows:

$$\mu_2(\alpha) = \begin{cases} \alpha \text{ if } \alpha \in \mathcal{T}' \\ \phi(\alpha) \text{ if } \alpha \in \mathcal{T} \backslash \mathcal{T}'. \end{cases}$$

Since $\mathrm{r}(\mathcal{T}_\alpha) \leq \beta$ this is order preserving and so is a fragmentation map. Thus $\mathcal{T}$ and $\mathcal{T}'$ are fragment equivalent, but the fragmentation map $\mu_1 \colon \mathcal{T}' \to \mathcal{T}$ is injective but not surjective. □

As an example, note that Reduction 1 can be used to remove any non-base element of a term tree if it is labelled by 1.

If none of the three reductions of Proposition 2.8 can be applied to a term tree $\mathcal{T}$, then we say that $\mathcal{T}$ is *reduced*; otherwise $\mathcal{T}$ is *reducible*. While it is clear that every term tree is fragment equivalent to a reduced term tree, we now want to prove that $K(\mathcal{T})$ contains a unique reduced element.

We first prove a technical, but crucial lemma.

Lemma 2.9. *Let $\mathcal{A}$ be a term tree that is a fragment of a subtree $\mathcal{A}'$ of itself. Then one of the following is true:*

(i) $\mathcal{A} = \mathcal{A}'$ *(that is,* $\mathrm{b}(\mathcal{A}') = \mathrm{b}(\mathcal{A})$*);*

(ii) $\mathrm{b}(\mathcal{A}')$ *is the unique atom of $\mathcal{A}$ and $\ell(\mathrm{b}(\mathcal{A}))$ is a proper subterm of $\ell(\mathrm{b}(\mathcal{A}'))$;*

(iii) *$\mathcal{A}$ is reducible.*

Proof. We show that if conditions (i) and (iii) in the lemma do not hold then condition (ii) does hold.

Let α' be the base of $\mathcal{A}'$, and α be the base of $\mathcal{A}$, and let $\mu\colon \mathcal{A} \to \mathcal{A}'$ be a fragmentation map (by assumption, $\alpha \neq \mu(\alpha) = \alpha'$). Our first step is to note that every element in $\mathcal{A}\backslash\mathcal{A}'$ is ordered in a chain below α'. Indeed, if this is not the case, then there is a proper subtree $\mathcal{A}''$ with base α'' that is disjoint from $\mathcal{A}'$ and whose root is ordered below α'. But then Reduction 3 of Proposition 2.8 applies, contradicting the assumption that condition (iii) is false. Thus we may assume that the underlying tree of $\mathcal{A}$ consists of the disjoint union of the elements of $\mathcal{A}'$ with some elements $\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ with $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_{n-1}$ and $\alpha_{n-1} \prec \alpha'$. We show that $n = 1$.

Assume $n > 1$, that is, α' is not an atom of $\mathcal{A}$. We are going to find an element γ in $\mathcal{A}$ such that $\mu(\gamma)$ is a cover of γ. We construct a pair of chains $\beta_1, \beta_2, \ldots$ and $\beta_1', \beta_2', \ldots$ inductively as follows. First let $\beta_1 = \mu(\alpha_1)$ and let β_1' be the unique cover of α_1 that is beneath or equal to β_1 (in this particular case it is evident that $\beta_1' = \alpha_2$ or α'). Now for $i \geq 1$, let $\beta_{i+1} = \mu(\beta_i')$ and, if it exists, let β_{i+1}' be the unique cover of β_i' that is beneath or equal β_{i+1} (notice that since $\beta_{i+1}' \succ \beta_i'$, $\mu(\beta_{i+1}') \geq \mu(\beta_i')$). If this does not exist it follows that β_{i+1} actually equals β_i' and therefore $\mu(\beta_{i-1}') = \beta_i'$, a cover of β_{i-1}' as desired. Indeed since every chain in $\mathcal{A}$ contains finitely many elements, it follows that there always exists an i such that $\mu(\beta_{i-1}') = \beta_i'$. The element β_{i-1}' is then our desired element γ.

Now $\mu(\gamma)$ is a cover of γ and $\ell(\gamma) \leqslant \ell(\mu(\gamma))$. We are assuming that condition (iii) does not hold and so Reduction 2 of Proposition 2.8 implies there must be a second cover δ of γ (since γ is not the base of $\mathcal{A}$). Now $\mu(\delta) \geq \mu(\gamma)$ because $\delta > \gamma$ and hence $\mathcal{A}_\delta$ is a fragment of $\mathcal{A}_{\mu(\delta)}$ while the root of $\mathcal{A}_\delta$ is $\gamma < \mu(\gamma) \leq \mu(\delta)$. As δ and $\mu(\delta)$ are incomparable, we may apply Reduction 3 of Proposition 2.8, contradicting the assumption that condition (iii) does not hold. Therefore the case

$n > 1$ is impossible. To complete the proof, note that if $\ell(\alpha') = \ell(\alpha)$ then we may apply Reduction 1 of Proposition 2.8, contradicting the assumption that condition (iii) does not hold. □

Lemma 2.10. *Let $\mathcal{A}$ be a term tree for which $\mu\colon \mathcal{A} \to \mathcal{A}$ is a fragmentation map that is not the identity map. Then $\mathcal{A}$ is reducible.*

Proof. The proof is by induction on the height of $\mathcal{A}$. Let $\mathcal{A}$ be of height 1 and μ be a fragmentation map other than the identity map. Our goal is to show that one of the three reductions described in Proposition 2.8 applies. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the atoms of $\mathcal{A}$. Now $\mu(\mathrm{b}(\mathcal{A})) = \mathrm{b}(\mathcal{A})$ so we must have either $\mu(\alpha_i) = \mathrm{b}(\mathcal{A})$ for some $i \leq n$ or there is an $i \leq n$ such that $\mu(\alpha_i) = \alpha_j$ for some $j \neq i$. In the first case, $\ell(\alpha_i) \leqslant \ell(\mathrm{b}(\mathcal{A}))$ and so Reduction 1 applies. In the second case $\mathcal{A}_{\alpha_i}$ (which equals α_i) is a fragment of $\mathcal{A}_{\alpha_j}$ (which equals α_j) and the root of α_i is the base of $\mathcal{A}$ which is lower than the base of α_j. Hence Reduction 3 applies. Thus by Proposition 2.8, in each case we have the desired term tree $\mathcal{B}$.

Now assume that the result is true for all term trees of height at most k and consider the case when $\mathcal{A}$ is of height $k+1$ and with non-identity fragmentation map $\mu\colon \mathcal{A} \to \mathcal{A}$. As before we let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the atoms of $\mathcal{A}$. Note that one of the following must be true: μ is the identity map on $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$; $\mu(\alpha_i) = \mathrm{b}(\mathcal{A})$ for some $i \leq n$; $\mu(\alpha_i) \geq \alpha_j$ for some $i \neq j$ and $i, j \leq n$; or $\mu(\alpha_i) > \alpha_i$ for some $i \leq n$.

If $\mu(\alpha_i) = \alpha_i$ for each i then because μ is not the identity map there must be an $i \leq n$ such that $\mu{\restriction}_{\mathcal{A}_{\alpha_i}}\colon \mathcal{A}_{\alpha_i} \to \mathcal{A}_{\alpha_i}$ is a non-identity fragmentation map. By the induction hypothesis, $\mathcal{A}_{\alpha_i}$ is reducible and so $\mathcal{A}$ is reducible.

Now consider the case when $\mu(\alpha_i) \neq \alpha_i$ for some i. As in the case when $k = 1$, if $\mu(\alpha_i) = \mathrm{b}(\mathcal{A})$ then Reduction 1 of Proposition 2.8 completes the proof. Now consider the case when $\mu(\alpha_i) \geq \alpha_j$ where $i \neq j$. In this case, $\mathcal{A}_{\alpha_i}$ is a fragment of $\mathcal{A}_{\mu(\alpha_i)}$ and the root of $\mathcal{A}_{\alpha_i}$ is the base of $\mathcal{A}$. Since $i \neq j$ we have α_i and $\mu(\alpha_i)$ are incomparable and so Reduction 3 of Proposition 2.8 completes the proof. Lastly we have to consider the case when $\mu(\alpha_i) > \alpha_i$. In this case, $\mathcal{A}_{\alpha_i}$ is a fragment of a proper subtree of itself. By Lemma 2.9, either $\mathcal{A}_{\alpha_i}$, whence $\mathcal{A}$, is reducible or $\mu(\alpha_i)$ is the unique cover of α_i in $\mathcal{A}$. But then we may apply Reduction 2 of Proposition 2.8 (since α_i is not the base of $\mathcal{A}$).

Thus by induction, the proof is complete. □

Corollary 2.11. *A term tree $\mathcal{A}$ is reduced if and only if whenever $\mathcal{B} \in K(\mathcal{A})$ and $\mu\colon \mathcal{B} \to \mathcal{A}$ is a fragmentation map, then μ is surjective.*

Proof. If $\mathcal{A}$ is reducible, then the final statement of Proposition 2.8 implies there is a fragment equivalent restriction of $\mathcal{A}$ that is a fragment of $\mathcal{A}$ under some non-surjective fragmentation map. Now say that $\mathcal{B}$ is fragment equivalent to $\mathcal{A}$ and

$\mu_1 \colon \mathcal{B} \to \mathcal{A}$ is a non-surjective fragmentation map. Let $\mu_2 \colon \mathcal{A} \to \mathcal{B}$ be any fragmentation map. Then the fragmentation map $\mu_1 \circ \mu_2 \colon \mathcal{A} \to \mathcal{A}$ is non-surjective. By Lemma 2.10, $\mathcal{A}$ is reducible. □

Theorem 2.12. *Let $\mathcal{T}$ be a term tree. Then (up to label-isomorphism), $K(\mathcal{T})$ contains a unique reduced term tree.*

Proof. First note that any term tree with a minimal number of elements (as a semilattice) in $K(\mathcal{T})$ is reduced. As term trees are finite there is always at least one member, $\mathcal{A}$, of $K(\mathcal{T})$ with minimal size. Now let $\mathcal{B} \in K(\mathcal{T})$ be reduced. By Corollary 2.11, $\mathcal{B}$ is also of minimal size. Therefore fragmentation maps between $\mathcal{A}$ and $\mathcal{B}$ are order isomorphisms and hence label-isomorphisms. □

An elementary but useful variation of the main idea of the proof of this theorem is contained in the following.

Lemma 2.13. *If $\mathcal{T}$ is a reduced term tree, then any fragmentation map from $\mathcal{T}$ into a fragment equivalent term tree is injective.*

Proof. If μ is a non-injective fragmentation map from a reduced term tree $\mathcal{T}$ into a term tree $\mathcal{S}$ then every map $\mu' \colon \mathcal{S} \to \mathcal{T}$ is such that $\mu' \circ \mu$ is a non-injective (and therefore non-surjective) map from $\mathcal{T}$ into itself. Hence μ' cannot be a fragmentation map and $\mathcal{S}$ is not fragment equivalent to $\mathcal{T}$. □

We can now define the desired normal form for CSL terms, modulo commutativity.

Definition 2.14. A CSL term is in *normal form* if $\Xi^{-1}(t)$ is a reduced term tree.

Evidently, to find a normal form term from a given CSL term t it suffices to construct the corresponding term tree $\Xi^{-1}(t)$ and then make as many applications of the reductions in Proposition 2.8 as is possible. The resulting term tree, say $\mathcal{A}$, is the unique reduced fragment equivalent term tree and $\Xi(\mathcal{A})$ is the normal form for t.

It is shown in Lemma 2.6 that if $\mathcal{A}$ and $\mathcal{B}$ are fragment equivalent then $\Xi(\mathcal{A}) \approx \Xi(\mathcal{B})$ is a CSL law. To complete the proof of Theorem 2.5 we are going to show that if $\mathcal{A}$ and $\mathcal{B}$ are two reduced, but not fragment equivalent term trees then $\Xi(\mathcal{A}) \approx \Xi(\mathcal{B})$ is not a CSL law. To do this we show how to construct the structure of a CSL on the set of reduced term trees over $\mathbf{F}_{\mathrm{SL}}$. The resulting CSL will satisfy $\Xi(\mathcal{A}) \approx \Xi(\mathcal{B})$ if and only if $\mathcal{A}$ is label isomorphic to $\mathcal{B}$ and then Theorem 2.5 will follow. In fact the constructed CSL is, up to isomorphism, the CSL freely generated by the alphabet of the term tree labels.

Let $\mathbf{T}(X)$ be the set of all isomorphism classes of reduced term trees over the free meet semilattice with identity freely generated by X and let $\mathcal{S}, \mathcal{T} \in \mathbf{T}(X)$ be two reduced term trees. Let S and T be the set of maximal proper subtrees of $\mathcal{S}$ and $\mathcal{T}$ (respectively) not containing the base of $\mathcal{S}$ and $\mathcal{T}$. Define $\mathcal{S}\times\mathcal{T}$ to be the term tree whose base is labelled by the semilattice product of the base labels for $\mathcal{S}$ and $\mathcal{T}$ and whose set of maximal proper subtrees is the union of S and T. Then we define the product $\mathcal{S}\cdot\mathcal{T}$ of $\mathcal{S}$ and $\mathcal{T}$ to be the reduced term tree that is fragment equivalent to $\mathcal{S}\times\mathcal{T}$. Note that $\mathcal{S}$ and $\mathcal{T}$ are fragments of $\mathcal{S}\times\mathcal{T}$ and therefore of $\mathcal{S}\cdot\mathcal{T}$.

If the base of $\mathcal{T}$ is labelled by 1, then we define the closure (denoted by $C(\mathcal{T})$) of $\mathcal{T}$ to be itself. Otherwise, the closure of $\mathcal{T}$ is the term tree obtained from $\mathcal{T}$ by adjoining a new base (below the existing one) and labelling it by 1 (note that we have defined this for not necessarily reduced term trees).

Proposition 2.15. *The set* $\mathbf{T}(X)$ *endowed with the above operations is, up to isomorphism, the free CSL generated by the set of one element term trees with labels from* X*.*

Proof. We first show that $\mathbf{T}(X)$ is a CSL. The associativity of the operation $\times$ is obvious and so the corresponding result for $\cdot$ follows from the transitivity of the $\sim$ relation. Likewise, it is easily seen that $\mathcal{T}\times\mathcal{T}\sim\mathcal{T}$ and so $\mathcal{T}\cdot\mathcal{T}=\mathcal{T}$ by the uniqueness of reduced term trees. Hence $\mathbf{T}(X)$ is a semilattice under the $\cdot$ operation.

We now need to verify that the described unary operation is a closure in the sense of closure semilattices. The equality $C(C(\mathcal{T}))=C(\mathcal{T})$ follows trivially from the definition of C, and the equality $\mathcal{T}\cdot C(\mathcal{T})=\mathcal{T}$ follows since $\mathcal{T}$ is trivially a fragment of itself. It remains to show that $C(\mathcal{T}\cdot\mathcal{S})=C(\mathcal{T}\cdot\mathcal{S})\cdot C(\mathcal{S})$ holds. Now $C(\mathcal{T}\times\mathcal{S})$ is certainly a fragment of $C(\mathcal{T}\times\mathcal{S})\times C(\mathcal{S})$ and $C(\mathcal{S})$ is a fragment of the term tree $C(\mathcal{T}\times\mathcal{S})$ with fragmentation map μ, say. Let ϕ be the identity map on $C(\mathcal{S}\times\mathcal{T})$. Then we can define a fragmentation map $\mu\vee\phi\colon C(\mathcal{S}\times\mathcal{T})C(\mathcal{S})\to C(\mathcal{S}\times\mathcal{T})$ by $(\mu\vee\phi)(\alpha)=\phi(\alpha)$ if $\alpha\in C(\mathcal{S}\times\mathcal{T})$ and $(\mu\vee\phi)(\alpha)=\mu(\alpha)$ if $\alpha\in C(\mathcal{S})\backslash \mathrm{b}(C(\mathcal{S}))$. Thus $C(\mathcal{T}\times\mathcal{S})\sim C(\mathcal{T}\times\mathcal{S})\times C(\mathcal{S})$ and so equality holds. Therefore $\mathbf{T}(X)$ is a CSL.

To complete the proof we need to show that if s and t are CSL terms with $\Xi^{-1}(s)$ not fragment equivalent to $\Xi^{-1}(t)$, then $\mathbf{T}(X)\not\models s\approx t$. First observe that for any CSL terms r_1 and r_2, we have $\Xi^{-1}(r_1)\times\Xi^{-1}(r_2)$ is label-isomorphic to the term tree $\Xi^{-1}(r_1r_2)$ and $C(\Xi^{-1}(r_1))$ is label isomorphic to $\Xi^{-1}(C(r_1))$. Thus if we assign to each variable x in $s\approx t$, the one element term tree with label x, we have that the terms s and t take as their values, the reduced term trees for $\Xi^{-1}(t)$ and $\Xi^{-1}(s)$ respectively. As these are distinct by assumption, we find that $s\approx t$ fails on $\mathbf{T}(X)$ as required. □

As a corollary to this theorem, it is easily established that every non-closed CSL term in two variables is equivalent up to a permutation in variable names to one of the following terms

$$x,\ xy,\ xC(y),\ xC(xy),\ xC(yC(xC(\cdots yC(x)\cdots))),$$
$$xC(yC(xC(\cdots yC(xC(y))\cdots)))$$

and that the closed terms are equivalent to one of

$$C(x), C(xy), C(xC(yC(xC(\cdots yC(x)\cdots)))), C(xC(yC(xC(\cdots yC(xC(y))\cdots)))),$$

or

$$\underbrace{C(xC(yC(\cdots xC(yC(x))\cdots)))}_{2n+1 \text{ closures}}\underbrace{C(yC(xC(\cdots yC(xC(y))\cdots)))}_{2n+1 \text{ closures}}$$

or

$$\underbrace{C(xC(yC(\cdots xC(y)\cdots)))}_{2n \text{ closures}}\underbrace{C(yC(xC(\cdots yC(x)\cdots)))}_{2n \text{ closures}}.$$

Combining these facts we obtain the Hasse diagram representation for the free CSL on two generators in Figure 1. Here and elsewhere we use “hollow” vertices to represent closed elements and “filled” vertices otherwise.

3. CSL’s and closure algebras

A connection between CSL’s and the closure algebras of [17] is discussed above where it is claimed that the variety of CSL’s is generated by the class of all reducts (to meet and closure) of closure algebras. Since every closure algebra is certainly a CSL, this is equivalent to there being no “exotic” CSL laws satisfied by the variety of closure algebras. In this section we prove this claim by constructing an “embedding” of any given CSL into a closure algebra. In particular this construction will work for the free CSL’s and thus any CSL identity that fails on a free CSL will also fail on some closure algebra. The construction is an elementary variation on standard techniques for representing partial orders as systems of partially ordered sets.

First recall that a closure algebra $\mathbf{B}$ is an algebra $\langle B, \vee, \wedge, ', C, 0, 1\rangle$ where $\langle B, \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra and C is a unary operation such that $C(1) \approx 1$, $C(0) \approx 0$, $C(C(x)) \approx C(x)$, $x \wedge C(x) \approx x$, $C(x \vee y) \approx C(x) \vee C(y)$ hold for every $x, y \in B$. Given any topological space $(X, \mathcal{T})$ one may construct a closure algebra by taking the Boolean algebra of subsets of X endowed with the closure operator associated with $\mathcal{T}$. This closure algebra will be called the *closure algebra of* $(X, \mathcal{T})$. Every closure algebra is embeddable in the closure algebra of some topological space [17].

Definition 3.1. Let $\mathbf{S} := \langle S, \cdot, C_S\rangle$ be a CSL and let $\mathcal{L} := \{\vee, \wedge, ', C, 0, 1\}$ be the language of closure algebras. Then the *closure algebra on the power set of* S,

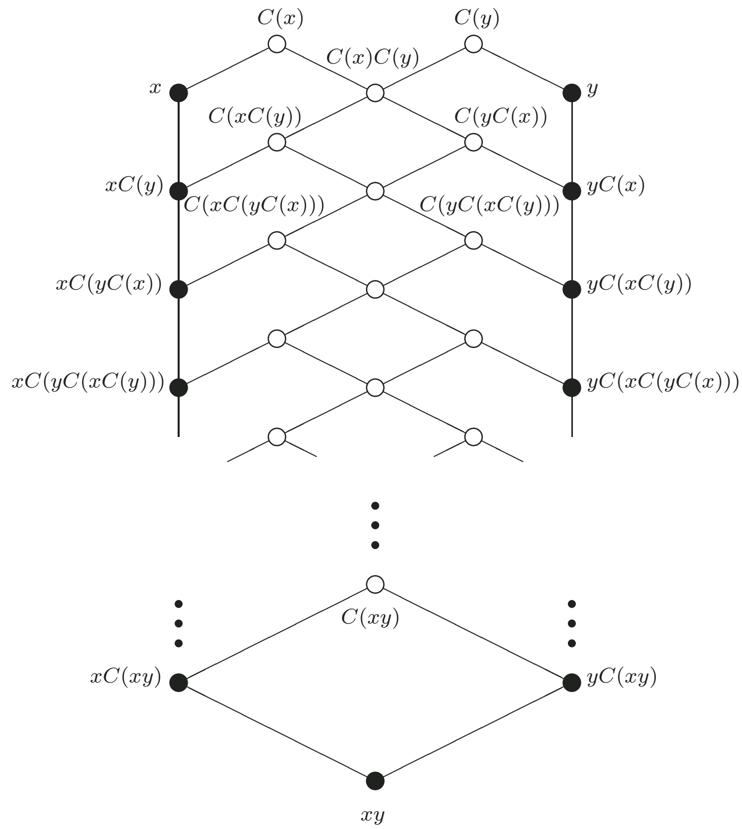

FIGURE 1. The free CSL generated by the elements x and y.

$\langle 2^S, \vee, \wedge, ', C, 0, 1\rangle$ (denoted by $\mathbf{2^S}$) is the $\mathcal{L}$-algebra consisting of the usual Boolean algebra of subsets of the universe S of $\mathbf{S}$ and with

$$C(X) := \{x \in S : x \le C_S(y),\ \text{for some } y \in X\}$$

for each $X \subseteq S$ (here $\le$ is the usual semilattice order on $\mathbf{S}$).

Proposition 3.2. *If* $\mathbf{S}$ *is a CSL then* $\mathbf{2^S}$ *is a closure algebra.*

Proof. By construction, $\mathbf{2^S}$ is certainly a Boolean algebra and so it remains to show that the closure axioms are satisfied by $\mathbf{2^S}$.

Firstly, $C(\emptyset) = \emptyset$ follows immediately from the definition of C on $\mathbf{2^S}$ and $C(S) = S$ follows since for every $s \in S$, $s \le C_S(s)$.

Let X be any subset of S and let x be contained in $C(C(X))$. So there exists y in $C(X)$ and $z \in X$ such that $x \leq C_S(y)$ and $y \leq C_S(z)$. Therefore $C_S(y) \leq C_S(C_S(z)) = C_S(z)$ giving $x \leq C_S(y) \leq C_S(z)$ and hence $x \in C(X)$. Therefore $C(C(X)) \subseteq C(X)$.

Now let x be contained in $C(X)$. There exists $y \in X$ such that $x \leq C_S(y)$. However $C_S(y) \leq C_S(y)$ and so $C_S(y) \in C(X)$. Therefore $x \in C(C(X))$ since $x \leq C_S(y) = C_S(C_S(y))$ where $C_S(y) \in C(X)$. Therefore $C(C(X)) = C(X)$ for every $X \subseteq S$.

If $x \in X \subseteq S$ then since $x \leq C_S(x)$ in $\mathbf{S}$ we have $x \in C(X)$. Hence $\mathbf{2^S}$ satisfies $X \wedge C(X) = X$ for every $X \subseteq S$. It remains to check that $C(X \vee Y) = C(X) \vee C(Y)$ for each $X, Y \subseteq S$. Now $x \in C(X \vee Y)$ if and only if there exists a y such that $y \in X \vee Y$ and $x \leq C_S(y)$ in S. This is possible if and only if x is contained in $C(X)$ or $C(Y)$, that is, if and only if $x \in C(X) \vee C(Y)$. Hence $C(X) \vee C(Y) = C(X \vee Y)$. Therefore $\mathbf{2^S}$ satisfies all defining identities of closure algebras and so is a closure algebra itself. □

It is routinely verified that if $\mathbf{S}$ is a CSL then the closure algebra $\mathbf{2^S}$ is in fact the closure algebra of a topological space on S whose closed topology is the set of closed elements of 2^S as a closure algebra. This space has the property that arbitrary unions of closed sets are open and that no pair of nonempty closed sets are disjoint.

Definition 3.3. A *CSL-embedding* of a CSL $\mathbf{S} := \langle S, \cdot, C_S \rangle$ into a closure algebra $\mathbf{B} := \langle B, \vee, \wedge, ', C, 0, 1 \rangle$ is an injective map $\xi \colon S \to B$ satisfying $\xi(x \cdot y) = \xi(x) \wedge \xi(y)$ and $\xi(C_S(x)) = C(\xi(x))$.

Lemma 3.4. *The map $\xi \colon S \to 2^S$ defined by $\xi(s) := \{x \in S : x \leq s\}$ is a CSL-embedding of $\mathbf{S}$ into $\mathbf{2^S}$.*

Proof. Let X_s denote the set $\{x \in S : x \leq s\}$ (that is, the image of s under ξ). Now $X_x \wedge X_y = \{z \in S : z \leq x \text{ and } z \leq y\}$. Since $\mathbf{S}$ is a semilattice, $z \leq x$ and $z \leq y$ if and only if $z \leq xy$. Hence $X_x \wedge X_y = X_{x \cdot y}$, that is, $\xi(x) \wedge \xi(y) = \xi(x \cdot y)$.

Now $\xi(C_S(x)) = X_{C_S(x)} = \{s \in S : s \leq C_S(x)\}$. If $s \leq C_S(x)$, then $s \in C(X_x)$ by the definition of $C(X)$ on $\mathbf{2^X}$ (since $x \in X_x$). Therefore $\xi(C_S(x)) \subseteq C(\xi(x))$. Now let s be an element of $C(\xi(x))$. That is, there is a $y \in \xi(x) = X_x$ such that $s \leq C_S(y)$. Since $y \leq x$ we have $C_S(y) \leq C_S(x)$ and therefore $s \leq C_S(y) \leq C_S(x)$. It follows that $s \in X_{C_S(x)} = \xi(C_S(x))$. So $\xi(C_S(x)) = C(\xi(x))$ as required. □

Corollary 3.5. *Let K be the class of subreducts of closure algebras to the operations of meet and closure. Then K is a variety that coincides with $\mathcal{CSL}$.*

Since for a CSL $\mathbf{S}$, the set S becomes a topological space under our construction we get the following results.

Proposition 3.6. (i) *Every CSL can be representated as a semilattice of subsets of some topological space closed under finite intersections and taking closures (where the CSL product corresponds to intersection and the closure corresponds to topological closure). Conversely, every such collection of subsets of a topological space is a CSL with respect to intersection and closure.*

(ii) *There is a topological space $(X, \mathfrak{T})$ for which there are two subsets $A \subseteq X$ and $B \subseteq X$ such that the number of sets obtainable from A and B by taking intersections and closures is infinite.*

(iii) *There is no identity involving only intersection and closure that distinguishes topological closure operators from general closure operators.*

For part (ii), the space on S where $\mathbf{S}$ is the free CSL on two generators suffices. It follows from Theorem 5.10 of [17], that the following spaces (with their usual topologies) also have this property: $\mathbb{R}$, $\mathbb{Q}$, the Cantor discontinuum. Indeed this theorem shows that the CSL of all subsets of any one of these generates the variety of all CSL's.

Note that while Lemma 3.4 shows that any CSL identity that fails on a CSL $\mathbf{S}$ must also fail on $\mathbf{2}^{\mathbf{S}}$, the reverse implication is not true. For example, it is obvious that the identity $C(x) \approx C(y)$ is equivalent to $x \approx y$ in the variety of closure algebras, so $\mathbf{2}^{\mathbf{M}}$ does not satisfy $C(x) \approx C(y)$ even though $\mathbf{M}$ does.

4. The base of the lattice of CSL varieties: entropic CSL's

In this section we describe several other varieties near the base of the lattice of subvarieties of $\mathcal{CSL}$. We begin by looking at some varieties of RC-semigroups that are central to the investigations of [11]: those given (within the variety of RC-semigroups) by $C(x)y \approx yC(xy)$, $C(C(x)y) \approx C(xy)$ and $C(x)y \approx yC(C(x)y)$. These varieties are called the *variety of twisted RC-semigroups*, the *variety of RC-semigroups satisfying the right congruence condition* and the *variety of translucent semigroups* respectively. In fact the variety of twisted RC-semigroups is exactly the meet of the other two varieties [11]. Restricted to the class of CSL's however, we have the following result (here, and elsewhere, if Σ is a list of CSL identities, then we let $[\![\Sigma]\!]$ denote the variety of CSL's defined by Σ).

Theorem 4.1. *The varieties $[\![C(x)y \approx yC(xy)]\!]$ and $[\![C(C(x)y) \approx C(xy)]\!]$ are both equivalent to the variety $[\![C(xy) \approx C(x)C(y)]\!]$. The variety $[\![C(x)y \approx yC(C(x)y)]\!]$ is the same as $\mathcal{CSL}$.*

Proof. Firstly Theorem 2.5 implies that $C(x)y \approx yC(C(x)y)$ is a law of every CSL. Since the lattice of subvarieties of $\mathcal{CSL}$ is a homomorphic image of the lattice of subvarieties of all RC-semigroups, we must have $[\![C(x)y \approx yC(xy)]\!] =$

$[\![C(C(x)y) \approx C(xy)]\!] \wedge [\![C(x)y \approx yC(C(x)y)]\!] = [\![C(C(x)y) \approx C(xy)]\!]$. It remains to show that a twisted CSL satisfies $C(xy) \approx C(x)C(y)$ (for certainly every CSL satisfying $C(xy) \approx C(x)C(y)$ satisfies $C(C(x)y) \approx C(xy)$). We have $C(x)C(y) \approx C(C(x)C(y)) \approx C(xC(y)) \approx C(C(y)x) \approx C(yx)$ as required. □

Algebras for which each fundamental operation is a homomorphism are known as *entropic* algebras. Accordingly, we will denote the variety $[\![C(xy) \approx C(x)C(y)]\!]$ of entropic CSL's by $\mathcal{E}$.

Theorem 4.2. *The variety $\mathcal{E}$ is the join of the atomic varieties $\mathcal{I}$ and $\mathcal{M}$ (given by $\{x \approx C(x)\}$ and $\{C(x) \approx C(y)\}$ respectively) and every nontrivial CSL variety other than $\mathcal{I}$ and $\mathcal{M}$ contains $\mathcal{E}$. In particular, $\mathcal{E}$ is the unique cover of the atoms $\mathcal{I}$ and $\mathcal{M}$ in the lattice of subvarieties of $\mathcal{CSL}$.*

Proof. We first show that any identity that is not an identity of $\mathcal{E}$ is not an identity of one of $\mathcal{I}$ or $\mathcal{M}$. Since $\mathbf{I}$ and $\mathbf{M}$ are both entropic (as is easily confirmed), it will follow that $\mathcal{E}$ is the join of $\mathcal{I}$ and $\mathcal{M}$.

It is an elementary consequence of the CSL axioms and the identities $C(xy) \approx C(x)C(y)$ and $C(xC(y)) \approx C(xy)$ that every CSL term t is equivalent (for $\mathcal{E}$) to one of the form $x_1x_2\cdots x_nC(y_1)C(y_2)\cdots C(y_m)$, where $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_m\}$ are disjoint sets of variables. Now say that

$$\mathcal{E} \models x_1\cdots x_nC(y_1)\cdots C(y_m) \approx x'_1\cdots x'_{n'}C(y'_1)\cdots C(y'_{m'}).$$

This must also be satisfied by $\mathbf{I}$ and $\mathbf{M}$ since these algebras are contained in $\mathcal{E}$ (they both satisfy $C(xy) \approx C(x)C(y)$). By considering $\mathbf{I}$ we find that $x_1x_2\cdots x_n$ and $x'_1x'_2\cdots x'_{n'}$ are identical (modulo semilattice laws). By considering $\mathbf{M}$, we find that $x_1\cdots x_ny_1\cdots y_m$ and $x'_1\cdots x'_{n'}y'_1\cdots y'_{m'}$ are identical (modulo semilattice laws). Since the variables labelled by y are all distinct from variables labelled by x, it follows that $y_1\cdots y_m$ is identical to $y'_1\cdots y'_{m'}$ and hence the identity is a tautology (or more precisely, equivalent modulo semilattice laws). Hence $\mathcal{E} = \mathcal{I} \vee \mathcal{M}$. To complete the proof we show that any CSL not contained in one of the (atomic) varieties $\mathcal{I}$ or $\mathcal{M}$ contains $\mathcal{I} \vee \mathcal{M}$.

Let $\mathbf{S}$ be any CSL that does not lie in one of the varieties $\mathcal{I}$ or $\mathcal{M}$. Therefore $\mathbf{S}$ contains a non-closed element x and at least two closed elements $C(a)$ and $C(b)$ (we may assume without loss of generality that $C(a) \not\leq C(b)$). Then the sub-CSL $\{x, C(x)\}$ is isomorphic to $\mathbf{M}$ and the sub-CSL $\{C(a), C(a)C(b)\}$ is isomorphic to $\mathbf{M}$. Hence $\mathbf{S}$ generates a variety that contains $\mathcal{E} = \mathcal{I} \vee \mathcal{M}$. □

We now obtain a description of the entropic CSL's that is similar in flavour to the celebrated description of distributive lattices (Birkhoff) and modular lattices (Dedekind) in terms of excluded subalgebras (see [5] for example).

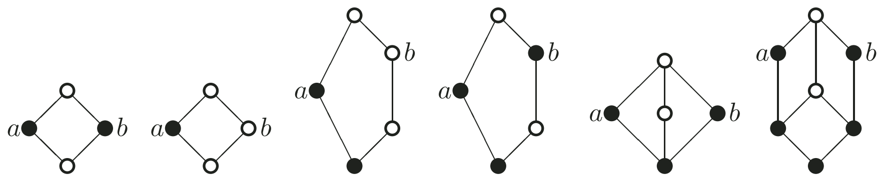

FIGURE 2. The class of minimal forbidden subalgebras for $\mathcal{E}$.

Theorem 4.3. *A CSL,* $\mathbf{S}$*, is entropic if and only if none of the six CSL's in Figure* 2 *is embeddable in* $\mathbf{S}$*.*

Proof. Our approach is to consider a CSL in which the identity $C(xy) \approx C(x)C(y)$ fails. Let S be a CSL containing two elements a and b such that $C(ab) \neq C(a)C(b)$ and let $\mathbf{A}$ be the sub-CSL generated by a and b. Notice that $a \not\leq b$ and $b \not\leq a$ since, for example, $a \leq b$ implies that $C(ab) = C(ab)C(b) = C(a)C(b)$. This means that $C(a) \not\leq b$ and $C(b) \not\leq a$ as well. Furthermore, it cannot be that both $a = C(a)$ and $b = C(b)$ for then $C(ab) = C(C(a)C(b)) = C(a)C(b)$. Now $\mathbf{A}$ is two generated and therefore is a homomorphic image of the two-generated free CSL depicted in Figure 1. This shows for example that ab is the (multiplicative) zero of $\mathbf{A}$ and that $C(ab)$ is a zero for $C(A)$, the set of closed elements in A. By comparing possible collapses of the top part of this Hasse diagram with the above facts concerning $\mathbf{A}$ one arrives at five possible categories into which the algebra $\mathbf{A}$ can be classified (up to switching the letter name a with b and b with a). They are:

(1) a, b, $C(a)$, $C(b)$, $C(a)C(b)$ are all distinct;
(2) a, b, $C(a)$, $C(b)$ are all distinct but $C(a)C(b) = C(b)$;
(3) a, b and $C(a)$ are distinct but $C(a) = C(b)$
(of course in this case $C(a) = C(a)C(b)$ as well);
(4) a, b, $C(a)$ and $C(a)C(b)$ are all distinct but $b = C(b)$.
(5) a, b and $C(a)$ are distinct but $b = C(b) = C(a)C(b)$.

We will consider each case separately. The first is the most difficult and we consider it in two separate subcases.

Case 1, Subcase 1. $C(a)C(b)$ is below neither a nor b: Now since $a \not\leq C(b)$, it follows (from Figure 1) that $aC(b)$ is covered by a. If $C(aC(b))$ is distinct from $C(a)C(b)$ then the sub-CSL of $\mathbf{A}$ on the elements

$$\{a, C(a), C(a)C(b), aC(b), C(aC(b))\}$$

is isomorphic to the third algebra in Figure 2. By symmetry, a similar result holds for $C(bC(a))$ and $C(a)C(b)$. Now assume that $C(aC(b)) = C(a)C(b) =$

$C(bC(a))$. It follows from the Hasse diagram in Figure 1 (or by an elementary equational deduction) that $aC(b) = aC(bC(a)) = aC(bC(aC(b))) = \cdots$ and $bC(a) = bC(aC(b)) = bC(aC(bC(a))) = \cdots$. The only elements of the free CSL on two generators remaining to be considered are ab, $C(ab)$, $aC(ab)$ and $bC(ab)$, so we now examine possible identifications between these.

We are assuming that $C(ab)$ does not equal $C(a)C(b)$ and so $aC(ab) \neq aC(b)$ and $bC(ab) \neq bC(a)$ (otherwise $C(a)C(b) = C(aC(b)) = C(aC(ab)) = C(ab)$).

If $aC(ab) = ab = bC(ab)$ then the sub-CSL of $\mathbf{A}$ generated by the elements $\{aC(b), bC(a)\}$ is isomorphic to the first or fifth CSL in Figure 2 (depending on whether or not $C(ab) = ab$).

If $bC(ab) = C(ab)$ then it follows that $aC(ab) = abC(ab) = ab$. Thus in this case, either all of ab, $aC(ab)$, $bC(ab)$ are equal (which is already considered above) or the subalgebra on $\{aC(b), bC(a), C(a)C(b), bC(ab) = C(ab), ab = aC(ab)\}$ is isomorphic to the fourth CSL in Figure 2.

There is a dual argument if $aC(ab) = C(ab)$ and so the remaining case is when all four of ab, $C(ab)$, $aC(ab)$, and $bC(ab)$ are distinct. But then, the sub-CSL generated by $aC(b)$ and $bC(a)$ is easily seen to be the last of the CSL's in Figure 2. The proof of Subcase 1 is complete.

Case 1, Subcase 2. $C(a)C(b) < a$: In this case it cannot be that $C(a)C(b) \leq b$ for then $C(a)C(b) \leq ab$ and by Figure 1, $C(a)C(b) = C(ab)$. Now $C(a)C(b) < a$ implies that $bC(a) = bC(a)C(b) \leq ab$ which by Figure 1 implies that $bC(a) = ab$. Hence the sub-CSL of $\mathbf{A}$ on the elements $\{C(a)C(b), b, C(b), ab, C(ab)\}$ is isomorphic to the second or third of the CSL's in Figure 2 (depending on whether or not $ab = C(ab)$). The proof in Subcase 2 is complete which (by symmetry) completes the proof for Case 1.

Case 2. First note that $bC(a) = b$. If $aC(b) \not\leq b$ then either $C(aC(b)) = aC(b)$, or $C(aC(b)) = C(a)C(b) = C(b)$, or $C(aC(b))$ is distinct from $aC(b)$ and $C(a)C(b)$. In the first and third of these cases, the sub-CSL of $\mathbf{A}$ generated by

$$\{a, C(a), C(b), aC(b), C(aC(b))\}$$

is isomorphic to the second or third of the CSL's in Figure 2 (depending on whether $aC(b) = C(aC(b))$ or not). In the second case it follows that the identity $C(xy) = C(x)C(y)$ fails when x is assigned the value b and y is assigned the value $aC(b)$ (since $C(b)C(aC(b)) = C(a)C(b)$ and $C(baC(b)) = C(ab)$ in $\mathbf{A}$). Therefore $C(xy) \approx C(x)C(y)$ fails on the sub-CSL of $\mathbf{A}$ generated by $\{b, aC(b)\}$, which is of the form covered by Case 3 below.

Now consider when $aC(b) \leq b$. Then $aC(b) \leq ab$ and so it must be the case that $ab = aC(b)$. This also means that $aC(ab) = aC(b)C(ab) = abC(ab) = ab$. Thus, amongst the four bottom elements of Figure 1, we have only four further collapses to consider: when $ab = bC(ab) = C(ab)$; when $bC(ab) = C(ab) \neq ab$; and when $bC(ab) = ab \neq C(ab)$; and when all three of $ab, bC(ab), C(ab)$ are distinct. In the first case, $\{a, C(a), C(b), C(ab)\}$ is a sub-CSL isomorphic to the second CSL in Figure 2. In the remaining three cases $\{a, C(a), C(b), C(ab), ab\}$ is a sub-CSL isomorphic to the third CSL in Figure 2. The proof in Case 2 is complete.

Case 3. In this case $aC(b) = a$ and $bC(a) = b$ and it follows that $\mathbf{A}$ is a quotient of the final CSL in Figure 2. It is routine to verify that all proper quotients of this CSL with $C(ab) \neq C(a)C(b)$ are isomorphic to either the first, fourth or fifth CSL in Figure 2. This completes the proof in Case 3.

Case 4. Note that $C(aC(a)C(b)) = C(aC(b)) = C(ab)$ hence $C(xy) \approx C(x)C(y)$ fails on the subalgebra generated by a and $C(a)C(b)$. This subalgebra falls under Case 5 to follow and so the proof in Case 4 will be complete when the proof of Case 5 is complete.

Case 5. In this case $aC(b) = ab$, $bC(a) = C(b)C(a) = b$ and $bC(ab) = C(b)C(ab)$ $= C(ab)$. Therefore $aC(ab) = ab$. If $ab \neq C(ab)$ then $\mathbf{A}$ is isomorphic to the third CSL in Figure 2. Otherwise, $\mathbf{A}$ is isomorphic to the second CSL in Figure 2. This completes the proof of Case 5 and indeed the proof of the "only if" part of the lemma.

To complete the "if" part of the lemma it remains to show that any CSL in which one of the six CSL's of Figure 2 can be embedded is not entropic. This follows immediately once it has been verified that none of the six are themselves entropic, which is routine and will be omitted (simply note that the closure operation is not a homomorphism). □

What we have shown is that the six CSL's in Figure 2 are the class of minimal forbidden subalgebras for the class $\mathcal{E}$. It is now easy to describe the minimal forbidden divisors of $\mathcal{E}$ (recall that a *divisor* of an algebra $\mathbf{A}$ is any homomorphic image of a subalgebra of $\mathbf{A}$). First note that a sub-CSL of a CSL $\mathbf{S}$ is a divisor of $\mathbf{S}$. Thus the minimal forbidden divisors of $\mathcal{E}$ must be a subset of the collection of six CSL's in Figure 2. A simple examination reveals that each of the final four CSL's in this figure have a quotient isomorphic to at least one of the first two CSL's in Figure 2. This proves the following theorem.

Theorem 4.4. *A CSL* $\mathbf{S}$ *is entropic if and only if neither of the first two CSL's in Figure* 2 *are divisors of* $\mathbf{S}$. *That is, the first two CSL's of Figure* 2 *are* (*up to isomorphism*) *exactly the class of forbidden divisors for* $\mathcal{E}$.

Corollary 4.5. *A CSL variety,* $\mathcal{V}$, *properly contains* $\mathcal{E}$ *if and only if one of the first two CSL's in Figure* 2 *is contained in* $\mathcal{V}$.

Denote the first of the CSL's in Figure 2 by $\mathbf{B}$ and the second by $\mathbf{C}$.

Lemma 4.6. *The CSL's* **B** *and* **C** *generate distinct varieties, neither of which is a subvariety of the other.*

Proof. It is easily verified that $\mathbf{B} \models C(xC(y)) \approx C(x)C(y)$. However assigning x the value a in **C** and y the value b, we obtain $C(xC(y)) = C(aC(b)) = C(ab) \neq C(a)C(b) = C(x)C(y)$. Hence **C** does not satisfy $C(xC(y)) \approx C(x)C(y)$ and $\mathbf{C} \notin \mathbb{V}(\mathbf{B})$.

Similarly, it is easily verified that $\mathbf{C} \models C(xy) \approx C(xC(y))C(yC(x))$. However assigning x the value a in **B** and y the value b we have that $C(xy) = C(ab) = ab \neq C(a)C(b) = C(aC(b))C(bC(a)) = C(xC(y))C(yC(x))$. Hence **B** does not satisfy $C(xy) \approx C(xC(y))C(yC(x))$ and $\mathbf{B} \notin \mathbb{V}(\mathbf{C})$. □

Corollary 4.5 and Lemma 4.6 imply the following theorem.

Theorem 4.7. *The variety $\mathcal{E}$ has exactly two covers in the lattice of subvarieties of $\mathcal{CSL}$, the variety generated by $\mathcal{B}$ and the variety generated by $\mathcal{C}$. Furthermore, every supervariety of $\mathcal{E}$ contains at least one of these two varieties.*

5. The base of the lattice of CSL varieties: normal CSL's

Let $\mathcal{B}$ and $\mathcal{C}$ denote the varieties generated by **B** and **C** respectively. We are going to show that the variety $\mathcal{B} \vee \mathcal{C}$ is a cover of $\mathcal{B}$ in the lattice of subvarieties of $\mathcal{CSL}$, however only slightly more work can give us a complete description of the subvarieties of the variety $[\![C(xC(y)) \approx C(x)C(y)]\!] \vee \mathcal{C}$. Following [12], we will refer to CSL's satisfying the identity $C(xC(y)) \approx C(x)C(y)$ as *normal* CSL's.

The following lemma is useful.

Lemma 5.1. *Let $\mathcal{V}$ be a CSL variety.*

(i) *If $\mathcal{V}$ contains the variety of monoidal CSL's then it can be defined* (*within* $\mathcal{CSL}$) *by a set of identities between closed CSL terms.*

(ii) *If $\mathcal{V}$ contains the variety of identity CSL's, then every identity $u \approx v$ satisfied by $\mathcal{V}$ has the property that u and v contain the same variables.*

Proof. For (i), let $\Sigma := \{s_i \approx t_i : i \in I\}$ be a defining set of identities for $\mathcal{V}$. If one of the identities $s_i \approx t_i$ is not a closed term then it can be written in the form $w_1C(v_1) \approx w_2C(v_2)$ (where w_1, w_2 are semilattice terms and v_1, v_2 are CSL terms). Since $\mathcal{V}$ contains $\mathcal{M}$, we may assume that w_1 is identical to w_2. The identity $w_1C(v_1) \approx w_1C(v_2)$ implies the identity $C(w_1C(v_1)) \approx C(w_1C(v_2))$. However the identity $C(w_1C(v_1)) \approx C(w_1C(v_2))$ implies that $w_1C(w_1C(v_1)) \approx w_1C(w_1C(v_2))$ which gives $w_1C(v_1) \approx w_1C(v_2)$. Hence we may assume without loss of generality that each identity in Σ is between closed CSL terms.

For part (ii), note that this is a property of $\mathcal{I}$ and hence of every variety containing $\mathcal{I}$. □

Proposition 5.2. *The variety $\mathcal{C}$ is equal to*

$$[\![C(xy) \approx C(xC(y))C(yC(x)),\ C(xyz) \approx C(xy)C(yz)C(zx),$$
$$C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z))]\!].$$

Proof. First note that the identity $C(xyz) \approx C(xy)C(yz)C(zx)$ implies the identity $C(xC(y)C(z)) \approx C(xC(y))C(xC(z))$ and then we get

$$\begin{aligned} &C(xC(yz)) \\ &\approx C(xC(yC(z))C(zC(y))) \quad \text{(by the first identity in Proposition 5.2)} \\ &\approx C(xC(yC(z)))C(xC(zC(y))) \quad (\text{by } C(xC(y)C(z)) \approx C(xC(y))C(xC(z))) \\ &\approx C(xC(y))C(yC(z))C(xC(z))C(xC(z))C(zC(y))C(xC(y)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the third identity)} \\ &\approx C(xC(y))C(xC(z))C(yz) \quad \text{(by CSL axioms).} \end{aligned}$$

Repeated applications from left to right of the identities derived so far (along with those in the statement of the proposition) now reduce every closed term to one of the form

$$C(u_1C(v_1))C(u_2C(v_2))\cdots C(u_nC(v_n)) \tag{1}$$

where either v_i is empty and u_i is of length 1 (that is, a single variable) or u_i and v_i are both single (and distinct) variables. By Theorem 2.5, we may assume that the semilattice term u_iv_i equals a term u_jv_j only if $i = j$ or both $u_i = v_j$ and $v_i = u_j$.

Now let s and t be two closed CSL terms that have been reduced into the form of Expression (1) using the above method. We are going to show that if $\mathbf{C} \models s \approx t$ then s is identical to t (modulo commutativity). Lemma 5.1 implies that we may assume that s and t contain the same variables. We will now assume that s is not equivalent modulo commutativity to t and derive a contradiction. Without loss of generality, we may assume that there is a subterm of s equal to $C(xC(y))$ but no such subterm exists in t. Assign all variables except x and y the value $C(x)$. Under this assignment, t takes on a value ordered above or equal to $C(yC(x))$, while s takes on a value below or equal to $C(xC(y))$. A contradiction is obtained from the fact that $C(xC(y)) \geq C(yC(x))$ fails on $\mathbf{C}$ by choosing $x = a$ and $y = b$.

Hence in every case, a contradiction is obtained and it therefore follows that s is equivalent to t modulo commutativity. Since an easy verification shows that $\mathbf{C}$ satisfies all of the identities described in the statement of the proposition, it follows that these form an identity basis for the equational theory of $\mathbf{C}$. □

For each $n \geq 2$ let $\mathbf{w}_n$ denote the term $C(x_1x_2\cdots x_n)$ and for each $i \leq n$, let $\mathbf{w}_{n,i}$ denote the term $C(x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n)$. Denote the set of variables appearing in a CSL term t by $\mathrm{cont}(t)$ (the *content* of t).

Proposition 5.3. *Let $\mathcal{V}$ be a normal variety of CSL's containing the variety $\mathcal{E}$. Then either $\mathcal{V}$ is the variety of all normal CSL's or there is an $n > 2$ such that $\mathcal{V} = [\![\mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2}\cdots\mathbf{w}_{n,n}, C(xC(y)) \approx C(x)C(y)]\!]$.*

Proof. Let $\mathcal{V}$ be a variety properly contained within the variety of all normal CSL's and also containing the variety of entropic CSL's. We let Σ be a basis for the identities of $\mathcal{V}$ within the variety of normal CSL's and show how to replace each identity in Σ by one of the form $\mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2}\cdots\mathbf{w}_{n,n}$ for some n. Since for $n > m$, $\mathbf{w}_m \approx \mathbf{w}_{m,1}\mathbf{w}_{m,2}\cdots\mathbf{w}_{m,m} \vdash \mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2}\cdots\mathbf{w}_{n,n}$ and since $\mathcal{V}$ properly contains the variety of entropic CSL's, there is an $n \geq 2$ such that $\mathcal{V} = [\![\mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2}\cdots\mathbf{w}_{n,n}]\!]$.

Let $u \approx v$ be an identity from Σ. We may assume that $u \approx v$ does not follow from the CSL axioms and the identity $C(xC(y)) \approx C(x)C(y)$ and (by Lemma 5.1) that both u and v are closed terms with the same content. In the presence of the identity $C(xC(y)) \approx C(x)C(y)$ and by Theorem 2.5, we may assume that $u = C(u_1)C(u_2)\cdots C(u_n)$ and $v = C(v_1)C(v_2)\cdots C(v_m)$ where each u_i (and v_i) are closure free terms and for each distinct i and j, u_i is not a subterm of u_j (and likewise for v_i, v_j). Thus without loss of generality, we may assume there is an $i \leq n$ such that u_i is not equal to v_j for any $j \leq m$. We may further assume that there is $i \leq n$ such that u_i is not a subterm of any v_j. Indeed, if u_i is not equal to any v_j, but is nevertheless a subterm of some v_j, then v_j is not a subterm of any u_k, and so we may switch the choice of u with v.

For every $i \leq n$ let n_i denote the length of the semilattice term u_i and let ℓ_i denote the maximum number j such that for every j element subset A of $\mathrm{cont}(u_i)$, there is a v_j such that $A \subseteq \mathrm{cont}(v_j)$. This exists since every single variable in $\mathrm{cont}(u_i)$ occurs somewhere in v. Note that $1 \leq \ell_i \leq n_i$. Let $\ell := \min\{\ell_i : \ell_i < n_i\}$, which is well defined since there is an i for which u_i is not a subterm of any v_j. We may assume without loss of generality that the corresponding number constructed from the term v is not smaller than ℓ. We now show that $u \approx v$ is equivalent (within normal CSL's) to the law $\mathbf{w}_{\ell+1} \approx \mathbf{w}_{\ell+1,1}\cdots\mathbf{w}_{\ell+1,\ell+1}$.

Let k be such that $\ell_k = \ell$ and let $X := \{x_1, \dots, x_k\}$ be the alphabet of u_k. By deleting all the variables in $\mathrm{cont}(u)$ except for $x_1, x_2, \dots, x_{\ell+1}$ (this can be done by assigning $C(x_1)$ to unwanted variables) and reducing modulo the CSL axioms, an identity $\bar{u} \approx \bar{v}$ is obtained where $\bar{u}$ is the term $C(x_1\cdots x_{\ell+1})$. By the choice of ℓ and k such that $\ell_k = \ell$ we have that $\bar{v}$ is equal (modulo commutativity) to the term $C(x_1\cdots x_\ell)C(x_1\cdots x_{\ell-1}x_{\ell+1})\cdots C(x_2\cdots x_{\ell+1})$ as required.

This shows that $u \approx v \vdash \mathbf{w}_{\ell+1} \approx \mathbf{w}_{\ell+1,1}\cdots\mathbf{w}_{\ell+1,\ell+1}$ within the variety of normal CSL's. To show the reverse, note that using $\mathbf{w}_{\ell+1} \approx \mathbf{w}_{\ell+1,1}\cdots\mathbf{w}_{\ell+1,\ell+1}$ and the normal CSL axioms we may obtain identities of the form $u \approx C(u'_1)\cdots C(u'_{n'})$ and $v \approx C(v'_1)\cdots C(v'_{m'})$ where each u'_i and v'_i is closure free, contains at most

ℓ distinct letters and for each $i \neq j$, u'_i is not a subterm of u'_j (and v'_i is not a subterm of v'_j). The choice of ℓ now implies that u'_i is a subterm of some v'_j and vice versa. Therefore $C(u'_1) \cdots C(u'_{n'}) \approx C(v'_1) \cdots C(v'_{m'})$ is a tautology (modulo commutativity). This completes the proof. □

Note that $\mathbf{B} \models C(xyz) \approx C(xy)C(yz)C(zx)$ but does not satisfy $C(xy) \approx C(x)C(y)$ and so a basis for its identities (within the variety of CSL's) is

$$\{C(xC(y)) \approx C(x)C(y), C(xyz) \approx C(xy)C(yz)C(zx)\}.$$

Proposition 5.4. *The lattice of varieties of normal CSL's consists of the four subvarieties of $\mathcal{E}$ and the normal variety defined by the identities $\mathbf{w}_n \approx \mathbf{w}_{n,1} \cdots \mathbf{w}_{n,n}$ for each $n \geq 2$. Each of these varieties are distinct.*

Proof. It remains to show that the varieties defined by $\mathbf{w}_n \approx \mathbf{w}_{n,1} \cdots \mathbf{w}_{n,n}$ for each $n \geq 3$ are distinct. For each $n \in \mathbb{N}$, let $\mathbf{B_n}$ denote the 2^n element Boolean algebra (considered as a meet semilattice written multiplicatively) with closure given by $C(x) = 1$ if $x \neq 0$, and $C(0) = 0$ (note that $\mathbf{B}$ is the same as $\mathbf{B_2}$). Now $\mathbf{B_n} \models C(xC(y)) \approx C(x)C(y)$ since if either of x or y is assigned 0, then both sides equal 0; otherwise both sides equal 1.

Now $\mathbf{B_n} \not\models \mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2} \cdots \mathbf{w}_{n,n}$ since the collection of n anti-atoms of $\mathbf{B_n}$ have the property that the product of all these elements is 0 while the product of any proper subset of these is non-zero. Hence assigning each of these to a distinct variable x_i, gives $\mathbf{w}_n$ the value 0, but each $\mathbf{w}_{n,i}$ takes the value 1.

Now let m be greater than n and consider a substitution of values from B_n to the variables $x_1, \ldots, x_m$; say $\theta(x_i) = y_i \in B_n$. Theorem 2.5 implies that $\mathbf{w}_m \leq \mathbf{w}_{m,1} \cdots \mathbf{w}_{m,m}$ and so if the identity is to fail then $\theta(\mathbf{w}_m) = 0$, and $\theta(\mathbf{w}_{m,1} \cdots \mathbf{w}_{m,m}) = 1$. Let $\theta(\mathbf{w}_m) = 0$; that is, $y_1 \cdots y_m = 0$. If some $y_i = 0$ then obviously $\theta(\mathbf{w}_{m,1} \cdots \mathbf{w}_{m,m}) = 0$ also. If some $y_i = 1$, then $0 = \theta(\mathbf{w}_n) = \theta(\mathbf{w}_{n,i})$ showing $\theta(\mathbf{w}_n) = \theta(\mathbf{w}_{n,1} \cdots \mathbf{w}_{n,n})$. Otherwise, let $a_1, \ldots, a_k$ be the (at most $n-1$) atoms beneath (or equal to) y_1. There is no atom that is beneath the product $y_1 \cdots y_m$ so for each $j \leq k$ there is a number $1_j \leq m$ such that a_j is not beneath y_{1_j}. Therefore no atom is beneath the product $y_1 y_{1_1} \cdots y_{1_k}$ and so $y_1 y_{1_1} \cdots y_{1_k} = 0$. Since $k < n$ and $n < m$ there is a number $j \leq m$ with $j \notin \{1, 1_1, \ldots, 1_k\}$ which implies that $\theta(\mathbf{w}_{m,j}) = 0$. Hence $\theta(\mathbf{w}_m) = \theta(\mathbf{w}_{m,1} \cdots \mathbf{w}_{m,m})$ as required. □

At this point it is worth mentioning the similarity between the varieties of normal CSL's and the varieties of binary linear codes, as described in [20]. A binary linear code $\mathbf{V}$ can be described as an algebra $\langle V, +, \underline{0}, ' \rangle$, where $\langle V, +, \underline{0} \rangle$ is a vector space over GF(2) and $\mathbf{V}$ satisfies $\underline{0}' = \underline{0}$, $x'' \approx x'$ and $(x + y')' \approx x' + y'$. Aside from some similarities to normal CSL's in the defining identities, the lattice of binary linear code varieties is in fact isomorphic to that just found for normal CSL's.

For $n > 2$ we will denote the variety generated by $\mathbf{B}_n$ by $\mathcal{B}_n$ and the variety of all normal CSL's by $\mathcal{B}_\omega$.

Lemma 5.5. *If $\mathcal{V}$ is a non-normal variety then $\mathbf{C} \in \mathcal{V}$. That is, the varieties not containing $\mathbf{C}$ are exactly $\mathcal{B}_n$ for $n \in \{2, 3, 4, 5, \ldots, \omega\}$ and the four entropic varieties in $\mathcal{E}$.*

Proof. We prove the first statement; the second follows as a corollary of this and Proposition 5.4.

Let $\mathbf{S}$ be a CSL for which the identity $C(xC(y)) \approx C(x)C(y)$ fails; say, at the pair a, b. Thus $C(xC(y)) \approx C(x)C(y)$ fails on the two generated sub-CSL of $\mathbf{S}$ generated by a and b. We may assume that $b = C(b)$ in this CSL since $C(aC(C(b))) = C(aC(b))$ and $C(a)C(C(b)) = C(a)C(b)$. Now $aC(b) = ab$, $bC(a) = C(b)C(a)$, $bC(ab) = C(b)C(ab) = C(ab)$ and $aC(ab) = aC(aC(b)) = aC(b) = ab$, and so $\mathbf{A}$ is isomorphic to the second (that is, $\mathbf{C}$) or third of the CSL's in Figure 2. Since the third CSL in Figure 2 has a quotient isomorphic to $\mathbf{C}$, it follows that the variety generated by $\mathbf{S}$ contains the variety generated by $\mathbf{C}$. □

This also shows that if $\mathcal{V}$ is a CSL variety properly containing $\mathcal{B}_n$ but not $\mathcal{B}_{n+1}$ then $\mathcal{V}$ contains $\mathcal{B}_n \vee \mathcal{C}$. Therefore, for each $n > 1$ the only covers of $\mathcal{B}_n$ in the lattice of CSL varieties are $\mathcal{B}_n \vee \mathcal{C}$ and $\mathcal{B}_{n+1}$. The only cover of $\mathcal{B}_\omega$ is $\mathcal{B}_\omega \vee \mathcal{C}$.

Lemma 5.6. *An equational basis for the variety $\mathcal{C} \vee \mathcal{B}_n$ (for $n \in \{2, 3, 4, \ldots\}$) is given by*

$$\begin{aligned}\Sigma_n := \{&\mathbf{w}_{n+1} \approx \mathbf{w}_{n+1,1} \cdots \mathbf{w}_{n+1,n+1},\\ &C(xC(yz)) \approx C(xC(y))C(xC(z))C(yz),\\ &C(xyC(z)) \approx C(xC(z))C(yC(z))C(xy)\}.\end{aligned}$$

Proof. It is easily verified that Σ_n is satisfied by $\mathbf{C}$ and by $\mathbf{B}_n$. We now show that any identity that does not follow from these is not satisfied by either $\mathbf{C}$ or $\mathbf{B}_n$. By Lemma 5.1 we may restrict our attention to identities between closed terms.

First note that the identity $C(aC(bC(c))) \approx C(aC(b))C(aC(c))C(bC(c))$ follows from the identity $C(xC(yz)) \approx C(xC(y))C(xC(z))C(yz)$ by taking $x = a$, $y = b$ and $z = C(c)$. Also for any i, j we have

$$\begin{aligned}C(x_1 \cdots x_i C(y_1 \cdots y_j)) &\approx C(x_1 \cdots x_i C(y_1)) \cdots C(x_1 \cdots x_i C(y_j))C(y_1 \cdots y_j)\\ &\approx \Big(\prod_{k,l} C(x_k C(y_l))\Big) C(y_1 y_2 \cdots y_j) C(x_1 x_2 \cdots x_j).\end{aligned}$$

Using these identities and $\mathbf{w}_{n+1} \approx \mathbf{w}_{n+1,1} \cdots \mathbf{w}_{n+1,n+1}$, every closed term can be reduced to one that is of the form $C(u_1) \cdots C(u_m)$ where each u_i is either a closure free term in at most n variables or is of the form $xC(y)$ for some distinct variables x, y. Furthermore, we may assume that if $u_i = xC(y)$ then there is no *semilattice* subterm u_j with $x, y \in \mathrm{cont}(u_j)$ and that no semilattice subterm u_i is a subterm of any other different semilattice subterm. For the duration of this proof, such a term will be called *reduced*.

Let $u \approx v$ be an identity between two reduced terms that is satisfied $\mathcal{C} \vee \mathcal{B}_n$. If $u \approx v$ does not follow from the CSL laws then we may assume without loss of generality that there is a closed subterm of u that is not a closed subterm of v. The possibilities are that $x_1 \cdots x_k$ (with $k \leq n$) is a subterm of u but not a subterm of v or that $C(xC(y))$ is a subterm of u but not of v. The first case is impossible since if $C(x_1 \cdots x_k)$ occurs in u then (since $\mathcal{B}_n \models u \approx v$) we must have that for every at most n element subset A of $\{x_1, \ldots, x_k\}$ there is a semilattice subterm v' of v with $A \subseteq \mathrm{cont}(v')$. Since $k \leq n$ this implies that $x_1 \cdots x_k \leqslant v'$ and (because $C(x_1 \cdots x_k)$ is not a subterm of v) the content of v' properly contains the set $\{x_1, \ldots, x_k\}$. But then $C(v')$ is a closed subterm of v that does not occur in u (as u is reduced); repeating the above argument then gives a contradiction. In the second case, assign every variable except x and y the value $C(x)$. Under this assignment, u reduces to $C(xC(y))$ or $C(xC(y))C(yC(x))$ while v must reduce to one of $C(yC(x))$, $C(xy)$, or $C(x)C(y)$. Of the six possible resulting identities, $\mathcal{B}_n \not\models C(xC(y))C(yC(x)) \approx C(xy)$ while the rest are not satisfied by $\mathcal{C}$, another contradiction.

Therefore $u \approx v$ follows from the CSL axioms and so Σ_n is a basis for the identities of $\mathcal{C} \vee \mathcal{B}_n$ within the variety of all CSL's. □

With slightly more work, this proof also gives the result that the variety $\mathcal{B}_{n+1} \vee \mathcal{C}$ is a cover of the variety $\mathcal{B}_n \vee \mathcal{C}$. Indeed if $\mathcal{V}$ is a variety containing $\mathcal{B}_n \vee \mathcal{C}$ and properly contained in $\mathcal{B}_{n+1} \vee \mathcal{C}$ then $\mathcal{V}$ satisfies an identity $u \approx v$ that does not follow from Σ_{n+1} in Lemma 5.6. Since $\mathcal{B}_n$ and $\mathcal{C}$ are subvarieties of $\mathcal{V}$, $u \approx v$ cannot fail for either of these varieties. It then follows from the proof of Lemma 5.6 that $u \approx v$ implies an identity of the form $C(x_1 \cdots x_k) \approx v'$ where $k \leq n+1$ and v' contains only semilattice subterms of length less than k. Because $\mathcal{B}_n$ is a subvariety of $\mathcal{V}$, it cannot be that $k < n+1$. Hence $k = n+1$ and so $u \approx v$ implies the identity $\mathbf{w}_{n+1} \approx \mathbf{w}_{n+1,1} \cdots \mathbf{w}_{n+1,n+1}$. Therefore $\mathcal{V}$ is a subvariety of, and hence equal to, the variety $\mathcal{B}_n \vee \mathcal{C}$. This proves the following proposition.

Proposition 5.7. *For each $n \geq 2$, the variety $\mathcal{B}_{n+1} \vee \mathcal{C}$ is a cover of both $\mathcal{B}_{n+1}$ and $\mathcal{B}_n \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ is a cover of $\mathcal{B}$ and $\mathcal{C}$.*

We have shown that the CSL varieties not containing **C** are exactly the normal varieties of CSL's, which are completely described above. We now show that there are infinitely many varieties containing **C** but not containing **B**.

Lemma 5.8. *There are infinitely many CSL varieties containing* **C** *that do not contain* **B**.

Proof. First note that if $\mathbf{S}$ is a CSL and $e \in C(S)$ is a closed element, then the set $\{x \in S : x \leq e\}$ is an ideal I that is closed under taking closures and hence determines a congruence whose classes are exactly the singletons of $S\backslash I$ along with I. We will call this the Rees quotient of $\mathbf{S}$ with respect to I (after the almost identical concept in semigroup theory). Denote the term $C(xC(yC(x\cdots)))C(yC(xC(y\cdots)))$ by c_n (where there are exactly $2n$ applications of the C operation in the term) and let $\mathbf{C_n}$ denote the Rees quotient of the free CSL on two generators by the principal ideal of the c_n. It is routinely verified that $\mathbf{C_n} \models c_n \approx C(xy)$ and contains a sub-CSL isomorphic to $\mathbf{C}$, while in the equational theory of $\mathbf{B}$, the term c_n is equivalent to $C(x)C(y)$ and so $\mathbf{B} \not\models c_n \approx C(xy)$. As $\mathbf{C_n}$ obviously does not satisfy $c_m \approx C(xy)$ when $m < n$, it follows that each $\mathbf{C_n}$ generates a distinct supervariety of $\mathcal{C}$ that does not containing $\mathbf{B}$. $\square$

The results so far in this section combine to give the partial Hasse diagram in Figure 3 of the lattice of all subvarieties of $\mathcal{CSL}$. Note that by Lemma 5.5 all other varieties appear to the left of the diagram.

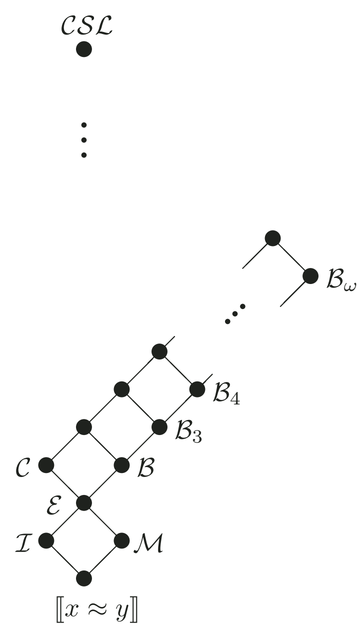

FIGURE 3. The lattice of subvarieties of $\mathcal{CSL}$.

Figure 3 is suggestive of a rather basic general structure for the lattice of subvarieties of $\mathcal{CSL}$. We now show however that the "square lattice" structure does not continue far beyond what we have described above.

Example 5.9. Let $\mathbf{D}_1$ denote the left of the two CSL's in Figure 4. The variety generated by $\mathbf{D}_1$ properly contains the variety $\mathcal{B}\vee\mathcal{C}$ and satisfies the laws $C(xyz) \approx C(xy)C(xz)C(yz)$ and $C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z))$.

Proof. Let $\mathcal{D}_1$ denote the variety generated by $\mathbf{D}_1$. The subalgebra of $\mathbf{D}_1$ on $\{a, b, ab, 1\}$ is isomorphic to $\mathbf{B}$ while the subalgebra on $\{c, ab, 0, 1\}$ is isomorphic to $\mathbf{C}$. Hence $\mathcal{D}_1 \supseteq \mathcal{B} \vee \mathcal{C}$. It is routinely verified that the identities $C(xyz) \approx C(xy)C(xz)C(yz)$ and $C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z))$ are satisfied by $\mathbf{D}_1$. Now note that the identity $C(xC(yz)) \approx C(xC(y))C(xC(z))C(yz)$ fails on $\mathbf{D}_1$ since $C(cC(ab)) = 0$ and $C(cC(a))C(cC(b))C(ab) = C(ab)$. Since $\mathcal{B}\vee\mathcal{C}$ satisfies this identity it follows that $\mathcal{D}_1$ properly contains $\mathcal{C} \vee \mathcal{B}$. □

Example 5.10. Let $\mathbf{D}_2$ denote the right of the two CSL's in Figure 4. The variety generated by $\mathbf{D}_2$ properly contains the variety $\mathcal{C}$ and satisfies the laws $C(xyz) \approx C(xy)C(xz)C(yz)$ and $C(xC(yC(x)))C(yC(xC(y))) \approx C(xy)$. The variety generated by $\mathbf{D}_2$ does not contain $\mathbf{B}$ and does not satisfy

$$C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z)).$$

Proof. Let $\mathcal{D}_2$ denote the variety generated by $\mathbf{D}_2$. The subalgebra of $\mathbf{D}_2$ on $\{a, 1, C(b), aC(b)\}$ is isomorphic to $\mathbf{C}$ and so $\mathcal{D}_2 \supseteq \mathcal{C}$. It is routinely verified that the identities $C(xyz) \approx C(xy)C(xz)C(yz)$ and $C(xC(yC(x)))C(yC(xC(y))) \approx C(xy)$ are satisfied. Now since $\mathbf{B}$ is a normal (but not entropic) CSL, it satisfies the law $C(xC(yC(x)))C(yC(xC(y))) \approx C(x)C(y)$ but not $C(xy) \approx C(x)C(y)$. Hence $\mathbf{B} \not\models C(xC(yC(x)))C(yC(xC(y))) \approx C(xy)$ and so $\mathbf{B} \notin \mathcal{D}_2$. The law $C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z))$ fails on $\mathbf{D}_2$ since $C(bC(aC(b))) = 0$ while $C(bC(a))C(aC(b))C(bC(a)) = aC(b) \neq 0$. □

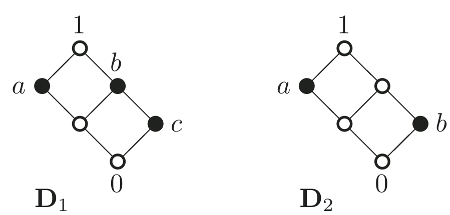

FIGURE 4. The CSL's $\mathbf{D}_1$ and $\mathbf{D}_2$.

Proposition 5.11. *The lattice of CSL varieties is not modular.*

Proof. Let $\mathcal{D}$ denote the variety $\mathcal{D}_1 \vee \mathcal{D}_2$. Examples 5.9 and 5.10 show that $\mathcal{D}$ properly contains both $\mathcal{D}_1$ and $\mathcal{D}_2$ and that $\mathcal{D}_2$ does not contain $\mathcal{B} \vee \mathcal{C}$ while $\mathcal{D}_1$

does. We now show that $\mathcal{D}_1 \wedge \mathcal{D}_2$ equals $\mathcal{C}$. This will imply that the sublattice of the lattice of CSL varieties on $\{\mathcal{D}, \mathcal{C}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{B}\vee\mathcal{C}\}$ is isomorphic to the lattice $\mathbf{N_5}$, which is a forbidden subalgebra for modular lattices. First note that $\mathcal{D}_1 \wedge \mathcal{D}_2$ contains $\mathcal{C}$ and that it satisfies the identities $C(xyz) \approx C(xy)C(xz)C(yz)$, $C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z))$ and $C(xC(yC(x)))C(yC(xC(y))) \approx C(xy)$. We now show that it also satisfies $C(xC(y))C(yC(x)) \approx C(xy)$ which by Proposition 5.2 shows that $\mathcal{D}_1 \wedge \mathcal{D}_2 = \mathcal{C}$. We have

$$\begin{aligned} C(xy) &\approx C(xC(yC(x)))C(yC(xC(y))) \\ &\approx C(xC(y))C(yC(x))C(xC(x))C(yC(x))C(xC(y))C(yC(y)) \\ &\approx C(xC(y))C(yC(x)) \end{aligned}$$

as required. □

6. Simple CSL's are normal

In this section we show that the class of simple (that is, congruence free) CSL's lies inside the variety of normal CSL's. The key tool is the notion of a Rees quotient of a CSL (as introduced above) by an ideal closed under taking closures. If S is a CSL and e is any element of $C(S)$, then the ideal consisting of $\{x \in S : x \leq e\}$ will be denoted by I_e. If e is a non-zero closed element that is not the identity element then the Rees congruence with respect to I_e is a proper, nontrivial congruence. Hence any simple CSL must contain at most two closed elements; a closed zero and a closed identity element. (We note in comparison that there are congruence free RC-semigroups with arbitrary numbers of closed elements [11].)

Proposition 6.1. *Every simple CSL is contained in the variety $\mathcal{B}_\omega$ of normal CSL's.*

Proof. We have shown that a simple CSL must contain at most two closed elements. If $\mathbf{S}$ is a simple CSL with only one closed element then $\mathbf{S} \models C(xC(y)) \approx C(x)C(y)$ trivially.

Now say that $\mathbf{S}$ has two closed elements (a closed zero and a closed identity element) and consider two elements $a, b \in S$. If $C(a) = C(b)$ then certainly $C(aC(b)) = C(aC(a)) = C(a) = C(a)C(b)$. If $C(a) < C(b)$ then $C(aC(b)) = C(a) = C(a)C(b)$. If $C(b) < C(a)$ then $C(b) = 0$ and so $C(aC(b)) = C(a0) = 0 = C(a)0 = C(a)C(b)$. Therefore $\mathbf{S}$ is normal. □

If $\mathbf{S}$ is a CSL with zero, then we will say that two elements a, b are *separable* if there is an element c such that either $ac = 0$ and $bc \neq 0$ or $bc = 0$ and $ac \neq 0$. A CSL with closed 0 will be called separable if every pair of distinct elements is separable.

Lemma 6.2. *If* $\mathbf{S}$ *is a separable CSL with* $C(S) = \{0, 1\}$ *then* $\mathbf{S}$ *is simple.*

Proof. Let θ be a proper congruence on S and let a and b be distinct θ-related elements. Without loss of generality there is a c such that $ac = 0$ and $bc \neq 0$. Hence $0\ \theta\ bc \neq 0$. But $C(bc) = 1$ and $C(0) = 0$. Hence $0\ \theta\ 1$, from which it easily follows that $\theta = \nabla_S$. □

Lemma 6.3. *If* $\mathbf{S}$ *is a non-separable CSL with* $C(S) = \{0, 1\}$ *then* $\mathbf{S}$ *is not simple.*

Proof. Let $a, b \in S$ be non-separable, so that for every $c \in S$, $ac = 0$ if and only if $bc = 0$. Since all elements of S except for 0 have closure equal to 1, any semilattice congruence on $\mathbf{S}$ in which $\{0\}$ is a singleton congruence class is equal to a CSL congruence on $\mathbf{S}$. Let θ be the principal semilattice congruence $\Theta(a, b)$ on the semilattice reduct of $\mathbf{S}$. If $x\ \theta\ y$ then for some $c \in S$, $x = ac$ and $y = bc$. Hence $x\ \theta\ 0$ if and only if $x = 0$ and so θ is equal to a CSL congruence on $\mathbf{S}$ that is not ∇. □

Corollary 6.4. *A CSL* $\mathbf{S}$ *is simple if and only if* $C(S) = \{0, 1\}$ *and it is separable.*

Theorem 6.5. *The class of simple CSL's generates the variety of all normal CSL's.*

Proof. Any Boolean algebra considered as a meet semilattice is separable and so each of the CSL's $\mathbf{B}_n$ and $\mathbf{B}_\omega$ are simple CSL's. These algebras generate the variety of normal CSL's so by Proposition 6.1, the variety generated by all simple CSL's is exactly the variety of normal CSL's. □

Other examples of simple CSL's of arbitrary size can be constructed by taking any antichain S not including 0 and 1 and then making $S \cup \{0, 1\}$ a CSL by putting $C(S \cup \{0, 1\}) = \{0, 1\}$ and defining $0 \leq x \leq 1$ for all $x \in S$. This construction gives a simple CSL in the variety $\mathcal{B}$.

7. Finiteness properties of varieties of CSL's

The identities $\mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2} \cdots \mathbf{w}_{n,n}$ play an interesting role in relation to the number of elements of a finite CSL. If $\mathbf{S}$ is an n-element CSL then any collection of n elements contains a pair a and b for which $a \geq b$. Hence a product of any n-elements is equivalent to a product of an $n-1$ element subset of the elements. This shows that under every assignment of values from S, $\mathbf{w}_n$ takes on a value that is equal to some $\mathbf{w}_{n,i}$. Therefore $\mathbf{S} \models \mathbf{w}_n \geq \mathbf{w}_{n,1} \cdots \mathbf{w}_{n,n}$. However, Theorem 2.5 shows that $\mathbf{w}_n \leq \mathbf{w}_{n,1} \cdots \mathbf{w}_{n,n}$ holds in every CSL and hence $\mathbf{S} \models \mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2} \cdots \mathbf{w}_{n,n}$. We have shown the following result.

Lemma 7.1. *If* $\mathbf{S}$ *is an* n*-element CSL then* $\mathbf{S} \models \mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2} \cdots \mathbf{w}_{n,n}$.

We now construct a second collection of identities, the n^{th} member of which is satisfied by any CSL with at most n elements. For each $n \geq 2$, let $\mathbf{v}_n$ denote the term

$$C(x_1 C(x_2 C(\cdots C(x_n)\cdots)))$$

and $\mathbf{v}_{n,i}$ denote the term

$$C(x_1 C(x_2 C(\cdots C(x_{i-1} C(x_{i+1} C(\cdots C(x_n)\cdots)))\cdots))).$$

Lemma 7.2. *If* $\mathbf{S}$ *is an n-element CSL then* $\mathbf{S} \models \mathbf{v}_n \approx \mathbf{v}_{n,1}\mathbf{v}_{n,2}\cdots\mathbf{v}_{n,n}$.

Proof. We may assume that $\mathbf{S}$ contains at most $n-1$ closed elements; otherwise it satisfies $x \approx C(x)$ and the result holds trivially. Now say that each x_i has been assigned some value a_i in $\mathbf{S}$ by a substitution θ (recall that a substitution is a homomorphism from a free CSL over some suitable generating set X into $\mathbf{S}$ defined by its action on X). By Lemma 2.6, we have $C(a_i C(a_{i+1} C(\cdots C(a_n)\cdots))) \leq C(a_{i+1} C(\cdots C(a_n)\cdots))$. Since $\mathbf{S}$ has at most $n-1$ closed elements, at least one of the n inequalities of this kind must be an equality. Let

$$C(a_k C(a_{k+1} C(\cdots C(a_n)\cdots))) = C(a_{k+1} C(\cdots C(a_n)\cdots))$$

be such an equality. Then

$$\begin{aligned}
\theta(\mathbf{v}_n) &= C(a_1 C(a_2 C(\cdots C(a_k C(a_{k+1} C(\cdots C(a_n)\cdots)))\cdots))) \\
&= C(a_1 C(a_2 C(\cdots C(a_{k-1} C(a_{k+1} C(\cdots C(a_n)\cdots)))\cdots))) \\
&= \theta(\mathbf{v}_{n,k}) \\
&\geq \theta(\mathbf{v}_{n,1})\theta(\mathbf{v}_{n,2})\cdots\theta(\mathbf{v}_{n,n})
\end{aligned}$$

However by Theorem 2.5, $\mathbf{v}_n \leq \mathbf{v}_{n,1}\mathbf{v}_{n,2}\cdots\mathbf{v}_{n,n}$ is satisfied by every CSL and hence $\theta(\mathbf{v}_n) = \theta(\mathbf{v}_{n,1}\mathbf{v}_{n,2}\cdots\mathbf{v}_{n,n})$ in $\mathbf{S}$. Because θ was arbitrary, the lemma is proved. $\square$

Notice that $\mathcal{B}_\omega$ does not satisfy $\mathbf{w}_n \approx \mathbf{w}_{n,1}\mathbf{w}_{n,2}\cdots\mathbf{w}_{n,n}$ for any $n \in \mathbb{N}$. This means that any supervariety of $\mathcal{B}_\omega$ cannot be generated by a finite CSL. Such a variety is said to be inherently nonfinitely generated in $\mathcal{CSL}$. A variety which is finitely based, has finitely many subvarieties and is generated by a finite algebra is called a Cross variety. All subvarieties of $\mathcal{B}_\omega$ are Cross varieties and so we have the following result.

Corollary 7.3. *The variety* $\mathcal{B}_\omega$ *is a "just non-Cross" variety that is minimally inherently nonfinitely generated in the lattice of subvarieties of* $\mathcal{CSL}$.

We will now show that a CSL satisfying $\mathbf{v}_n \approx \mathbf{v}_{n,1}\cdots\mathbf{v}_{n,n}$ is necessarily locally finite. Let $\mathcal{F}_n$ be the CSL variety given by

$$\{\mathbf{w}_n \approx \mathbf{w}_{1,n}\cdots\mathbf{w}_{n,n}, \mathbf{v}_n \approx \mathbf{v}_{n,1}\cdots\mathbf{v}_{n,n}\}$$

and $\mathcal{LF}_n$ be the variety given by $\mathbf{v}_n \approx \mathbf{v}_{n,1}\cdots\mathbf{v}_{n,n}$.

Theorem 7.4. *The variety $\mathcal{LF}_n$ is locally finite.*

Proof. We need to show that the finitely generated $\mathcal{LF}_n$-free CSL's in the variety are finite. Fix some finite alphabet of generators, say A with ℓ elements. We use the term tree description of Section 2 and show that every term tree whose labels come from the alphabet of A can be reduced using the laws of $\mathcal{LF}_n$ to one that is a product of certain trees of bounded size. We then use an elementary combinatorial argument to give a finite bound on the number of such trees.

First note that the identity $\mathbf{v}_n \approx \mathbf{v}_{n,1} \cdots \mathbf{v}_{n,n}$ can be used to reduce every term to one whose height is at most $n-1$ (since each term $\mathbf{v}_{n,i}$ has a term tree of height $n-1$). Now let $\mathcal{T}$ be a term tree of height at most $n-1$ and for which there is an element α that is not the base of $\mathcal{T}$ and has at least $n-1$ covers in $\mathcal{T}$, say $\alpha_1, \alpha_2, \ldots, \alpha_m$ (where $m \geq n-1$). Without loss of generality, we may assume that we have chosen α to be of maximal height in $\mathcal{T}$ amongst those elements with at least $n-1$ covers.

Let β be the element of $\mathcal{T}$ covered by α, let the labels of α, β and each α_i be a, b and a_i respectively, and let each $\Xi(\mathcal{T}_{\alpha_i})$ be denoted by a_i. So $\Xi(\mathcal{T})$ contains the subterm $bC(aC(a_1)C(a_2)\cdots C(a_m))$. Now $bC(aC(a_1)C(a_2)\cdots C(a_m)) =$

$$bC(aC(C(a_1)C(C(a_2)C(\cdots C\Big(C(a_{n-2})C\big(C(a_{n-1})C(a_n)\cdots C(a_m)\big)\Big)\cdots))))$$

which is of the form required to apply $\mathbf{v}_n \approx \mathbf{v}_{n,1} \cdots \mathbf{v}_{n,n}$. By letting e denote the subterm $C(a_{n-1})C(a_n)\cdots C(a_m)$ we get

$$\begin{aligned} &bC(aC(a_1)C(a_2)\cdots C(a_m)) \\ &= bC\Big(aC(a_1)C(a_2)\cdots C(a_{n-2})\Big)C\Big(aC(a_1)C(a_2)\cdots C(a_{n-3})e\Big)\cdots \\ &\qquad\qquad\qquad\qquad \cdots C\Big(aC(a_2)C(a_3)\cdots C(a_{n-2})e\Big). \end{aligned}$$

Replace the subtree of $\mathcal{T}$ corresponding to $bC(aC(a_1)C(a_2)\cdots C(a_m))$ by the tree corresponding to this new term and let the new term tree be called $\mathcal{S}$. While β has many more covers in $\mathcal{S}$ than in $\mathcal{T}$, each of the new covers of β in $\mathcal{S}$ has one fewer cover than that which α did in $\mathcal{T}$. Performing this procedure repeatedly for each cover of β that has more than $n-2$ covers, we eventually we arrive at a term tree in which the element β has many covers, but no element ordered higher than β has more than $n-2$ covers. Continuing in this way, one arrives at a term tree $\mathcal{T}'$ with a large number of atoms, say k, but for which every element ordered equal to or above an atom has at most $n-2$ covers. Thus the term corresponding to $\mathcal{T}$ has been reduced to one that is the product of a possibly empty semilattice term w (the label of the base of $\mathcal{T}'$) and k closed terms whose term trees have the following properties: they are of height at most $n-1$; each element in these trees has at most $n-2$ covers; the base element (with label 1) has a unique cover (an atom of $\mathcal{T}'$);

each label is a semilattice term in the given ℓ-element alphabet. Term trees with these four properties will be called *n-atomic trees*.

Let $\mathcal{S}$ be an n-atomic term tree in the alphabet A. The underlying tree of $\mathcal{S}$ (that is, without the labels) can be obtained by removing some limbs from the tree with 1 atom and 1 element of height 1, $n-2$ elements of height 2, $(n-2)^2$ elements of height 3, and so on, with $(n-2)^{(n-2)}$ elements of height $n-1$. We will call this tree $\mathcal{P}_n$. The total number of elements of $\mathcal{P}_n$ excluding the base is

$$\sum_{i=0}^{n-2}(n-2)^i = \frac{(n-2)^{(n-1)}-1}{n-3}, \text{ for } n>3$$

or 2 when $n=3$. Denote this number by σ_n. The underlying tree of $\mathcal{S}$ is obtained by deleting some subset of the elements of $\mathcal{P}_n$ and so there are at most 2^{σ_n} choices for the underlying tree of $\mathcal{S}$ (the actual number of possibilities is clearly much smaller than this). It is well known that the free semilattice on A has exactly $2^\ell-1$ elements (since $|A|=\ell$) and so there are exactly this number of potential labels for each element of a term tree. A labelling of a term tree is a mapping from the underlying tree into a free semilattice with identity. As the base label of an atomic term tree is the only label of the value 1, the total number of labellings of an n-atomic term tree $\mathcal{S}$ in the alphabet A is at most $(2^\ell-1)^{\sigma_n}$. This gives a (generous) upper bound for the total number of distinct n-atomic term trees $(2^\ell-1)^{\sigma_n}2^{\sigma_n}$.

We have showed above that every CSL term can be reduced using the identity $\mathbf{v}_n \approx \mathbf{v}_{n,1}\cdots\mathbf{v}_{n,n}$ to one whose term tree is a product of a possibly empty semilattice term and a collection of the so called n-atomic trees. Since CSL's are idempotent and there are only finitely many different choices for the semilattice term and for the n-atomic trees, it follows that there are only finitely many such products and consequently only finitely many distinct elements of the ℓ-generated $\mathcal{LF}_n$-free CSL; indeed a (generous) upper bound can be given by $2^{(2^\ell-1)^{\sigma_n}2^{\sigma_n}}$. □

The variety $\mathcal{B}_\omega$ is the same as $\mathcal{LF}_2$ and so Corollary 7.3 shows that $\mathcal{LF}_n$ is not finitely generated for any $n>1$. Also, since $\mathcal{F}_n$ is a subvariety of $\mathcal{LF}_n$, we have the following result.

Corollary 7.5. *The variety $\mathcal{F}_n$ is locally finite.*

In contrast with Theorem 7.4, the identities $\mathbf{w}_n \approx \mathbf{w}_{1,n}\cdots\mathbf{w}_{n,n}$ by themselves do not guarantee local finiteness. Indeed it is not difficult to verify that the free CSL on 2 generators, $\mathbf{F}_2$ (depicted in Figure 1) satisfies $C(xyz) \approx C(xy)C(yz)C(zx)$ (essentially, the closure of the product of three incomparable elements always equals the closure of the product of just two of the elements).

Now we are going to use the ideas in the proof of Theorem 7.4 to prove the main result of this section:

Theorem 7.6. *If* $\mathbf{S}$ *is a finite CSL with identity element* 1*, then the variety generated by* $\mathbf{S}$ *can be defined by a single identity within the variety of CSL's and has only finitely many subvarieties within the class of CSL monoids.*

This will be proved over a number of lemmas. Unless otherwise stated, the CSL's we consider below will all be assumed to contain an identity element 1. Note that 1 is always a closed element in a CSL.

Lemma 7.7. *If* $\mathcal{V}$ *is a nontrivial variety of CSL's other than* $\mathcal{I}$ *and defined by finitely many identities, then* $\mathcal{V}$ *can be defined by a single identity between closed terms along with the CSL axioms.*

Proof. We have that $\mathcal{V}$ contains $\mathcal{M}$. Let $\Sigma := \{s_1 \approx t_1, \ldots, s_n \approx t_n\}$ be a defining set of identities for $\mathcal{V}$ within the variety of all CSL's. Lemma 5.1 shows that we may assume that Σ is a set of identities between closed terms. We may further assume that each identity in Σ is written in a distinct alphabet of variables and therefore the single identity $s_1 s_2 \cdots s_n \approx t_1 t_2 \cdots t_n$ (which obviously follows from Σ) can be used to derive $s_i \approx t_i$ for all $i \leq n$ (for some variable x appearing in $s_i \approx t_i$, simply assign the value $C(x)$ to each variable not appearing in $s_i \approx t_i$). Hence $\{s_1 s_2 \cdots s_n \approx t_1 t_2 \cdots t_n\}$ is a basis (consisting of a single identity between closed terms) for $\mathcal{V}$ within CSL. □

In fact this proof did not require the presence of an identity element.

Definition 7.8. If X is a set of variables with non-empty subset X', let $\theta_{X'}$ denote the substitution that fixes each variable in $X \backslash X'$, and assigns 1 to each variable in X'.

The following lemma follows easily from Lemma 2.6.

Lemma 7.9. *If* t *is a closed CSL term and* $X' \subseteq \mathrm{cont}(t)$ *then the identity* $t \approx t\theta_{X'}(t)$ *holds in every CSL with identity.* (*Note that* $\theta_{X'}(t)$ *is also a closed term.*)

If $X' \neq X$, then $\theta_{X'}(t)$ reduces to a term with no occurrences of 1 and that is a fragment of a subterm of t. If $X' = X$ then the lemma simply states that $t \approx t$ is always a CSL identity. Hence Lemma 7.9 also holds for CSL's without identity element.

A term whose term tree is n-atomic will be called an *n-atomic term.*

The proof of Theorem 7.4 shows that every identity satisfied by a CSL monoid contained in $\mathcal{F}_n$ for some n, may be written (using the two defining identities of the variety $\mathcal{F}_n$) as an identity between a product of n-atomic terms. For the variety $\mathcal{LF}_n$ there are possibly infinitely many distinct term trees, even if changes in variable names are take into account. For example, the term $C(x_1 x_2 \cdots x_m)$ for each m is not reducible using the identity $\mathbf{v}_n \approx \mathbf{v}_{n,1} \cdots \mathbf{v}_{n,n}$ for $n \geq 2$. In the variety $\mathcal{F}_n$

however, we may always assume that the length of a maximal semilattice subterm of a closed term is at most $n-1$. Hence, up to a permutation of variable names, there is a finite number of distinct n-atomic terms trees relative to the variety $\mathcal{F}_n$. Because of this, there is an upper bound for the number of distinct variables that can appear in an n-atomic term relative to the variety $\mathcal{F}_n$. In fact, since the longest semilattice subterms of an n-atomic term are of length $n-1$, the largest number of distinct variables occurs when every element (except the base element) of the atomic term tree $\mathcal{P}_n$ is labelled with a distinct $n-1$ letter semilattice term. Hence the maximal number of distinct variables is exactly $N(n) := (n-1)\sigma_n$.

Within a variety $\mathcal{V} \subseteq \mathcal{F}_n$ it is always possible to reduce a closed term t to one of the form $t_1 t_2 \cdots t_m$, where each t_i is an n-atomic term and such that none of the identities $t_1 t_2 \cdots t_m \approx t_1 \cdots t_{i-1} t_{i+1} \cdots t_m$ follow from the CSL axioms. A term with this property or an identity between such terms will be said to be *n-reduced*.

The first claim of Theorem 7.6 now follows from Lemma 7.7 and the following.

Theorem 7.10. *Let $\mathcal{V}$ be a subvariety of $\mathcal{F}_n$ generated by a CSL with identity and let X be a fixed alphabet of $N(n)$ variables. Then the (finite) set Σ of all n-reduced identities in the variables X that are satisfied by $\mathcal{V}$ are a basis for the identities of $\mathcal{V}$ within $\mathcal{F}_n$.*

Proof. Let $s = s_1 s_2 \cdots s_{m_1} \approx t_1 t_2 \cdots t_{m_2} = t$ be an n-reduced identity between closed terms satisfied by $\mathcal{V}$ and let ℓ be a positive integer less than or equal to m_1 and $Y := \operatorname{cont}(s_\ell)$. We show how to replace the n-atomic term s_ℓ in s by a collection of n-atomic terms appearing in t using an identity from the set Σ defined in the statement of the theorem. Since the choice of ℓ is arbitrary, this shows that after m_1 applications of this procedure we can replace all n-atomic terms in the product $s_1 s_2 \cdots s_m$ by some of the n-atomic terms in t, showing that $\Sigma \vdash s \geq t$ modulo the CSL axioms. A dual argument then shows that $\Sigma \vdash t \geq s$ from which we can deduce that $\Sigma \vdash s \approx t$ as required.

Let $I_s := \{i \leq m_1 : \operatorname{cont}(s_i) \cap Y = \emptyset\}$, $J_s := \{i \leq m_1 : \operatorname{cont}(s_i) \cap Y \neq \emptyset$ and $\operatorname{cont}(s_i) \not\subseteq Y\}$ and $K_s := \{i \leq m_1 : \operatorname{cont}(s_i) \subseteq Y\}$. Let I_t, J_t and K_t be defined likewise for t but relative to the same set $Y = \operatorname{cont}(s_\ell)$. The CSL laws easily allow us to arrange $s_1 s_2 \cdots s_{m_1} \approx t_1 t_2 \cdots t_{m_2}$ as

$$\Big(\prod_{i\in I_s} s_i\Big)\Big(\prod_{i\in J_s} s_i\Big)\Big(\prod_{i\in K_s} s_i\Big) \approx \Big(\prod_{i\in I_t} t_i\Big)\Big(\prod_{i\in J_t} t_i\Big)\Big(\prod_{i\in K_t} t_i\Big).$$

Now apply the substitution θ given by $\theta_{\operatorname{cont}(s)\setminus Y}$ to both sides of this identity. Notice that $\theta\big(\prod_{i\in I_s} s_i\big)$ and $\theta\big(\prod_{i\in I_t} t_i\big)$ are both equal to 1. Therefore $\theta(s) \approx \theta(t)$ reduces to

$$\theta\Big(\prod_{i\in J_s} s_i\Big) \times \Big(\prod_{i\in K_s} s_i\Big) \approx \theta\Big(\prod_{i\in J_t} t_i\Big) \times \Big(\prod_{i\in K_t} t_i\Big).$$

Denote $\theta\left(\prod_{i\in J_s} s_i\right)$ by s'_ℓ and $\theta\left(\prod_{i\in J_t} t_i\right)$ by t'_ℓ. Since $s \approx t$ is an identity of $\mathcal{V}$, so must be the identity $s'_\ell \times \left(\prod_{i\in K_s} s_i\right) \approx t'_\ell \times \left(\prod_{i\in K_t} s_i\right)$ whose content is equal to that of s_ℓ. This identity is easily seen to be contained in Σ (after reduction and a suitable renaming of variables in Y). Now applying Lemma 7.9 we get the new law $\left(\prod_{i\in I_s} s_i\right)\left(\prod_{i\in J_s} s_i\right)\left(\prod_{i\in K_s} s_i\right) \approx \left(\prod_{i\in I_s} s_i\right)\left(\prod_{i\in J_s} s_i\right) s'_\ell \left(\prod_{i\in K_s} s_i\right) \approx \left(\prod_{i\in I_s} s_i\right)\left(\prod_{i\in J_s} s_i\right) t'_\ell \left(\prod_{i\in K_t} t_i\right)$.

This process may be performed for each choice of $\ell \leq m_1$. The combined effect of this gives a deduction from Σ of an identity $s \approx s'$ where s' is a product of n-atomic terms appearing in t (these were $\{t_i : i \in K_t\}$ in the case above) with terms that are greater than or equal to t under the usual semilattice ordering (these were $\theta(t_i) : i \in J_t\}$ in the above case). Thus $s' \geq t$. Likewise the same argument applied to the term t gives a term t' for which $\Sigma \vdash t \approx t'$ and $t' \geq s$. Hence we can deduce $s \approx s' \geq t \approx t' \geq s$ and so equality holds. Therefore every identity $s \approx t$ holding in $\mathcal{V}$ follows from the two $\mathcal{F}_n$ axioms, the (finite) set of $\mathcal{V}$-identities Σ and the set of CSL monoid axioms (also finite). □

To complete the proof of Theorem 7.6 note that Theorem 7.10 shows that each subvariety of $\mathcal{F}_n$ generated by a CSL with identity is defined by a subset of the (finite) set of all possible n-reduced identities in a fixed $N(n)$ letter alphabet. Hence there are only finitely many subvarieties of $\mathcal{F}_n$ generated by CSL monoids. In fact an upper bound for this number can be computed in terms of $N(n)$, though it is hopelessly large.

The proof of Theorem 7.10 also yields the following result.

Theorem 7.11. *If $\mathcal{V}$ is a variety of CSL's with identity then $\mathcal{V}$ is finitely generated* (*that is, generated by a finite CSL monoid*) *if and only if it is contained in $\mathcal{F}_n$ for some $n \in \mathbb{N}$.*

Proof. The "only if" direction follows since every n-element CSL is contained in $\mathcal{F}_n$. Now let $\mathcal{V}$ be a subvariety of $\mathcal{F}_n$ and let $\mathbf{S}$ be the $N(n)$-generated $\mathcal{V}$-free algebra. If $\mathbf{S} \models u \approx v$ then Theorem 7.10 implies that $u \approx v$ can be derived from the $N(n)$-variable identities of $\mathbf{S}$. However the $N(n)$-variable identities of $\mathbf{S}$ are exactly those of $\mathcal{V}$ and hence $\mathcal{V} \models u \approx v$ also. Therefore $\mathcal{V}$ is generated by its $N(n)$-generated free algebra, which is finite since $\mathcal{F}_n$ is locally finite. □

Corollary 7.12. *Every finite CSL monoid generates a finitely based variety defined by finitely many identities and with finitely many subvarieties* (*that is, a Cross variety*).

Other examples of varieties whose finitely generated varieties are all Cross are the classes of groups [18], rings [13, 15] and lattices [16]. All of these classes contain (necessarily not-finitely generated) subvarieties without a finite basis of identities (for example see [19], [3] and [16] respectively) or even uncountably many such subvarieties. This makes the following questions of interest.

Question 1. (i) Is there a CSL variety without a finite basis of identities?
(ii) Are there uncountably many varieties of CSL's?

A further question raised by the results of this section is whether or not there exists a (finitely generated) CSL variety that cannot be generated by a CSL with identity. If no such variety exists, then the results of this section immediately extend to all finitely generated CSL's.

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References

[1] A. Batbedat, *γ-demi-groups, demi-modules, produit demi-directs*, in Semigroups, Proceedings, Oberwolfalch, Germany, 1978, Lecture Notes in Mathematics 855, Springer-Verlag (1981), pp. 1–18 (French).
[2] A. Batbedat and J. B. Fountain, *Connections between left adequate semigroups and γ-semigroups*, Semigroup Forum **22** (1981), 59–65.
[3] A. Ya. Belov, *Counterexamples to the Specht problem*, Mat. Sb. **191** (2000), 13–24. (Russian); English translation in Mat. Sb. **191** (2000), 329–340.
[4] W. J. Blok, *The free closure algebra on finitely many generators*, Proc. Kon. Nederl. Akad. Wetensch. Series A **80** (1977), 362–198.
[5] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics **78**, Springer Verlag, 1980.
[6] T. Evans, *The lattice of semigroup varieties*, Semigroup Forum **2** (1971), 1–43.
[7] J. Fountain, *Adequate semigroups*, Proc. Edinburgh Math. Soc. **22** (1979), 113–125.
[8] J. Fountain, *Abundant semigroups*, Proc. London Math. Soc. **44** (1982), 103–129.
[9] B. J. Gardner and T. E. Stokes, *Closure rings*, Comment. Math. Univ. Carolin. **40** (1999), 413–427.
[10] G. M. S. Gomes and V. Gould, *Proper weakly left ample semigroups*, Internat. J. Algebra Comput. **9** (1999), 721–739.
[11] M. Jackson and T. Stokes, *An invitation to C-semigroups*, Semigroup Forum **62** (2001), 279–310.
[12] M. Jackson and T. Stokes, *Agreeable semigroups*, J. Algebra **266** (2003), 393–417.
[13] R. Kruse, *Identities satisfied in a finite ring*, J. Algebra **26** (1973), 298–318.
[14] M. Lawson, *Semigroups of ordered categories I. The reduced case.* J. Algebra **141** (1991), 422–462.
[15] I. V. L'vov, *Varieties of associative rings I*, Algebra i Logika **12** (1973), 269–297; English translation in Algebra and Logic **12** (1973), 156–167.
[16] R. McKenzie, *Equational bases for lattice theories*, Math. Scand. **27** (1970), 24–38.
[17] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, *Ann. Math.* **45** (1944), 141–191.
[18] S. Oates and M. B. Powell, *Identical relations in finite groups*, J. Algebra **1** (1964), 11–39.

[19] A. Ju. Ol'šanskiĭ, *The finite basis problem for identities in groups*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 376–384 (Russian); English translation in Math. USSR Izv. **4** (1970).
[20] R. Quackenbush, *Varieties of binary linear codes*, Algebra Universalis **42** (1999), 141–149.

Marcel Jackson
Department of Mathematics, La Trobe University, Victoria 3086, Australia
e-mail: m.g.jackson@latrobe.edu.au