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RESIDUAL BOUNDS FOR COMPACT TOTALLY DISCONNECTED ALGEBRAS

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Abstract. It is well known that every compact, totally disconnected semigroup (*Boolean topological* semigroup), group or ring $\underset{\sim}{\mathbf{A}}$ is topologically residually finite; that is, for every pair of distinct elements a, b in the underlying set A, there is a continuous homomorphism ϕ from $\underset{\sim}{\mathbf{A}}$ into a (discretely topologised) finite algebra with $\phi(a) \neq \phi(b)$. We examine the possible residual bounds for Boolean topological algebras in relation to their non-topological residual bound, with particular emphasis given to groups and completely simple semigroups. Amongst the results is the undecidability of the problem of determining if all Boolean topological models of a finite set of identities are profinite.

1. Introduction

Recall that a *Boolean space* (sometimes called a *Stone space*) is a topological space that is compact, Hausdorff and totally disconnected. A *Boolean topological algebra* is a general algebra whose underlying set is a Boolean space and whose operations are continuous. Boolean topological algebras arise in numerous settings, particularly in the study of natural dualities (see [4] and [12]) as well as the study of profinite algebras (see [2], [12] and [21] for example).

Let $\mathfrak{K}$ and $\mathfrak{J}$ be classes of algebras of the same type. We say that the members of $\mathfrak{K}$ are residually-$\mathfrak{J}$ if for every $\mathbf{A} \in \mathfrak{K}$ and every pair $a, b \in A$ with $a \neq b$ there is an algebra $\mathbf{B} \in \mathfrak{J}$ and a surjective homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ with $\phi(a) \neq \phi(b)$

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(see [9, Section II.7] for example). We may define the same notions for classes of Boolean topological algebras if we insist that ϕ is a continuous homomorphism.

For a cardinal λ, and type $\mathcal{F}$ we let $\mathcal{F}_\lambda$ denote the class of all $\mathcal{F}$-algebras (that is, algebras of type $\mathcal{F}$) of cardinality less than λ, and $\underset{\sim}{\mathcal{F}}_\lambda$ the class of all Boolean topological $\mathcal{F}$-algebras of cardinality less than λ. We say that a class $\mathbfcal{K}$ of $\mathcal{F}$-algebras [Boolean topological $\mathcal{F}$-algebras] is residually-λ, or has *residual bound* λ, if $\mathbfcal{K}$ is residually-$\mathcal{F}_\lambda$ [residually-$\underset{\sim}{\mathcal{F}}_\lambda$, respectively]. If $\mathbfcal{K}$ is residually-$\aleph_0$, then we also say that $\mathbfcal{K}$ is *residually finite*. The class $\mathbfcal{K}$ is *residually very finite* if it is residually-n for some $n \in \mathbb{N}$.

If a class $\mathbfcal{K}$ of algebras or of Boolean topological algebras is residually-α for some cardinal α, we will let $\operatorname{rb}(\mathbfcal{K})$ denote the smallest residual bound for $\mathbfcal{K}$; that is, the smallest cardinal λ such that $\mathbfcal{K}$ is residually-λ. Otherwise we write $\operatorname{rb}(\mathbfcal{K}) = \infty$. In the case where $\mathbfcal{K}$ is a singleton with member $\mathbf{A}$, we also write $\operatorname{rb}(\mathbf{A})$ and say that $\mathbf{A}$ is residually-λ, and so on.

Residual bounds of algebras and classes of algebras are of deep importance in algebra and have been well studied. Residual finiteness in particular has many connections with other finiteness properties and algorithmic decidability (see [14, Connection 2.7] for example). Topological residual bounds are of similar interest; we give some applications and connections in Section 2.

The ability to admit a compatible Boolean topology is quite a restrictive property and algebras satisfying it tend to be quite well behaved from the perspective of residual bounds. As an example, we note that if $\mathbfcal{K}$ is the class of Boolean topological semigroups [groups, rings] then $\operatorname{rb}(\mathbfcal{K}) = \aleph_0$, while the class of all non-topological semigroups [groups, rings respectively] have no residual bound. More generally, Taylor [23] has shown that if $\mathbfcal{K}$ is a class of Boolean topological algebras of finite type, then $\operatorname{rb}(\mathbfcal{K}) \leq \left(2^{\aleph_0}\right)^+$ (the successor cardinal of $2^{\aleph_0}$). On the other hand, examples of non-topologically residually finite Boolean topological algebras are known, including a countable Boolean topological modular lattice found by Clinkenbeard [8] (a number of further examples can be found in [5] and [12]).

In this paper, we concentrate on investigating the following two questions:

Question 1. *Let* $\underset{\sim}{\mathbf{X}}$ *be a Boolean topological algebra, and* $\mathbf{X}$ *its underlying non-topological algebra.*

(1) *What can be said about* $\operatorname{rb}(\underset{\sim}{\mathbf{X}})$*?*
(2) *What is the relationship between* $\operatorname{rb}(\underset{\sim}{\mathbf{X}})$ *and* $\operatorname{rb}(\mathbf{X})$*?*

Of the two questions, the first has received the most attention in the literature. The second question relates to the notion of standardness introduced in [6], and has apparent connections with finite axiomatisability of quasivarieties. For part (2), it is trivial that $\mathrm{rb}(\mathbf{X}) \leq \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq |X|^+$ (the successor cardinal of $|X|$) and so we will mostly be interested in finding when $\mathrm{rb}(\mathbf{X})$ can be very small (such as finite, or $\aleph_0$) compared to $\mathrm{rb}(\underset{\sim}{\mathbf{X}})$.

We begin with two preliminary sections which introduce notation, important concepts, and establish the interplay between the concepts encountered in this article. In particular, Section 2 illustrates the connection between Question 1 and the problems of axiomatising topological quasivarieties and with characterising profinite algebras amongst Boolean topological algebras. Section 3 recalls some very general conditions under which we can guarantee a countable upper bound for the residual bound of the topological algebras from a class.

We embark on the main part of the article in Section 4, where we investigate the possible upper values for the residual bounds of Boolean topological algebras (Section 4). Upper bounds for $\mathrm{rb}(\underset{\sim}{\mathbf{X}})$ were given by Taylor [23] and we give a number of examples illustrating the sharpness of these. Here we are able to use easy examples of unary algebras (algebras whose operations are of arity 1) to demonstrate our results.

The case of finite type is somewhat more elusive, but in Section 5 we give an example that lies toward the extreme end of possibilities with regards to Question 1(2): this example (with a choice of either one or two binary operations) has algebraic residual bound 5, but smallest topological residual bound $\aleph_1$. The example also solves a problem from [6].

We then (Section 6) show for any cardinal $\mathfrak{n}$ between 3 and $(2^{\aleph_0})^+$, the problem of determining for a given finite set Σ of identities, whether or not the class of Boolean topological models of Σ have residual bound $\mathfrak{n}$ is undecidable.

To finish, we turn our attention to the more familiar class of completely simple semigroups (a class that includes the variety of groups). Here the answer of Question 1(1) is well known; $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$. Our main result is to provide a reasonable answer to Question 1(2). We find many Boolean topological groups $\underset{\sim}{\mathbf{G}}$ for which $\mathrm{rb}(\mathbf{G})$ is finite but $\mathrm{rb}(\underset{\sim}{\mathbf{G}}) = \aleph_0$. More precisely, we show that in the quasivariety of any finite group $\mathbf{G}$ containing a non-abelian Sylow subgroup, there is a (necessarily infinite) residually-$(|G|+1)$ group admitting a Boolean topology giving it smallest residual bound $\aleph_0$ (see Theorem 7.4 and Theorem 7.12). These results can be contrasted with the behaviour of some other classes such as lattices, where a finite algebraic residual bound implies that the algebraic and topological residual bounds coincide (Corollary 3.8).

2. Preliminaries: Boolean topological algebras, profiniteness and standardness

While questions regarding residual bounds for topological algebras are of interest in their own right, a central motivation for the investigations in this article is the notion of standardness, as introduced in [6]. Very roughly, a class of Boolean topological algebras is standard if it admits a purely algebraic axiomatisation. To make the notion precise and to establish the link with residual bounds, we need some notation. For proofs of the elementary model theoretic facts we refer the reader to a text such as [3] or [9]. Appendix B of [4] contains some useful facts about Boolean spaces; refer to a topology text such as [13] for more general topological facts.

We begin by weakening our notion of residually-$\boldsymbol{\mathcal{K}}$. We say that a [Boolean topological] algebra $\mathbf{X}$ is *weakly residually-*$\boldsymbol{\mathcal{K}}$ if we can separate elements of X by [continuous] homomorphisms from $\mathbf{X}$ into (rather than onto) members of $\boldsymbol{\mathcal{K}}$. If $\boldsymbol{\mathcal{K}}$ is hereditary (closed under taking subalgebras, or closed subalgebras in the topological case) then the properties of being residually-$\boldsymbol{\mathcal{K}}$ and weakly residually-$\boldsymbol{\mathcal{K}}$ coincide (in the topological case, this depends on the fact that a continuous map between two Boolean topological spaces is a closed map). So from the perspective of calculating the residual bound of an algebra, we can use either of the weak or the usual notion.

In the following well known lemma, we let $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ denote respectively the class operators corresponding to taking isomorphic copies, subalgebras and direct products.

Lemma 2.1. *Let* $\boldsymbol{\mathcal{K}}$ *be a class of algebras of some type* $\mathcal{F}$. *The following are equivalent for an* $\mathcal{F}$*-algebra* $\mathbf{X}$:

1. $\mathbf{X}$ *is weakly residually-*$\boldsymbol{\mathcal{K}}$;
2. $\mathbf{X}$ *is residually-*$\mathbb{S}(\boldsymbol{\mathcal{K}})$;
3. $\mathbf{X} \in \mathbb{ISP}(\boldsymbol{\mathcal{K}})$.

Proof. (1)⇔(2) is discussed above. For (3)⇒(2) use projection maps, while (2)⇒(3) is essentially [3, Theorem II.7.15]. □

We now wish to observe a similar statement for Boolean topological algebras. We let $\mathbb{S}_c$ denote the class operator corresponding to taking closed subalgebras. The class operators $\mathbb{I}$ and $\mathbb{P}$ are as before, however we now interpret them in the category of Boolean topological algebras. So direct products inherit the product topology and isomorphisms are maps that are simultaneously algebraic isomorphisms and topological homeomorphisms. The class operator $\mathbb{P}_{\text{fin}}$ is the restriction of $\mathbb{P}$ to finitary direct products. Importantly, the class of Boolean topological spaces is closed under the operators $\mathbb{I}$, $\mathbb{S}_c$, $\mathbb{P}$ and $\mathbb{P}_{\text{fin}}$.

Before we state a Boolean topological version of Lemma 2.1, we recall one further notion. An *inverse system* $\boldsymbol{\mathcal{A}}$ of algebras is a family $\{\mathbf{A}_i \mid i \in \mathbf{D}\}$ of similar algebras indexed by a directed set $\mathbf{D} = \langle D; \leq\rangle$ (for technical reasons we allow the possibility that D is empty, giving an empty inverse system), together with a homomorphism (continuous, if the $\mathbf{A}_i$ are topological algebras) $\phi_{i,j} : \mathbf{A}_i \to \mathbf{A}_j$ whenever $i \geq j$ in D and such that $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ whenever $i \geq j \geq k$. The concrete construction for the *inverse limit* (or *projective limit*) $\varprojlim \boldsymbol{\mathcal{A}}$ of the inverse system $\boldsymbol{\mathcal{A}}$ is the algebra on the subuniverse of $\prod_{i\in D} \mathbf{A}_i$ consisting of the elements $\{x \mid i \geq j \Rightarrow \phi_{i,j}(x(i)) = x(j)\}$. In the case when the $\mathbf{A}_i$ are Boolean topological algebras (for example, if they are finite and discretely topologised), then $\varprojlim \boldsymbol{\mathcal{A}}$ is a topologically closed (and non-empty) subalgebra of $\prod_{i\in D} \mathbf{A}_i$ and hence is a Boolean topological algebra. The one element algebra is the inverse limit of an empty inverse system.

For a class of topological algebras $\boldsymbol{\mathcal{K}}$, we use the notation $\mathbb{I}\text{nv}(\boldsymbol{\mathcal{K}})$ to denote the class of topological algebras arising as isomorphic copies of inverse limits over members of $\boldsymbol{\mathcal{K}}$. We let $\mathbb{P}\text{ro}(\boldsymbol{\mathcal{K}})$ denote the members of $\boldsymbol{\mathcal{K}}$ that are isomorphic to inverse limits of finite (discrete) algebras; these are usually called the *profinite* members of $\boldsymbol{\mathcal{K}}$.

The following is a Boolean topological version of Lemma 2.1.

Lemma 2.2. *Let $\boldsymbol{\mathcal{K}}$ be a class of Boolean topological algebras of some type $\mathcal{F}$. The following are equivalent for a Boolean topological algebra $\underset{\sim}{\mathbf{X}}$ of type $\mathcal{F}$:*

(1) $\underset{\sim}{\mathbf{X}}$ *is weakly residually-*$\boldsymbol{\mathcal{K}}$;
(2) $\underset{\sim}{\mathbf{X}}$ *is residually-*$\mathbb{S}_c(\boldsymbol{\mathcal{K}})$;
(3) $\underset{\sim}{\mathbf{X}} \in \mathbb{I}\mathbb{S}_c\mathbb{P}(\boldsymbol{\mathcal{K}})$;
(4) $\underset{\sim}{\mathbf{X}} \in \mathbb{I}\text{nv}(\mathbb{I}\mathbb{S}_c\mathbb{P}_{\text{fin}}(\boldsymbol{\mathcal{K}}))$.

Proof. (1)⇔(2)⇔(3) is proved in essentially the same way as Lemma 2.1, except in the category of Boolean topological algebras. The equivalence (3)⇒(4) is also well known; a full proof in the more general setting of structures is given in [7, Lemma 2.3] for example. □

Lemmas 2.1 and 2.2 offer a new perspective to Question 1. We are interested in the relationship between $\text{rb}(\underset{\sim}{\mathbf{X}})$ and $\text{rb}(\mathbf{X})$, but these lemmas invite us to examine the relationship between $\mathbb{I}\mathbb{S}\mathbb{P}$-classes and $\mathbb{I}\mathbb{S}_c\mathbb{P}$-classes. To make this more precise, we introduce some notation. For a class $\mathscr{J}$ of algebras (possibly containing infinite members), we write $\mathscr{J}_{\mathcal{T}}$ (or sometimes $[\mathscr{J}]_{\mathcal{T}}$ to avoid ambiguity) to denote the Boolean topological algebras whose underlying algebra lies in $\mathscr{J}$. So for example, $\mathbb{IS}_c\mathbb{P}(\mathscr{J}_{\mathcal{T}}) \subseteq [\mathbb{ISP}(\mathscr{J})]_{\mathcal{T}}$, since the left hand side consists of Boolean topological algebras whose underlying algebra lies in $\mathbb{ISP}(\mathscr{J})$, while the right hand side consists of all such objects. We let $\mathscr{J}_{\mathrm{fin}}$ denote the finite members of $\mathscr{J}$.

Let us consider a Boolean topological algebra $\underset{\sim}{\mathbf{X}}$; as usual we let $\mathbf{X}$ denote the underlying algebra of $\underset{\sim}{\mathbf{X}}$. Let $\mathscr{K}$ denote some class of Boolean topological algebras of the same type as $\underset{\sim}{\mathbf{X}}$, with the class of underlying algebras denoted by $\mathscr{K}^\flat$ (note that it is possible that $\mathscr{K} \subsetneq [\mathscr{K}^\flat]_{\mathcal{T}}$ because there may be more than one way for a member of $\mathscr{K}^\flat$ to be given a compatible Boolean topology). For notational convenience we assume that $\mathscr{K}$ and $\mathscr{K}^\flat$ are hereditary. Now Lemma 2.1 shows that $\mathbf{X}$ is residually-$\mathscr{K}^\flat$ if and only if $\underset{\sim}{\mathbf{X}} \in [\mathbb{ISP}(\mathscr{K}^\flat)]_{\mathcal{T}}$, while Lemma 2.2 shows that $\underset{\sim}{\mathbf{X}}$ is residually-$\mathscr{K}$ if and only if $\underset{\sim}{\mathbf{X}} \in \mathbb{IS}_c\mathbb{P}(\mathscr{K})$. Hence any discrepancy between the residual properties of $\underset{\sim}{\mathbf{X}}$ and those of $\mathbf{X}$ can be captured by the inequality $\mathbb{IS}_c\mathbb{P}(\mathscr{K}) \subsetneq [\mathbb{ISP}(\mathscr{K}^\flat)]_{\mathcal{T}}$ (for suitable $\mathscr{K}$). The possible inequality of these two classes is precisely the kind of issue that concerns the topic of standardness as introduced in [6].

Recall that a *quasi-identity* is a universally quantified formula of the form

$$\mathop{\&}_{1\leq i\leq n} p_i \approx q_i \rightarrow p_0 \approx q_0$$

for some non-negative integer n. (We interpret the case that $n = 0$ as the *identity* $p_0 \approx q_0$.) Classes defined by quasi-identities are known as *quasivarieties*, while classes defined by identities are called *varieties*. We let $\mathrm{Th}_{\mathrm{qi}}(\mathscr{K})$ denote the quasi-equational theory of $\mathscr{K}$; that is the set of quasi-identities satisfied by $\mathscr{K}$. In general, when Σ is a set of first order sentences, we let $\mathrm{Mod}(\Sigma)$ denote the set of all models of Σ. The notation $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ abbreviates $[\mathrm{Mod}(\Sigma)]_{\mathcal{T}}$, the *Boolean topological models of* Σ.

In the case when $\mathscr{K}$ is a finite set of finite algebras, a famous characterisation due to Maltsev shows that the class $\mathbb{ISP}(\mathscr{K})$ is equal to the quasivariety $\mathrm{Mod}(\mathrm{Th}_{\mathrm{qi}}(\mathscr{K}))$ generated by $\mathscr{K}$. Note that we have $[\mathbb{ISP}(\mathscr{K})]_{\mathcal{T}} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{qi}}(\mathscr{K}))$ and so the possible inequality of $\mathbb{IS}_c\mathbb{P}(\mathscr{K}_{\mathcal{T}}) \subsetneq [\mathbb{ISP}(\mathscr{K})]_{\mathcal{T}}$ translates to the natural question of whether or not $\mathbb{IS}_c\mathbb{P}(\mathscr{K}_{\mathcal{T}})$ is equal to the class of *all* Boolean topological models of $\mathrm{Th}_{\mathrm{qi}}(\mathscr{K})$—in symbols $\mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{qi}}(\mathscr{K}))$. If $\mathbb{IS}_c\mathbb{P}(\mathscr{K}_{\mathcal{T}}) = [\mathbb{ISP}(\mathscr{K})]_{\mathcal{T}}$, then we have a direct analogue of the Maltsev result, and can utilise all of the usual benefits of an axiomatic description. We say that $\mathscr{K}$ (or $\mathscr{K}_{\mathcal{T}}$ or $\mathbb{IS}_c\mathbb{P}(\mathscr{K}_{\mathcal{T}})$)

is *standard* if $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{qi}}(\mathcal{K}))$; in other words, if the standard axiomatic description of the quasivariety of $\mathcal{K}$ also describes $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$. When $\underset{\sim}{\mathbf{X}}$ is a Boolean topological algebra of finite type and $\mathrm{rb}(\mathbf{X})$ is finite then there is a minimal finite class $\mathcal{K}$ of finite quotients of $\mathbf{X}$ for which $\mathbf{X}$ is residually-$\mathcal{K}$; whence $\mathbf{X} \in \mathbb{ISP}(\mathcal{K})$ by Lemma 2.1. If $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) > \mathrm{rb}(\mathbf{X})$ then $\mathcal{K}$ is non-standard because Lemma 2.2 shows that $\underset{\sim}{\mathbf{X}} \notin \mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$. Of particular interest is if $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \geq \aleph_0$, for then any finite class of finite algebras that contains $\mathcal{K}$ is non-standard.

(We note that in [6] and [7] the notion of standardness is defined in terms of the universal Horn class generated by $\mathcal{K}$ instead of quasivariety generated by $\mathcal{K}$. These two classes differ by at most a 1-element algebra, a distinction that is useful in certain contexts, but not here. Importantly, the two definitions of standardness coincide.)

The restriction to finite $\mathcal{K}$ is not all that artificial. One of the main motivations of the notion of standardness comes from the theory of natural dualities, where one attempts to construct, in a canonical fashion, a dual equivalence between a quasivariety and a class such as $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}})$, where $\underset{\sim}{\mathbf{M}}$ is a single finite discretely topologised algebra; see [4]. Even in this situation, the question of standardness appears very difficult to answer. (More generally, the theory of natural dualities allows $\underset{\sim}{\mathbf{M}}$ to also have relations and partial operations in its type.)

The case where $\mathcal{K}$ consists of finitely many finite algebras also arises whenever we have an algebra of finite type and finite residual bound. Indeed, if $\mathcal{F}$ is a finite type and n is a positive integer then the class $\mathcal{F}_n$ is finite (up to isomorphisms).

Now let us move to a situation when $\mathcal{K}$ contains an infinite algebra, or consists of infinitely many finite algebras. In this case, the class $\mathbb{ISP}(\mathcal{K})$ does not necessarily admit any first order axiomatisation. In fact, the Maltsev characterisation of the quasivariety of $\mathcal{K}$ becomes: $\mathrm{Mod}(\mathrm{Th}_{\mathrm{qi}}(\mathcal{K})) = \mathbb{ISPP}_{\mathrm{u}}(\mathcal{K})$, where $\mathbb{P}_{\mathrm{u}}$ is the class operator corresponding to taking ultraproducts. Because of this, it might seem rather too optimistic to hope for an axiomatic characterisation of $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$. However it turns out that non-trivial ultraproducts frequently fail to admit any compatible compact (let alone Boolean) topology[1] and the class $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ exhibits properties that are often very similar to $[\mathrm{Mod}(\mathrm{Th}_{\mathrm{qi}}(\mathcal{K}))]_{\mathcal{T}}$. For example, if $\mathcal{G}$ denotes the class of groups and Γ is the usual set of group axioms, then all of the following equalities follow from well known properties of Boolean topological

[1]This follows, for example, from the fact that a compact topological algebra of finite type is residually-$\left((2^{\aleph_0})^{\aleph_0}\right)^+$ in the category of compact topological algebras [23, Theorem 7.7], however there are simple algebras of arbitrary cardinality that arise as ultraproducts of finite simple algebras of finite type.

groups (see [21] for example):
(2.1)

$$\mathbb{IS}_c\mathbb{P}([\mathcal{G}_{\text{fin}}]_{\mathcal{T}}) = [\mathbb{ISP}(\mathcal{G}_{\text{fin}})]_{\mathcal{T}} = [\mathbb{ISPP}_u(\mathcal{G}_{\text{fin}})]_{\mathcal{T}} = \text{Mod}_{\mathcal{T}}(\text{Th}_{\text{qi}}(\mathcal{G}_{\text{fin}})) = \text{Mod}_{\mathcal{T}}(\Gamma).$$

This is despite the fact that (without topology) we have:

$$\mathbb{ISP}(\mathcal{G}_{\text{fin}}) \subsetneq \mathbb{ISPP}_u(\mathcal{G}_{\text{fin}}) = \text{Mod}(\text{Th}_{\text{qi}}(\mathcal{G}_{\text{fin}})) \subsetneq \text{Mod}(\Gamma).$$

In contrast, if $\mathcal{L}$ denotes the class of lattices, Λ the usual lattice axioms and $\mathcal{U}$ the class of unars (algebras with a single unary operation), then

$$\mathbb{IS}_c\mathbb{P}([\mathcal{L}_{\text{fin}}]_{\mathcal{T}}) \subseteq [\mathbb{ISP}(\mathcal{L}_{\text{fin}})]_{\mathcal{T}} \subsetneq \text{Mod}_{\mathcal{T}}(\Lambda) \tag{2.2}$$

by the Clinkenbeard example [8], while it follows from results in [7] that for every set of sentences Σ we have

$$\text{Mod}_{\mathcal{T}}(\Sigma) \neq \mathbb{IS}_c\mathbb{P}([\mathcal{U}_{\text{fin}}]_{\mathcal{T}}) \subsetneq [\mathbb{ISP}(\mathcal{U}_{\text{fin}})]_{\mathcal{T}} \subsetneq [\mathcal{U}]_{\mathcal{T}}. \tag{2.3}$$

We note that it is not known if the inclusion $\mathbb{IS}_c\mathbb{P}([\mathcal{L}_{\text{fin}}]_{\mathcal{T}}) \subseteq [\mathbb{ISP}(\mathcal{L}_{\text{fin}})]_{\mathcal{T}}$ is strict. The following definitions capture some of the possibilities. (As we have allowed empty direct products, our definitions are subtly different from those given in [7], however they coincide if we replace $\mathcal{K}$ by $\mathcal{K} \cup \{\mathbf{1}\}$, where $\mathbf{1}$ denotes the one element algebra of appropriate type.)

Definition 1. [7] Let $\mathcal{K}$ be a class of finite structures of the same similarity type.

1. $\mathcal{K}$ is *pre-standard* if $[\mathbb{ISP}(\mathcal{K})]_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$.
2. $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ is *first order definable* (with respect to Σ) if there is a set Σ of first order formulæ such that $\text{Mod}_{\mathcal{T}}(\Sigma) = \mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$.
3. $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ is *universally definable* if it is first order definable with respect to a set of universal sentences.
4. $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ is *q-definable* if it is first order definable with respect to a set of quasi-identities.[2]

The equalities (2.1) show that the class $\mathcal{G}_{\text{fin}}$ is pre-standard and $\mathbb{IS}_c\mathbb{P}([\mathcal{G}_{\text{fin}}]_{\mathcal{T}})$ is q-definable (and so universally definable and first order definable as well). This is also known to be true of the class of finite semigroups and of finite rings. The inequalities in (2.3), show that $\mathbb{IS}_c\mathbb{P}([\mathcal{U}_{\text{fin}}]_{\mathcal{T}})$ is neither pre-standard nor even first order definable. The inequalities in (2.2) leave unsolved the problem of whether or not the class $\mathcal{L}_{\text{fin}}$ is pre-standard or first order definable. It is shown in [7] that

[2]The notion of q-definable is not given in [7], however there is an almost identical definition corresponding to *universal Horn formulæ*, the natural extension of quasi-identities to classes closed under taking isomorphic copies of subalgebras of *non-empty* direct products of ultraproducts.

every finite discrete lattice generates an $\mathbb{IS}_c\mathbb{P}$-class that is first order definable, although not necessarily pre-standard or universally definable.

The following result details the known interdependencies between the properties just described.

Proposition 2.3. [7] *Let $\mathcal{K}$ be a class of finite algebras of the same type. Then the following are related by*

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \begin{cases} \text{(iii)} \\ \text{(iv)} \Leftrightarrow \text{(v)} \Leftrightarrow \text{(vi)} \end{cases} :$$

(i) $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_\mathcal{T})$ *is q-definable*;
(ii) $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_\mathcal{T})$ *is universally definable*;
(iii) $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_\mathcal{T})$ *is first order definable*;
(iv) $\mathcal{K}_\mathcal{T}$ *is pre-standard*;
(v) $[\mathbb{ISP}(\mathcal{K})]_\mathcal{T} = \mathbb{I}\mathrm{nv}(\mathbb{ISP}_{\mathrm{fin}}(\mathcal{K}))$;
(vi) $[\mathbb{ISP}(\mathcal{K})]_\mathcal{T}$ *is residually-*$\mathbb{S}_c(\mathcal{K}_\mathcal{T})$.

When $\mathcal{K}$ is also finite, then (i), (ii), (iv), (v) and (vi) are all equivalent to $\mathcal{K}$ standard and all imply (iii).

As already mentioned, condition (iii) does not in general imply any of the other conditions in Proposition 2.3.

The following example illustrates some consequences of the observations in this section with regards to Question 1.

Example 2.4. *If $\mathcal{V}$ is a variety then the implication* $\mathrm{rb}(\mathbf{X}) \leq \aleph_0 \Rightarrow \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ *holds for all Boolean topological algebras $\underset{\sim}{\mathbf{X}}$ in $[\mathcal{V}]_\mathcal{T}$ if and only if $\mathcal{V}_{\mathrm{fin}}$ is pre-standard.*

Proof. For the "if" direction, assume that $\mathcal{V}_{\mathrm{fin}}$ is pre-standard and that $\mathrm{rb}(\mathbf{X}) \leq \aleph_0$. So $\mathbf{X} \in \mathbb{ISP}(\mathcal{V}_{\mathrm{fin}})$ by Lemma 2.1 and then $\underset{\sim}{\mathbf{X}} \in [\mathbb{ISP}(\mathcal{V}_{\mathrm{fin}})]_\mathcal{T}$. By pre-standardness we have $[\mathbb{ISP}(\mathcal{V}_{\mathrm{fin}})]_\mathcal{T} = \mathbb{IS}_c\mathbb{P}([\mathcal{V}_{\mathrm{fin}}]_\mathcal{T})$ and then by Lemma 2.2 we have $\underset{\sim}{\mathbf{X}}$ is residually-$[\mathcal{V}_{\mathrm{fin}}]_\mathcal{T}$. So $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ as well.

For the "only if" direction, assume that the implication $\mathrm{rb}(\mathbf{X}) \leq \aleph_0 \Rightarrow \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ holds and let $\underset{\sim}{\mathbf{Y}} \in [\mathbb{ISP}(\mathcal{V}_{\mathrm{fin}})]_\mathcal{T}$. So $\mathbf{Y} \in \mathbb{ISP}(\mathcal{V}_{\mathrm{fin}})$ and then $\mathrm{rb}(\mathbf{Y}) \leq \aleph_0$ by Lemma 2.1. Hence $\mathrm{rb}(\underset{\sim}{\mathbf{Y}}) \leq \aleph_0$ and so $\underset{\sim}{\mathbf{Y}}$ is residually-$[\mathcal{V}_{\mathrm{fin}}]_\mathcal{T}$ (as all finite Boolean images of $\underset{\sim}{\mathbf{Y}}$ are in $[\mathcal{V}_{\mathrm{fin}}]_\mathcal{T}$). Hence $\underset{\sim}{\mathbf{Y}} \in \mathbb{IS}_c\mathbb{P}([\mathcal{V}_{\mathrm{fin}}]_\mathcal{T})$ by Lemma 2.2. □

We will say that a finite algebra $\mathbf{M}$ is *inherently non-standard* if whenever $\mathcal{K}$ is a finite class of finite algebras with $\mathbf{M} \in \mathbb{ISP}(\mathcal{K})$ then $\mathcal{K}$ is non-standard. The following lemma is obvious.

Lemma 2.5. *Let* $\mathbf{M}$ *be a finite algebra. If* $\mathrm{rb}([\mathbb{ISP}(\mathbf{M})]_{\mathcal{T}}) \geq \aleph_0$ *then* $\mathbf{M}$ *is inherently non-standard.*

This article contains the first known examples of inherently non-standard algebras.

3. Preliminaries: Topological residual finiteness and finitely determined syntactic congruences

The most general way of guaranteeing that the Boolean topological algebras in a variety are topologically residually finite is via the notion of finitely determined syntactic congruences.

Let $\mathcal{L}$ be a type of algebras and let T_x denote the set of all $\mathcal{L}$-terms with a distinguished variable x. For any $\mathcal{L}$-algebra $\mathbf{X}$, equivalence θ on X, and subset $F \subseteq T_x$ we can define an equivalence θ_F as follows:

$$a\,\theta_F\,b \text{ if for all } t(x,\vec{z}) \in T_x \text{ and } \vec{c} \text{ in } X \text{ we have } t(a,\vec{c})\,\theta\,t(b,\vec{c}).$$

In the case where $F = T_x$ this is a congruence called the *syntactic congruence* and is denoted $\mathrm{syn}(\theta)$. The syntactic congruence is the largest congruence contained in θ.

Let $\mathbf{X}$ be an algebra. If there is a finite subset $F \subseteq T_x$ such that for each equivalence θ on X we have $\theta_F = \mathrm{syn}(\theta)$, then we say that $\mathbf{X}$ has *finitely determined syntactic congruences* (FDSC). We say that a class of algebras $\boldsymbol{\mathcal{K}}$ has FDSC if there is a finite set F such that each member of $\boldsymbol{\mathcal{K}}$ has FDSC with respect to F. Many familiar varieties have FDSC. The following examples are surveyed or proved in [5].

Example 3.1. *The following varieties have FDSC*: *semigroups*; *groups*; *rings*; *the variety generated by a finite unary algebra*; *a finitely generated congruence distributive variety* (*for example, a finitely generated variety of lattices*); *a variety with definable principal congruences.*

The concept of FDSC and its connection with topological residual finiteness has been discovered by a number of authors (see [12, Section VI.2] for further details and early developers) and is extensively developed in [5]. The fundamental connection is given in the following (see [5] or [12, Lemma VI.2.7] for proof); here a *clopen equivalence* is an equivalence whose classes are clopen sets.

Clopen Equivalence Lemma 3.2. *Let* $\underset{\sim}{\mathbf{X}} = \langle X; G, \mathcal{T}\rangle$ *be a Boolean topological algebra and let* θ *be a clopen equivalence relation on* X. *If* $F \subseteq T_x$ *is a finite set of terms of* $\mathbf{X}$, *then* θ_F *is also a clopen equivalence relation on* X.

Of course, in a Boolean topological algebra, there is a plentiful supply of clopen equivalences; in particular every pair $a \neq b$ in X is separated by some clopen equivalence. When the set F in the Clopen Equivalence Lemma is sufficient to determine syntactic congruences on $\underset{\sim}{\mathbf{X}}$, then the clopen equivalence θ_F is a congruence, and the corresponding quotient map provides a continuous morphism onto a finite discretely topologised algebra. We get the following result (see [5] for example).

Theorem 3.3. *If $\underset{\sim}{\mathbf{X}}$ is a Boolean topological algebra with FDSC, then* $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$.

Theorem 3.3 and Example 3.1 give us the following.

Example 3.4. *Let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra. If $\mathbf{X}$ is contained in any of the following classes, then $\underset{\sim}{\mathbf{X}}$ is* (*topologically*) *residually finite*: *semigroups*; *groups*; *rings*; *a variety generated by a finite unary algebra*; *a finitely generated congruence distributive variety*; *a variety with definable principal congruences.*

Recall that $\mathbb{H}$ denotes the class operator corresponding to taking homomorphic images and that the variety generated by a class $\mathcal{K}$ is equal to $\mathbb{HSP}(\mathcal{K})$. The following result follows easily from Theorem 3.3.

FDSC-HSP Theorem 3.5. [5] *Let $\mathcal{K}$ be a class of finite algebras for which $\mathbb{HSP}(\mathcal{K})$ has FDSC and $\mathbb{ISP}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$. If $\underset{\sim}{\mathbf{X}} \in [\mathbb{HSP}(\mathcal{K})]_{\mathcal{T}}$ then $\underset{\sim}{\mathbf{X}}$ is residually-$\mathbb{S}_{\mathrm{c}}(\mathcal{K}_{\mathcal{T}})$ and $\mathcal{K}$ is standard.*

Theorem 3.3 and the FDSC-HSP Theorem 3.5 supply us with some strong results concerning the relationships between topological and algebraic residual bounds.

Theorem 3.6. *Let $\mathcal{V}$ be any variety of finite type whose finite members each generate varieties with FDSC. Then for $\underset{\sim}{\mathbf{X}} \in [\mathcal{V}]_{\mathcal{T}}$ the following implication holds*:

$$\mathrm{rb}(\mathbf{X}) < \aleph_0 \Rightarrow \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0.$$

Proof. If $\mathrm{rb}(\mathbf{X}) = n \in \mathbb{N}$ then $\mathbf{X}$ is in the $\mathbb{ISP}$-closure of the class $\mathcal{K}$ of all members of $\mathcal{V}$ in at most n-elements. Because $\mathcal{V}$ is of finite type, we may assume that $\mathcal{K}$ is finite (by including members only up to isomorphism). Hence the variety $\mathbb{HSP}(\mathcal{K})$ generated by $\mathcal{K}$ is the variety generated by the finite member of $\mathcal{V}$ created by forming the direct product $\mathbf{D}$ of members of $\mathcal{K}$. By assumption, this variety has FDSC and contains $\mathbf{X}$. By Theorem 3.3, we have $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$. □

We will see below (Proposition 4.3) that the assumption of finite type is necessary, as is the assumption of FDSC (see Section 5). Of course, Theorem 3.3 and

Example 3.4 show that semigroups, groups and rings all satisfy the implication of Theorem 3.6. One of the main results below will be a characterisation when the inequality $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ in Theorem 3.6 is strict for groups, and more generally completely simple semigroups.

The statement $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ is known to fail in general for Boolean topological algebras amongst congruence distributive varieties or varieties of unary algebras. However, the finite algebras from these varieties have FDSC and so the implication $\mathrm{rb}(\mathbf{X}) = n \Rightarrow \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq \aleph_0$ does hold for these classes. In the case of congruence distributive varieties we can say even more; first we observe a more general result.

Theorem 3.7. *Let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra. If there is a finite class $\mathcal{K}$ of finite algebras such that*

(1) $\mathbf{X}$ *is residually-*$\mathbb{S}(\mathcal{K})$,
(2) $\mathbb{ISP}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$,
(3) $\mathbb{HSP}(\mathcal{K})$ *has FDSC*,

then $\underset{\sim}{\mathbf{X}}$ is residually-$\mathbb{S}_{\mathrm{c}}(\mathcal{K}_{\mathcal{T}})$.

Proof. We have by Lemma 2.1 that $\mathbf{X} \in \mathbb{ISP}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$. Now use the FDSC-HSP Theorem 3.5. □

Corollary 3.8. *Let $\mathcal{V}$ be a congruence distributive variety and $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra with $\mathbf{X} \in \mathcal{V}$. The following equivalence holds*:

$$\mathrm{rb}(\mathbf{X}) \leq n \Leftrightarrow \mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq n.$$

Proof. The $\Leftarrow$ direction is trivial. For the other direction, let $\mathcal{V}_n$ denote the set of members of $\mathcal{V}$ with fewer than n elements. So $\mathbf{X}$ is residually-$\mathbb{S}(\mathcal{V}_n)$. By Jónsson's Theorem (see [3, Theorem 6.8] for example), all subdirectly irreducible members of $\mathbb{HSP}(\mathcal{V}_n)$ lie in $\mathbb{HSP}_{\mathrm{u}}(\mathcal{V}_n) = \mathcal{V}_n$ and so $\mathbb{HSP}(\mathcal{V}_n) = \mathbb{ISP}(\mathcal{V}_n)$. Also, because finitely generated congruence distributive varieties have FDSC (see Example 3.1), Theorem 3.7 (with $\mathcal{K} = \mathcal{V}_n$) shows that $\underset{\sim}{\mathbf{X}}$ is residually-$[\mathcal{V}_n]_{\mathcal{T}}$. □

4. Upper bounds

The following fundamental theorem due to W. Taylor gives an upper bound for the answer to Question 1(1).

Theorem 4.1. [23] *Let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra and $\mathfrak{m}$ denote $\aleph_0$ plus the number of operations of $\underset{\sim}{\mathbf{X}}$. The following hold*:

(1) $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq (2^{\mathfrak{m}})^+$;

(2) *there is an inverse system* $\boldsymbol{\mathcal{A}}$ *of Boolean topological algebras with* $\underset{\sim}{\mathbf{X}} \cong \varprojlim \boldsymbol{\mathcal{A}}$ *and such that each algebra in* $\boldsymbol{\mathcal{A}}$ *has underlying topological space equal to a subspace of the usual generalised Cantor space on* $2^{\mathfrak{m}}$.

In this section we show that the bound $(2^{\mathfrak{m}})^+$ of Theorem 4.1 is sharp, and we also show that arbitrary topological residual bounds can be obtained even amongst topological structures whose underlying algebras are residually very finite. We are going to consider the case of finite type and of infinite type separately, because different behaviour will be observed.

Proposition 4.2. *Let* $\mathfrak{m}$ *be an infinite cardinal. There is a Boolean topological algebra* $\underset{\sim}{\mathbf{U}}_{\mathfrak{m}}$ *with* $\mathfrak{m}$ *operations and cardinality* $2^{\mathfrak{m}}$, *with* $\mathbf{U}_{\mathfrak{m}}$ *subdirectly irreducible and hence* $\mathrm{rb}(\underset{\sim}{\mathbf{U}}_{\mathfrak{m}}) = (2^{\mathfrak{m}})^+$.

Proof. Let X be a set with $|X| = \mathfrak{m}$, and define a unary algebra $\mathbf{U}_{\mathfrak{m}}$ on the set $2^{X\times\{0,1\}}$ as follows. For each $x \in X$ there are two unary operations f_x and g_x. For $t \in 2^{X\times\{0,1\}}$, we define $g_x(t)$ by $g_x(t)(y,i) := t(x,0)$ and $f_x(t)$ by

$$f_x(t)(y,i) = \begin{cases} t(y,1-i) & \text{if } x = y \\ t(y,i) & \text{otherwise.} \end{cases}$$

It is not hard to see that $\mathbf{U}_{\mathfrak{m}}$ is subdirectly irreducible with monolith $\Theta(\underline{0},\underline{1})$. Indeed, let θ be any congruence. Say that $s \neq t$ have $(s,t) \in \theta$. As $s \neq t$ there is $(x,i) \in X \times \{0,1\}$ such that $s(x,i) \neq t(x,i)$. Then $\{g_x f_x^i(s), g_x f_x^i(t)\} = \{\underline{0},\underline{1}\}$ (where f_x^0 denotes the identity function).

Now $\mathbf{U}_{\mathfrak{m}}$ also inherits a natural topology as a power of the two element discrete space $\underset{\sim}{\mathbf{2}}$. This topology is Boolean and the operations of $\mathbf{U}_{\mathfrak{m}}$ are easily seen to be continuous. □

One can achieve a similar result amongst algebraically residually finite algebras. Recall that every Boolean space is (homeomorphic to) a closed subspace of a power of $\underset{\sim}{\mathbf{2}}$. Let the *spectral magnitude* of a Boolean space $\underset{\sim}{\mathbf{X}}$ be the smallest cardinal $\mathfrak{n}$ for which $\underset{\sim}{\mathbf{X}}$ embeds as a closed subspace of $\underset{\sim}{\mathbf{2}}^Y$ for some set Y with $|Y| = \mathfrak{n}$. (This is the cardinality of the Stone dual of $\underset{\sim}{\mathbf{X}}$, or in other words, the cardinality of the set of all clopen sets in $\underset{\sim}{\mathbf{X}}$.)

Proposition 4.3. *Let* $\mathfrak{m}$ *be any infinite cardinal. There is a Boolean topological algebra* $\underset{\sim}{\mathbf{V}}_{\mathfrak{m}}$ *with* $\mathfrak{m}$ *operations and cardinality* $2^{\mathfrak{m}}$ *such that*:

(1) $\mathrm{rb}(\mathbf{V}_{\mathfrak{m}}) = 4$; *and*
(2) $\mathrm{rb}(\underset{\sim}{\mathbf{V}}_{\mathfrak{m}}) \geq \mathfrak{m}^+$ *and* $\underset{\sim}{\mathbf{V}}_{\mathfrak{m}}$ *is not residually of spectral magnitude* $\mathfrak{n}$ *for any* $\mathfrak{n} < \mathfrak{m}$.

Proof. We fix the cardinal $\mathfrak{m}$ and let S be any set of cardinality $\mathfrak{m}$. Define an algebra $\mathbf{V}$ on the set $2^S \cup \{\infty\}$ with operations $\{f_\lambda : \lambda \in S\}$ satisfying (for each $x \in 2^S$ and $\lambda, \gamma \in S$):

- $f_\lambda(\infty) = \infty$;
- $[f_\lambda(x)](\gamma) = \begin{cases} 1 - x(\gamma) & \text{if } \lambda = \gamma, \\ x(\gamma) & \text{otherwise.} \end{cases}$

Note that for each $x \in 2^S$ and $\lambda \in S$ we have $f_\lambda^{-1}(x)$ is a singleton subset of 2^S. Thus the operations $\{f_\lambda : \lambda \in S\}$ are continuous if we give this algebra the topology of the one point compactification of the discretely topologised set 2^S by the compactification point ∞. We denote the corresponding Boolean topological algebra by $\underset{\sim}{\mathbf{V}}$.

Now let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra such that there is $\phi : \underset{\sim}{\mathbf{V}} \twoheadrightarrow \underset{\sim}{\mathbf{X}}$. First observe that the topology on $\underset{\sim}{\mathbf{X}}$ is also a one point compactification of a discrete space; this can be proved directly or by using Stone duality and noting that a Boolean space is a one point compactification of a discrete space if and only if its dual Boolean algebra is generated by its atoms. We claim that either $|X| = 1$ or $|X| \geq \mathfrak{m}$. Say that $|X| \neq 1$. So there is $x \in \underset{\sim}{\mathbf{V}}$ such that $\phi(x) \neq \phi(\infty)$. Therefore $\{\phi(x)\}$ is a clopen singleton and so $\phi^{-1}(\{\phi(x)\})$ is a clopen set containing x but not ∞. In particular, $\phi^{-1}(\{\phi(x)\})$ is a finite subset of 2^S.

Let Sg denote the closure operator on an algebra returning the subalgebra generated by a given set (see [3]). On the underlying algebra $\mathbf{V}$ of $\underset{\sim}{\mathbf{V}}$ we have $\mathrm{Sg}^{\mathbf{V}}(\{x\}) \cap \phi^{-1}(\infty) = \varnothing$, because $y \in \mathrm{Sg}^{\mathbf{V}}(\{x\})$ implies that there is a finite sequence $\lambda_1, \dots, \lambda_n \in X$ such that $f_{\lambda_1} \dots f_{\lambda_n}(y) = x$, while $f_{\lambda_1} \dots f_{\lambda_n}(\infty) = \infty$. Now, as $\gamma \neq \lambda \Rightarrow f_\delta(x) \neq f_\gamma(x)$ we have $|\mathrm{Sg}^{\mathbf{V}}(x)| = \mathfrak{m}$. As the kernel of ϕ on its restriction to $\mathrm{Sg}^{\mathbf{V}}(x)$ has finite classes, we have that $|X| \geq |\phi(\mathrm{Sg}^{\mathbf{V}}(x))| = |\mathrm{Sg}^{\mathbf{V}}(x)| = \mathfrak{m}$. Therefore $|X| \geq \mathfrak{m}$.

This shows that $\mathrm{rb}(\underset{\sim}{\mathbf{V}}) \geq \mathfrak{m}^+$. To observe that $\underset{\sim}{\mathbf{V}}$ is not residually of spectral magnitude less than $\mathfrak{m}$, observe that in the above argument, $\phi(\mathrm{Sg}^{\mathbf{V}}(x))$ consists of $\mathfrak{m}$ clopen singletons in the topology on $\underset{\sim}{\mathbf{X}}$. Hence $\underset{\sim}{\mathbf{X}}$ cannot embed into $2^{\mathfrak{n}}$ for any $\mathfrak{n} < \mathfrak{m}$ (because this space has only $\mathfrak{n}$ clopen sets).

We now verify that $\mathbf{V}$ is residually-4. For each $\lambda \in S$ we let F_λ denote the 3-element algebra on $\{0, 1, \infty\}$ with operations $\{f_\gamma^{F_\lambda} : \gamma \in S\}$ defined by (where $i = 0, 1$ and $\gamma \in S$):

- $f_\gamma^{F_\lambda}(\infty) = \infty$;
- $f_\gamma^{F_\lambda}(i) = \begin{cases} 1 - i & \text{if } \lambda = \gamma, \\ i & \text{otherwise.} \end{cases}$

The algebra $\mathbf{F}_\lambda$ is easily seen to be a homomorphic image of $\mathbf{V}$ by projecting onto the λ-coordinate of 2^S and fixing ∞. Homomorphisms of this kind can be used to separate arbitrary points of $\mathbf{V}$. This shows that $\mathbf{V}$ is residually-4. $\square$

We note that the finite algebras in the variety of $\mathbf{V}$ are finite unary algebras and hence have FDSC by Example 3.1. So Theorem 3.6 fails if we relax the restriction to finite type.

Propositions 4.2 and 4.3 give a reasonable picture of the possible solutions to Question 1 parts (1) and (2) in the case of infinite type. We now examine the situation for finite type.

The following proposition is a finite type version of Proposition 4.2.

Proposition 4.4. *For every* $n \in \mathbb{N}$ *there is a Boolean topological algebra* $\underset{\sim}{\mathbf{X}}$ *with* n *operations and with* $\mathrm{rb}(\mathbf{X}) = (2^{\aleph_0})^+$.

Proof. There are many easy examples demonstrating this for small values of n (including $n = 1$). These can be extended to examples with larger numbers of operations, by simply duplicating the operations, or adding on projections. Here are some easy examples.

As observed in [5], the Ockham algebra on $2^{\mathbb{N}}$ is subdirectly irreducible and the obvious Boolean topology is compatible (so has residual bound $(2^{\aleph_0})^+$). This algebra has three operations: the usual pointwise meet and join, as well as the unary operation c given by $c(x) := \sigma(x)'$, where σ is the shift map (that is, $\sigma(x)(i) := x(i+1)$) and $'$ is the usual pointwise Boolean complement. For a Boolean topological unary algebra with (algebraic and hence topological) residual bound $(2^{\aleph_0})^+$, one can take the space $\underset{\sim}{\mathbf{2}}^{\mathbb{N}}$ with the shift map σ and the operation g defined by $g(x)(i) = x(1)$; the algebras in Proposition 4.2 are modelled on this. For a Boolean topological algebra with a single binary operation that is subdirectly irreducible as a non-topological algebra and of cardinality $2^{\aleph_0}$, let $\cdot$ be defined on $\underset{\sim}{\mathbf{2}}^{\mathbb{N}}$ by

$$[x \cdot y](i) = \begin{cases} y(i+1) & \text{if } x(i) = 0 \\ 1 & \text{if } x(i) = 1. \end{cases}$$

$\square$

The question now arises as to whether a finite type version of Proposition 4.3 is possible. While we will prove that the answer is yes, the required examples are much harder to find: for example, Theorem 3.6 shows that they cannot be found amongst semigroups, groups, rings, algebras in congruence distributive varieties or unary algebras. We postpone the construction of the desired example until

Section 5. Instead we complete this section by presenting an easier example that is residually finite as an algebra (but not residually-n for any $n \in \mathbb{N}$) but as a Boolean topological structure lies at the upper limit of possible residual bounds. We first observe some easy facts on Boolean spaces that are also useful in later sections. Recall that a Boolean space is dense-in-itself if and only if it contains no clopen singletons.

Lemma 4.5. (i) *Let $\phi : \underset{\sim}{\mathbf{X}} \twoheadrightarrow \underset{\sim}{\mathbf{Y}}$ be a continuous surjective map from a compact metric space onto a Boolean space $\underset{\sim}{\mathbf{Y}}$. Then $\underset{\sim}{\mathbf{Y}}$ is a metric space.*

(ii) *Let $\phi : \underset{\sim}{\mathbf{X}} \twoheadrightarrow \underset{\sim}{\mathbf{Y}}$ be a continuous surjective map from a compact metric space onto a dense-in-itself Boolean space $\underset{\sim}{\mathbf{Y}}$. Then $\underset{\sim}{\mathbf{Y}}$ is homeomorphic to the Cantor space $\underset{\sim}{\mathbf{2}}^{\mathbb{N}}$.*

Proof. In this proof we denote the Cantor space on $2^{\mathbb{N}}$ by C. For (i), we note that as $\underset{\sim}{\mathbf{X}}$ is a compact metric space, a well known theorem (see [13, p. 166] for example) shows that there is a continuous map from C onto $\underset{\sim}{\mathbf{X}}$. Hence there is a continuous map from C onto $\underset{\sim}{\mathbf{Y}}$.

For a Boolean space $\underset{\sim}{\mathbf{Z}}$, let $E(\underset{\sim}{\mathbf{Z}})$ denote the Stone dual, and let $D(\mathbf{B})$ denote the Stone dual of a Boolean algebra $\mathbf{B}$. By Stone duality, we have that $E(\underset{\sim}{\mathbf{Y}})$ embeds into $E(C)$, which is isomorphic to the denumerably generated free Boolean algebra. Hence $E(\underset{\sim}{\mathbf{Y}})$ is countable and so because $E(C)$ is free, there is a surjective homomorphism from $E(C)$ onto $E(\underset{\sim}{\mathbf{Y}})$. Dualising again, we find that there is a continuous injective map from $D(E(\underset{\sim}{\mathbf{Y}})) \cong \underset{\sim}{\mathbf{Y}}$ into $D(E(C)) \cong C$. Hence $\underset{\sim}{\mathbf{Y}}$ is metrisable.

Part (ii) follows from (i) because a metrisable, dense-in-itself Boolean space is homeomorphic to C. □

A standard example of a non-topologically residually finite Boolean topological algebra is the unary algebra $\underset{\sim}{\mathbf{S}} := \langle 2^{\mathbb{Z}}; \sigma, \mathcal{T}\rangle$, where σ denotes the shift on $2^{\mathbb{Z}}$ and $\mathcal{T}$ is the usual topology (see for example, [23]). As we now observe, this example gives a reasonable finite type version of Proposition 4.3.

Example 4.6. *The Boolean topological algebra $\underset{\sim}{\mathbf{S}}$ has $\mathrm{rb}(\underset{\sim}{\mathbf{S}}) = (2^{\aleph_0})^+$ but $\mathrm{rb}(\mathbf{S}) = \aleph_0$.*

Proof. We first check that $\mathbf{S}$ is residually-$\aleph_0$. Let x and y be distinct elements of $2^{\mathbb{Z}}$. We check the case where one of x and y is not periodic under σ. The case where both are periodic is similar and left to the reader.

Assume that for all $i \in \mathbb{N}$ there is no $j \in \mathbb{N}$ for which $\sigma^i(x) = \sigma^{i+j}(x)$. For each $i \in \mathbb{Z}$ let $X_i \subset 2^{\mathbb{Z}}$ denote $\{z : \sigma^i(z) = x\}$, or equivalently, $\sigma^{-i}(\{x\})$. Because the orbit of x under σ is non-periodic, it follows that the sets in the family $\{X_i : i \in \mathbb{Z}\}$ are pairwise disjoint; also $x \in X_0$. Notice also that $X := \cup_{i\in\mathbb{Z}} X_i$ is a subuniverse of $\mathbf{S}$ with the property that $\sigma^i(z) \in X$ if and only if $z \in X$ and such that $\sigma^i(X_j) = X_{i+j}$. Now because $x \neq y$ and x is non-periodic under σ, there is a number $k > 0$ such that $\sigma^{ik}(x) \neq y$ for all $i \in \mathbb{N}$. Let $\mathbf{F}_k$ denote the unary algebra on the $k+1$-element set $\mathbb{Z}/k\mathbb{Z} \cup \{\infty\}$ with the single unary operation $i \mapsto i+1 \mod k$ and $\infty \mapsto \infty$. Then the map defined by $X_i \mapsto i \mod k$ and $z \mapsto \infty$ if $z \notin X_i$ for any $i \in \mathbb{Z}$ is a homomorphism separating x and y.

Now we must check that $\mathrm{rb}(\underset{\sim}{\mathbf{S}}) = (2^{\aleph_0})^+$. It will suffice to show that all non-trivial Boolean continuous homomorphic images of $\underset{\sim}{\mathbf{S}}$ have cardinality $2^{\aleph_0}$. (The proof is slightly shorter if we assume the continuum hypothesis, but the argument we give avoids this.) We use Lemma 4.5. Let $\phi : \underset{\sim}{\mathbf{S}} \to \underset{\sim}{\mathbf{X}}$ be a continuous homomorphism into a Boolean topological unar $\underset{\sim}{\mathbf{X}}$. Because ϕ maps $\underset{\sim}{\mathbf{S}}$ onto a closed subalgebra of $\underset{\sim}{\mathbf{X}}$, we may assume that ϕ is surjective (or otherwise replace $\underset{\sim}{\mathbf{X}}$ by $\phi(\underset{\sim}{\mathbf{S}})$).

Assume $\underset{\sim}{\mathbf{X}}$ is not dense-in-itself; that is, there is an isolated point $w \in X$. We have that $\phi^{-1}(w)$ is a clopen subset of $\underset{\sim}{\mathbf{S}}$ on which ϕ is constant. There is a finite subset $F \subset \mathbb{Z}$ and a (finite) set $G \subseteq 2^F$ such that $x \in \phi^{-1}(w)$ if and only if $x{\restriction}_F \in G$. Let $z \in 2^{\mathbb{Z}}$ be arbitrary. Because F is finite, we can find $u, v \in \phi^{-1}(w)$ such that there is $k \in \mathbb{N}$ for which $\sigma^k(u){\restriction}_F = z{\restriction}_F$ and $\sigma^k(v) \in \phi^{-1}(w)$. Because $\phi(\sigma^k(u)) = \sigma^k(\phi(u)) = \sigma^k(\phi(v)) = \phi(\sigma^k(v)) = w$, it follows that $z{\restriction}_F = \sigma^k(u){\restriction}_F \in G$ and then $\phi(z) = w$. In other words, $|\phi(\underset{\sim}{\mathbf{S}})| = 1$. Hence if $\underset{\sim}{\mathbf{X}}$ is non-trivial, then $\underset{\sim}{\mathbf{X}}$ is dense-in-itself and Lemma 4.5 shows that $|X| = 2^{\aleph_0}$ as required. □

5. Inherent non-standardness for a finite algebra

We are now going to present an example that lies near the boundary of possibilities with regards to Question 1(2). We begin by constructing a four element algebra $\mathbf{F}$ and then find in its quasivariety a countably infinite algebra $\mathbf{E}$ admitting a Boolean topology making $\underset{\sim}{\mathbf{E}}$ non-topologically residually finite. Hence $\underset{\sim}{\mathbf{E}}$ satisfies $\mathrm{rb}(\mathbf{E}) = 5$ and $\mathrm{rb}(\underset{\sim}{\mathbf{E}}) = \aleph_1$. This example resolves a number of lingering questions that arise from results above and elsewhere. First, it shows that the elusive behaviour observed for algebras of infinite type in Proposition 4.3 can be found amongst algebras of finite type. Second, it shows that $\mathbf{F}$ is not contained in the $\mathbb{ISP}$-closure of any pre-standard class of finite algebras (Corollary 5.7), which is the first finite (even locally finite) example of this kind. Lastly, we will show that $\mathbf{F}$ has a finitely axiomatisable quasi-equational theory (Proposition

5.8), which solves Problem 3.11 of [6]. This problem asked if every finitely q-based finite algebra is standard.

Our algebra $\mathbf{F}$ is based on the set $\{0,1,2,3\}$ and has two binary operations $\wedge$ and $\cdot$. In fact most of the results in this section can be achieved with only the multiplication $\cdot$; the operation $\wedge$ is used to ensure that $\mathbf{F}$ has finitely axiomatisable quasi-equational theory. With respect to the operation $\wedge$, the algebra $\mathbf{F}$ is a height one semilattice with 0 as the bottom element (often called a *flat semilattice*). The operation $\cdot$ is the same as $\wedge$ except for the two products $1 \cdot 3 = 3 \cdot 1 = 2$. We will often use concatenation in place of $\cdot$.

Let $\mathbf{E}$ be the subalgebra of $\mathbf{F}^{\mathbb{Z}}$ generated by the constant element $\underline{0}$ and the elements

$$A_n := \ldots, 1, 1, \overset{n}{3}, 2, 2, \ldots$$

$$B_n := \ldots, 1, 1, \overset{n}{2}, 2, 2, \ldots$$

Proposition 5.1. *The algebra* $\mathbf{E} \in \mathbb{ISP}(\mathbf{F})$ *admits a Boolean topology making it non-residually finite.*

Proof. This result will follow from a number of easy lemmas. First, for $i < j \in \mathbb{Z}$, let $E_{i,j} := \{x \in E \mid k \leq i \Rightarrow x(k) = 1 \text{ and } \ell \geq j \Rightarrow x(\ell) = 2\}$.

Lemma 5.2. $E \backslash \{\underline{0}\} = \bigcup_{i<j} E_{i,j}$.

Proof. Clearly $\cup_{i<j} E_{i,j}$ contains all non-zero generators of $\mathbf{E}$, but also the set $E_{i,j}$ is a subuniverse of $\mathbf{E}$. ☐

Lemma 5.3. *For every* $a, b \in E \backslash \{\underline{0}\}$, *the set* $\{i \in \mathbb{Z} : a(i) = b(i)\}$ *is cofinite.*

Proof. Lemma 5.2 shows that there exists $i < j$ with $a, b \in E_{i,j}$. ☐

Lemma 5.4. *If* $a \in E$ *has* $a(i) = 3$ *for some* $i \in \mathbb{Z}$, *then* $a = A_i$.

Proof. This follows easily from the definition of the non-zero generators of $\mathbf{E}$ and the fact that $\{0,1,2\}$ is an absorbing ideal of $\mathbf{F}$. ☐

For $d \in E \backslash \{\underline{0}\}$, let $L_d := \min\{i : d(i+1) \neq 1\}$ and $R_d := \max\{i : d(i-1) \neq 2\}$. It is obvious from Lemma 5.2 that these numbers are well-defined.

Lemma 5.5. *Let* $a \in E \backslash \{\underline{0}\}$. *The set* $\{b : (\exists c)\ bc = a\}$ *is finite.*

Proof. We show that $bc = a$ implies $L_a \leq L_b, L_c$ and $R_a \geq R_b - 1, R_c - 1$.

Say that $bc = a$. If $i < L_a$ then $b(i)c(i) = 1$ which implies that $b(i) = c(i) = 1$. So $L_a \leq L_b, L_c$.

If $i \geq R_a$ then $b(i)c(i) = 2$ which implies that

$$\text{either} \quad b(i) = c(i) = 2 \text{ or } \{b(i), c(i)\} = \{1, 3\}. \tag{5.1}$$

If the second case does not occur for any $i \geq R_a$ then we are done. Now say that the second case does occur for some $i \geq R_a$. It follows from Lemma 5.4 that one of b, c is the element A_i. Without loss of generality, say that $b = A_i$ (and so $c(i) = 1$). Then $b(i-1) = 1$ and $c(i-1) \neq 3$ so by (5.1) we have $i - 1 \ngeq R_a$. Hence $i = R_a = R_b - 1$, and for all $j \geq R_b = i + 1$ we have $b(j) = 2$. Then (5.1) implies $b(j) = c(j) = 2$. Hence $R_a + 1 \geq R_b, R_c$ as required.

These facts show that $|\{b : (\exists c)\ bc = a\}| \leq 4^{R_a+1-L_a}$ which proves the lemma. □

Lemma 5.6. *Let $a \in E\backslash\{\underline{0}\}$. The set $\{b : (\exists c)\ b \wedge c = a\}$ is finite.*

Proof. This is similar to the previous lemma only much easier. The proof is left to the reader. □

Let $\underset{\sim}{\mathbf{E}}$ be the result of giving $E\backslash\{\underline{0}\}$ the one-point compactification by $\{\underline{0}\}$, a Boolean topology. Lemmas 5.5 and 5.6 show that $\underset{\sim}{\mathbf{E}}$ is a Boolean topological algebra.

We now prove that $\underset{\sim}{\mathbf{E}}$ is not residually finite (of course, as $\mathbf{E} \in \mathbb{ISP}(\mathbf{F})$, we have that $\mathbf{E}$ is residually-5).

The kernel θ of any continuous homomorphism ϕ onto a finite space has clopen equivalence classes. The only clopen sets containing $\underline{0}$ are cofinite and hence $0/\theta$ must contain an element B_k from $\{B_j : j \in \mathbb{N}\}$. However

$$B_0 = A_0(A_1(\ldots(A_{k-1}B_k)\ldots))\ \theta\ A_0(A_1(\ldots(A_{k-1}0)\ldots)) = 0.$$

Hence every continuous morphism into a finite algebra maps B_0 to the same place as $\underline{0}$, which shows that $\mathrm{rb}(\underset{\sim}{\mathbf{E}}) = \aleph_1$ (as $|E| = \aleph_0$). This completes the proof of Proposition 5.1. □

Lemma 2.5 shows that the algebra $\mathbf{F}$ is inherently non-standard. In fact $\mathbf{F}$ satisfies an even stronger property.

Corollary 5.7. *Let $\mathcal{K}$ be any class of finite algebras of type $\langle 2, 2\rangle$ and for which $\mathbf{F} \in \mathbb{ISP}(\mathcal{K})$. Then $\mathcal{K}_{\mathcal{T}}$ is not pre-standard and $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ is not universally definable.*

Proof. $\mathcal{K}$ is not pre-standard because $\underset{\sim}{\mathbf{E}} \in [\mathbb{ISP}(\mathbf{F})]_{\mathcal{T}} \subseteq [\mathbb{ISP}(\mathcal{K})]_{\mathcal{T}}$ while $\underset{\sim}{\mathbf{E}}$ is not residually-$[\mathbb{S}(\mathcal{K})]_{\mathcal{T}}$, whence $\underset{\sim}{\mathbf{E}} \notin \mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ by Lemma 2.2. By Proposition 2.3 $\mathbb{IS}_c\mathbb{P}(\mathcal{K}_{\mathcal{T}})$ is not universally definable. □

The operation $\wedge$ has so far played no role in the results. So the algebra $\mathbf{F}^\flat := \langle F; \cdot\rangle$ and corresponding algebra $\mathbf{E}^\flat$ also supply a binary version (that is, in the type $\langle 2\rangle$) of Proposition 5.1 and Corollary 5.7.

Proposition 5.8. *The algebra* $\mathbf{F}$ *has a finite basis of quasi-identities.*

Proof. For this we use a very general result due to Maroti and McKenzie [17]. Let $\mathbf{M}$ be a finite algebra whose operations include a semilattice operation. Corollary 5.7 of [17] shows that $\mathbf{M}$ generates a finitely axiomatisable quasivariety provided that $\mathbb{HS}(\mathbf{M}) \subseteq \mathbb{ISP}(\mathbf{M})$. This property is routinely verified to hold for $\mathbf{M} = \mathbf{F}$, hence $\mathbf{F}$ has a finite basis for its quasi-identities. □

We can prove that the identities of $\mathbf{F}$ are less well behaved; there is no locally finite variety containing $\mathbf{F}$ whose identities have a finite basis. Because this claim is not of direct importance to the results of this section we omit the proof (however we note that it follows easily in a similar manner to the proof that the algebra $\mathbf{A}$ of [16] has this property).

6. Profiniteness is undecidable

We now show how to modify the constructions of Section 4 to prove the following theorem.

Theorem 6.1. *Let* $\mathfrak{n}$ *be a cardinal with* $3 \leq \mathfrak{n} \leq 2^{\aleph_0}$. *The following problems are undecidable. Given a finite set* Σ *of identities,*

(1) *are all members of* $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ *profinite* (*residually finite*)*?*
(2) *is* $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ *residually-*$\mathfrak{n}$*?*
(3) *does* Σ *define a variety with FDSC?*

Proof. We use the undecidability of the triviality problem for finitely presented groups. We let $\mathbf{G}_{\langle A:R\rangle}$ denote the group defined by a group presentation $\langle A{:}R\rangle$. Rabin [20] proved that there is no algorithm to determine when given a finite presentation $\langle A{:}R\rangle$, whether or not $\mathbf{G}_{\langle A:R\rangle}$ is the trivial group (this was also independently proved by Adyan [1]). To be more precise, let $\langle X{:}E\rangle$ be a fixed finite group presentation with undecidable word problem (necessarily infinite). For any group word w in the alphabet X, Rabin provides an effective procedure to construct a presentation $\Pi_w = \langle X'{:}E_w\rangle$ with the property that if $w = 1$ in $\mathbf{G}_{\langle X:E\rangle}$, then $\mathbf{G}_{\Pi_w}$ is trivial, while if $w \neq 1$, then $\mathbf{G}_{\Pi_w}$ embeds $\mathbf{G}_{\langle X:E\rangle}$ and so is infinite.

Now we translate these facts to the language of monoids. For a given monoid presentation $\langle A{:}R\rangle$, let $\mathbf{S}_{\langle A:R\rangle}$ denote the monoid defined by $\langle A{:}R\rangle$. Every finite presentation $\langle A{:}R\rangle$ in the variety of groups has an equivalent monoid presentation in the variety of monoids; namely the presentation over $A^\sharp := A \cup \{a^{-1} : a \in A\}$ (assumed to be a disjoint union) with extra relations $R^\sharp$ formed by adjoining the relations $aa^{-1} = a^{-1}a = 1$ for each $a \in A$ and $x \in A \cup \{a^{-1} : a \in A\}$. The monoid presentation $\langle A^\sharp{:}R^\sharp\rangle$ defines the same group (as a monoid) as the group presentation $\langle A{:}R\rangle$. Note that, modulo left to right applications of the group law $(xy)^{-1} \approx y^{-1}x^{-1}$, a group word in the alphabet A is a semigroup word in the alphabet $A^\sharp$.

For the remainder of this section we fix a group presentation $\langle X{:}E\rangle$ with undecidable word problem and let $\Pi^\sharp_w$ be the monoid presentation obtained from Rabin's construction $\Pi_w = \langle X'{:}E_w\rangle$ for w a group word in the alphabet X. Summarising the above properties, we have if $w = 1$ holds in $\mathbf{G}_{\langle X:E\rangle}$ then $\mathbf{S}_{\Pi^\sharp_w}$ is trivial, while if $w \neq 1$ in $\mathbf{G}_{\langle X:E\rangle}$ then $\mathbf{S}_{\Pi^\sharp_w}$ embeds $\mathbf{G}_{\langle X:E\rangle}$ as a monoid. For notational convenience we will abbreviate $\mathbf{S}_{\Pi^\sharp_w}$ to $\mathbf{S}(w)$, and let Y denote the generator set $(X')^\sharp$ in $\Pi^\sharp_w$.

There is a well known method attributed to Maltsev that translates semigroup or monoid presentations into unary algebra identities (see [3, Exercise II.1.6] for example). Given a monoid presentation $\langle A{:}R\rangle$, we will let $U(\langle A{:}R\rangle)$ denote the variety of unary algebras with operation symbols $\{a : a \in A\}$ and defining identities $\Sigma(\langle A{:}R\rangle)$ given by $a_{i_1}(\dots(a_{i_n}(x))\dots) \approx a_{j_1}(\dots(a_{j_m}(x))\dots)$ whenever $a_{i_1}\dots a_{i_n} = a_{j_1}\dots a_{j_m}$ is a relation in R.

Claim 1. Let $\langle A{:}R\rangle$ be a monoid presentation. If u is a semigroup word in A such that $u = 1$ holds in $\mathbf{S}_{\langle A:R\rangle}$, then $\Sigma(\langle A{:}R\rangle) \vdash u(x) \approx x$.

Proof. See [3, Exercise II.1.6] for example. □

Now let w be an arbitrary semigroup word in the alphabet X.

Claim 2. If $w = 1$ in $\mathbf{G}_{\langle X:E\rangle}$ then:

(1) the variety $U(\Pi^\sharp_w)$ is generated by a two element unary algebra;
(2) all Boolean topological models of $\Sigma(\Pi^\sharp_w)$ are profinite;
(3) $\mathrm{rb}(\mathrm{Mod}_{\mathfrak{T}}(\Pi^\sharp_w)) = 3$;
(4) the variety $U(\Pi^\sharp_w)$ has FDSC.

Proof. Because $w = 1$, we have that $\mathbf{S}(w)$ is trivial and so all words in the generators of $\Pi^\sharp_w$ are equal to 1 in $\mathbf{S}(w)$. By Claim 1 we have $U(\Pi^\sharp_w) \vDash a(x) \approx x$ for every fundamental operation $a \in Y$. Let $\mathbf{F}_2$ denote the two element $U(\Pi^\sharp_w)$-free algebra. Clearly $\mathbf{F}_2$ has two elements, and because every unary algebra identity involves at most two variables, it follows that $\mathbb{HSP}(\mathbf{F}_2) = U(\Pi^\sharp_w)$. But also, any map from a model of $\{a(x) \approx x \mid a \in Y\}$ to $\mathbf{F}_2$ is a homomorphism and so by Lemma 2.1 we have

$$U(\Pi_w^\sharp) = \mathrm{Mod}(\{a(x) \approx x \mid a \in Y\}) \subseteq \mathbb{ISP}(\mathbf{F}_2) \subseteq U(\Pi_w^\sharp),$$

showing that $\mathbf{F}_2$ generates $U(\Pi_w^\sharp)$ as a quasivariety as well. It is also clear that the single term $\{x\}$ determines the syntactic congruences in $U(\Pi_w^\sharp)$. Hence parts 1 and 4 of Claim 2 hold. Parts 2 and 3 follow from the FDSC-HSP-Theorem 3.5. □

Claim 3. If $w \neq 1$, in $\mathbf{G}_{\langle X:E\rangle}$ then:

(1) there is a Boolean topological model $\underset{\sim}{\mathbf{U}}$ of $\Sigma(\Pi_w^\sharp)$ with $\mathrm{rb}(\underset{\sim}{\mathbf{U}}) = (2^{\aleph_0})^+$;
(2) the variety $U(\Pi_w^\sharp)$ does not have FDSC.

Proof. Because $w \neq 1$, we have that $\mathbf{S}(w)$ is an infinite group. We define an action of the elements $x \in Y$ on the set $2^{S(w)}$ by setting (for $\underline{a} : S(w) \to \{0,1\}$ and $g \in S(w)$)

$$[x(\underline{a})](g) := \underline{a}(xg).$$

We denote this unary algebra (with operations $\{x : x \in Y\}$) by $\mathbf{U}_w$. It is trivial to verify that $\Sigma(\Pi_w^\sharp)$ is satisfied by $\mathbf{U}_w$.

Now note that the operations $x \in Y$ are continuous with respect to the product topology on U_w (as a power of the discrete topologised set $\{0,1\}$). Thus we may write $\underset{\sim}{\mathbf{U}}_w$ to denote the corresponding Boolean topological algebra. We now show that $\underset{\sim}{\mathbf{U}}_w$ is not residually-$2^{\aleph_0}$.

Let $\phi : \underset{\sim}{\mathbf{U}}_w \to \underset{\sim}{\mathbf{Z}}$ be a continuous homomorphism into a Boolean topological algebra $\underset{\sim}{\mathbf{Z}}$ of cardinality less than $2^{\aleph_0}$. Now $\phi(\underset{\sim}{\mathbf{U}}_w)$ is a closed subalgebra of $\underset{\sim}{\mathbf{Z}}$ and so is also a Boolean topological algebra of cardinality less than $2^{\aleph_0}$. By Lemma 4.5(ii), $\phi(\underset{\sim}{\mathbf{U}}_w)$ is not dense-in-itself and so we may proceed as in the proof of Example 4.6 to show that $|\phi(\underset{\sim}{\mathbf{U}}_w)| = 1$. Let a be an isolated point in $\phi(\underset{\sim}{\mathbf{U}}_w)$. So $V_a := \phi^{-1}(a)$ is clopen and we may find a finite subset $H \subset S(w)$ and (finite) set $J \subseteq 2^H$ such that for $\underline{b} \in 2^{S(w)}$, we have $\underline{b} \in V_a$ if and only if $\underline{b}\restriction_H \in J$. Let $\underline{b} \in 2^{S(w)}$ be arbitrary. We want to show that $\underline{b} \in V_a$. Because H is finite and $\mathbf{S}(w)$ is an infinite group, there is an element $g \in S(w)$ such that $gH \cap H = \varnothing$. Now say that $\underline{c} \in V_a$.

Define $\underline{c}_b \in 2^{S(w)}$ by $\underline{c}_b(f) := \underline{b}(g^{-1}f)$ if $f \in gH$ and $\underline{c}_b(f) := \underline{c}(f)$ otherwise. Now define $\underline{c}_c(f) := \underline{c}(g^{-1}f)$ if $f \in gH$ and $\underline{c}_c(f) := \underline{c}(f)$ otherwise. Note that as $\underline{c}_b\restriction_H = \underline{c}_c\restriction_H = \underline{c}\restriction_H \in J$, we have $\phi(\underline{c}_b) = \phi(\underline{c}_c) = \phi(\underline{c}) = a$. Also, for $h \in H$ we have $[g(\underline{c}_c)](h) = \underline{c}_c(gh) = \underline{c}(g^{-1}gh) = \underline{c}(h)$, as $gh \in gH$. So $[g(\underline{c}_c)]\restriction_H = \underline{c}\restriction_H$ and $\phi(g(\underline{c}_c)) = a$. Similarly, we have $[g(\underline{c}_b)]\restriction_H = \underline{b}\restriction_H$. Then $\phi(g(\underline{c}_b)) = g^{\mathbf{Z}}(\phi(\underline{c}_b)) = g^{\mathbf{Z}}(\phi(\underline{c}_c)) = \phi(g(\underline{c}_c)) = a$, so that $[g(\underline{c}_b)]\restriction_H \in J$. But $[g(\underline{c}_b)]\restriction_H = \underline{b}\restriction_H$, showing that $\phi(\underline{b}) = a$ as well. Therefore $|\phi(\underset{\sim}{\mathbf{U}}_w)| = 1$ as required. This establishes part 1 of Claim 3. Part 2 follows from part 1 by Theorem 3.3. □

Because it is undecidable whether or not $w = 1$ in the original presentation $\langle X{:}E\rangle$, Claims 2 and 3 show that the three problems stated in Theorem 6.1 are undecidable. □

7. Residual bounds for completely simple semigroups

For our final results we are going to investigate Question 1(2) for a more familiar class of algebras, namely completely simple semigroups. A finite completely simple semigroup is a semigroup with a unique ideal, namely itself. For example, every finite group is a completely simple semigroup. We now recall the powerful Rees-Suskevich structural theorem for completely simple semigroups. (For proofs of general semigroup theoretic facts, the reader is directed to a text such as [11].)

Let $\mathbf{G}$ be a finite group with identity element e, let n, m be positive integers and P be an $n \times m$ matrix (the *sandwich matrix*) whose entries come from $\mathbf{G}$. The *Rees matrix semigroup* $\mathcal{M}(\mathbf{G}, n, m, P)$ is the semigroup on the set $\{(i, g, j) : 1 \leq i \leq n, 1 \leq j \leq m, g \in G\}$ with product $(i, g, j)(k, h, \ell) = (i, gP_{k,j}h, \ell)$. Every finite completely simple semigroup is isomorphic to one constructed in this way (for the infinite case, we allow $\mathbf{G}$ to be infinite and let n and m be arbitrary index sets; but this is not equivalent to having a unique ideal). The choice of $\mathbf{G}$, n and m are unique in this representation, but the choice of P is not. In particular, it is known that P can always be chosen so that $P_{i,j} = e$ whenever $1 \in \{i, j\}$. Such a matrix is said to be *normalised*. We note also that the variety generated by a finite set of finite completely simple semigroups consists entirely of completely simple semigroups. Indeed, the variety of completely simple semigroups over groups whose exponent divides n is given by the identities

$$\{x^{n+1} \approx x, (xyx)^n \approx x^n\}.$$

The following two lemmas provide a complete description of a number of finiteness properties for completely simple semigroups. The first combines the two main results of Ol'shanskii from [18] and [19], while the second combines results from Golubov and Sapir [10] and McKenzie [15] with those of Sapir [22].

Lemma 7.1. [18, 19] *Let* $\mathbf{G}$ *be a finite group. The following are equivalent*:

(1) *every Sylow subgroup of* $\mathbf{G}$ *is abelian*;
(2) $\mathbf{G}$ *has a finitely axiomatisable quasi-equational theory*;
(3) *there is a finite algebra* $\mathbf{A}$ *with* $\mathbf{G} \in \mathbb{ISP}(\mathbf{A})$ *and* $\mathbf{A}$ *has finitely axiomatisable quasi-equational theory*;

(4) $\mathrm{rb}(\mathbb{HSP}(\mathbf{G}))$ *is finite*;
(5) $\mathrm{rb}(\mathbb{HSP}(\mathbf{G})) \neq \infty$.

Recall that a completely simple semigroup is *orthodox* if its idempotents form a subsemigroup.

Lemma 7.2. [10, 15, 22] *Let $\mathcal{M}(\mathbf{G}, n, m, P)$ be a finite completely simple semigroup with normalised sandwich matrix. The following are equivalent*:

(1) $P_{i,j} = e$ *for every* i, j *and Sylow subgroups of* $\mathbf{G}$ *are abelian*;
(2) $\mathcal{M}(\mathbf{G}, n, m, P)$ *is orthodox and Sylow subgroups of* $\mathbf{G}$ *are abelian*;
(3) $\mathcal{M}(\mathbf{G}, n, m, P)$ *has a finitely axiomatisable quasi-equational theory*;
(4) *there is a finite algebra* $\mathbf{A}$ *with* $\mathcal{M}(\mathbf{G}, n, m, P) \in \mathbb{ISP}(\mathbf{A})$ *and* $\mathbf{A}$ *has finitely axiomatisable quasi-equational theory*;
(5) $\mathrm{rb}(\mathbb{HSP}(\mathcal{M}(\mathbf{G}, n, m, P)))$ *is finite*;
(6) $\mathrm{rb}(\mathbb{HSP}(\mathcal{M}(\mathbf{G}, n, m, P))) \neq \infty$.

(We note that $1 \Leftrightarrow 2$ for P normalised is an easy exercise using the definition of the Rees matrix semigroup.) The property of simultaneously being orthodox and having all Sylow subgroups abelian will appear many times below. We denote it by $\heartsuit$. It is easily verified that the class of finite completely simple semigroups satisfying property $\heartsuit$ is closed under taking finitary direct products and subsemigroups.

We can now prove the following result.

Theorem 7.3. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological completely simple semigroup. If $\mathrm{rb}(\mathbf{X}) = n$ and all finite quotients of $\mathbf{X}$ satisfy $\heartsuit$ then $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq f(n)$.*

Proof. Say that all finite quotients of $\mathbf{X}$ satisfy $\heartsuit$ and $\mathrm{rb}(\mathbf{X}) = n \in \mathbb{N}$. There are only finitely many non-isomorphic finite completely simple semigroups with fewer than n-elements and satisfying $\heartsuit$. Therefore for each n, the direct product of representatives from each isomorphism class of such semigroups is finite and also satisfies $\heartsuit$. We denote this semigroup by $\mathbf{G}_n$. Now by Lemma 7.2(5), there is a finite bound on the number (and therefore on the size) of subdirectly irreducibles in $\mathbb{HSP}(\mathbf{G}_n)$. Let $\boldsymbol{\mathcal{K}}_n$ denote a set consisting of one copy (up to isomorphism) of each subdirectly irreducible semigroup in $\mathbb{HSP}(\mathbf{G}_n)$, and let $f(n)$ denote 1 plus the cardinality of the largest of the members of $\boldsymbol{\mathcal{K}}_n$. Now $\mathbb{ISP}(\boldsymbol{\mathcal{K}}_n) = \mathbb{HSP}(\boldsymbol{\mathcal{K}}_n)$ and $\mathbf{X}$ is residually-$\mathbb{S}(\boldsymbol{\mathcal{K}}_n)$. So, because semigroups have FDSC, it follows from the FDSC-HSP Theorem 3.5 that $\underset{\sim}{\mathbf{X}}$ is residually-$[\mathbb{S}(\boldsymbol{\mathcal{K}}_n)]_{\mathcal{T}}$. Hence $\mathrm{rb}(\underset{\sim}{\mathbf{X}}) \leq f(n)$ as required. $\square$

The rest of this section is devoted to proving that Theorem 7.3 cannot be substantially improved. The main theorem is the following (Lemma 7.2 provides many other equivalent conditions).

Theorem 7.4. *Let* $\mathbf{S}$ *be a finite completely simple semigroup. The following are equivalent*:

(1) $\mathbf{S}$ *satisfies* $\heartsuit$;
(2) *every member of* $[\mathbb{ISP}(\mathbf{S})]_{\mathcal{T}}$ *is residually very finite*;
(3) $[\mathbb{ISP}(\mathbf{S})]_{\mathcal{T}}$ *is residually very finite*;
(4) $\mathbf{S}$ *is contained in a standard quasivariety generated by a finite set of finite algebras* (*that is*, $\mathbf{S}$ *is* not *inherently non-standard*).

Proof. (4) $\Rightarrow$ (3) follows from (iv)$\Rightarrow$(vi) of Proposition 2.3, while (3) $\Rightarrow$ (2) is trivial. Next we show (1) $\Rightarrow$ (4).

If (1) holds, then by Lemma 7.2(5) there is a finite bound on the size of the subdirectly irreducible members of $\mathbb{HSP}(\mathbf{S})$. Let $\mathcal{K}$ be a set of subdirectly irreducible members of $\mathbb{HSP}(\mathbf{S})$, containing precisely one copy of each subdirectly irreducible member of $\mathbb{HSP}(\mathbf{S})$ up to isomorphism. We have $\mathbb{ISP}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K}) = \mathbb{HSP}(\mathbf{S})$. Let $\mathbf{T}$ be the direct product $\prod_{\mathbf{A}\in\mathcal{K}} \mathbf{A}$, so that $\mathbb{ISP}(\mathbf{T}) = \mathbb{HSP}(\mathbf{T}) = \mathbb{ISP}(\mathcal{K}) = \mathbb{HSP}(\mathbf{S})$. We have proved that $\mathbf{S} \in \mathbb{ISP}(\mathbf{T})$, so implication (1)$\Rightarrow$(4) will be proved if we can show that $\mathbf{T}$ is standard (it is certainly finite, as $\mathcal{K}$ is a finite set of finite semigroups). Let $\underset{\sim}{\mathbf{X}} \in [\mathbb{ISP}(\mathbf{T})]_{\mathcal{T}}$. The variety of semigroups has FDSC so by the FDSC-HSP Theorem 3.5, we have $\underset{\sim}{\mathbf{X}}$ is residually-$[\mathbb{S}(\mathcal{K})]_{\mathcal{T}} \subseteq [\mathbb{S}(\mathbf{T})]_{\mathcal{T}}$. By Proposition 2.3 we find that $\mathbf{T}$ is standard. Thus (4) holds.

The proof of Theorem 7.4 will be complete if we can show $\neg(1) \Rightarrow \neg(2)$. If (1) fails, then either $\mathbf{S}$ contains a subgroup with a non-abelian subgroup, or there are idempotents whose product is not idempotent. These two cases are quite involved and are considered in two subsections. Property $\neg(2)$ in the first case is established in Theorem 7.6, while the second case is established in Theorem 7.13. □

Theorem 7.4 demonstrates that Theorem 7.3 cannot be substantially improved. Indeed if $\underset{\sim}{\mathbf{X}}$ is a Boolean topological completely simple semigroup $\underset{\sim}{\mathbf{X}}$ with $\mathrm{rb}(\mathbf{X}) \leq n$ but there is a finite quotient of $\underset{\sim}{\mathbf{X}}$ failing $\heartsuit$, then $\underset{\sim}{\mathbf{X}}$ does not lie in the quasivariety generated by a completely simple semigroup satisfying $\heartsuit$ and Theorem 7.4 shows that $\mathrm{rb}(\underset{\sim}{\mathbf{X}})$ may be $\aleph_0$.

The following lemma is the second of several inverse limit techniques for showing non-standardness established in [7] and is our main tool in the coming subsections.

Second Inverse Limit Technique 7.5. [7] *Let* $\{\mathbf{X}_n : n \in \mathbb{N}\}$ *be a family of finite algebras such that for each* $n \in \mathbb{N}$ *there is a surjective homomorphism* $\phi_n : \mathbf{X}_{n+1} \to \mathbf{X}_n$. *Let* $\mathcal{K}$ *and* $\mathcal{L}$ *be classes of finite discrete algebras of the same signature as the* $\mathbf{X}_n$, *with* $\mathcal{K}$ *finite and with* $\mathbb{ISP}(\mathcal{K}) \subseteq \mathbb{ISP}(\mathcal{L})$. *Assume that*

(i) *for all* $n \in \mathbb{N}$, *every* n*-generated subalgebra of* $\mathbf{X}_n$ *is in* $\mathbb{ISP}(\mathcal{K})$,
(ii) *there is a system of pairs* $x_n, y_n \in X_n$ *for* $n \in \mathbb{N}$ *such that the following conditions hold for all* $n \in \mathbb{N}$: $x_n \neq y_n$; $\phi_n(x_{n+1}) = x_n$, $\phi_n(y_{n+1}) = y_n$; *and* $\chi(x_n) = \chi(y_n)$ *for every morphism* $\chi : \mathbf{X}_n \to \mathbf{M}$ *with* $\mathbf{M} \in \mathcal{L}$.

Then the inverse limit of the inverse system given by the $\mathbf{X}_i$ *and the maps* ϕ_i *is a Boolean topological algebra* $\underset{\sim}{\mathbf{X}}$ *with* $\mathbf{X} \in \mathbb{ISP}(\mathcal{K})$ *and* $\underset{\sim}{\mathbf{X}} \notin \mathbb{IS}_c\mathbb{P}(\mathcal{L})$. *Consequently, every class of finite algebras* $\mathcal{P}$ *with* $\mathbb{ISP}(\mathcal{K}) \subseteq \mathbb{ISP}(\mathcal{P}) \subseteq \mathbb{ISP}(\mathcal{L})$ *is not prestandard* (*and by Proposition 2.3*, $\mathbb{IS}_c\mathbb{P}(\mathcal{P})$ *is not universally-definable*).

7.1. **Groups with non-abelian Sylow subgroups.** We are going to prove the following theorem (which can also be stated in the other types $\langle 2, 1\rangle$, $\langle 2, 1, 0\rangle$ and so on).

Theorem 7.6. *Let* $\mathcal{J}$ *be any class of finite algebras of type* $\langle 2\rangle$ (binars). *If* $\mathbb{ISP}(\mathcal{J})$ *contains a finite group* $\mathbf{H}$ *with a non-abelian Sylow subgroup*, *then there is a group* $\mathbf{G}$ *that is residually-*$\mathbb{S}(\mathbf{H})$ (*so has residual bound* $|H| + 1$) *admitting a Boolean topology so that* $\mathrm{rb}(\underset{\sim}{\mathbf{G}}) = \aleph_0$. *If* $\mathcal{J}$ *is finite*, *then* $\mathcal{J}$ *is non-standard.*

Theorem 7.6 (whose proof covers all of this subsection) shows that the implication $\neg 1 \Rightarrow \neg 2$ of Theorem 7.4 holds in the case where $\mathbf{S}$ is a group.

Most of the work in proving Theorem 7.6 is covered by the following lemma.

Lemma 7.7. *Let* k *be a fixed positive integer and* $\mathbf{G}$ *be a finite group containing a non-abelian Sylow subgroup. Then* $[\mathbb{ISP}(\mathbf{G})]_{\mathcal{T}}$ *contains a Boolean topological group* $\underset{\sim}{\mathbf{C}}_k$ *that is not residually-*k.

Proof. It will suffice to prove the lemma in the case when $k \geq |G|$. Fix such a k and let $\mathcal{B}_k$ be the class of all binars of cardinality at most k. In particular, $\mathbf{G} \in \mathcal{B}_k$ and $\mathbb{ISP}(\mathbf{G}) \subseteq \mathbb{ISP}(\mathcal{B}_k)$. We use SILT 7.5 using an inverse system of some groups $\{\mathbf{C}_k(t) : t \in \mathbb{N}\}$ and with $\mathcal{K} := \{\mathbf{G}\}$ and $\mathcal{L} := \mathcal{B}_k$. A substantial part of the work can be taken from Ol'shanskii's proof in [19] of $\neg(1) \Rightarrow \neg(3)$ in Lemma 7.1. In particular, this proof supplies us with the algebras $\mathbf{C}_k(t)$ and a proof that part (i) of the SILT 7.5 is satisfied (t-generated subgroups of $\mathbf{C}_k(t)$ are in the quasivariety of $\mathbf{G}$). We need only verify part (ii) and provide the connecting homomorphisms $\phi_t : \mathbf{C}_k(t+1) \to \mathbf{C}_k(t)$.

Ol'shanskii notes that we can find in $\mathbf{G}$ a subgroup $\mathbf{H}$ satisfying the following properties:

($1_{\mathbf{G}}$) $\mathbf{H}$ is of nilpotency class 2;
($2_{\mathbf{G}}$) $[\mathbf{H},\mathbf{H}]$ is a cyclic group of order p for some prime p.

We let $\mathbf{F}_n$ denote the n-generated relatively free group in $\mathbb{HSP}(\mathbf{H})$ with free generators $f_1,\dots,f_n$. Routine group calculations give

($3_{\mathbf{G}}$) $[\mathbf{F}_n,\mathbf{F}_n]$ is a relatively free group in the class of abelian groups of exponent p and $\{[f_i,f_j] : 1\le i<j\le n\}$ are free generators. Equivalently,

$$\text{if } \prod_{1\le i<j\le n}[f_i,f_j]^{r_{i,j}} = e \text{ then } r_{i,j}\equiv 0 \mod (p) \text{ for each } 1\le i<j\le n.$$

Ol'shanskii proves the following result [19, Lemma 1].

Lemma 7.8. *Let $m,\ell\in\mathbb{N}$ satisfy $\ell>4m+3$. Then $[\mathbf{F}_\ell,\mathbf{F}_\ell]$ contains an element that cannot be written in the form $[w_1,w_2]\dots[w_{2m-1},w_{2m}]v^p$, for $w_1,\dots,w_{2m},v\in\mathbf{F}_\ell$.*

Our eventual choice of the element x_n of the SILT 7.5 will be based on a particular element of the kind shown to exist in Lemma 7.8. An explicit description of such an element is supplied by the following lemma.

Lemma 7.9. *Let $m\in\mathbb{N}$ and $n:=\binom{4m+4}{2}$. Then in $[\mathbf{F}_n,\mathbf{F}_n]$, the element $[f_1,f_2][f_3,f_4]\dots[f_{n-1},f_n]$ cannot be written in the form $[w_1,w_2]\dots[w_{2m-1},w_{2m}]v^p$ for $w_1,\dots,w_{2m},v\in\mathbf{F}_n$.*

Proof. By Lemma 7.8, there is an element $a\in[\mathbf{F}_{4m+4},\mathbf{F}_{4m+4}]$ that cannot be written in the form $[w_1,w_2]\dots[w_{2m-1},w_{2m}]v^p$ for any $w_1,\dots,w_{2m},v\in\mathbf{F}_{4m+4}$. By property ($3_{\mathbf{G}}$), the element a can be written uniquely as $\prod_{1\le i<j\le 4m+4}[f_i,f_j]^{r_{ij}}$, where $0\le r_{ij}\le p-1$. Now, in a group of nilpotency class of 2, a power of a commutator is a commutator (for example, because $[x,y]$ is central, it is easily established by induction on n that in such a group, the law $[x,y]^n\approx[x^n,y]$ holds for each n). Hence there are elements $g_1,\dots,g_n\in F_{4m+4}$ such that $a=\prod_{1\le i<j\le 4m+4}[f_i,f_j]^{r_{ij}}=[g_1,g_2]\dots[g_{n-1},g_n]$. Now assume (to obtain a contradiction) that there are elements $u_1,\dots,u_{2m},u\in F_n$ such that

$$[f_1,f_2][f_3,f_4]\dots[f_{n-1},f_n]=[u_1,u_2]\dots[u_{2m-1},u_{2m}]u^p.$$

As $\mathbf{F}_n$ is relatively free, there is a homomorphism $\phi : \mathbf{F}_n \to \mathbf{F}_{4m+4}$ with $\phi(f_i) = g_i$. But then

$$\begin{aligned} a &= [g_1, g_2] \dots [g_{n-1}, g_n] \\ &= [\phi(f_1), \phi(f_2)] \dots [\phi(f_{n-1}), \phi(f_n)] \\ &= \phi([f_1, f_2] \dots [f_{n-1}, f_n]) \\ &= \phi([u_1, u_2] \dots [u_{2m-1}, u_{2m}]u^p) \\ &= [\phi(u_1), \phi(u_2)] \dots [\phi(u_{2m-1}), \phi(u_{2m})]\phi(u)^p \end{aligned}$$

which contradicts the choice of a, because $\phi(u)$ and each $\phi(u_i)$ lie in $\mathbf{F}_{4m+4}$. □

Let $t \in \mathbb{N}$ be arbitrary and choose $m = m(t) = \binom{t}{2}$ and $n = n(t) = \binom{4m+4}{2}$. Choose $s = s(k) = 2^k + 1$. We let $\mathbf{D}(t)$ denote the direct power $(\mathbf{F}_n)^s$ and let $a_t := [f_1, f_2] \dots [f_{n-1}, f_n] \in \mathbf{F}_n$ and $a_{k,t} := \langle a_t, e, e, \dots \rangle \in \mathbf{D}(t)$. Now let $N(t)$ denote the normal subgroup of $\mathbf{D}(t)$ generated by the elements

$$x_{i,j}(\ell) = \begin{cases} a_t & \text{if } \ell = i, \\ a_t^{-1} & \text{if } \ell = j, \\ e & \text{otherwise} \end{cases} \qquad \text{for each } 1 \le i, j \le s \text{ with } i \ne j.$$

We now let $\mathbf{C}(t) := \mathbf{D}(t)/N(t)$.

We apply SILT 7.5 with $\mathbf{X}_t := \mathbf{C}(t)$ and with the element $x_t := a_{k,t}N(t)$ and y_t as the identity element $N(t)$ of $\mathbf{C}(t)$. Ol'shanskii [19] shows that t-generated subgroups of $\mathbf{C}(t)$ are in $\mathbb{ISP}(\mathbf{H}) \subseteq \mathbb{ISP}(\mathbf{G})$ and that every normal subgroup J of index at most k in $\mathbf{C}(t)$ has $a_{k,t}J = J$. So every homomorphism from $\mathbf{C}(t)$ into a member of $\mathfrak{B}_k$ identifies x_t and y_t. To complete the proof that the SILT 7.5 applies, it remains to find connecting homomorphisms $\phi_t : \mathbf{C}(t+1) \to \mathbf{C}(t)$ with $\phi_t(a_{k,t+1}N(t+1)) = a_{k,t}N(t)$.

We begin by letting $\psi_t : \mathbf{F}_{n(t+1)} \to \mathbf{F}_{n(t)}$ be defined by setting

$$\psi_t(f_i) := \begin{cases} f_i & \text{if } i \le n(t), \\ e & \text{otherwise.} \end{cases}$$

Clearly, $\psi_t(a_{t+1}) = a_t$. Define $\bar{\psi}_t := \langle \psi_t, \dots, \psi_t \rangle : \mathbf{D}(t+1) \to \mathbf{D}(t)$. Note that $\bar{\psi}_t(a_{k,t+1}) = a_{k,t}$.

We need the following easy lemma, essentially one of the fundamental isomorphism theorems.

Lemma 7.10. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *be algebras and* $\xi_{A,B} : \mathbf{A} \twoheadrightarrow \mathbf{B}$, $\xi_{A,C} : \mathbf{A} \twoheadrightarrow \mathbf{C}$ *and* $\xi_{B,D} : \mathbf{B} \twoheadrightarrow \mathbf{D}$ *be surjective homomorphisms. Suppose further that* $a \in A$,

$b \in B$, $c \in C$ and $d \in D$ are such that $\xi_{A,C}(a) = c$, $\xi_{A,B}(a) = b$ and $\xi_{B,D}(b) = d$. If $\ker(\xi_{B,D} \circ \xi_{A,B}) \geq \ker(\xi_{A,C})$ then the map defined by $\xi_{C,D} : \xi_{A,C}(x) \mapsto (\xi_{B,D} \circ \xi_{A,B})(x)$ is a well defined surjective homomorphism from $\mathbf{C}$ to $\mathbf{D}$ with $\xi_{C,D}(c) = d$.

Proof. Now $\xi_{C,D}$ is well defined because $\xi_{A,C}(x) = \xi_{A,C}(y)$ implies $(\xi_{B,D} \circ \xi_{A,B})(x) = (\xi_{B,D} \circ \xi_{A,B})(y)$. Furthermore, $\xi_{C,D}$ is trivially seen to be a surjective homomorphism. Lastly, $\xi_{C,D}(c) = \xi_{C,D}(\xi_{A,C}(a)) = (\xi_{B,D} \circ \xi_{A,B})(a) = d$. $\square$

For $i \in \mathbb{N}$, let $\nu_i : \mathbf{D}(i) \to \mathbf{C}(i)$ denote the natural map. As $\bar{\psi}_t(N(t+1)) = N(t)$, we have $\ker(\nu_t \circ \bar{\psi}_t) \geq \ker(\nu_{t+1})$. Thus Lemma 7.10 (with $\mathbf{A} := \mathbf{D}(t+1)$, $\mathbf{B} := \mathbf{D}(t)$, $\mathbf{C} := \mathbf{C}(t+1)$ and $\mathbf{D} := \mathbf{C}(t)$ and so on) shows that the map $\phi_t : \mathbf{C}(t+1) \to \mathbf{C}(t)$ defined by $xN(t+1) \mapsto \bar{\psi}_t(x)N(t)$ is a well defined surjective homomorphism with $a_{k,t+1}N(t+1) \mapsto a_{k,t}N(t)$. This completes the verification that the SILT 7.5 applies to the algebras $\{\mathbf{C}(t) : t \in \mathbb{N}\}$, with connecting homomorphisms $\phi_t : \mathbf{C}(t+1) \to \mathbf{C}(t)$. By the SILT 7.5, the inverse limit $\underset{\sim}{\mathbf{C}}_k$ is a Boolean topological group that is not residually-k. $\square$

Lemma 7.11. *Let $\boldsymbol{\mathfrak{J}}$ be a set of Boolean topological semigroups. Then $\prod_{\{\underset{\sim}{\mathbf{X}} \in \boldsymbol{\mathfrak{J}}\}} \underset{\sim}{\mathbf{X}}$ is a Boolean topological semigroup with $\mathrm{rb}(\boldsymbol{\mathfrak{J}}) = \mathrm{rb}(\prod_{\{\underset{\sim}{\mathbf{X}} \in \boldsymbol{\mathfrak{J}}\}} \underset{\sim}{\mathbf{X}})$.*

Proof. It is well known that a profinite semigroup contains an idempotent element. For $\underset{\sim}{\mathbf{X}} \in \boldsymbol{\mathfrak{J}}$, let e_X denote some idempotent element of $\underset{\sim}{\mathbf{X}}$. Then for $\underset{\sim}{\mathbf{Y}} \in \boldsymbol{\mathfrak{J}}$ the map $\iota_Y : \underset{\sim}{\mathbf{Y}} \to \prod_{\{\underset{\sim}{\mathbf{X}} \in \boldsymbol{\mathfrak{J}}\}} \underset{\sim}{\mathbf{X}}$ that associates each $y \in Y$ with the $\boldsymbol{\mathfrak{J}}$-tuple given by

$$\iota_Y(y)(\underset{\sim}{\mathbf{X}}) = \begin{cases} y & \text{if } \underset{\sim}{\mathbf{X}} = \underset{\sim}{\mathbf{Y}}, \\ e_X & \text{otherwise,} \end{cases}$$

is a continuous embedding of $\underset{\sim}{\mathbf{Y}}$ into $\prod_{\{\underset{\sim}{\mathbf{X}} \in \boldsymbol{\mathfrak{J}}\}} \underset{\sim}{\mathbf{X}}$. $\square$

Theorem 7.12. *Let $\mathbf{G}$ be a finite group with a non-abelian Sylow subgroup. There is a Boolean topological group $\underset{\sim}{\mathbf{C}} \in [\mathbb{ISP}(\mathbf{G})]_{\mathcal{T}}$ with $\mathrm{rb}(\underset{\sim}{\mathbf{C}}) = \aleph_0$.*

Proof. By Lemma 7.7, there are groups $\underset{\sim}{\mathbf{C}}_1, \underset{\sim}{\mathbf{C}}_2, \dots$ with $\underset{\sim}{\mathbf{C}}_i$ not residually-i and $\underset{\sim}{\mathbf{C}}_i \in \mathrm{Mod}_{\mathcal{T}}(\mathbb{ISP}(\mathbf{G}))$. Then Lemma 7.11 shows that $\underset{\sim}{\mathbf{C}} := \prod_{i \in \mathbb{N}} \underset{\sim}{\mathbf{C}}_i$ has $\mathrm{rb}(\underset{\sim}{\mathbf{C}}) = \aleph_0$. Finally, $\underset{\sim}{\mathbf{C}} \in \mathrm{Mod}_{\mathcal{T}}(\mathbb{ISP}(\mathbf{G}))$ as $\mathrm{Mod}_{\mathcal{T}}(\mathbb{ISP}(\mathbf{G}))$ is closed under direct products. $\square$

The first part of Theorem 7.6 is an immediate corollary of Theorem 7.12. The second part follows because $\mathbf{C} \in \mathbb{ISP}(\mathbf{G}) \subseteq \mathbb{ISP}(\boldsymbol{\mathcal{K}})$ but $\underset{\sim}{\mathbf{C}}$ is not residually-$\mathbb{S}(\boldsymbol{\mathcal{K}})$.

7.2. **Completely simple semigroups whose idempotents do not form a subsemigroup.** We now turn our attention from groups to the more general class of completely simple semigroups. This section is similar to the previous subsection except that we are going to work closely from Sapir's proof in [22] of the implication $\neg(1) \Rightarrow \neg(4)$ of Lemma 7.2 above. The main theorem is the following.

Theorem 7.13. *Let $\mathfrak{J}$ be any class of finite binars. If $\mathbb{ISP}(\mathfrak{J})$ contains a completely simple semigroup whose idempotents do not form a subsemigroup, then there is a completely simple semigroup $\mathbf{T}$ that is residually-$\mathbb{S}(\mathfrak{J})$ (so has residual bound $1 + \max\{|A| : \mathbf{A} \in \mathfrak{J}\}$) admitting a Boolean topology so that $\operatorname{rb}(\underset{\sim}{\mathbf{T}}) = \aleph_0$. If $\mathfrak{J}$ is finite, then $\mathfrak{J}$ is non-standard.*

Most of the proof is contained in the following lemma.

Lemma 7.14. *Let k be a fixed positive integer and $\mathbf{S}$ be a finite completely simple semigroup whose idempotents do not form a subsemigroup. Then $[\mathbb{ISP}(\mathbf{S})]_{\mathcal{T}}$ contains a Boolean topological semigroup $\underset{\sim}{\mathbf{T}}_k$ that is not residually-k.*

Proof. Fix some $k \geq |S|$ and let $\mathfrak{B}_k$ denote the class of finite binars of cardinality at most k. We will prove the lemma using SILT 7.5 with $\mathfrak{K} = \{\mathbf{S}\}$ and $\mathfrak{L} = \mathfrak{B}_k$.

In [22], Sapir observes that there is a subsemigroup of $\mathbf{S}$ isomorphic to

$$\mathbf{H} := \mathcal{M}\left[\mathbf{C}_p, 2, 2, \begin{pmatrix} e & e \\ e & g \end{pmatrix}\right],$$

where $\mathbf{C}_p$ is the cyclic group (as a multiplicative semigroup) of order p generated by g and with identity element e. For each $t \in \mathbb{N}$, let $\mathbf{G}_t$ denote the relatively free group in $\mathbb{HSP}(\mathbf{C}_p)$ freely generated by the set $\{x_{i,j} : 1 \leq i, j \leq t\}$. As $\mathbf{C}_p$ is abelian, it follows that each element of $\mathbf{G}_t$ can be written uniquely in the form $x_{1,1}^{r_{1,1}} \dots x_{t,t}^{r_{t,t}}$ for some numbers $0 \leq r_{1,1}, r_{1,2}, \dots, r_{t,t} < p$. For $x = x_{1,1}^{r_{1,1}} \dots x_{t,t}^{r_{t,t}} \in \mathbf{G}_t$, we say that the *$(i,j)$-projection* of x (denoted $\pi_{(i,j)}(x)$) is the number $r_{i,j}$.

Now we consider the Rees matrix semigroup $\mathcal{M}[\mathbf{G}_t, t+1, t+1, P^{(t)}]$ where $P^{(t)}$ is the matrix with entries $P^{(t)}_{i,j} = e$ if $1 \in \{i,j\}$ and $P^{(t)}_{i,j} = x_{i-1,j-1}$ otherwise. Sapir shows that $\mathcal{M}[\mathbf{G}_t, t+1, t+1, P^{(t)}] \in \mathbb{ISP}(\mathbf{S})$.

Now let $n = n(k) = 2^k + 1$. We define a new completely simple semigroup $\mathbf{V}_t$ to be the direct product of n copies of $\mathcal{M}[\mathbf{G}_t, t+1, t+1, P^{(t)}]$. There is a convenient Rees matrix description of $\mathbf{V}_t$ for which the sandwich matrix, denoted $Q^{(t)}$, is a $(t+1)^n$ by $(t+1)^n$ matrix of elements of the direct power $\mathbf{G}_t^n := (\mathbf{G}_t)^n$. To aid the description of this semigroup, it is convenient to let $Q^{(t)}$ be indexed by

$\{1,\dots,t+1\}^n \times \{1,\dots,t+1\}^n$ (instead of $\{1,\dots,(t+1)^n\} \times \{1,\dots,(t+1)^n\}$). Then the entry $Q^{(t)}_{(i_1,\dots,i_{t+1}),(j_1,\dots,j_{t+1})}$ equals $\left(P^{(t)}_{i_1,j_1},\dots,P^{(t)}_{i_{t+1},j_{t+1}}\right)$.

Some notation is also convenient for considering elements of $\mathbf{G}_t^n$. Let $\{i_1,\dots,i_\ell\}$ be a subset of $\{1,\dots,n\}$. For $b_1,\dots,b_\ell \in \mathbf{G}_t$, we let $\underline{e}^{b_1,\dots,b_\ell}_{i_1,\dots,i_\ell} \in \mathbf{G}_t^n$ be defined by

$$\underline{e}^{b_1,\dots,b_\ell}_{i_1,\dots,i_\ell}(j) = \begin{cases} b_r & \text{if } j = i_r, \\ e & \text{otherwise.} \end{cases}$$

In $\mathbf{G}_t$, we let the element a_t denote $\prod_{1\leq i,j\leq t} x_{i,j}$. We let $a_{k,t} = (\underline{1}, \underline{e}^{a_t}_1, \underline{1}) \in \mathbf{V}_t$ (where $\underline{1}$ denotes the n-tuple $(1,\dots,1)$). Let N_t denote the normal subgroup of $\mathbf{G}_t^n$ generated by

$$\{\underline{e}^{a_t,a_t^{-1}}_{i,\ j} : i \neq j \text{ and } 1 \leq i,j \leq n\}$$

and let $\mathbf{C}_t := \mathbf{G}_t^n/N_t$. The normal subgroup N_t determines a natural congruence θ_t on $\mathbf{V}_t$ with $(i_1,g,j_1)\,\theta_t\,(i_2,h,j_2)$ if $i_1 = i_2$, $j_1 = j_2$ and $gN_t = hN_t$. We let $\mathbf{T}(t) := \mathbf{V}_t/\theta_t$.

We are going to apply the SILT 7.5 with $\mathbf{X}_t$ chosen as the semigroup $\mathbf{T}(t+1)$. As the element x_t we choose $a_{k,t+1}/\theta_{t+1}$ and as y_t we choose $(\underline{1},\underline{e},\underline{1})/\theta_{t+1}$. The choice of connecting maps will be made at a later part of this proof.

Sapir proves that every congruence θ on $\mathbf{T}(t+1)$ with index at most k has $x_t\,\theta\,y_t$, which establishes part of the SILT 7.5 (ii). He also proves the SILT 7.5 (i) by showing that subsemigroups of $\mathbf{T}(t)$ generated by fewer than t elements lie in $\mathbb{ISP}(\mathbf{S})$. We give here a thorough exposition of Sapir's proof of this last fact (the first fact is more easily proved and is given in sufficient detail in [22]).

We first note that there is an easy way to visualise $\mathbf{T}(t)$ as a Rees matrix semigroup. Fix some $n-1$ tuple z of elements of $\{(i,j) : 1 \leq i,j \leq t\}$.

Claim 1. For each $g \in G_t^n$ there is a (unique) element $\mathrm{rep}_z(g) \in gN_t$ (the *z-representative*) with the property that for each $i < n$, the $z(i)^{\text{th}}$ projection of $\mathrm{rep}_z(g)(i)$ is 0.

Proof. Recall that the element g is an n-tuple. We can choose $\mathrm{rep}_z(g)$ to be the element obtained by multiplying g by $\left(\underline{e}^{a_t,a_t^{-1}}_{n,i}\right)^{\pi_{z(i)}(g(i))} \in N_t$ for every $i < n$. For uniqueness, it suffices to note that the identity element $\underline{e}$ is the only element of N_t such that on each of the coordinates $1,2,\dots,n-1$ there is some (i,j)-projection equal to 0. □

For fixed z, the subset $\mathrm{rep}_z(\mathbf{G}_t^n) := \{\mathrm{rep}_z(g) : g \in G_t^n\}$ is routinely verified to be a subgroup of $\mathbf{G}_t^n$ and because it contains one and only one element from each coset modulo N_t, we have $\mathrm{rep}_z(\mathbf{G}_t^n) \cong \mathbf{G}_t^n/N_t$. Let $\mathrm{rep}_z(Q^{(t)})$ denote the matrix obtained from $Q^{(t)}$ by replacing each entry by its z-representative. Then $\mathbf{T}(t) \cong \mathcal{M}[\mathrm{rep}_z(\mathbf{G}_t^n), (t+1)^n, (t+1)^n, \mathrm{rep}_z(Q^{(t)})]$. We let this isomorph of $\mathbf{T}(t)$ be denoted $\mathrm{rep}_z(\mathbf{T}(t))$.

Note that $\mathrm{rep}_z(Q^{(t)})$ agrees with $Q^{(t)}$ on some entries and disagrees on others, depending on the choice of z. The idea is that for a small enough subsemigroup of $\mathbf{T}(t)$, we can find a tuple z so that required elements of $\mathrm{rep}_z(Q^{(t)})$ actually agree with those in $Q^{(t)}$.

Let $m < t$ and $\mathbf{Y}$ be an m-generated subsemigroup of $\mathbf{T}(t)$ and $\{(i_\ell, g_\ell N_t, j_\ell) : 1 \leq \ell \leq m\}$ be a set of generators. We want $\mathbf{Y} \in \mathbb{ISP}(\mathbf{S})$. For $1 \leq r, s \leq m$, let $h_{r,s}$ denote the matrix entry $Q^{(t)}_{i_r, j_s}$. For each $\ell < n$, let K_ℓ denote the set of all pairs (i,j) for which there is r, s such that $\pi_{(i,j)}(h_{r,s}(\ell)) \neq 0$.

For any $\ell < n$ observe that on the ℓ^{th} coordinate, an element $h_{r,s}$ is either e, or one of the generators of $\mathbf{G}_t$. Hence for each $h_{r,s}$, there is at most one possible pair (i,j) for which $\pi_{(i,j)}(h_{r,s}(\ell)) \neq 0$. So $|K_\ell| \leq m^2 < t^2$. This implies that for each $\ell < n$ we can find a pair $(\gamma_\ell, \lambda_\ell)$ such that the ℓ^{th} coordinate of each $h_{r,s}$ has $(\gamma_\ell, \lambda_\ell)$ projection equal to 0. Let z denote an $n-1$ tuple of such choices. Then for each $h_{r,s}$, we have $\mathrm{rep}_z(h_{r,s}) = h_{r,s}$. Let $\mathrm{rep}_z(\mathbf{Y})$ be the subsemigroup of $\mathrm{rep}_z(\mathbf{T}(t))$ corresponding to $\mathbf{Y}$. The submatrix of $\mathrm{rep}_z(Q^{(t)})$ corresponding to the left and right indices of elements from $\mathrm{rep}_z(\mathbf{Y})$ are the elements $\mathrm{rep}_z(h_{r,s}) = h_{r,s}$. Hence this submatrix corresponds to the same submatrix of $Q^{(t)}$ and therefore $\mathrm{rep}_z(\mathbf{Y})$ is also a subsemigroup of $\mathbf{V}_t$, which lies in $\mathbb{ISP}(\mathbf{S})$ as required. So part (i) of the SILT 7.5 is established.

We now need to supply suitable connecting maps. We first let $\psi_t : \mathbf{G}_{t+1} \to \mathbf{G}_t$ be the surjective homomorphism defined by $x_{i,j} \mapsto x_{i,j}$ if $t+1 \notin \{i,j\}$ and $x_{i,j} \mapsto e$ otherwise. Let $\bar{\psi}_t$ denote the surjective homomorphism $\langle \psi_t, \dots, \psi_t \rangle : \mathbf{G}_{t+1}^n \to \mathbf{G}_t^n$. Note that as $\bar{\psi}_t(\underline{e}_{i,j}^{a_{t+1}, a_{t+1}^{-1}}) = \underline{e}_{i,j}^{a_t, a_t^{-1}}$ we have $\bar{\psi}_t(N_{t+1}) = N_t$. Also note that $\bar{\psi}(\underline{e}_1^{a_{t+1}}) = \underline{e}_1^{a_t}$. For each $i \in \mathbb{N}$, let $\nu_i : \mathbf{G}_i^n \to \mathbf{G}_i^n / N_i$ denote the natural map. Now $\ker(\nu_t \circ \bar{\psi}_t) \geq \ker \nu_{t+1}$ (because $\bar{\psi}_t(N_{t+1}) = N_t$), so by Lemma 7.10 we have that the map $\rho_t : \mathbf{G}_{t+1}^n / N_{t+1} \to \mathbf{G}_t^n / N_t$ defined by $x N_{t+1} \mapsto \bar{\psi}_t(x) N_t$ is a surjective homomorphism mapping $\underline{e}_1^{a_{t+1}} N_{t+1}$ to $\underline{e}_1^{a_t} N_t$.

Now recall again that a quotient $\mathbf{G}/N$ of a group $\mathbf{G}$ induces a corresponding quotient of a completely simple semigroup over $\mathbf{G}$. So the quotient of $\mathbf{C}_{t+1}$ corresponding to the homomorphism ρ_t onto $\mathbf{C}_t$ induces a corresponding quotient of $\mathbf{T}(t+1)$. This quotient corresponds to the map $(i, g, j) \mapsto (i, \rho_t(g), j)$ into the Rees matrix semigroup obtained from $\mathbf{T}(t{+}1)$ by replacing the group $\mathbf{C}_{t+1}$ by $\mathbf{C}_t$, and replacing each entry of the sandwich matrix of $\mathbf{T}(t+1)$ by its image under ρ_t. Let us denote this semigroup by $\mathbf{T}^\flat(t+1)$ and the corresponding surjective homomorphism by η_t. Now $\mathbf{T}^\flat(t+1)$ is not quite isomorphic to $\mathbf{T}(t)$—although the underlying group $\rho_t(\mathbf{C}_{t+1}) = \mathbf{C}_t$ is the same, the sandwich matrix of $\mathbf{T}^\flat(t+1)$ has dimension $(t+2)^n \times (t+2)^n$, while that of $\mathbf{T}(t)$ is only $(t+1)^n \times (t+1)^n$. In fact, we observe that $\mathbf{T}(t)$ is the subsemigroup of $\mathbf{T}^\flat(t+1)$ consisting of those elements whose left and right indices do not equal $t+2$ on any coordinate.

Let $\mu : \{1, \dots, t+2\} \to \{1, \dots, t+1\}$ be defined by $\mu(t+2) = 1$ and $\mu(i) = i$ otherwise and then let $\bar{\mu} : \{1, \dots, t+2\}^n \to \{1, \dots, t+1\}^n$ be defined by $\bar{\mu}(\underline{i})(j) := \mu(\underline{i}(j))$. Now we may define $\gamma_t : \mathbf{T}^\flat(t+1) \to \mathbf{T}(t)$ by

$$\gamma_t(i, g, j) = (\bar{\mu}(i), g, \bar{\mu}(j)).$$

Recall that $\underline{1}$ denotes the constant n-tuple $\langle 1, \dots, 1 \rangle$.

Claim 2. γ_t is a surjective homomorphism fixing $(\underline{1}, \underline{e}_n^{a_t} N_t, \underline{1})$.

PROOF. This is certainly a surjective map and it fixes the element $(\underline{1}, \underline{e}_n^{a_t} N_t, \underline{1})$ because the left and right index $\underline{1}$ is fixed by $\bar{\mu}$ (indeed, γ_t fixes all of $\mathbf{T}(t)$). We need to show that it is a homomorphism.

Consider

$$\begin{aligned} \gamma_t((i_1, g, j_1) \cdot (i_2, h, j_2)) &= \gamma_t((i_1, g\rho_t(Q^{(t+1)}_{i_2,j_1} N_{t+1})h, j_2)) \\ &= (\bar{\mu}(i_1), g\rho_t(Q^{(t+1)}_{i_2,j_1} N_{t+1})h, \bar{\mu}(j_2)) \\ &= (\bar{\mu}(i_1), g\bar{\psi}_t(Q^{(t+1)}_{i_2,j_1})N_t h, \bar{\mu}(j_2)) \end{aligned}$$

while

$$\begin{aligned} \gamma_t((i_1, g, j_1)) \cdot \gamma_t((i_2, h, j_2)) &= (\bar{\mu}(i_1), g, \bar{\mu}(j_1)) \cdot (\bar{\mu}(i_2), h, \bar{\mu}(j_2)) \\ &= (\bar{\mu}(i_1), gQ^{(t)}_{\bar{\mu}(i_2),\bar{\mu}(j_1)} N_t h, \bar{\mu}(j_2)) \end{aligned}$$

Therefore it suffices to show that

$$\bar{\psi}_t(Q^{(t+1)}_{i_2,j_1}) = Q^{(t)}_{\bar{\mu}(i_2),\bar{\mu}(j_1)}. \tag{7.1}$$

Now let $\ell \leq n$ and consider $i_2(\ell)$ and $j_1(\ell)$. We have

$$\begin{aligned} Q^{(t)}_{\bar{\mu}(i_2),\bar{\mu}(j_1)}(\ell) &= P^{(t)}_{\mu(i_2(\ell)),\mu(j_1(\ell))} \\ &= \begin{cases} P^{(t)}_{i_2(\ell),j_1(\ell)} & \text{if } t+2 \notin \{i_2(\ell), j_1(\ell)\}, \\ e & \text{otherwise,} \end{cases} \end{aligned}$$

while

$$\begin{aligned}\bar{\psi}_t(Q^{(t+1)}_{i_2,j_1})(\ell) &= \psi_t(Q^{(t+1)}_{i_2,j_1}(\ell))\\ &= \psi_t(P^{(t+1)}_{i_2(\ell),j_1(\ell)})\\ &= \begin{cases} P^{(t)}_{i_2(\ell),j_1(\ell)} & \text{if } t+2 \notin \{i_2(\ell), j_1(\ell)\},\\ e & \text{otherwise.}\end{cases}\end{aligned}$$

Hence we see that Equation (7.1) holds. So γ_t is a homomorphism and the proof of Claim 2 is complete. □

Now we find that the map $\phi_t : \mathbf{T}(t+2) \to \mathbf{T}(t+1)$ defined by $\gamma_{t+1} \circ \eta_{t+1}$ satisfies all of the conditions of the SILT 7.5, with $\boldsymbol{\mathcal{K}} = \{\mathbf{S}\}$, $\boldsymbol{\mathcal{L}} := \boldsymbol{\mathcal{B}}_k$, $\mathbf{X}_t := \mathbf{T}(t+1)$, $x_t := (\underline{1}, \underline{e}_1^{a_{t+1}} N_{t+1}, \underline{1})$ and $y_t := (\underline{1}, N_{t+1}, \underline{1})$. Hence the inverse limit, $\underset{\sim}{\mathbf{T}}_k$, is not residually-$k$, while $\mathbf{T}_k \in \mathbb{ISP}(\mathbf{S})$. □

The following Theorem now follows from Lemma 7.14 and Lemma 7.11 in the same way that Theorem 7.12 follows from Lemma 7.7 and Lemma 7.11.

Theorem 7.15. *Let* $\mathbf{S}$ *be a finite completely simple semigroup failing* ♡*. There is a Boolean topological completely simple semigroup* $\underset{\sim}{\mathbf{T}} \in [\mathbb{ISP}(\mathbf{S})]_{\mathcal{T}}$ *with* $\mathrm{rb}(\underset{\sim}{\mathbf{T}}) = \aleph_0$.

This theorem also completes the proof of Theorem 7.13.

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