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On locally finite varieties with undecidable equational theory

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In several recent papers [2, 3, 4, 10, 14] (and in the doctoral thesis of B. Wells [13]) examples have been found of varieties, \mathbf{V} , in which every finitely presented algebra has a decidable word problem but for which no uniform algorithm exists which solves the word problem in an arbitrary algebra from \mathbf{V} (the uniform word problem). A second kind of example presented in the above papers are varieties \mathbf{V} in which every finitely generated \mathbf{V} -free algebra has a decidable word problem but the equational theory of \mathbf{V} is undecidable. Varieties with this property are said to be *pseudorecursive* [14]. A further variation on these ideas are pseudorecursive varieties in which every finitely presented algebra has decidable word problem; we will call varieties with these properties *strongly pseudorecursive*. Examples of strongly pseudorecursive varieties are also presented in the above papers. It is well known that the undecidability of the equational theory of a variety implies the undecidability of the uniform word problem for that variety. Thus a strongly pseudorecursive variety is also a variety of the first kind described above. Of particular note is the example of Delić [4] which is a *finitely based* strongly pseudorecursive variety though its basis is very complicated. On the other hand the identity basis for the strongly pseudorecursive variety presented in [3] is infinite but of quite a simple form. The reader is referred to [14] for an interesting discussion on some of the implications of the existence of pseudorecursive varieties.

As noted in [14] (see Remark 11.2.4) it is easy to establish the *existence* of many (strongly) pseudorecursive varieties, even varieties of groups. However all explicitly described examples of strongly pseudorecursive varieties existing in the literature are either non associative varieties of groupoids or varieties of groupoids (sometimes associative) with additional operations. Here we combine a method used in many of the above papers with some existing results to construct comparatively basic strongly pseudorecursive varieties of semigroups (of type $\langle 2 \rangle$) and a strongly pseudorecursive variety of groups (of type $\langle 2, 1, 0 \rangle$). Furthermore, rather than presenting particular examples, we describe a large class of examples.

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1. The method

The following facts are well known (see Remark 11.2.2 [14] for example). The first part follows immediately from the fact that a finitely generated, locally finite algebra is finite; the second part holds because locally finite varieties are generated by their finite members.

Lemma 1. *Every finitely presented algebra in a locally finite variety has a decidable word problem. If a locally finite variety is finitely based then it has decidable equational theory.*

This lemma implies that amongst locally finite varieties, the properties of being pseudorecursive and strongly pseudorecursive are equivalent.

We now attempt to encapsulate the method referred to above. Let \mathbf{V} be a variety of type \mathcal{F} and $Id(\mathbf{V})$ be the set of identities of \mathbf{V} in some fixed countably infinite set of variables X . If $t(x)$ is a term involving a variable x (and possibly others) then an infinite list $T = \{t(x), t_1, t_2, \dots\}$ of type \mathcal{F} terms involving increasing numbers of variables x_1, x_2, \dots is said to be *strongly independent* with respect to \mathbf{V} (or strongly independent with respect to Σ , a basis for $Id(\mathbf{V})$) if for every distinct pair of subsets $P, Q \subseteq \mathbb{N}$ the identities $\Sigma \cup \{t(x) \approx t(y)\} \cup \{t_n \approx t(x) : n \in P\}$ and $\Sigma \cup \{t(x) \approx t(y)\} \cup \{t_n \approx t(x) : n \in Q\}$ determine distinct subvarieties of \mathbf{V} (here $t(y)$ is the term obtained from $t(x)$ by replacing every occurrence of x in $t(x)$ by the variable y).

Lemma 2. *Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a primitive recursive function such that $A = \{\phi(n) : n \in \mathbb{N}\}$ is a recursively enumerable but not recursive set and let \mathbf{V} be a variety of type \mathcal{F} algebras with recursive basis of identities Σ . If $T = \{t(x), t_1, t_2, \dots\}$ is a countably infinite recursive list of type \mathcal{F} terms strongly independent with respect to \mathbf{V} , then the set of identities*

$$\Phi(\phi, \Sigma, T) = \Sigma \cup \{t(x) \approx t(y)\} \cup \{t_{\phi(n)} \approx t(x_n) : n \in \mathbb{N}\}$$

is recursive and determines a subvariety of \mathbf{V} with undecidable equational theory. (If we assume that at least one term of the form $t_{\phi(n)}$ does not contain the variable x_n then the identity $t(x) \approx t(y)$ is an obvious consequence and is not required in the definition of $\Phi(\phi, \Sigma, T)$.)

Proof. Firstly $\Phi(\phi, \Sigma, T)$ is recursive: since Σ is recursive we need only check identities of the form $t_m \approx t(x_n)$ and clearly such an identity is contained in $\Phi(\phi, \Sigma, T)$ if and only if m is the number $\phi(n)$, which can be effectively calculated.

The identities $\{t(z) \approx t_{\phi(n)} : n \in \mathbb{N}\}$ (where z is a fixed variable) are easily seen to be a consequence of $\Phi(\phi, \Sigma, T)$. Since T is strongly independent with respect to \mathbf{V} , $t(z) \approx t_m \in Id(\mathbf{V}')$ if and only if $m \in A$. However the set A is not recursive, and so neither can be $Id(\mathbf{V}')$. \square

The notation $\Phi(\phi, \Sigma, T)$ of the above lemma (where ϕ is a fixed primitive recursive function with non recursive range) will be retained below.

By combining Lemma 1 with the previous lemma we obtain the following

Corollary 1. *If \mathbf{V} is a locally finite variety with recursive basis Σ and T is a countably infinite, recursive set of terms T that is strongly independent with respect to \mathbf{V} then $\Phi(\phi, \Sigma, T)$ determines a recursively based, strongly pseudorecursive subvariety of \mathbf{V} .*

2. Applying the method

In [6], countably infinite, recursive sets of terms are shown to be strongly independent with respect to many varieties generated by finite semigroups. By Corollary 1 we correspondingly obtain many examples of strongly pseudorecursive semigroup varieties. We present just one of the possibilities below, however all results in [6] concerning locally finite varieties with uncountably many subvarieties provide corresponding examples of strongly pseudorecursive varieties. Recall that the index, if it exists, of a semigroup \mathbf{S} is the smallest number n such that $\mathbf{S} \models x^n \approx x^{n+m}$ for some m ; if \mathbf{S} generates a locally finite variety then it is easily seen to have an index. Let \mathcal{T} be the list of semigroup words $\{xy_1y_2y_3y_4xx_1x_2x_1x_3x_4x_3 \cdots x_{2n-1}x_{2n}x_{2n-1}yy_5y_6y_7y_8y : n \in \mathbb{N}\}$.

Corollary 2. *If \mathbf{S} is a monoid generating a locally finite semigroup variety with basis of identities Σ and with index more than two then the following are equivalent:*

- (i) *both $xyx \approx yxx$ and $xyx \approx xxy$ fail on \mathbf{S} ;*
- (ii) *\mathbf{S} generates a variety with a nonfinitely based subvariety;*
- (iii) *\mathcal{T} is strongly independent with respect to the variety generated by \mathbf{S} ;*
- (iv) *there is a pseudorecursive subvariety of the variety generated by \mathbf{S} ;*
- (v) *$\Phi(\phi, \Sigma, \mathcal{T})$ determines a strongly pseudorecursive variety (and is recursive basis if and only if Σ is recursive).*

Proof. The equivalence of conditions (i), (ii) and (iii) is shown in [6]. The result now follows from Corollary 1 and the second part of Lemma 1. \square

One of many possible varieties satisfying the conditions of Corollary 2 is that defined by $\Xi = \{(xy)z \approx x(yz), x^3 \approx x^4, xy_1xy_2x \approx y_1y_2xxx, yxxx \approx xyxx \approx xxyx \approx xxxy\}$. This is generated by a 16 element semigroup $(S(\{abab, aabb, abba\}))$ in the terminology of [7] with identity element and with index of three (these facts are proved in [7]). Thus by Corollary 2, $\Phi(\phi, \Xi, \mathcal{T})$ determines a recursively based, strongly pseudorecursive variety.

The existence of a (strongly) pseudorecursive group variety is noted in [14] and the possible existence of a recursively based example is an open question in [14]

(see Remark 11.4.2). As we now show, Corollary 1 is easily combined with existing results to provide such an example. Let $[x, y]$ be the group theoretic commutator $x^{-1}y^{-1}xy$ and for $n \geq 3$, let $[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$. In [12] it is shown that the group words

$$\mathcal{W} = \{x^{16}, [[y_1, y_2, y_3], [x_1, x_2], [x_3, x_4], \dots, [x_{2i-1}, x_{2i}], [y_1, y_2, y_3]] : i \in \mathbb{N}\}$$

are strongly independent with respect to the (locally finite) group variety \mathbf{V} defined within the variety of groups by the identity $w \approx 1$ where w is the word $x^{16}[[z_1, z_2, z_3], [z_4, z_5, z_6], [z_7, z_8]]$ (a similar result is found in [11]). Therefore Corollary 2 implies that $\Phi = \Phi(\phi, \{(xy)z \approx x(yz), x1 \approx x, xx^{-1} \approx x^{-1}x \approx w \approx 1\}, \mathcal{W})$ determines a strongly pseudorecursive variety.

Using the main result of [5], an equivalent variety to that given by Φ can be described (recursively) using an infinite set of groupoid identities all of some particular general form (that is, without any “sporadic” identities). In this description, the groupoid operation, “ \div ” say, models right division in the corresponding groups and the standard group operations can be recovered by replacing $x \div y$ with xy^{-1} . First let \mathbf{V}_i be the variety defined (as a group variety) by the identity $v_i[[y_1, y_2, y_3], [x_1, x_2], [x_3, x_4], \dots, [x_{2\phi(i)-1}, x_{2\phi(i)}], [y_1, y_2, y_3]] \approx 1$ where v_i is the group word obtained from w by replacing the variable x by x_i . So the variety defined by the identities Φ above is simply the variety $\bigwedge_{i \in \mathbb{N}} \mathbf{V}_i$. However by the result of [5], each variety \mathbf{V}_i is equivalent to a groupoid variety (of type $\langle 2 \rangle$), \mathbf{V}'_i say, given by a single identity $p_i \approx y$ where p_i is a groupoid term, y is a variable and p_i depends only on the form of the corresponding identity defining \mathbf{V}_i within the variety of groups. It follows that the variety defined by Φ is equivalent to the variety $\bigwedge_{i \in \mathbb{N}} \mathbf{V}'_i$ defined by the identities $\{p_i \approx y : i \in \mathbb{N}\}$. Another possible description of this variety is as a semigroup variety with just one “sporadic” identity (necessary to ensure associativity).

3. The membership problem

There is an obvious algorithm that determines membership in the class \mathbf{V}_{fin} of finite members of a finitely based variety \mathbf{V} (simply check the basis identities for satisfaction). In [8] however, Jeżek constructs a variety \mathbf{V} with *recursive* equational theory but such that \mathbf{V}_{fin} has undecidable membership problem. To finish this note, we briefly examine a connection between this problem and the ideas presented above. In a strongly pseudorecursive variety, there can be no general method of constructing the n -generated free algebra for arbitrary n ; this would constitute a uniform algorithm for determining when an identity is satisfied by the variety. Now let \mathbf{V} be a (locally finite) variety such that there is a computable function $f_{\mathbf{V}} : \mathbb{N} \rightarrow \mathbb{N}$ with the property that for all $n \in \mathbb{N}$, $f_{\mathbf{V}}(n)$ is greater than the cardinality

of the n generated \mathbf{V} -free algebra (if \mathbf{V} is a subvariety of a variety generated by a finite algebra \mathbf{A} then we can take $f_{\mathbf{V}}(n) = |A|^{|A|^n}$; see Exercise II.11.5 of [1]). If there is an algorithm for determining membership in \mathbf{V}_{fin} then we can effectively list all algebras of the same type as \mathbf{V} which have fewer than $f_{\mathbf{V}}(n)$ elements, decide which are members of \mathbf{V} , then decide which of these are generated by at most n elements. Up to isomorphism, the n -generated \mathbf{V} -free algebra can then be found by selecting the largest remaining algebra. It follows that a strongly pseudorecursive subvariety of a finitely generated variety has an undecidable membership problem. The variety $\Phi(\phi, \Xi, \mathcal{T})$ described above is one such example which has a recursive basis of identities. On the other hand, if \mathbf{V} is a variety (not necessarily locally finite) for which \mathbf{V}_{fin} has decidable uniform word problem then we can decide which finite algebras are contained in \mathbf{V} using the following algorithm: for an arbitrary finite algebra \mathbf{A} of the same type as \mathbf{V} , construct the set $\{r_1, r_2, \dots, r_m\}$ of all possible relations of the form $\rho(a_1, \dots, a_n) = b$, where $a_1, a_2, \dots, a_n, b \in A$, ρ is an n -ary operation of \mathbf{A} and $\rho(a_1, \dots, a_n) = b$ in \mathbf{A} ; \mathbf{A} is contained in \mathbf{V} if and only if *none* of the quasi-identities $(r_1 \ \& \ r_2 \ \& \ \dots \ \& \ r_m) \rightarrow a \approx b$, where a and b are distinct elements of \mathbf{A} , are satisfied by \mathbf{V}_{fin} . This can be effectively decided since the decidability of the uniform word problem for \mathbf{V}_{fin} is equivalent to the decidability of the quasi-identity theory of \mathbf{V}_{fin} (see [9] for example). Hence if the membership problem is undecidable for the class of finite members of a variety \mathbf{V} , then the uniform word problem is undecidable for \mathbf{V}_{fin} as well.

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