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A note on *HSI*-algebras and counterexamples to Wilkie's identity

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DEFINITION 1. Let the following set of "High School Identities"

$$x+y\approx y+x,\qquad x+(y+z)\approx(x+y)+z,$$
$$x\cdot y\approx y\cdot x,\qquad x\cdot(y\cdot z)\approx(x\cdot y)\cdot z,$$
$$x\cdot 1\approx x,\qquad x\cdot(y+z)\approx x\cdot y+x\cdot z,$$
$$1^x\approx 1,\qquad x^1\approx x,$$
$$x^{y+z}\approx x^y\cdot x^z,\qquad (x\cdot y)^z\approx x^z\cdot y^z,$$
$$(x^y)^z\approx x^{y\cdot z}.$$

be denoted by *HSI* and let $\mathscr{L}$ be the language $\{+,\cdot,\uparrow,1\}$. An *HSI*-algebra is a $\mathscr{L}$-algebra satsifying *HSI* and in particular, let $\mathbf{N}$ denote $\langle N,+,\cdot,\uparrow,1\rangle$, the *HSI*-free algebra freely generated by $\emptyset$.

In 1980, Wilkie [5] showed that the identity $W(x,y)=$

$$((1+x)^x+(1+x+x^2)^x)^y\cdot((1+x^3)^y+(1+x^2+x^4)^y)^x$$
$$\approx((1+x)^y+(1+x+x^2)^y)^x\cdot((1+x^3)^x+(1+x^2+x^4)^x)^y,$$

satisfied by $\mathbf{N}$, was derivable from *HSI*, providing a negative answer to a question of Tarski's regarding whether *HSI* was a basis for the equational theory of $\mathbf{N}$ (Tarski's High School Problem). A simple proof of this result can be provided by the existence of finite models of *HSI* that do not satisfy $W(x,y)$. Gurevic [2] provided the first such example with a rather complicated 59-element algebra and quite a few further reductions have been made since then. At this stage it is not

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known what the smallest counterexample to $W(x, y)$ is, but I will improve the bounds of between 7 and 15 as established by Burris and Lee in [1] to be between 8 and 14. Following the notation of [1], HSI-algebras not satisfying $W(x, y)$ will be called G-algebras (after Gurevic) and the polynomial parts of $W(x, y)$ will be denoted P, Q, R, S respectively.

Central to the arguments deployed in [1] is the behaviour of elements (called integers) of the form

$$\overbrace{1 + 1 + \cdots + 1}^{n}$$

within HSI-algebras. In particular, the integers of an HSI-algebra will be isomorphic to the quotients of $\mathbf{N}$ modulo Δ or

$$\theta_{a,k} = \{\langle m, n\rangle \in N \times N\colon\ m = n,\ or\ a \leq m, n\ and\ m \equiv n \bmod k\}$$

(written $\mathbf{N}_{\mathbf{a,k}}$) for an unusual collection of $a, k \in N$.

1. The smallest G-algebra has at least 8 elements

The following is one of five problems outlined in [1] (p. 172):

> If $\mathbf{A}$ is an HSI-algebra with an element a such that a generates the reduct of $\mathbf{A}$ to the language $\{+, \cdot, 1\}$ (i.e., every element of $\mathbf{A}$ is a polynomial in a), then does it follow that $\mathbf{A} \models W(a, y) \forall y$?

The following Lemma establishes this result.

LEMMA 1. *Let* $y = n + n_1 x + n_2 x^2 + \cdots + n_m x^m$, *where* $n, n_1, \ldots, n_m$ *are integers. Then* $HSI \vdash W(x, y)$.

Proof. First note that $PS \approx QR \approx 1 + x + x^2 + x^3 + x^4 + x^5$.

Let $u = n_1 + n_2 x + \cdots + n_m x^{m-1}$. Then $y = n + ux$. So

$$\begin{aligned}
&(P^x + Q^x)^y (R^y + S^y)^x \\
&\quad \approx (P^x + Q^x)^n (P^x + Q^x)^{ux} (R^{n+ux} + S^{n+ux})^x \\
&\quad \approx (P^x + Q^x)^n ((P^x + Q^x)^u (R^n R^{ux} + S^n S^{ux}))^x \\
&\quad \approx (P^x + Q^x)^n ((P^x + Q^x)^u R^n R^{ux} + (P^x + Q^x)^u S^n S^{ux})^x
\end{aligned}$$

$$
\begin{aligned}
&\approx (P^x + Q^x)^n(((PR)^x + (QR)^x)^u R^n + ((PS)^x + (QS)^x)^u S^n)^x \\
&\approx (P^x + Q^x)^n(((PR)^x + (PS)^x)^u R^n + ((QR)^x + (QS)^x)^u S^n)^x \\
&\approx (P^x + Q^x)^n(P^{ux}(R^x + S^x)^u R^n + Q^{ux}(R^x + S^x)^u S^n)^x \\
&\approx (P^x + Q^x)^n(R^x + S^x)^{ux}(P^{ux}R^n + Q^{ux}S^n)^x \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^x)^i(Q^x)^{n-i}\right)(R^x + S^x)^{ux}(P^{ux}R^n + Q^{ux}S^n)^x \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^iQ^{n-i})^x(P^{ux}R^n + Q^{ux}S^n)^x\right)(R^x + S^x)^{ux} \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^iQ^{n-i}P^{ux}R^n + P^iQ^{n-i}Q^{ux}S^n)^x\right)(R^x + S^x)^{ux} \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^iQ^{n-i}R^{n-i}P^{ux}R^i + P^iS^iQ^{n-i}Q^{ux}S^{n-i})^x\right)(R^x + S^x)^{ux} \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^iP^{n-i}S^{n-i}P^{ux}R^i + Q^iR^iQ^{n-i}Q^{ux}S^{n-i})^x\right)(R^x + S^x)^{ux} \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(P^nS^{n-i}P^{ux}R^i + Q^nR^iQ^{ux}S^{n-i})^x\right)(R^x + S^x)^{ux} \\
&\approx \left(\sum_{i=0}^{n} \tbinom{n}{i}(S^x)^{n-i}(R^x)^i(P^nP^{ux} + Q^nQ^{ux})^x\right)(R^x + S^x)^{ux} \\
&\approx (R^x + S^x)^n(P^nP^{ux} + Q^nQ^{ux})^x(R^x + S^x)^{ux} \\
&\approx (P^{n+ux} + Q^{n+ux})^x(R^x + S^x)^{n+ux} \\
&\approx (P^y + Q^y)^x(R^x + S^x)^y. \qquad \square
\end{aligned}
$$

THEOREM 2. *If* **A** *is a* G*-algebra, then* $|A| \geq 8$.

This uses several results established in [1]:

- A G-algebra must have at least 3 integers,
- If $W(x, y)$ fails at a, b then a and b are distinct non integer elements,
- If **A** is a 3 integer G-algebra (i.e., **A** has exactly 3 integers) such that $W(x, y)$ fails at a, b, then there are at least 7 distinct elements in **A** that are polynomials in a,
- If **A** is a 4 integer G-algebra such that $W(x, y)$ fails at a, b, then either there are 7 distinct elements in **A** that are polynomials in a, or there are 6 distinct polynomials in a and b is not among them.

One needs only consider G-algebras with 3, 4, or 5 integers, since the second point above ensures that a 6 integer G-algebra must have at least 8 elements. In the case of 3 integer G-algebras, Lemma 1 shows that b must be another distinct element, so the result holds. To establish the case for 4 integer G-algebras, one needs to show that there is always a seventh distinct polynomial in a, and then the result follows immediately from Lemma 1. Finally in the case of 5 integer G-algebras, one again needs to show that there is a seventh element that is a polynomial in a and then apply Lemma 1. To establish these two cases, I proceeded in essentially the same manner as Burris and Lee follow to establish the last two points above. This part of the proof is agonizingly tedious (particularly in the case of 4 integers) though fairly simple and will not be given.

2. A 14 element G-algebra

The upper bound of 14 is established by the following 14 element G-algebra:

DEFINITION 2. Define **F** to be the following algebra:

+	1	2	3	4	a	b	c	d	e	f	g	h	i	j
1	2	3	4	4	2	3	d	3	4	4	4	4	4	4
2	3	4	4	4	3	4	3	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
a	2	3	4	4	b	3	b	3	4	4	4	4	4	4
b	3	4	4	4	3	4	3	4	4	4	4	4	4	4
c	d	3	4	4	b	3	b	3	4	4	4	4	4	4
d	3	4	4	4	3	4	3	4	4	4	4	4	4	4
e	4	4	4	4	4	4	4	4	4	g	4	4	4	4
f	4	4	4	4	4	4	4	4	g	4	4	i	4	4
g	4	4	4	4	4	4	4	4	4	4	4	4	4	4
h	4	4	4	4	4	4	4	4	4	i	4	4	4	4
i	4	4	4	4	4	4	4	4	4	4	4	4	4	4
j	4	4	4	4	4	4	4	4	4	4	4	4	4	4

·	1	2	3	4	a	b	c	d	e	f	g	h	i	j
1	1	2	3	4	a	b	c	d	e	f	g	h	i	j
2	2	4	4	4	b	4	b	4	4	4	4	4	4	4
3	3	4	4	4	3	4	3	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	b	3	4	c	b	c	b	4	f	4	4	4	4
b	b	4	4	4	b	4	b	4	4	4	4	4	4	4
c	c	b	3	4	c	b	c	b	4	f	4	4	4	4
d	d	4	4	4	b	4	b	4	4	4	4	4	4	4
e	e	4	4	4	4	4	4	4	4	4	4	j	4	4
f	f	4	4	4	f	4	f	4	4	4	4	4	4	4
g	g	4	4	4	4	4	4	4	4	4	4	4	4	4
h	h	4	4	4	4	4	4	4	j	4	4	4	4	4
i	i	4	4	4	4	4	4	4	4	4	4	4	4	4
j	j	4	4	4	4	4	4	4	4	4	4	4	4	4

↑	1	2	3	4	a	b	c	d	e	f	g	h	i	j
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	e	4	4	4	4	4	4	4	4	4
3	3	4	4	4	f	4	4	4	f	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
a	a	c	c	c	c	c	c	c	c	c	c	c	c	c
b	b	4	4	4	4	4	4	4	4	4	4	4	4	4
c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
d	d	4	4	4	4	4	4	4	h	4	4	4	4	4
e	e	4	4	4	4	4	4	4	4	4	4	4	4	4
f	f	4	4	4	4	4	4	4	4	4	4	4	4	4
g	g	4	4	4	4	4	4	4	h	4	4	4	4	4
h	h	4	4	4	4	4	4	4	4	4	4	4	4	4
i	i	4	4	4	e	4	4	4	4	4	4	4	4	4
j	j	4	4	4	4	4	4	4	4	4	4	4	4	4

I used a computer to verify *HSI* on **F** and $W(x, y)$ fails for the pair a, e. The algebra was found via a simple reduction of the 15 element G-algebra of Lee presented in [1]. In particular the second generator in Lee's model (represented in [1] by b) has been removed in **F** and its role replaced by the element e.

3. The existence of a G-algebra whose integers are $N_{a,k}$ for $k > 1$

A further problem outlined in [1] is the following (p. 166):

Is there a finite G-algebra **A** with the integers of **A** isomorphic to $N_{a,k}$ for some $k > 1$?

DEFINITION 3. Let **G** be type $\mathscr{L}$-algebra defined by the tables below.

+	1	2	3	4	5	a	b	c	d	e	f	g	h	i	j	k	l
1	2	3	4	5	4	2	3	d	3	5	4	5	4	4	4	4	4
2	3	4	5	4	5	3	4	3	4	4	5	4	5	5	5	5	5
3	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
4	5	4	5	4	5	5	4	5	4	4	5	4	5	5	5	5	5
5	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
a	2	3	4	5	4	b	3	b	3	5	4	5	4	4	4	4	4
b	3	4	5	4	5	3	4	3	4	4	5	4	5	5	5	5	5
c	d	3	4	5	4	b	3	b	3	5	4	5	4	4	4	4	4
d	3	4	5	4	5	3	4	3	4	4	5	4	5	5	5	5	5
e	5	4	5	4	5	5	4	5	4	4	h	4	5	5	5	5	5
f	4	5	4	5	4	4	5	4	5	h	4	i	4	4	4	4	4
g	5	4	5	4	5	5	4	5	4	4	i	4	5	5	5	5	5
h	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
i	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
j	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
k	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4
l	4	5	4	5	4	4	5	4	5	5	4	5	4	4	4	4	4

·	1	2	3	4	5	a	b	c	d	e	f	g	h	i	j	k	l
1	1	2	3	4	5	a	b	c	d	e	f	g	h	i	j	k	l
2	2	4	4	4	4	b	4	b	4	4	4	4	4	4	4	4	4
3	3	4	5	4	5	3	4	3	4	4	5	4	5	5	5	5	5
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	4	5	4	5	5	4	5	4	4	5	4	5	5	5	5	5
a	a	b	3	4	5	c	b	c	b	4	f	4	5	5	5	5	5
b	b	4	4	4	4	b	4	b	4	4	4	4	4	4	4	4	4
c	c	b	3	4	5	c	b	c	b	4	f	4	5	5	5	5	5
d	d	4	4	4	4	b	4	b	4	4	4	4	4	4	4	4	4
e	e	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
f	f	4	5	4	5	f	4	f	4	4	5	4	5	5	5	5	5
g	g	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
h	h	4	5	4	5	5	4	5	4	4	5	4	5	5	5	5	5
i	i	4	5	4	5	5	4	5	4	4	5	4	5	5	5	5	5
j	j	4	5	4	5	5	4	5	4	4	5	4	5	5	5	l	5
k	k	4	5	4	5	5	4	5	4	4	5	4	5	5	l	5	5
l	l	4	5	4	5	5	4	5	4	4	5	4	5	5	5	5	5

↑	1	2	3	4	5	a	b	c	d	e	f	g	h	i	j	k	l
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	4	e	4	4	4	4	4	4	4	4	4	4	4
3	3	5	5	5	5	f	5	5	5	f	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
a	a	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
b	b	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
d	d	4	4	4	4	4	4	4	4	g	4	4	4	4	4	4	4
e	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
f	f	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
g	g	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
h	h	5	5	5	5	5	5	5	5	j	5	5	5	5	5	5	5
i	i	5	5	5	5	k	5	5	5	5	5	5	5	5	5	5	5
j	j	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
k	k	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
l	l	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

THEOREM 3. **G** *is a G-algebra whose integers are isomorphic to* $\mathbf{N}_{4,2}$.

To check that **G** was indeed a *G*-algebra, I used a computer to exhaustively verify the axioms of *HSI*. $W(x, y)$ again fails at a, e. □

To construct such a model, first consider the following notions of 'oddness' and 'evenness' of terms as follows:

DEFINITION 4. For a term t of type $\mathscr{L}$ in variables $x_1, x_2, \ldots, x_n$, let $|t|$ denote the value of the function on **N** represented by t at the point $(1, 1, \ldots, 1)$. Let t be called an odd term if $|t|$ is an odd number, and even otherwise.

This notion of oddness and evenness was attached to each subterm of $W(x, y)$ and taking these as elements (such as in [2] or [3]), an algebra was constructed around the integers $\mathbf{N}_{5,2}$, at all times preserving the normal behaviour of oddness and evenness under addition, multiplication and exponentiation. This led to a 29 element *G*-algebra of the required form and then significant reduction led to **G**. Reductions were done by hand and checked for validity by computer, however the behaviour of those elements of **G** expressible as polynomials in a is based on that between analogous elements in **F** and Lee's model.

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