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Generalising congruence regularity for varieties

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Numerous authors have given complete descriptions of those varieties whose members have all congruences determined by the congruence class containing a given nullary element [2], [3], [4], [8] (see also [1]); such varieties are called 0-*regular varieties*. The well known 0-regularity of groups generalises quite successfully to inverse semigroups where every congruence is determined by the congruence classes containing idempotents. In this note we investigate similarly generalised notions of congruence regularity.

Let $\mathbb{V}$ be a variety and $[s,t]$ be a pair of terms in the signature of $\mathbb{V}$ with arity n_1 and n_2 respectively. We say that $\mathbb{V}$ is $[s,t]$-*regular* if the congruences of every algebra $\mathbf{A} \in \mathbb{V}$ are determined by their restriction to the set $\{(s^A(\underline{a}), t^A(\underline{b})) : \underline{a} \in A^{n_1}, \underline{b} \in A^{n_2}\} \subseteq A \times A$. When s is simply a variable we say that $\mathbb{V}$ is t-*regular* and when s is the same term as t, we say that $\mathbb{V}$ is *strongly* t-*regular*. Note that if t is a nullary term, say 0, then the definition of 0-regularity coincides with the standard definition given above.

Theorem 1. *Let $\mathbb{V}$ be a variety and s, t be n_1-ary and n_2-ary terms in the signature of $\mathbb{V}$. The following are equivalent:*

(i) $\mathbb{V}$ *is* $[s,t]$*-regular;*

(ii) *there are natural numbers n and m, binary terms $f_{i,j}$ and $g_{i,k}$ (for $1 \le i \le n$, $1 \le j \le n_1$, $1 \le k \le n_2$), and $2n+2$-ary terms p_ℓ ($1 \le \ell \le m$) such that $\mathbb{V}$ satisfies*

$$s_i(x,x) \approx t_i(x,x),$$
$$x \approx p_1(x,y,s_1(x,y),\dots,s_n(x,y),t_1(x,y),\dots,t_n(x,y)),$$
$$p_i(x,y,t_1(x,y),\dots,t_n(x,y),s_1(x,y),\dots,s_n(x,y))$$
$$\approx p_{i+1}(x,y,s_1(x,y),\dots,s_n(x,y),t_1(x,y),\dots,t_n(x,y)),$$
$$p_m(x,y,t_1(x,y),\dots,t_n(x,y),s_1(x,y),\dots,s_n(x,y)) \approx y,$$

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where s_i *denotes* $s(f_{i,1},\ldots,f_{i,n_1})$ *and* t_i *denotes* $t(g_{i,1},\ldots,g_{i,n_2})$*;*

(iii) *there is a natural number* n *such that* $\mathbb{V}$ *satisfies*

$$\mathop{\&}_{1\le i\le n} s(f_{i,1}(x,y),\ldots,f_{i,n_1}(x,y)) \approx t(g_{i,1}(x,y),\ldots,g_{i,n_2}(x,y)) \;\leftrightarrow\; x\approx y.$$

Proof. The proof is an elementary variation of the standard techniques used to establish conditions for 0-regularity. Let $\mathbb{V}$ be $[s,t]$-regular and $F(\{x,y\})$ be the 2-generated $\mathbb{V}$-free algebra. By $[s,t]$-regularity, the principle congruence $\Theta\langle x,y\rangle$ is equal to

$$\bigvee_{(s(f_1(x,y),\ldots,f_{n_1}(x,y)),t(g_1(x,y),\ldots,g_{n_2}(x,y)))\in\Theta\langle x,y\rangle} \Theta\langle s(f_1(x,y),\ldots,f_{n_1}(x,y)),t(g_1(x,y),\ldots,g_{n_2}(x,y))\rangle$$

which by the compactness of $\Theta\langle x,y\rangle$ is equivalent to

$$\Theta\langle x,y\rangle = \bigvee_{1\le i\le n} \Theta\langle s(f_{i,1}(x,y),\ldots,f_{i,n_1}(x,y)),t(g_{i,1}(x,y),\ldots,g_{i,n_2}(x,y))\rangle \tag{1}$$

for some finite subset $\{(s(f_{i,1}(x,y),\ldots,f_{i,n_1}(x,y)),t(g_{i,1}(x,y),\ldots,g_{i,n_2}(x,y))) : 1\le i\le n\}$ of the $\Theta\langle x,y\rangle$-related pairs.

Now notice that $F(\{x,y\})/\Theta\langle x,y\rangle$ is exactly the 1-generated free algebra in $\mathbb{V}$ and so we must have $s(f_{i,1}(x,x),\ldots,f_{i,n_1}(x,x)) = t(g_{i,1}(x,x),\ldots,g_{i,n_2}(x,x))$ in $F(\{x\})$. That is, $s(f_{i,1}(x,x),\ldots,f_{i,n_1}(x,x)) \approx t(g_{i,1}(x,x),\ldots,g_{i,n_2}(x,x))$ is satisfied by $\mathbb{V}$. The standard techniques on equation (1) now give the remaining equations for part (ii).

To prove (iii) from (ii), note that setting

$$s(f_{i,1}(x,y),\ldots,f_{i,n_1}(x,y)) = t(g_{i,1}(x,y),\ldots,g_{i,n_2}(x,y))$$

in the identities of (ii) yields $x=y$. Thus, taking the same $f_{i,j}$ and $g_{i,j}$ in part (iii), one half of the implication holds. The reverse implication follows immediately from the first series of identities in part (ii).

Finally (i) follows from (iii) almost trivially since for any congruence θ we have $a\theta b$ if and only if $\&_{1\le j\le n} s(f_{i,1}(a,b),\ldots,f_{i,n_1}(a,b)) = t(g_{i,1}(a,b),\ldots,g_{i,n_2}(a,b))$. □

Note that if t is a nullary term, say 0, then the conditions of (iii) for t-regularity coincide with the implicational form of Fichtner's description [3] of 0-regular varieties as presented in [1] and [5]. Note also that every variety is strongly x-regular (x is a variable) and this is reflected in the theorem by taking $f_{1,1}(x,y)=x$, $g_{1,1}(x,y)=y$. In general we will say that a variety is *trivially* $[s,t]$*-regular* (where s and t are n_1-ary and n_2-ary terms respectively) if for every $\mathbf{A}\in\mathbb{V}$ and every $a\in A$ there are $\underline{b}\in A^{n_1}$ and $\underline{c}\in A^{n_2}$ such that $s^A(\underline{b}) = a = t^A(\underline{c})$.

An I-semigroup is an algebra $\mathbf{A} := \langle A, \cdot, ^{-1} \rangle$ of type $\langle 2, 1\rangle$ such that $\langle A, \cdot \rangle$ is a semigroup and $\mathbf{A} \models \{xx^{-1}x \approx x, (x^{-1})^{-1} \approx x\}$ [6]. Every semigroup satisfying $(\forall x)(\exists y)\ xyx \approx x$ (this property is also called regular) can be made into a I-semigroup (while every I-semigroup has a semigroup reduct satisfying the law $(\forall x)(\exists y)\ xyx \approx x$). The following result is well known.

Example 2. *The variety of I-semigroups is $x^{-1}x$-regular.*

Take $f_{1,1}(x, y) := x^{-1}x$, $f_{2,1}(x, y) := xy^{-1}$, $g_{1,1}(x, y) := y$ and $g_{2,1}(x, y) := y^{-1}$ and use part (iii); or use these and the terms $p_1(x, y, a, b, c, d) := xa$ and $p_2(x, y, a, b, c, d) := by$ in part (ii). □

The class of I-semigroups includes the class of inverse semigroups and completely regular semigroups (that is, unions of groups) considered in the type $\langle 2, 1\rangle$. For completely regular semigroups of period n, the inverse x^{-1} can be written as x^{n-1} and so these are x^n-regular when considered in the type $\langle 2\rangle$. This is the 'if' part of the following.

Proposition 3. *A semigroup variety $\mathbb{V}$ is non-trivially $[s, t]$-regular for some terms s, t if and only if $\mathbb{V}$ consists of completely regular semigroups with period dividing some fixed $n \in \mathbb{N}$.*

Proof. First note that a semigroup variety is a variety of completely regular semigroups of period dividing some number n if and only if it satisfies $x \approx x^{n+1}$. The identity $x \approx p_1(x, y, s_1, \ldots, s_n, t_1, \ldots, t_n)$ in part (ii) of the theorem gives $x \approx x^k$ for some $k \geq 1$. If $k > 1$ we are done, while if $k = 1$, then we may assume without loss of generality that either $s_i(x, y) = x$ or $t_i(x, y) = x$ for some i. If both terms equal x, then both s and t are single variables and $\mathbb{V}$ is trivially $[s, t]$-regular. Otherwise the identity $s_i(x, x) \approx t_i(x, x)$ yields $x \approx x^{k'}$ where $k' > 1$. □

Proposition 3 can also be proved by considering congruences on the 1-generated semigroups in a variety.

Example 4. *Inverse semigroups with the natural order forming a $\wedge$-semilattice are strongly $x^{-1}x$-regular.*

These form a variety in the signature $\{\cdot, \wedge, ^{-1}\}$ of type $\langle 2, 2, 1\rangle$ given by the usual inverse semigroup (an I-semigroup satisfying $x^{-1}xy^{-1}y \approx y^{-1}yx^{-1}x$) and semilattice axioms along with the laws $\{(x \wedge y)z^{-1}z \approx xz^{-1}z \wedge yz^{-1}z, x \wedge y \approx x(x \wedge y)^{-1}(x \wedge y)\}$. The theorem applies using $f_{1,1}(x, y) := x$, $f_{2,1}(x, y) := y$, $g_{1,1}(x, y) := g_{2,1}(x, y) := x \wedge y$ (with $p_1(x, y, a, b, c, d) := xa$ and $p_2(x, y, a, b, c, d) := yd$). □

Example 5. *EQ-monoids are $[1, x * y]$-regular.*

EQ-monoids [7] are the class (in fact a variety) of monoids with a subsemilattice L (including the identity 1) for which $\max\{e \in L : xe = ye\}$ exists for all x, y and is denoted by $x * y$. The theorem applies since $x * y = 1$ if and only if $x = y$ (thus we may take $f_{1,1}(x, y) := x$ and $f_{1,2}(x, y) := y$). Note that since an EQ-monoid $\mathbf{A}$ may have $\{1\} \subsetneq L \subsetneq A$, the $[1, x * y]$-regularity condition is stronger than strong $x * y$-regularity or 1-regularity. □

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