

## FLAT ALGEBRAS AND THE TRANSLATION OF UNIVERSAL HORN LOGIC TO EQUATIONAL LOGIC

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**Abstract.** We describe which subdirectly irreducible flat algebras arise in the variety generated by an arbitrary class of flat algebras with absorbing bottom element. This is used to give an elementary translation of the universal Horn logic of algebras, partial algebras, and more generally still, partial structures into the equational logic of conventional algebras. A number of examples and corollaries follow. For example, the problem of deciding which finite algebras of some fixed type have a finite basis for their quasi-identities is shown to be equivalent to the finite identity basis problem for the finite members of a finitely based variety with definable principal congruences.

**§1. Introduction.** This paper has arisen out of a proof in [22] showing that the variety of Clifford semigroups whose natural order is a meet semilattice order has undecidable equational theory. The proof is by an interpretation of semigroup quasi-identities as identities in the enlarged signature involving the semilattice  $\wedge$ . In this paper we are going to describe how this result is a part of a completely general translation of universal Horn classes (henceforth, *uH classes*) of partial algebras into varieties of conventional (that is, not partial) algebras. More precisely we find (Corollary 5.4) an isomorphism from the lattice of all uH classes of a given similarity type to the lattice of subvarieties of a particular variety. This isomorphism translates the main global properties of the uH class to the corresponding variety properties (such finite axiomatisability, local finiteness and so on; see Theorem 5.12). The constructed varieties are hereditarily simple, semisimple and have definable principal congruences (Theorem 5.3). So we have a very precise interpretation of uH logic inside the equational logic of some apparently quite well behaved varieties. Furthermore, the constructed varieties are generated by flat algebras which have become of increasing interest since McKenzie's landmark articles on undecidability, residual bounds and axiomatisability [33, 32, 34].

We begin (Section 3) by giving a general investigation into varieties generated by flat algebras whose bottom element is absorbing, which we call *sink algebras*, or simply *sinks*. (Flat algebras with absorbing bottom element were referred to

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as *M-algebras* by Willard [45], but the word sink is more descriptive.) As we now briefly explain, this section can be seen as an extension of the easier part of [45] (itself, an explanation of the methods of McKenzie in [33, 32, 34]). In [45] a description is given for the finite subdirectly irreducible algebras in the equational class (or *variety*)  $V(\mathbf{F})$  generated by a sink  $\mathbf{F}$  with absorbing bottom element  $\infty$ . Willard characterises the finite subdirectly irreducible members of  $V(\mathbf{F})$  in terms of partial algebras relating (via restrictions, substructures and finite direct powers) to the partial algebra on  $F \setminus \{\infty\}$  obtained by removing the flat semilattice operation, and restricting all other operations accordingly. In Theorem 3.1 we give an axiomatisation for the variety generated by the class of all sinks of a given similarity type, and then in Theorem 3.3 show how to extend Willard's result to a characterisation of the subdirectly irreducibles of arbitrary cardinality within the variety generated by any class  $\mathcal{C}$  of sinks of the same type. Corollary 3.11 takes this further, describing which sinks lie in the variety generated by a class of sinks. These results cement the relationship between the subdirectly irreducibles of  $V(\mathcal{C})$  and certain partial algebra restrictions of the universal Horn class of the associated partial algebras. In many natural cases where  $\mathcal{C}$  consists of all partial algebras or all algebras of a given type—the variety is not finitely axiomatisable (Proposition 3.2). Also, we show that if there is an operation that is of arity greater than 1 (other than the flat semilattice operation  $\wedge$ ), and if the partial algebras associated with  $\mathcal{C}$  (by removing  $\infty$  and  $\wedge$  as above) are actually totally defined, then  $V(\mathcal{C})$  is not finitely axiomatisable provided that the universal Horn class of  $\mathcal{C}$  is not finitely axiomatised (Corollary 3.14(2)).

In Section 4 we examine ways of refining the connection between universal Horn classes of partial algebras and varieties generated by sinks. The idea culminates in Section 5, where we introduce the *pointed semidiscriminator extension* of a (possibly partial) algebra and prove the precise translation of uH logic into equational logic alluded to above (see Corollaries 5.4 and Theorem 5.12 for example). More specifically, we show how subdirectly irreducible algebras in a variety generated by pointed semidiscriminator algebras correspond in a bijective way to the members of a corresponding universal Horn class. This facilitates a translation of universal Horn sentences in one language to identities in the language of pointed semidiscriminator algebras, in a way that preserves the logical entailment relation  $\vdash$  (Corollary 5.11).

Many interesting examples arise as corollaries; we examine these in Section 7. For example, we find that there is a 3-element algebra generating a variety whose lattice of subvarieties is large enough to have any possible lattice of quasivarieties or lattice of varieties (in any finite type) as a factor. We also give a complete description of the flat extensions of groups with a finite basis of identities. The flat extensions of groups turn out to generate the variety of Clifford inverse algebras as investigated by Leech [30] and the undecidability of the equational theory of this class (as proved in [22]) becomes a direct corollary of the undecidability of the quasi-equational theory of the variety of groups (which is equivalent to the well known uniform word problem for groups). We extend this result to the class of Brandt inverse semigroups with natural semilattice order.

A further interesting insight from our translation is that the problem of deciding when a finite algebra has a finite basis for its quasi-identities is equivalent to the corresponding problem for identities within some apparently quite well-behaved

varieties. (Analogous statements hold for a number of related axiomatisability problems.)

**§2. Partial algebras.** In this article we adopt a standard universal algebraic perspective on algebra: the book by Burris and Sankappanavar [11] is a good guide to the conventions we adopt. In particular, we do not consider the empty set to be an algebra, but consider the one element algebra to be subdirectly irreducible.<sup>1</sup> While we do assume a basic familiarity with the universal algebra of algebras, in this section we will give the reader a brief introduction to some algebraic facts about partial algebras, with particular emphasis on universal Horn classes. For a more detailed treatment of the model theory of partial algebras see Burmeister’s survey article [9], the article by Andréka, Burmeister and Némethi [3], or the book by Burmeister [10]. See also the article of McNulty [35] for the general theory of uH logic of conventional algebras.

Throughout the article we use  $\omega := \{0, 1, 2, \dots\}$  and  $\mathbb{N} := \omega \setminus \{0\}$ . Recall that an ( $n$ -ary) *partial operation* on a set  $A$  is a map from a subset of  $A^n$  into  $A$ . A *similarity type*, or *signature* is a set of symbols along with a function mapping each symbol to a non-negative integer, the *arity* of the symbol. A partial algebra of type  $\mathcal{F}$  is a tuple of the form  $\mathbf{P} = \langle P; F \rangle$  where  $P$  is a set and  $F$  is a collection of partial operations on  $P$  which can be indexed by the operation symbols of  $\mathcal{F}$  in a way that each  $n$ -ary partial operation in  $F$  is indexed by a corresponding operation symbol of arity  $n$ . A similarity type is determined by the associated list of arities, and (as is usual), we often simply list these to describe a similarity type. In the case where we have a list of operation symbols for which the associated arities are implicit, we will refer to the set of these as a *language*. For example, the language of groups (considered as partial algebras) might be given as  $\{\cdot, {}^{-1}\}$  or as  $\{\cdot, {}^{-1}, 1\}$ , which are of similarity type  $\langle 2, 1 \rangle$  or  $\langle 2, 1, 0 \rangle$  respectively (under the suggested ordering  $\cdot, {}^{-1}, 1$ ). An  $n$ -ary partial operation on  $P$  whose domain is  $P^n$  is said to be *total*. We use the term *total algebra* to describe a partial algebra whose partial operations are all total. Note that a nullary operation (an operation on  $P$  of domain  $P^\emptyset = \{\emptyset\}$ ; also known as a “constant”) is not necessarily total. *We allow the empty algebra as a partial algebra* (even though we do not consider it to be an algebra). The empty algebra is a total algebra unless there are nullaries in the signature (there can be no total function from  $\emptyset^\emptyset \rightarrow \emptyset$ , since  $\emptyset^\emptyset = \{\emptyset\}$ ). We use  $\mathbf{O}$  to denote the class operator for partial algebras, that adjoins the empty structure to  $\mathcal{K}$ .

We can extend all of these definitions to partial structures, where in addition to partial operations, we have relations. In this case, each relation symbol  $r$  in a similarity type has an associated arity  $n \in \mathbb{N}$  and its interpretation on a partial structure  $\langle P; F, R \rangle$  must be as a subset of  $P^n$  (an  *$n$ -ary relation*).

In this article we use *algebras* to denote members of the class  $\mathcal{A}_{\mathcal{F}}$  of (nonempty) type  $\mathcal{F}$  algebras for some signature  $\mathcal{F}$ , sometimes using *conventional algebras* to emphasize the fact that they are not being considered as partial algebras. We use *partial algebras* to denote members of the class  $\mathcal{P}_{\mathcal{F}}$  of all type  $\mathcal{F}$  partial algebras in some signature  $\mathcal{F}$ . While formally speaking the classes  $\mathcal{A}_{\mathcal{F}}$  and  $\mathcal{P}_{\mathcal{F}}$  intersect to

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<sup>1</sup>Readers who prefer other conventions will find it easy to make very minor adjustments to the statements of the results and their proofs.

the class of nonempty total algebras, in this article it is convenient to adopt the convention restricting our use of the words *total algebras* to refer only to members of  $\mathcal{P}_{\mathcal{F}}$  (for some  $\mathcal{F}$ ) and restricting the word *algebras* (and *conventional algebras*) to refer only to members of  $\mathcal{A}_{\mathcal{F}}$  (for some  $\mathcal{F}$ ). We use  $\mathcal{F}$ -algebras, partial  $\mathcal{F}$ -algebras etc. if we need to be specific about the similarity type.

Before we recall some basic constructions for partial algebras, we fix some convenient notation. For a symbol  $s$  we use the notation  $\vec{s}$  to denote the finite sequence  $s_1, s_2, \dots, s_k$  for some  $k$ ; the value of  $k$  will be clear from the context. For example,  $\vec{a}_i$  will denote the finite sequence  $a_{i,1}, a_{i,2}, \dots, a_{i,k}$ . We also use  $\vec{s}/\theta$  to denote  $s_1/\theta, \dots, s_k/\theta$  (when  $\theta$  is an equivalence relation on a set containing the  $s_i$ ) and  $\vec{s}(x)$  to abbreviate  $s_1(x), \dots, s_k(x)$  (when the  $s_i$  are functions being applied to some point  $x$ ).

Let  $\mathbf{P} = \langle P; F \rangle$  be a partial algebra. A subuniverse of  $\mathbf{P}$  is a subset  $S \subseteq P$  satisfying the property: if  $(\vec{a}) \in S^n \cap \text{dom}(f^{\mathbf{P}})$  for some  $n$ -ary partial operation  $f^{\mathbf{P}}$  on  $\mathbf{P}$ , then  $f^{\mathbf{P}}(\vec{a}) \in S$ . The subalgebra  $\mathbf{S}$  on a subuniverse  $S$  (sometimes called the *relative subalgebra*; [9]) has the obvious partial operations inherited from  $\mathbf{P}$ , with domains  $\text{dom}(f^{\mathbf{S}}) = S^n \cap \text{dom}(f^{\mathbf{P}})$ . We use  $\mathbf{S}$  to denote the class operator taking a class  $\mathcal{K}$  to the class of all subalgebras of  $\mathcal{K}$ . Observe that  $\text{SO}(\mathcal{K}) = \mathbf{S}(\mathcal{K})$  holds if (and only if) there is some  $\mathbf{P} \in \mathcal{K}$  with all nullaries in the type undefined on  $\mathbf{P}$ .

A nonempty direct product of a family of similar partial algebras  $\{A_i : i \in I\}$  (with  $I \neq \emptyset$ ) is formed on the universe  $\prod_{i \in I} A_i$  with operations defined pointwise and with  $(\vec{a}) \in \text{dom}(f^{\mathbf{D}})$  if  $(\vec{a}(i)) \in \text{dom}(f^{A_i})$  for all  $i \in I$ . If some  $A_i$  is empty then this direct product is the empty algebra. If  $I$  is empty, then the direct product  $\prod_{i \in I} \emptyset$  is defined to be the one element structure with all operations total (if there are relations in the type, then each will be the universal relation of appropriate arity on  $\prod_{i \in I} \emptyset$ ). We use the usual symbol  $\mathbf{P}$  to denote the class operator taking a class  $\mathcal{K}$  to the class of all direct products over indexed families of members of  $\mathcal{K}$ . We use  $\mathbf{P}^+$  to denote the class operator  $\mathbf{P}$  restricted to nonempty products.

Now we define the reduced product for partial algebras. Let  $\mathcal{A} := \{A_i : i \in I\}$  be a nonempty family of similar nonempty partial algebras and let  $\mathcal{F}$  be a filter of the Boolean algebra  $2^I$ . Let  $A$  denote the cartesian product of sets  $\prod_{i \in I} A_i$ . For a tuple  $(\vec{a}) \in A^n$  and a partial operation symbol  $f$  of arity  $n$  we write

$$\llbracket (\vec{a}) \in \text{dom}(f^A) \rrbracket$$

to abbreviate the set

$$\{i \in I \mid (\vec{a}(i)) \in \text{dom}(f^{A_i})\}.$$

We introduce an equivalence  $\theta_{\mathcal{F}}$  on  $A$  by setting  $a \theta_{\mathcal{F}} b$  if  $\llbracket a = b \rrbracket := \{i \in I \mid a(i) = b(i)\} \in \mathcal{F}$ . The reduced product,  $\prod_{I/\mathcal{F}} \mathcal{A}$  (sometimes written  $\prod_{I/\mathcal{F}} A_i$ ) will be defined on the universe  $A/\theta_{\mathcal{F}}$ .

For each fundamental  $f$  of arity  $n$ , we let  $(\vec{a}/\theta_{\mathcal{F}}) \in \text{dom}(f^{\prod_{I/\mathcal{F}} \mathcal{A}})$  if  $\llbracket (\vec{a}) \in \text{dom}(f^A) \rrbracket \in \mathcal{F}$ . The usual filter properties (closure under finite intersections) ensures that this is well-defined: that is, if  $\vec{a}/\theta_{\mathcal{F}} = \vec{b}/\theta_{\mathcal{F}}$  then  $\llbracket (\vec{a}) \in \text{dom}(f^A) \rrbracket \in \mathcal{F}$  if and only if  $\llbracket (\vec{b}) \in \text{dom}(f^A) \rrbracket \in \mathcal{F}$ . Now, if  $(\vec{a}/\theta_{\mathcal{F}}) \in \text{dom}(f^{\prod_{I/\mathcal{F}} \mathcal{A}})$  then we may find an element  $b \in A$ , satisfying  $\{i \in I \mid b(i) = f^{A_i}(\vec{a}(i))\} \in \mathcal{F}$ . If

$c \in A$  is a second element satisfying the same property, then  $\llbracket b = c \rrbracket \in \mathcal{F}$  and hence we can uniquely define  $f^{\prod_{I/\mathcal{F}} \mathcal{A}}(\vec{a}) = b/\theta_{\mathcal{F}}$ .

The resulting partial algebra  $\prod_{I/\mathcal{F}} \mathcal{A}$  is the *reduced product* of the family  $\{A_i : i \in I\}$  relative to  $\mathcal{F}$ . In the case where  $\mathcal{F}$  is an ultrafilter, we say that  $\prod_{I/\mathcal{F}} \mathcal{A}$  is the *ultraproduct*.

We now extend these notions of reduced (and ultra) products to include the situation where some of the  $A_i$  in  $\mathcal{A}$  are the empty algebra. If the set  $J_{\emptyset} := \{i \in I \mid |A_i| > 0\}$  is in  $\mathcal{F}$ , then we can identify  $\prod_{I/\mathcal{F}} \mathcal{A}$  with  $\prod_{J_{\emptyset}/\mathcal{F}} \{A_j \mid j \in J_{\emptyset}\}$  (where  $\mathcal{F}$  is restricted in the obvious way to a filter on  $J_{\emptyset}$ ). Otherwise,  $\prod_{I/\mathcal{F}} \mathcal{A}$  is the empty algebra.<sup>2</sup> The class operators  $P_r$  and  $P_u$  stand respectively for reduced products and ultraproducts of members of a class.

Let  $A$  and  $B$  be similar partial algebras. A map  $\phi : A \rightarrow B$  is an *isomorphism* if it is a bijection and for every fundamental partial operation  $f$  we have:

- (1)  $\vec{a} \in \text{dom}(f^A) \Leftrightarrow \phi(\vec{a}) \in \text{dom}(f^B)$ ;
- (2)  $\vec{a} \in \text{dom}(f^A) \Rightarrow \phi(f^A(\vec{a})) = f^B(\phi(\vec{a}))$ .

The class operator for taking isomorphic copies is  $I$ . The notion of a homomorphism between partial algebras is a little more complicated than for conventional algebras. The weakest form is simply a map satisfying the forward implication of condition (1) and condition (2) just given (which coincides with the usual notion amongst algebras). If the full equivalence of (1) is enforced (as well as (2)), then a *closed homomorphism* is obtained ([9]; sometimes these are called *strong* homomorphisms). In this article, we only require explicit discussion of isomorphisms between partial algebras and so omit any further discussion of other kinds of homomorphisms.

The operator  $O$  has no meaning on the class of algebras, but each of the operators  $I, S, P, P^+, P_r, P_u$  have familiar meanings (that are consistent with our definitions given above), and we extend their usage to this setting. We will also use the usual class operator  $H$  for the closure of a class of algebras under taking homomorphic images, and the operator  $V$  to abbreviate the class operator  $HSP$  returning the variety generated by a class of algebras. The notation  $V_{s.i.}(\mathcal{K})$  denotes the subdirectly irreducible members in the variety  $V(\mathcal{K})$ . We do not need to give any meaning to the operators  $H, V, V_{s.i.}$  for partial algebras.

We now define universal Horn formulæ for partial algebras and algebras (the definition will be the same). An *atomic expression* (or *identity*) is one of the form  $s \approx t$  where  $s$  and  $t$  are terms built from the fundamental operations in the usual way. We adopt the convention that the negated atomic formula  $\neg u \approx v$  can equivalently be written  $u \not\approx v$ . Universal Horn formulæ are universally quantified expressions of one of the two forms

$$\bigwedge_{1 \leq i \leq n} \Phi_i \rightarrow \Phi_0$$

where the  $\Phi_i$  are atomic expressions or

$$\bigvee_{1 \leq i \leq n} \neg \Phi_i$$

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<sup>2</sup>This construction agrees up to isomorphism with the category theoretic treatment given in [9].

where the  $\Phi_i$  are atomic expressions. Formulæ of the first kind are often called *quasi-identities*. Identities can be considered as quasi-identities where the number  $n$  is equal to 0.

Satisfaction of uH formulæ by partial algebras is slightly more technical than for conventional algebras because we need to be precise about the meaning of equality between partial functions. Let  $\mathbf{P}$  be a partial algebra and  $\theta$  be a variable assignment; that is, a map from some fixed countably infinite set of variables  $X$  to the elements of  $P$ . If  $p \approx q$  is atomic, then we say that  $p \approx q$  takes the value T relative to  $\theta$  if the evaluation of  $p$  and  $q$  based on the values assigned to the variables by  $\theta$  are well defined and are equal. Otherwise we say that  $p \approx q$  takes the value F. These truth values are extended to arbitrary sentences in the usual way. The empty algebra satisfies any universal sentence, since there are no variable assignments from the infinite set  $X$  into  $\emptyset$  (see [9, Remark 2.6.iii]).

We use  $\text{Mod}_{\text{uH}}(\mathcal{K})$  to denote the set of all models of the uH formulæ satisfied by a class of similar partial algebras  $\mathcal{K}$ . The following result can be found in [9, Theorem 4.5] or [3, Theorem 5.1.5] for example.

**THEOREM 2.1.** *Let  $\mathcal{K}$  be a class of similar partial algebras. Then  $\text{Mod}_{\text{uH}}(\mathcal{K}) = \text{ISP}_r\text{O}(\mathcal{K}) = \text{ISP}^+\text{P}_u\text{O}(\mathcal{K})$ . When  $\mathcal{K}$  is a finite set of finite partial algebras then  $\text{ISP}^+\text{P}_u\text{O}(\mathcal{K}) = \text{ISP}^+\text{O}(\mathcal{K})$ .*

We now give some examples of uH classes of partial algebras.

**EXAMPLE 2.2.** The class of all total  $\mathcal{F}$ -algebras (with the partial empty algebra included, if  $\mathcal{F}$  contains nullaries).

This is a uH class of partial structures axiomatised by the atomic formulæ  $f(x_1, \dots, x_n) \approx f(x_1, \dots, x_n)$  for each fundamental operation  $f \in \mathcal{F}$ . In general we slightly abuse terminology and state that  $\mathcal{H}$  is a uH-class of total algebras if  $\mathcal{H}$  is a uH class satisfying these sentences (even though the empty algebra—which is necessarily contained in  $\mathcal{H}$ —may be partial).

**EXAMPLE 2.3.** The empty structure.

The class consisting of the empty structure is a uH class axiomatised by the contradiction  $x \not\approx x$ .

**EXAMPLE 2.4.** Brandt groupoids.

A category in which every morphism has an inverse is a *Brandt groupoid*. These can be considered as partial algebras with a partial multiplication  $\cdot$  and a total operation  $^{-1}$  of inverse. The usual axiomatisation for these structures can be written as follows:  $(xy)z \approx (xy)z \rightarrow (xy)z \approx x(yz)$ ,  $x(yz) \approx x(yz) \rightarrow (xy)z \approx x(yz)$ ,  $x^{-1} \approx x^{-1}$  (inverse is total),  $(x^{-1})^{-1} \approx x$ ,  $xx^{-1} \approx xx^{-1}$ ,  $xx^{-1}x \approx x$ ,  $xy \approx xy \rightarrow xyy^{-1} \approx x$ ,  $xy \approx xy \rightarrow x^{-1}xy \approx y$ .

Frequently in algebra we ask certain elements to have certain properties *if* they are present.

**EXAMPLE 2.5.** The class of monoids or semigroups.

We can allow for the distinguished 1 to be present, or not to be present. We want  $\cdot$  to be total:  $xy \approx xy$ , and associative:  $(xy)z \approx x(yz)$ , and *if* 1 is present, we want it to satisfy its usual multiplicative properties:  $1 \approx 1 \rightarrow x1 \approx x \approx 1x$ .

**2.1. Local finiteness.** When considering partial algebras, the reader should be warned against relying too heavily on intuition garnered from conventional algebra. The property of local finiteness for partial algebras is a good example of “unusual” behaviour and will be important later in this article. We recall three different possible definitions of local finiteness of a universal Horn class  $\mathcal{H}$  of partial algebras.

**LF:** (Locally finite.) Every finitely generated partial algebra in  $\mathcal{H}$  is finite.

**ULF:** (Uniformly locally finite.) There is a function  $u : \omega \rightarrow \omega$  such that for each  $n \in \omega$ , the  $n$ -generated partial algebras in  $\mathcal{H}$  each have at most  $u(n)$  elements.

**RLF:** (Regularly locally finite.) For each  $n$ , there are (up to isomorphism) only finitely many different  $n$ -generated partial algebras in  $\mathcal{H}$ , and all are finite.

Uniform local finiteness was introduced by Maltsev [31], while regular local finiteness was introduced by G. Bezhanishvili [8]<sup>3</sup>. The implications  $\text{RLF} \Rightarrow \text{ULF} \Rightarrow \text{LF}$  hold immediately from the definitions. For uH classes of (conventional) algebras the reverse implications also hold. For this one can use the fact that if  $\mathcal{H}$  consists of total (or conventional) algebras, the  $n$ -generated free algebra in  $\mathcal{H}$  maps homomorphically onto any  $n$ -generated algebra in  $\mathcal{H}$ . This proof fails in general for partial algebras (we do not explain this here), however the implications can be reversed provided the similarity type is finite.

**PROPOSITION 2.6.** *Let  $\mathcal{H}$  be a locally finite uH class of partial algebras of some finite similarity type  $\mathcal{F}$ . Then  $\mathcal{H}$  is regularly locally finite.*

**PROOF.** Assume that there is some number  $k$  for which there are infinitely many pairwise non-isomorphic  $k$ -generated finite partial algebras in  $\mathcal{H}$ . We prove that there is an infinite  $k$ -generated partial algebra in  $\mathcal{H}$ , so that  $\mathcal{H}$  is not locally finite.

Let  $\mathcal{F}^+$  denote the similarity type obtained by adjoining  $k$  new nullary symbols  $a_1, \dots, a_k$  to  $\mathcal{F}$ . Each finite partial algebra  $A$  of type  $\mathcal{F}$  can be made into a type  $\mathcal{F}^+$  partial algebra (in which the  $a_i$  are total) in  $|A|^k$  different ways. We let  $\mathcal{H}^+$  denote the uH class of type  $\mathcal{F}^+$  partial algebras in which the  $a_i$  are total, and whose  $\mathcal{F}$ -reducts are in  $\mathcal{H}$  (with the empty algebra included). We let  $\mathcal{H}_0^+$  denote the 0-generated members of  $\mathcal{H}^+$  and observe that the class of  $\mathcal{F}$ -reducts of  $\mathcal{H}_0^+$  coincides with the class of (at most)  $k$ -generated members of  $\mathcal{H}$ .

Let  $T$  be the set of all variable-free  $\mathcal{F}^+$ -terms. Define the *height*  $\text{ht}(t)$  of a term  $t \in T$  inductively as follows: the nullaries are of height 0;  $\text{ht}(f(t_1, \dots, t_m)) = 1 + \max\{\text{ht}(t_1), \dots, \text{ht}(t_m)\}$  (where  $m$  is the arity of the non-nullary fundamental operation symbol  $f$ ). For each  $n \in \omega$ , let  $T_n$  denote the set of terms of  $T$  whose height is equal to  $n$ .

For each  $n \in \mathbb{N}$ , let  $\Phi_n$  denote the (variable free) first order sentence in  $\mathcal{F}^+$  stating that there is a term in  $T_n$  that is defined, but not equal to any term in  $T$  of strictly smaller height. For example, if  $\mathcal{F}$  consists of two unary operations,  $f, g$  and  $k = 1$ , then  $T_0 = \{a_1\}$ ,  $T_1 = \{f(a_1), g(a_1)\}$  and  $\Phi_1$  could be

$$(f(a_1) \approx f(a_1) \ \& \ a_1 \not\approx f(a_1)) \vee (g(a_1) \approx g(a_1) \ \& \ a_1 \not\approx g(a_1)).$$

<sup>3</sup>In both [31] and [8], the ULF and RLF concepts are introduced primarily to study classes generating locally finite varieties (or uH classes) of conventional algebras, rather than as properties of varieties (or uH classes) themselves.

We claim that every finite subset of  $\{\Phi_n \mid n \in \mathbb{N}\}$  has a nonempty model in  $\mathcal{H}_0^+$ . Let  $\Xi$  be such a finite set, and let  $\ell$  be the largest number for which  $\Phi_\ell \in \Xi$ .

For each  $A \in \mathcal{H}_0^+$ , let  $r_n(A)$  (the  $n$ -restriction of  $A$ ) denote the set

$$\{t^A \mid \text{ht}(t) \leq n \text{ and } t^A \text{ is defined in } A\}.$$

If  $A \in \mathcal{H}_0^+$  has  $r_n(A) \subsetneq r_{n+1}(A)$ , then  $\Phi_{n+1}$  holds in  $A$ . For any partial  $\mathcal{F}^+$ -algebra  $P$  we have  $|r_n(P)| \leq |T_n|$  while a basic induction argument shows that if  $r_n(P) = r_{n+1}(P)$  then  $P = r_n(P)$ . As there is no bound on the size of the finite members of  $\mathcal{H}_0^+$ , it follows that there is some  $A_\Xi \in \mathcal{H}_0^+$  for which  $r_1(A_\Xi) \subsetneq r_2(A_\Xi) \subsetneq \dots \subsetneq r_\ell(A_\Xi)$ . So we have a nonempty model (namely,  $A_\Xi$ ) of  $\{\Phi_i \mid i \leq \ell\} \subseteq \Xi$  from  $\mathcal{H}_0^+$ .

The Compactness Theorem for nonempty partial algebras<sup>4</sup> shows that there is a nonempty model  $M$  of  $\{\Phi_n \mid n \in \mathbb{N}\}$  in  $\mathcal{H}^+$ . Without loss of generality, we may assume that  $M$  is 0-generated (because  $\{\Phi_n \mid n \in \omega\}$  are universal sentences). For distinct  $n, m \in \mathbb{N}$  (with, say,  $n < m$ ), the elements  $t_n^M$  (for some  $t_n \in T_n$ ) and  $t_m^M$  (for some  $t_m \in T_m$ ) guaranteed to exist in  $M$  by  $\Phi_n$  and  $\Phi_m$  (respectively) cannot be equal because  $\Phi_m$  states that  $t_m^M$  is not equal to the value of any term of height equal to  $n$ . Hence  $M$  is infinite and its  $\mathcal{F}$  reduct is a  $k$ -generated infinite partial algebra in  $\mathcal{H}$ , as required.  $\dashv$

The following example demonstrates that for infinite type uH classes of partial algebras, it is no longer true that  $\text{LF} \Rightarrow \text{ULF}$ , or that  $\text{ULF} \Rightarrow \text{RLF}$ .

**EXAMPLE 2.7.** Let  $\mathcal{F}$  be the infinite similarity type consisting of a countable infinity of unary operation symbols  $\{f_i \mid i \in \mathbb{N}\}$ . For each  $\sigma \in \mathbb{N}^{\mathbb{N}}$ , let  $\Xi_\sigma$  be the set consisting of the following uH sentences:

1.  $f_i(x) \not\approx f_i(x) \vee f_j(y) \not\approx f_j(y)$  whenever  $i \neq j$  and  $i, j \in \mathbb{N}$ ;
2.  $f_i^{\sigma(i)+1}(x) \not\approx f_i^{\sigma(i)+1}(x)$  for each  $i \in \mathbb{N}$ .

If  $\sigma$  is a bounded sequence, then  $\Xi_\sigma$  defines a uH class that is ULF but not RLF. If  $\sigma$  is unbounded, then  $\Xi_\sigma$  defines a uH class that is LF but not ULF.

**PROOF.** The sentences in (1) show that any model of  $\Xi_\sigma$  has at most one operation defined. When  $M$  is a model in which the operation  $f_i$  is defined somewhere, then the sentences in (2) ensure that at most  $\sigma(i)$  new elements can be generated from any point. If  $M$  is  $k$ -generated, then  $|M| \leq k(\sigma(i) + 1)$ . So  $\Xi_\sigma$  is locally finite. For each  $i \in \mathbb{N}$ , it is easy to find a 1-generated model of  $\Xi_\sigma$  on which  $f_i$  is defined and with precisely  $\sigma(i) + 1$  elements: simply consider the set  $\{0, 1, 2, \dots, \sigma(i)\}$  with  $f_i(j) = j + 1$  whenever  $j \leq \sigma(i) - 1$ , and all other operations undefined. These observations show that there are infinitely many finite non-isomorphic 1-generated models of  $\Xi_\sigma$  and that  $\Xi_\sigma$  is uniformly locally finite if and only if  $\sigma$  is bounded.  $\dashv$

**§3. Flat extensions of partial algebras.** We will say that a (conventional) algebra  $A$  is *flat* if it is a height one meet semilattice with respect to one of its operations, say  $\wedge$ . If the bottom element (with respect to  $\wedge$ ) is absorbing, then we say that  $A$  is a *sink algebra*, or simply a *sink*. We do not attempt to survey the literature on

<sup>4</sup>The usual ultraproduct proof of the Compactness Theorem carries through to the partial algebra setting.

flat algebras here, although articles of particular relevance are discussed as the need arises.

Let  $\mathbf{P} = \langle P; F \rangle$  be a partial algebra and let  $\infty \notin P$ . There is a natural way to form a flat algebra extending  $\mathbf{P}$  on the universe  $P \cup \{\infty\}$ , which we call the *flat (one point) extension* of  $\mathbf{P}$ . The signature is augmented by the addition of a new binary operation  $\wedge$  making  $P \cup \{\infty\}$  a flat semilattice with bottom element  $\infty$ , while the other operations of  $\mathbf{P}$  are extended in their definition on  $\mathbf{P}$  to include entries from  $P \cup \{\infty\}$  by letting all undefined values on  $P \cup \{\infty\}$  equal  $\infty$ . For nullary operations  $u$  that are not total (that is, undefined on  $\mathbf{P}$  but in the type of  $\mathbf{P}$ ), this means that  $\infty$  is the value of  $u$  on the flat extension of  $\mathbf{P}$ ; obviously,  $\mathfrak{b}(\mathbf{P})$  is a sink. Note that the flat extension of the empty algebra is the one element algebra in the extended type. It does no harm to assume that unless otherwise specified, the symbol  $\wedge$  is not in the signature of any partial algebra, that unless otherwise specified, the flat semilattice operation in any sink is denoted by  $\wedge$ , and that the absorbing element in a sink is denoted by  $\infty$ .

We denote the flat extension of  $\mathbf{P}$  by  $\mathfrak{b}(\mathbf{P})$ ; moreover, if  $\mathcal{K}$  is a class of partial algebras, then  $\mathfrak{b}(\mathcal{K})$  will denote the class of flat extensions members of  $\mathcal{K}$ . Note that  $\mathfrak{b}(\mathbf{P})$  is an algebra, even if  $\mathbf{P}$  is a partial algebra. We note the the partial algebras in  $\mathcal{P}_{\mathcal{F}}$  are the objects of a category  $Par_{\mathcal{F}}$  whose morphisms are the closed homomorphisms. The flat extension construction forms part of a functor from  $Par_{\mathcal{F}}$  to the usual category of algebras of similarity type  $\mathcal{F} \cup \{\wedge\}$  with the morphism  $\phi : \mathbf{P} \rightarrow \mathbf{Q}$  extending to a homomorphism  $\phi_b : \mathfrak{b}(\mathbf{P}) \rightarrow \mathfrak{b}(\mathbf{Q})$  by  $\infty \mapsto \infty$ .

Of course every sink algebra  $\mathbf{A}$  is isomorphic to the flat extension of a partial algebra on the set  $A \setminus \{\infty\}$ : remove the operation  $\wedge$  and make appropriate restrictions to the remaining operations (in domain and range). Hence we can describe any class of similar sink algebras as the class of flat extensions of some class of similar partial algebras. Throughout, we use  $\leq$  to denote the order induced by  $\wedge$ .

For a term  $t$  we often write  $t(x_0, \dots, x_{n-1})$  to indicate that the variables of  $t$  are amongst  $x_0, \dots, x_{n-1}$ . We say that  $t$  is *explicit in  $x_i$*  if  $x_i$  appears explicitly in the construction of  $t$  as a composite of fundamental operations and variables. If  $a$  is a symbol, we frequently use the notation  $\vec{a}$  to denote a finite sequence  $a_0, \dots, a_{n-1}$ , when the number  $n$  is unimportant. For partial algebra  $\mathbf{P}$ , a term  $t(x, \vec{y})$ , explicit in  $x$ , and elements  $\vec{c}$  in  $P$ , we say that the partial unary function  $t^{\mathbf{P}}(x, \vec{c})$  is a (unary) *translation* (often also called a unary *polynomial*). A *basic translation* is a translation built from the single application of a fundamental operation.

Let us say that an element  $a$  of  $\mathbf{P}$  *divides* an element  $b$  if there is a translation  $\lambda$  with  $\lambda(a) = b$ . We write  $a \rightsquigarrow b$ . The relation  $\rightsquigarrow$  is a quasi-order on  $\mathbf{P}$ .

In this section we investigate which sinks lie in the variety generated by a class of sinks. Our main result is Theorem 3.3, which extends a result of Willard [45, Theorem 1.2], itself a refinement of a method used by McKenzie in [33] and elsewhere. The following theorem is a first step toward the proof of Theorem 3.3, but is of interest in its own right. Recall  $\mathcal{P}_{\mathcal{F}}$  is the class of all type  $\mathcal{F}$  partial algebras.

**THEOREM 3.1.** *Let  $\mathcal{F}$  be a similarity type. Then*

1. *every flat algebra with absorbing bottom in  $\mathfrak{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  is a member of  $\mathfrak{I}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$ ,*
2. *every subdirectly irreducible algebra in the variety  $\mathfrak{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  is a member of  $\mathfrak{I}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$ ,*

3.  $\mathsf{V}(\mathsf{b}(\mathcal{P}_{\mathcal{F}}))$  is axiomatised by the following identities:
- (a) the semilattice axioms for  $\wedge$ ;
  - (b) for every pair of  $\mathcal{F} \cup \{\wedge\}$ -terms  $t(x, u_1, \dots, u_n)$  and  $s(x, v_1, \dots, v_m)$  (explicit in the listed variables), the identity

$$t(x \wedge y, u_1, \dots, u_n) \wedge s(x, v_1, \dots, v_m) \approx t(x \wedge y, u_1, \dots, u_n) \wedge s(y, v_1, \dots, v_m).$$

**PROOF.** We prove (3), deducing (2) and then (1) from our method.

We first observe that the identities described in (a) and (b) are satisfied by  $\mathsf{b}(\mathcal{P}_{\mathcal{F}})$ , whence  $\mathsf{V}(\mathsf{b}(\mathcal{P}_{\mathcal{F}}))$ . For the semilattice axioms, this is trivial. For the identities described in part (b), note that unless  $x$  takes the same value as  $y$ , both sides of the identity will take the value  $\infty$ . Now we show that every model of the identities in (a) and (b) is a subdirect product of sinks (with respect to the flat semilattice operation  $\wedge$ ) and hence lies in the variety  $\mathsf{V}(\mathsf{b}(\mathcal{P}_{\mathcal{F}}))$ .

Let  $\mathbf{S}$  satisfy the described identities and let  $a \neq b$  in  $\mathbf{S}$ . We find a congruence  $\theta$  on  $\mathbf{S}$  separating  $a$  and  $b$  and such that  $\mathbf{S}/\theta$  is flat with respect to the operation  $\wedge$ .

Without loss of generality we may assume that  $a \not\leq b$  (recall that  $\leq$  is the usual  $\wedge$  order). Define a binary relation  $\theta$  on  $\mathbf{S}$  by  $c \theta d$  if one of the following hold:

- (i) there is a translation  $\lambda$  such that  $\lambda(c \wedge d) = a$ ;
- (ii) both  $c \not\rightsquigarrow a$  and  $d \not\rightsquigarrow a$ .

This relation is obviously reflexive and symmetric. Now we show that it is transitive.

Say that  $c \theta d \theta e$ . If  $c \theta d$  because of property (ii) then property (ii) must hold for the pair  $d, e$  (as property (i) implies  $d \rightsquigarrow a$ ) and then also for the pair  $c, e$ . Hence it suffices to consider the case where  $c \wedge d \rightsquigarrow a$  and  $d \wedge e \rightsquigarrow a$ . There are terms  $t(x, \vec{y})$  and  $s(x, \vec{z})$ , explicit in  $x$ , and  $\vec{f}$  and  $\vec{g}$  in  $\mathbf{S}$  such that  $t^{\mathbf{S}}(c \wedge d, \vec{f}) = a$  and  $s^{\mathbf{S}}(d \wedge e, \vec{g}) = a$ . Let  $r(x, y, \vec{z})$  be the term  $s(x \wedge y, \vec{z})$ . So we have  $a = t^{\mathbf{S}}(c \wedge d, \vec{f}) \wedge r^{\mathbf{S}}(d, e, \vec{g})$ . By the appropriate axiom we get  $a = t^{\mathbf{S}}(c \wedge d, \vec{f}) \wedge r^{\mathbf{S}}(c, e, \vec{g})$ . Hence  $a = a \wedge r^{\mathbf{S}}(c, e, \vec{g})$  and then  $a = a \wedge s^{\mathbf{S}}(c \wedge e, \vec{g})$ . So  $c \theta e$  as well.

Now we prove that  $\theta$  is a congruence; that is, is stable under basic translations. Certainly property (ii) is stable under basic translations. Now say that property (i) holds for  $c, d$  and let  $\gamma$  be a basic translation. Assume that  $\gamma(c) \rightsquigarrow a$ . So there is a translation  $\eta$  with  $\eta(\gamma(c)) = a$ . So  $\lambda(c \wedge d) \wedge \eta(\gamma(c)) = a$ . Then

$$\begin{aligned} a &= \lambda(c \wedge d) \wedge \eta(\gamma(c)) \\ &= \lambda(c \wedge d) \wedge \eta(\gamma(c) \wedge \gamma(c)) \\ &= \lambda(c \wedge d) \wedge \eta(\gamma(c) \wedge \gamma(d)), \end{aligned}$$

by the appropriate axiom in part (3)(b) of the theorem statement. Hence  $\gamma(c) \wedge \gamma(d) \rightsquigarrow a$  and also  $\gamma(d) \rightsquigarrow a$ . By symmetry, we have either both  $\gamma(c) \not\rightsquigarrow a$  and  $\gamma(d) \not\rightsquigarrow a$  hold, or  $\gamma(c) \wedge \gamma(d) \rightsquigarrow a$ . So  $\gamma(c) \theta \gamma(d)$  as required. Hence  $\theta$  is a congruence on  $\mathbf{S}$ .

To complete the proof of (3) we must show that  $\mathbf{S}/\theta$  is flat with respect to  $\wedge$  and that  $a/\theta \neq b/\theta$ . The elements  $c$  for which  $c \not\rightsquigarrow a$  form a single  $\theta$ -class; we denote it by  $Z$ . Also, as  $c \in Z$  implies  $\lambda(c) \in Z$  for any translation  $\lambda$  we have that  $Z$  forms an absorbing element of  $\mathbf{S}/\theta$ . Now we show that as a semilattice,  $\mathbf{S}/\theta$  is of height 1. Say that  $c, d \notin Z$  with  $c \leq d$ . Then we must have  $c \rightsquigarrow a$  for some translation  $\lambda$ .

Then  $\lambda(c \wedge d) = a$  showing that  $c/\theta = d/\theta$ . So  $\mathcal{S}/\theta$  is a flat algebra with respect to  $\wedge$ .

Finally, we show that  $a/\theta \neq b/\theta$ . As  $a \rightsquigarrow a$  we have  $a \notin Z$ . If  $\lambda(a \wedge b) = a$  for some translation  $\lambda$ , then we have  $a = a \wedge a = \lambda(a \wedge b) \wedge a = \lambda(a \wedge b) \wedge b = a \wedge b$ , contradicting the choice of  $a \not\leq b$ . This proves (3).

Statement (2) now follows immediately. For part (1), let  $\mathbf{F} \in \mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  be a flat algebra with respect to some operation  $\perp$  and absorbing bottom, say  $0$ . We show that  $\mathbf{F}$  is also flat with respect to the operation  $\wedge$ . It suffices to show that each non- $0$  element is maximal in the  $\wedge$ -order. For each  $a \in \mathbf{F} \setminus \{0\}$ , we have that there is a congruence  $\theta$  satisfying  $a/\theta \neq 0/\theta$  and such that  $\mathbf{F}/\theta$  is flat with respect to  $\wedge$ ; this follows from part (2) or the proof of part (3). However, by considering the fact that  $\mathbf{F}$  is flat in  $\perp$ , we have that  $a/\theta$  is a singleton class. But then no element of  $\mathbf{F} \setminus \{a\}$  can be above  $a$  in the  $\wedge$ -order on  $\mathbf{F}$ . Hence  $a$  is maximal.  $\dashv$

The basis given in Theorem 3.1 is infinite. We now show that this is necessary. For the sake of familiarity, we have chosen the main construction  $\mathfrak{b}(\mathbf{M})$  in the first case of the proof to be (up to term equivalence) the 6 element flat algebra underlying the first stage of construction of McKenzie's algebra  $\mathbf{A}$  of [33] and his  $\mathbf{A}(\mathcal{F})$  algebras in [32].

**PROPOSITION 3.2.** *If  $\mathcal{F}$  is a similarity type with at least one operation symbol of arity strictly greater than 0, then  $\mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  is not finitely axiomatisable.*

**PROOF.** We show more: there is a reasonably "small" subvariety of  $\mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  such that any larger subvariety of  $\mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  is not finitely based.

First assume that there is an operation  $f$  of arity at least 2. Let  $\mathbf{M}$  be the partial algebra on the universe  $\{1, 2, h, c, d\}$  of type  $\mathcal{F}$  with all operations except  $f$  totally undefined and with

- $f(1, c, \vec{x}) = c$  for every  $\vec{x}$  in  $M$ ,
- $f(2, d, x) = f(h, c, \vec{x}) = d$  for every  $\vec{x}$  in  $M$ .

Because  $f$  is completely determined by its first two arguments, it is essentially binary, and we now treat it as a binary operation which we write as  $\cdot$ . The operations of  $\mathfrak{b}(\mathbf{M})$  consist of the flat  $\wedge$ , plus a binary operation  $\cdot$  and all other operations are constantly equal to  $\infty$ .

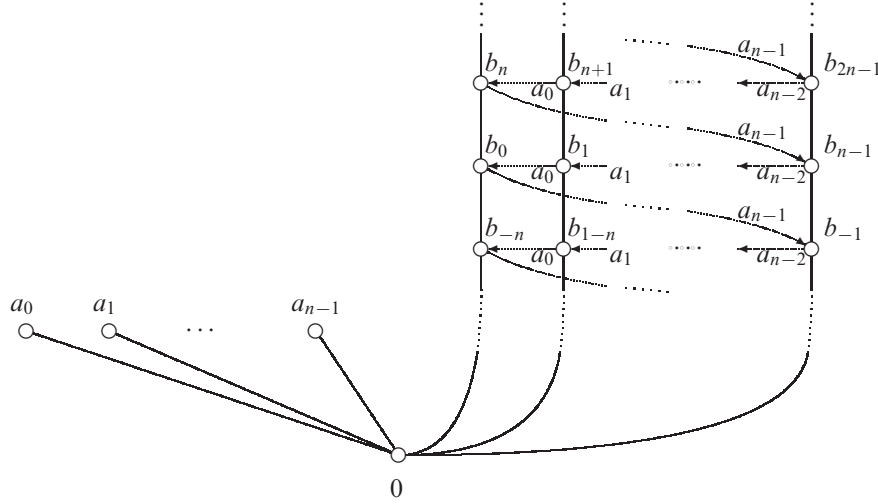
We now define, for each  $n > 1$ , an infinite algebra  $\mathbf{W}_n$  with the property that  $n - 1$ -generated subalgebras of  $\mathbf{W}_n$  lie in  $\mathcal{V}(\mathfrak{b}(\mathbf{M}))$ , but  $\mathbf{W}_n$  does not lie in the variety generated by any class of sinks. This will prove that any variety generated by sinks and containing  $\mathcal{V}(\mathfrak{b}(\mathbf{M}))$  is not finitely based.

The universe of  $\mathbf{W}_n$  is the set  $\{0, a_0, \dots, a_{n-1}\} \cup \{b_i \mid i \in \mathbb{Z}\}$ . We define a multiplication by giving all products the value 0 except the following:  $a_i \cdot b_{nj+i+1} = b_{nj+i}$  for all  $n, j \in \mathbb{Z}$  and  $i \in \{0, \dots, n-1\}$ . We now define an order  $\leq$  on  $\mathbf{W}_n$  by letting 0 be the bottom element,  $b_i \leq b_j$  if and only if  $i \leq j$  and  $i \equiv j \pmod{n}$ . All other elements are incomparable. We let  $\wedge$  denote the corresponding semilattice meet and include it as a fundamental operation. All other operations are set to constantly equal 0. See Figure 1.

**Claim 0.**  $\mathbf{W}_n$  is not contained in  $\mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$ .

**Proof of Claim 0.** We have a failure of

$$(x \wedge y) \wedge u_0(u_1(\dots(u_{n-1} \cdot x) \dots)) \approx (x \wedge y) \wedge u_0(u_1(\dots(u_{n-1} \cdot y) \dots))$$

FIGURE 1. The algebra  $W_n$ .

when  $x = b_0$ ,  $y = b_n$ ,  $u_i = a_i$ . Indeed, the left hand side of this identity equals  $b_{-n}$ , while the right hand side equals  $b_0$ . This completes the proof of Case 0.

**Claim 1.** Any  $n - 1$  generated subalgebra of  $W_n$  is contained in  $V(b(\mathcal{M}))$ .

Proof of Claim 1. Let  $V_n$  be the subset of  $W_n$  consisting of 0, along with all elements whose subscript is not one less than a multiple of  $n$ ; this is routinely seen to be a subuniverse. By applying a suitable automorphism if necessary, any set of  $n - 1$  elements in  $W_n$  lies in  $V_n$ . So it suffices to show that  $V_n \in V(b(\mathcal{M}))$  for each  $n \in \mathbb{Z}$ . We show that  $V_n$  is isomorphic to a quotient of a subalgebra of  $(b(\mathcal{M}))^{n \times \mathbb{Z}}$ . For  $0 \leq i \leq n - 2$  we define  $A_i \in (b(\mathcal{M}))^{n \times \mathbb{Z}}$  by

$$A_i(j, k) = \begin{cases} 1 & \text{if } j < i \\ h & \text{if } j = i \\ 2 & \text{if } j > i \end{cases}$$

and for  $0 \leq i \leq n - 2$  and  $m \in \mathbb{Z}$  we define  $B_{nm+i}$  by

$$B_{nm+i}(j, k) = \begin{cases} 0 & \text{if } k > m \\ c & \text{if } k \leq m \text{ and } j < i \\ d & \text{if } k \leq m \text{ and } j \geq i. \end{cases}$$

Let  $U_n$  be the subalgebra of  $(b(\mathcal{M}))^{n \times \mathbb{Z}}$  generated by  $\{A_i \mid 0 \leq i \leq n - 2\} \cup \{B_{nj+i} \mid 0 \leq i \leq n - 2, j \in \mathbb{Z}\}$  and let  $I$  be the subset of  $U_n$  consisting of the non-generators. The following computations are easily verified:

- $A_i \cdot B_{nm+i+1} = B_{nm+i}$ ;

- if  $p \leq q$  then  $B_{np+i} \wedge B_{nq+i} = B_{np+i}$ ;
- $A_i \wedge A_i = A_i$ ,

however every other product and every other application of  $\wedge$  gives to rise to a (non generator) element,  $C$  say, satisfying  $(\forall j \in \mathbb{Z})(\exists i \in n-1) C(i, j) = 0$ . Because 0 is an absorbing element, this shows that  $I$  is an absorbing ideal. The corresponding Rees quotient is isomorphic to  $V_n$ . This completes the proof of Claim 1, which in turn completes the proof of the proposition in the case that the similarity type includes an operation of arity greater than 1.

We now give a very brief sketch of how to adjust the above proof to the case where there is an operation of arity 1 but none of greater arity.<sup>5</sup> We replace  $\mathbf{M}$  by a total algebra  $\mathbf{M}'$  on  $\mathbb{Z}$  with the unary operation  $i \mapsto i-1$ . Replace  $W_n$  by the algebra  $W'_n$  on the elements  $\{b_k \mid k \in \mathbb{Z}\}$  (given the same order as before) but with the unary operation  $b_k \mapsto b_{k-1}$ . (In terms of Figure 1, we remove the elements  $a_0, \dots, a_{n-1}$  and the unary operation corresponds to the arrows.) A version of Claim 0 is easily established, however the obvious modification of Claim 1 does not hold. Instead, let  $V'$  denote the algebra on  $(\mathbb{Z} \times \mathbb{Z}) \cup \{0\}$  given the semilattice order making 0 the bottom element and  $(i, j) \leq (k, \ell)$  if and only if  $i \leq k$  and  $j = \ell$  and given the unary operation  $(i, j) \mapsto (i, j-1)$ . This is easily seen to lie in the variety generated by  $\mathfrak{b}(\mathbf{M}')$  using a similar method used to show that  $V_n \in \mathfrak{V}(\mathfrak{b}(\mathbf{M}))$ . Let  $\Sigma$  be any finite set of identities in the language of  $\mathfrak{b}(\mathbf{M}')$ . By choosing sufficiently large  $n$ , it is routinely verified that an evaluation in  $W'_n$  of the identities in  $\Sigma$ , takes place on a subset  $V'_n \subset W'_n$  on which  $\wedge$  and the unary operations act in an essentially identical way to a corresponding subset of  $V'$ . Hence the finite set  $\Sigma$  is satisfied by  $W'_n$ , showing that  $\Sigma$  is not a basis for  $\mathfrak{V}(\mathcal{P}_{\mathcal{F}})$ .  $\dashv$

Given a partial algebra  $\mathbf{P}$  and a point  $p \in P$ , we let  $D_p(\mathbf{P})$  denote the partial algebra on  $\{a \in P \mid a \rightsquigarrow p\}$  with the partial operations of  $\mathbf{P}$  restricted in domain and range to these elements. We let  $D(\mathbf{P})$  denote  $\{D_p(\mathbf{P}) \mid p \in P\}$ , or  $\{\mathbf{P}\}$  if  $\mathbf{P}$  is the empty structure. We extend  $D$  to a class operator in the obvious way. (Observe however that  $D$  is not in general a closure operator.)

The next result shows the close relationship between the uH class of a class of partial algebras and the variety generated by the corresponding flat extensions. In the case where  $\mathcal{K}$  consists of a single finite partial algebra, this result is proved for the finite members of  $\mathfrak{V}_{\text{s.i.}}(\mathfrak{b}(\mathcal{K}))$  by Willard [45, Theorem 1.2] (in this case, ultraproducts are not required in the statement). This case is also a simplified form of part of the method of McKenzie in [33, 32] (where the bottom element is not absorbing in all operations).

**THEOREM 3.3.** *Let  $\mathcal{K}$  be a class of similar partial algebras. Then  $\mathfrak{V}_{\text{s.i.}}(\mathfrak{b}(\mathcal{K})) = \mathfrak{I}(\mathfrak{b}(\text{DSP}_r\text{O}(\mathcal{K}))) = \mathfrak{I}(\mathfrak{b}(\text{DSP}^+\text{P}_u\text{O}(\mathcal{K})))$ .*

This theorem is proved over a number of lemmas.

**LEMMA 3.4.** *Let  $\mathcal{A} = \{A_i \mid i \in I\}$  be a nonempty family of similar partial algebras and  $\mathcal{U}$  be an ultrafilter on  $I$ . Then  $\mathfrak{b}(\prod_{\mathcal{U}} A) \cong \prod_{\mathcal{U}} \mathfrak{b}(A)$ .*

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<sup>5</sup>The method for algebras with operations of arity at most 1 is also easily adapted to give a proof that the variety generated by the class of flat extensions of *total* algebras (of non-trivial type) is not finitely axiomatised.

PROOF. By relabeling if necessary, we may assume that the symbol  $\infty$  does not appear in any of the algebras in  $\mathcal{K}$  or in  $A$ . Assume that  $A$  is an ultraproduct  $\prod_{I/\mathcal{U}} \mathcal{S}_i$  where  $\mathcal{S}_i \in \mathcal{K}$  and  $\mathcal{U}$  is an ultrafilter on  $I$ . We also let  $A^\sharp$  denote the set  $\prod_{i \in I} \mathcal{S}_i$  and let  $\theta$  denote the equivalence on  $A^\sharp$  for which  $A^\sharp/\theta = A$ .

Now consider  $\mathbf{B} := \prod_{I/\mathcal{U}} \mathfrak{b}(\mathcal{S}_i)$  using the same ultrafilter as before. The corresponding equivalence extends  $\theta$  and we will denote it by  $\rho$ . Also, let  $\mathbf{B}^\sharp$  denote  $\prod_{i \in I} \mathfrak{b}(\mathcal{S}_i)$  and  $\underline{\infty}$  denote the constant  $\underline{\infty}(i) = \infty$ . It is clear that the map  $\iota : \mathfrak{b}(A) \rightarrow \mathbf{B}$  defined by  $s/\theta \mapsto s/\rho$  and  $\underline{\infty} \mapsto \underline{\infty}/\rho$  is a well defined injective map. Furthermore,  $\iota$  is surjective because any  $s \in \mathbf{B}^\sharp$  with  $s/\rho \neq \underline{\infty}/\rho$  has  $J := \{i \in I \mid s(i) \in \mathcal{S}_i\} \in \mathcal{U}$ , so that any  $s' \in A^\sharp$  with  $s' \upharpoonright_J = s \upharpoonright_J$  has  $s'/\theta \mapsto s/\rho$ .

Now we check that  $\iota$  is an isomorphism. Let  $f^A$  be an  $n$ -ary partial operation on  $A$  and  $a_0, \dots, a_{n-1} =: \vec{a}$  in  $\mathfrak{b}(A)$ . We need to show that  $\iota(f^{b(A)}(\vec{a})) = f^{\mathbf{B}}(\iota(\vec{a}))$ , where  $\iota(\vec{a})$  abbreviates  $\iota(a_0), \dots, \iota(a_{n-1})$ . Note that each  $a_i$  is of the form  $b_i/\theta$  for some  $b_i \in A^\sharp$ , or is equal to  $\infty$ . If for some  $i \leq n$  we have  $a_i = \infty$ , then  $f^{b(A)}(\vec{a}) = \infty$  and as  $\iota(\infty) = \underline{\infty}/\rho$ , we also have  $f^{\mathbf{B}}(\iota(\vec{a})) = \underline{\infty}/\rho$ , as required. So let us assume that each  $a_i$  is of the form  $b_i/\theta$ . Recall that  $\vec{b}/\theta$  and  $\vec{b}$  abbreviate  $b_0/\theta, \dots, b_{n-1}/\theta$  and  $b_0, \dots, b_{n-1}$  respectively.

First assume that  $\vec{b}/\theta \in \text{dom}(f^A)$  so that there is  $c \in A^\sharp \subseteq \mathbf{B}^\sharp$  with  $f^A(\vec{b}/\theta) = c/\theta$ . As  $\{i \in I \mid f^{A_i}(\vec{b}(i)) = c(i)\} \in \mathcal{U}$  implies that  $\{i \in I \mid f^{\mathbf{B}_i}(\vec{b}(i)) = c(i)\} \in \mathcal{U}$ , we have  $f^{\mathbf{B}}(\vec{b}/\rho) = c/\rho$  so that  $\iota(f^{b(A)}(\vec{b}/\theta)) = c/\rho = f^{\mathbf{B}}(\iota(\vec{b}/\theta))$ .

Now assume that  $\vec{b}/\theta \notin \text{dom}(f^A)$ , so that  $f^{b(A)}(\vec{b}/\theta) = \infty$ . We need to show that  $f^{\mathbf{B}}(\vec{b}/\rho) = \underline{\infty}/\rho$ . However,  $\vec{b}/\theta \notin \text{dom}(f^A)$  implies that  $\{i \in I : \vec{b}(i) \notin \text{dom}(f^{A_i})\} \in \mathcal{U}$ , which implies that  $\{i \in I : f^{b(A_i)}(\vec{b}(i)) = \infty\} \in \mathcal{U}$ , which implies that  $f^{\mathbf{B}}(\vec{b}/\rho) = \underline{\infty}/\rho$ , as required. Hence  $\iota$  is an isomorphism.  $\dashv$

PROPOSITION 3.5. *Let  $A$  be a partial algebra and  $\mathcal{K}$  be a class of partial algebras of the same similarity type as  $A$ . Then  $A \in \text{ISP}^+ \text{P}_u \text{O}(\mathcal{K})$  implies  $\mathfrak{b}(A) \in \text{HSP}(\mathfrak{b}(\mathcal{K}))$ .*

PROOF. If  $A$  is the empty partial algebra, then  $\mathfrak{b}(A)$  is (isomorphic to) the one element algebra in  $\text{HSP}(\mathfrak{b}(\mathcal{K}))$ . Now say that  $A$  is not empty, so that  $A \in \text{ISP}^+ \text{P}_u(\mathcal{K})$ .

Let  $\mathcal{L}$  denote  $\text{P}_u(\mathcal{K})$ . We show that  $\mathfrak{b}(A) \in \text{HSP}(\mathfrak{b}(\mathcal{L}))$ . This will suffice because by the previous lemma we have  $\text{HSP}(\mathfrak{b}(\mathcal{L})) = \text{HSP}(\mathfrak{b}(\text{P}_u(\mathcal{K}))) \subseteq \text{HSP}(\text{IP}_u(\mathfrak{b}(\mathcal{K})) = \text{HSP}(\mathfrak{b}(\mathcal{K}))$ . Now  $A \in \text{ISP}^+ \text{P}_u(\mathcal{K})$  implies  $A \in \text{ISP}^+(\mathcal{L})$ . So without loss of generality we may assume there is a nonempty family  $\{\mathcal{S}_i : i \in I\}$  of members of  $\mathcal{L}$  such that  $A$  is a subalgebra of  $\prod_{i \in I} \mathcal{S}_i$ .

As sets we have  $A \subseteq \prod_{i \in I} \mathfrak{b}(\mathcal{S}_i)$ . Let  $\mathbf{B}$  be the subalgebra of  $\prod_{i \in I} \mathfrak{b}(\mathcal{S}_i)$  generated by  $A$ . It is easily verified that every element of  $\mathbf{B} \setminus A$  has a coordinate equal to  $\infty$ , while no element of  $A$  has this property. Hence  $\mathbf{B} \setminus A$  is an absorbing ideal. The corresponding quotient is routinely seen to be isomorphic to  $\mathfrak{b}(A)$ .  $\dashv$

LEMMA 3.6. *Let  $\mathbf{P}$  be a partial algebra and  $p \in P$ . For  $a \in \mathfrak{b}(P)$  we have  $a \rightsquigarrow p$  in  $\mathfrak{b}(P)$  if and only if  $a \in P$  and  $a \rightsquigarrow p$  in  $\mathbf{P}$ .*

PROOF. The direction  $\Leftarrow$  is trivial. Now say that  $\iota^{b(P)}(a, \vec{b}) = p$  for some term  $\iota(x, \vec{y})$  (explicit in  $x$ ) in the language of  $\mathfrak{b}(P)$  and elements  $\vec{b}$  in  $\mathfrak{b}(P)$ . As  $\infty$  is absorbing, we have that  $a \in P$  and may assume that  $\vec{b}$  lie in  $P$ . Now, starting with  $\iota(x, \vec{y})$ , repeatedly replace each maximal subterm of the form  $u(x, \vec{y}) \wedge v(x, \vec{y})$  by either  $u(x, \vec{y})$  or  $v(x, \vec{y})$ , with the choice made so that the resulting term remains

explicit in  $x$ . Eventually, all occurrences of  $\wedge$  are eliminated and so the resulting term  $s(x, \vec{y})$  is also a term in the language of  $\mathbf{P}$ . Now it is easy to verify directly, or from the laws in Theorem 3.1, that every term operation of a sink preserves the  $\wedge$  order. Hence  $s^{b(\mathbf{P})}(a, \vec{b}) \geq t^{b(\mathbf{P})}(a, \vec{b}) = p$ . Because  $b(\mathbf{P})$  is flat and  $p \neq \infty$ , we have  $s^{b(\mathbf{P})}(a, \vec{b}) = p$ . Hence  $s^{\mathbf{P}}(a, \vec{b}) = p$ ; that is,  $a \rightsquigarrow p$  in  $\mathbf{P}$ .  $\dashv$

**LEMMA 3.7.** *Let  $\mathcal{V}$  be a variety of algebras and  $\mathbf{B}$  be a partial algebra with  $p \in \mathbf{B}$ . If  $b(\mathbf{B}) \in \mathcal{V}$  then  $b(D_p(\mathbf{B})) \in \mathcal{V}$ .*

**PROOF.** Define the equivalence  $\theta$  on  $b(\mathbf{B})$  by  $a \theta b$  if  $a = b$  or neither  $a \rightsquigarrow p$  nor  $b \rightsquigarrow p$  in  $b(\mathbf{B})$  (or in  $\mathbf{B}$ , by Lemma 3.6). Say that  $a \theta b$  with  $a \neq b$  and let  $\lambda(x)$  be a translation in  $b(\mathbf{B})$ . By assumption, neither  $\lambda(a) \rightsquigarrow p$  nor  $\lambda(b) \rightsquigarrow p$ , and hence  $\lambda(a) \theta \lambda(b)$ . So  $\theta$  is a congruence. It is easily seen that  $b(\mathbf{B})/\theta \cong b(D_p(\mathbf{B}))$ .  $\dashv$

In the following lemma and elsewhere to follow, we use the notation  $\text{Cg}^A(a, b)$  to denote the congruence generated by a pair  $(a, b)$  in a given algebra  $A$  (we omit the superscript if  $A$  is implicit).

**LEMMA 3.8.** *Let  $\mathbf{C}$  be a nonempty partial algebra and  $p \in \mathbf{C}$ . If  $\mathbf{B} = D_p(\mathbf{C})$  then  $b(\mathbf{B})$  is subdirectly irreducible with monolith  $\text{Cg}(p, \infty)$ .*

**PROOF.** Say  $a \neq b \in b(\mathbf{B})$ ; we show that  $(p, \infty) \in \text{Cg}(a, b)$ . Without loss of generality we may assume that  $a \neq \infty$ , so that  $a \rightsquigarrow p$  in  $\mathbf{C}$ . Hence there is a term operation  $t^{\mathbf{C}}(x, \vec{y})$  of  $\mathbf{C}$  explicit in all listed variables, and  $\vec{b}$  in  $\mathbf{C}$ , such that  $t^{\mathbf{C}}(a, \vec{b}) = p$  in  $\mathbf{C}$ . Note that each  $b_i \rightsquigarrow p$  also, so  $t^{\mathbf{B}}(a, \vec{b}) = p$  in  $\mathbf{B}$  also. Hence in  $b(\mathbf{B})$  we have  $t^{b(\mathbf{B})}(a \wedge a, \vec{b}) = p$  while  $t^{b(\mathbf{B})}(b \wedge a, \vec{b}) = \infty$ , as required.  $\dashv$

Proposition 3.5 and Lemmas 3.7 and 3.8 give us one half of Theorem 3.3. Say that  $A$  is isomorphic to  $b(\mathbf{B})$  for some  $\mathbf{B} \in \text{DSP}^+ \text{P}_u \text{O}(\mathcal{K})$ . If  $\mathbf{B}$  is empty, then  $b(\mathbf{B})$  is the one element algebra and is in  $\text{V}_{\text{s.i.}}(b(\mathcal{K}))$ . If  $\mathbf{B}$  is nonempty there is  $\mathbf{C} \in \text{SP}^+ \text{P}_u(\mathcal{K})$  and  $p \in \mathbf{C}$  such that  $\mathbf{B} = D_p(\mathbf{C})$ . By Proposition 3.5 we have  $b(\mathbf{C}) \in \text{V}(b(\mathcal{K}))$  and by Lemma 3.7 we have  $b(\mathbf{B}) \in \text{V}(b(\mathcal{K}))$ . By Lemma 3.8,  $b(\mathbf{B}) \in \text{V}_{\text{s.i.}}(b(\mathcal{K}))$ . So  $\text{I}(b(\text{DSP}^+ \text{P}_u(\mathcal{K}))) \subseteq \text{V}_{\text{s.i.}}(b(\mathcal{K}))$ .

Now we must prove the other direction:  $\text{V}_{\text{s.i.}}(b(\mathcal{K})) \subseteq \text{I}(b(\text{DSP}^+ \text{P}_u \text{O}(\mathcal{K})))$ . We begin with a result giving the converse to Lemma 3.8.

**LEMMA 3.9.** *Let  $\mathbf{F} = b(\mathbf{A})$  be a subdirectly irreducible sink, with semilattice operation  $\wedge$  and bottom element  $\infty$  (where  $\mathbf{A}$  is a partial algebra of type  $\mathcal{F}$ , making  $\mathbf{F}$  of type  $\mathcal{F} \cup \{\wedge\}$ ). There is an element  $p \in A$  such that  $\mathbf{A} = D_p(\mathbf{A})$  and the monolith of  $\mathbf{F}$  is  $\text{Cg}(p, \infty)$ .*

**PROOF.** Let  $\text{Cg}(a, b)$  be the monolith. As  $\text{Cg}(a, b) = \text{Cg}(a, a \wedge b) \vee \text{Cg}(a \wedge b, b)$  we can assume without loss of generality that  $a < b$ . Note that  $a = \infty$  because  $\mathbf{F}$  is flat; now choose  $p := b$ . Let  $c \in A$  be arbitrary. As  $(p, \infty) \in \text{Cg}(c, \infty)$ , the Maltsev Lemma (see [11, Lemma V.3.1] for example) shows that there are elements  $p = c_0, \dots, c_{n-1} = \infty$  and translations  $\lambda_i$  in  $\mathbf{F}$  such that for each  $0 \leq i \leq n-2$  we have  $\{\lambda_i(c), \lambda_i(\infty)\} = \{c_i, c_{i+1}\}$ . But  $\lambda_0(\infty) = \infty$  so that  $\lambda_0(c) = p$  in  $\mathbf{F}$ . By Lemma 3.6 we have  $c \rightsquigarrow p$  in  $\mathbf{A}$ , showing that  $\mathbf{A} = D_p(\mathbf{A})$  as required.  $\dashv$

We now complete the proof of Theorem 3.3.

**LEMMA 3.10.** *Let  $\mathbf{S} \in \text{V}_{\text{s.i.}}(b(\mathcal{K}))$ . Then  $\mathbf{S} \cong b(\mathbf{P})$  for some  $\mathbf{P} \in \text{DSP}_r(\mathcal{K})$ .*

**PROOF.** By Theorem 3.1,  $\mathbf{S}$  is a sink (with respect to  $\wedge$ ). We let  $z$  denote the bottom element of  $\mathbf{S}$ . As  $\mathbf{S} \in \text{HSP}(b(\mathcal{K}))$ , there is  $\mathbf{B} \in \text{SP}(b(\mathcal{K}))$  such that  $\mathbf{B}/\theta \cong \mathbf{S}$

for some congruence  $\theta$ . We will assume without loss of generality that  $\mathbf{S} = \mathbf{B}/\theta$  (so  $z$  is a  $\theta$ -class). By Lemma 3.9 there is  $p \in \mathbf{B}$  such that  $\text{Cg}(p/\theta, z)$  is the monolith of  $\mathbf{S}$ .

Let  $\Lambda$  be a set indexing some  $A_i \in \mathcal{K}$  such that  $\mathbf{B} \leq \prod_{i \in \Lambda} b(A_i)$ . For  $a \in \prod_{i \in \Lambda} b(A_i)$  we let the *support* of  $a$  be the set  $\text{supp}(a) = \{i \in \Lambda \mid a(i) \neq \infty\}$ . Define a filter  $\mathcal{F}$  on  $2^\Lambda$  by  $I \in \mathcal{F}$  if  $I \supseteq \text{supp}(a)$  for some  $a \in p/\theta$ : this is obviously well defined because  $p/\theta$  is closed under taking (finite) meets. On  $\prod_{i \in \Lambda} b(A_i)$  define  $\rho$  by  $a \rho b$  if  $\llbracket a = b \rrbracket \in \mathcal{F}$ . Let  $A$  (a partial algebra) be the reduced product  $\prod_{\Lambda/\mathcal{F}} A_i$ . Note that  $\rho$  restricts to an equivalence  $\bar{\rho} := \rho \cap C^2$  on the set  $C := \prod_{i \in \Lambda} A_i$  and that  $C/\bar{\rho}$  is the universe of  $\prod_{\Lambda/\mathcal{F}} A_i$ .

**Claim 0.** If  $a \in B \setminus z$  then  $\text{supp}(a) \in \mathcal{F}$ .

**PROOF.** By Lemma 3.9 there is a translation  $\lambda$  of  $\mathbf{B}$  such that  $\lambda(a) \in p/\theta$ . Then  $\text{supp}(a) \supseteq \text{supp}(\lambda(a)) \in \mathcal{F}$ .  $\dashv$

**Claim 1.** For  $a \in B \setminus z$  and  $b \in B$  we have  $a \rho b \Leftrightarrow a \theta b$ .

**PROOF.** First observe that by Lemma 3.9 there is a translation  $\lambda(x)$  in  $\mathbf{B}$  such that  $\lambda(a) \in p/\theta$ . Now say  $a \rho b$ . By assumption we have  $\llbracket a = b \rrbracket \in \mathcal{F}$  and by Claim 0, we have  $\text{supp}(a) \in \mathcal{F}$ . So  $I := \llbracket a = b \rrbracket \cap \text{supp}(a) \in \mathcal{F}$ . Let  $q \in p/\theta$  be such that  $\text{supp}(q) \subseteq I$ . Then  $\lambda(a \wedge b) \wedge q \wedge \lambda(a) = q \wedge \lambda(a) \in p/\theta$ . By Lemma 3.9 we have  $a \wedge b \notin z$ , or equivalently,  $a/\theta \wedge b/\theta \neq z$ . Hence  $a/\theta = b/\theta$  as required.

For the other direction, assume that  $a \theta b$ . So  $\lambda(a \wedge b) \in p/\theta$  because  $\theta$  is a congruence. Because  $\infty$  is absorbing, we have  $\llbracket a = b \rrbracket \supseteq \text{supp}(a \wedge b) \supseteq \text{supp}(\lambda(a \wedge b))$ , showing that  $a \rho b$ .  $\dashv$

Now let  $P$  be the subset of  $\prod_{\Lambda/\mathcal{F}} A_i$  consisting of those elements  $a/\bar{\rho}$  such that there is  $a' \in B$  with  $a \rho a'$ ; or equivalently, such that  $a/\rho \cap B \neq \emptyset$ .

**Claim 2.** Let  $a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho} \in P$ , and  $a'_0, \dots, a'_{n-1} \in B$  have  $a_0 \rho a'_0, \dots, a_{n-1} \rho a'_{n-1}$ . If  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in \text{dom}(f^A)$  with  $f^A(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) = b/\bar{\rho}$  then  $b \rho f^B(a'_0, \dots, a'_{n-1})$ .

**PROOF.** Assume all the hypotheses of the claim. So there is  $I \in \mathcal{F}$  such that for  $i = 0, \dots, n-1$ , we have  $a_i|_I = a'_i|_I$  and  $\{i \in \Lambda \mid f^{A_i}(\vec{a}(i)) = b(i)\}$  contains  $I$ . Then  $I \subseteq \{i \in \Lambda \mid f^{B_i}(\vec{a}'(i)) = b(i)\}$  so that  $b \rho f^B(\vec{a}')$ .  $\dashv$

In particular,  $f^A(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) = b/\bar{\rho} \in P$  so that  $P$  is a subuniverse of  $A$ ; we write  $\mathbf{P}$  for the corresponding partial algebra.

**Claim 3.** Let  $a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho} \in P$ , and  $a'_0, \dots, a'_{n-1} \in B$  have  $a_0 \rho a'_0, \dots, a_{n-1} \rho a'_{n-1}$ . If  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \notin \text{dom}(f^A)$ , then  $f^B(a'_0, \dots, a'_{n-1}) \in z$ .

**PROOF.** We prove the contrapositive. Say that  $f^B(a'_0, \dots, a'_{n-1}) \notin z$ . By Claim 0, we have  $\text{supp}(f^B(a'_0, \dots, a'_{n-1})) \in \mathcal{F}$ . Therefore  $\{i \in \Lambda \mid (a'_0(i), \dots, a'_{n-1}(i)) \in \text{dom}(f^{A_i})\} \in \mathcal{F}$ . Choose  $I \in \mathcal{F}$  such that  $a'_i|_I = a_i|_I$  for each  $i \in I$  and  $I \subseteq \{i \in \Lambda \mid (a'_0(i), \dots, a'_{n-1}(i)) \in \text{dom}(f^{A_i})\}$ . So  $I \subseteq \{i \in \Lambda \mid (a_0(i), \dots, a_{n-1}(i)) \in \text{dom}(f^{A_i})\}$ , showing that  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in \text{dom}(f^A)$  as required.  $\dashv$

Let  $p^+$  denote some element of  $C$  with  $i \in \text{supp}(p) \Rightarrow p^+(i) = p(i)$ ; such an element obviously exists (simply define  $p^+(i)$  arbitrarily for  $i \notin \text{supp}(p)$ ). As  $\llbracket p = p^+ \rrbracket = \text{supp}(p)$  we have  $p^+ \rho p$  and so  $p^+ \in P$ . For notational convenience,

we let  $\mathbf{D}$  abbreviate  $D_{p^+/\bar{\rho}}(\mathbf{P})$ . We are going to prove that  $\mathbf{S} \cong \mathfrak{b}(\mathbf{D})$ . Let  $\mathbf{T}$  denote the partial algebra obtained from  $\mathbf{S}$  by removing the operation  $\wedge$  and restricting to  $S \setminus \{z\}$ . So our goal is to show that  $\mathbf{D} \cong \mathbf{T}$ .

By the definition of  $\mathbf{P}$  and Claim 1, the rule  $a/\bar{\rho} \mapsto (a/\rho \cap B)/\theta$  defines a map  $\iota$  from  $\mathbf{P}$  into  $S = B/\theta$  that is injective on  $D$ .

**Claim 4.**  $\iota(D) \supseteq T$  and  $\iota^{-1}(T) = D$ .

**PROOF.** Say that  $b/\theta \in T = S \setminus \{z\}$ . By Lemma 3.9, we have  $b/\theta \rightsquigarrow p/\theta$  in  $\mathbf{S}$ , and then by Lemma 3.6 we have  $b/\theta \rightsquigarrow p/\theta$  in  $\mathbf{T}$ . This means that there is a term  $\iota(x, \vec{y})$ , explicit in all variables and not involving  $\wedge$ , and elements  $c_0, \dots, c_{n-1}$  of  $B \setminus z$  such that  $\iota^T(b/\theta, \vec{c}/\theta) = p/\theta$ . By Claim 0 we can choose elements  $b^+, c_0^+, c_1^+, \dots$  of  $C$  and a set  $I \in \mathcal{F}$  such that each of the sets  $\llbracket b = b^+ \rrbracket$ ,  $\llbracket c_0 = c_0^+ \rrbracket, \dots, \llbracket c_{n-1} = c_{n-1}^+ \rrbracket$ ,  $\llbracket \iota^B(b, \vec{c}) = p \rrbracket$  and  $\llbracket p = p^+ \rrbracket$  all contain  $I$ . Then  $\iota^P(b^+/\bar{\rho}, \vec{c}^+/\bar{\rho}) = p^+/\bar{\rho}$ , showing that  $b^+/\bar{\rho} \in D$  and  $\iota(b^+/\bar{\rho}) = b/\theta$ .

This shows  $\iota(D) \supseteq T$ . The claim  $\iota^{-1}(T) = D$  follows because  $\iota(a/\bar{\rho}) = b/\theta$  implies  $b^+ \rho a$  which implies  $a \in D$  (here  $b^+$  is defined in previous paragraph).  $\dashv$

Let  $\bar{\iota}$  be the restriction of  $\iota$  to the domain  $D$ .

**Claim 5.**  $\bar{\iota}$  is an isomorphism between  $\mathbf{D}$  and  $\mathbf{T}$ .

**PROOF.** The proof that  $\bar{\iota}$  is surjective is of similar style to the proof of Claim 4, but easier (since  $\iota(D) \supseteq T$ , it suffices to prove that  $\iota(D) \subseteq T$ ). We leave it to the reader. The injectivity of  $\bar{\iota}$  has already been observed. Now we show that  $\bar{\iota}$  is an isomorphism.

Let  $f$  be a fundamental (partial) operation symbol of arity  $n$  in the language of  $\mathbf{P}$ . Say that  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in \text{dom}(f^D)$  and  $f^D(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) = b/\bar{\rho}$ . That is,  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in \text{dom}(f^P) \cap D^n$  and  $b/\bar{\rho} \in D$ . Now  $\bar{\iota}(a_0/\bar{\rho}), \dots, \bar{\iota}(a_{n-1}/\bar{\rho}), \bar{\iota}(b/\bar{\rho})$  all lie in  $T$  and by Claim 2 we have

$$f^S(\bar{\iota}(a_0/\bar{\rho}), \dots, \bar{\iota}(a_{n-1}/\bar{\rho})) = \bar{\iota}(b/\bar{\rho}) \in T,$$

so that  $f^T(\bar{\iota}(a_0/\bar{\rho}), \dots, \bar{\iota}(a_{n-1}/\bar{\rho})) = \bar{\iota}(b/\bar{\rho})$ .

Now say that  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in D^n \setminus \text{dom}(f^D)$ . So either  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \notin \text{dom}(f^P)$  or both  $(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) \in \text{dom}(f^P)$  and  $f^P(a_0/\bar{\rho}, \dots, a_{n-1}/\bar{\rho}) = b/\bar{\rho} \in P \setminus D$ . In the first case, Claim 3 shows that  $f^S(\bar{\iota}(a_0/\bar{\rho}), \dots, \bar{\iota}(a_{n-1}/\bar{\rho})) = z$  so that  $(\bar{\iota}(a_0/\bar{\rho}), \dots, \bar{\iota}(a_{n-1}/\bar{\rho})) \notin \text{dom}(f^T)$ . In the second case, Claim 2 shows that for  $a'_i \in \iota(a_i/\bar{\rho})$  we have  $f^B(a'_0, \dots, a'_{n-1}) \rho b$ . As  $b/\bar{\rho} \notin D$ , Claim 4 shows that  $f^B(a'_0, \dots, a'_{n-1})/\theta = z$ . Hence  $f^S(a'_0/\theta, \dots, a'_{n-1}/\theta) = z$  and  $(a'_0/\theta, \dots, a'_{n-1}/\theta) \notin \text{dom}(f^T)$  as required.  $\dashv$

So  $\mathbf{D} \cong \mathbf{T}$  and then  $\mathfrak{b}(D_{p^+/\bar{\rho}}(\mathbf{P})) = \mathfrak{b}(\mathbf{D}) \cong \mathfrak{b}(\mathbf{T}) \cong \mathbf{S}$ . This completes the proof, as  $\mathbf{P}$  is a subalgebra of a reduced product of members of  $\mathcal{K}$ .  $\dashv$

This also completes the proof Theorem 3.3.

**COROLLARY 3.11.** *Let  $\mathcal{K}$  be a class of partial algebras of type  $\mathcal{F}$  and  $\mathbf{P}$  be a partial algebra of type  $\mathcal{F}$ . Then  $\mathfrak{b}(\mathbf{P}) \in \mathcal{V}(\mathfrak{b}(\mathcal{K}))$  if and only if  $\mathfrak{D}(\mathbf{P}) \subseteq \text{IDSP}_r\mathcal{O}(\mathcal{K})$ .*

**PROOF.** It is routinely verified using Lemma 3.9 that  $\mathfrak{l}(\mathfrak{b}(\mathbf{D}(\mathbf{P})))$  consists of the class of all subdirectly irreducible homomorphic images of  $\mathfrak{b}(\mathbf{P})$ . Now, by the Birkhoff subdirect representation theorem (see [11, Theorem II.8.6]), we have  $\mathfrak{b}(\mathbf{P}) \in \mathcal{V}(\mathfrak{b}(\mathcal{K}))$  if and only if all subdirectly irreducible quotients of  $\mathfrak{b}(\mathbf{P})$  lie in  $\mathcal{V}_{\text{s.i.}}(\mathfrak{b}(\mathcal{K})) =$

$I(\mathfrak{b}(\text{DSP}_r\mathcal{O}(\mathcal{K})))$ . This in turn holds if and only if  $I(\mathfrak{b}(\mathbf{D}(\mathcal{P}))) \subseteq I(\mathfrak{b}(\text{DSP}_r\mathcal{O}(\mathcal{K})))$  which holds if and only if  $\mathbf{D}(\mathcal{P}) \subseteq \text{IDSP}_r\mathcal{O}(\mathcal{K})$ .  $\dashv$

The following easy example uses Corollary 3.11 to demonstrate that the converse of Proposition 3.5 is false in general.

**EXAMPLE 3.12.** Consider the uH class  $\mathcal{H}$  of total algebras in the similarity type of one unary operation  $\{f\}$  defined by  $\{f(x) \approx f(x), f^2(x) \not\approx x \vee f^3(y) \not\approx y\}$ . The total algebra  $\mathbf{F}$  on the 5 element set  $\{a_0, a_1, b_0, b_1, b_2\}$ , in which  $f(a_i) = a_{i+1 \bmod 2}$ ,  $f(b_i) := b_{i+1 \bmod 3}$  is not contained in  $\mathcal{H}$ , but the members of  $\mathbf{D}(\mathbf{F})$  are. Hence  $\mathfrak{b}(\mathbf{F}) \in \text{HSP}(\mathfrak{b}(\mathcal{H}))$ , even though  $\mathbf{F} \notin \text{IDSP}_r\mathcal{O}(\mathcal{H}) = \mathcal{H}$ .

**PROPOSITION 3.13.** *Let  $\mathcal{F}$  be a similarity type including at least one operation  $f$  of arity  $n > 1$ . Let  $\mathcal{H}$  be a uH class of partial  $\mathcal{F}$ -algebras and  $\mathbf{T}$  be a total  $\mathcal{F}$ -algebra. Then  $\mathfrak{b}(\mathbf{T}) \in \text{HSP}(\mathfrak{b}(\mathcal{H}))$  if and only if  $\mathbf{T} \in \mathcal{H}$ .*

**PROOF.** The ‘‘if’’ direction is Proposition 3.5. Now say that  $\mathbf{T} \notin \mathcal{H}$ . Hence there is an assignment  $\theta$  into  $\mathbf{T}$  causing some uH sentence  $\Phi$  of  $\mathcal{H}$  to fail. Since  $\mathbf{T}$  is total,  $\theta$  assigns values in  $T$  to all terms appearing in  $\Phi$ . Since  $\Phi$  is a finite expression, the set  $S \subseteq T$  of all such values is finite. By repeatedly applying  $f^T$  to the elements of  $S$  (in any way that eventually includes all elements of  $S$ ), we find an element  $p$  of  $T$  such that every element of  $S$  divides  $p$ . Now  $\Phi$  continues to fail in  $\mathbf{D}_p(\mathbf{T})$  under  $\theta$ , but moreover, all terms in  $\Phi$  are still given defined values by  $\theta$  in  $\mathbf{D}_p(\mathbf{T})$ . Hence any partial algebra  $\mathbf{A}$  for which  $\mathbf{D}_p(\mathbf{A}) = \mathbf{D}_p(\mathbf{T})$  fails  $\Phi$ . So  $\mathbf{D}(\mathbf{T}) \not\subseteq \mathcal{H}$  and Corollary 3.11 now shows that  $\mathfrak{b}(\mathbf{T}) \notin \text{HSP}(\mathfrak{b}(\mathcal{H}))$ .  $\dashv$

**COROLLARY 3.14.** *Let  $\mathcal{F}$  be a similarity type including at least one operation of arity strictly greater than 1.*

1. *Let  $\mathcal{K}$  be a class of partial  $\mathcal{F}$ -algebras such that there is a nonempty family  $\mathcal{T} := \{\mathbf{T}_i \mid i \in I\}$  of nonempty total  $\mathcal{F}$ -algebras disjoint from  $\text{ISP}_r\mathcal{O}(\mathcal{K})$  but for which some ultraproduct  $\prod_{I/\mathcal{U}} \mathcal{T}$  is contained in  $\text{ISP}_r\mathcal{O}(\mathcal{K})$ . Then  $\mathfrak{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable.*
2. *If  $\text{ISP}_r\mathcal{O}(\mathcal{K})$  is a not finitely axiomatisable uH class of total  $\mathcal{F}$ -algebras then  $\mathfrak{V}(\mathfrak{b}(\mathcal{K}))$  is a not finitely axiomatisable variety of algebras.*

**PROOF.** We use the fact that a uH class (of algebras or of partial algebras) is finitely axiomatisable if and only if the complement is closed under taking ultraproducts (see [11, Corollary V.2.18] for a proof in the model theory of conventional structures; this proof continues to hold in the present setting, provided suitable care is taken with the empty structure).

We now prove (1). Proposition 3.13 shows that  $\mathfrak{b}(\mathcal{T})$  is disjoint from  $\mathfrak{V}(\mathfrak{b}(\mathcal{K}))$ , and since  $\prod_{I/\mathcal{U}} \mathcal{T}$  is also total and contained in  $\text{ISP}_r\mathcal{O}(\mathcal{K})$ , Proposition 3.5 and Lemma 3.4 show that  $\prod_{I/\mathcal{U}} \mathfrak{b}(\mathcal{T}) \cong \mathfrak{b}(\prod_{I/\mathcal{U}} \mathcal{T}) \in \mathfrak{V}(\mathfrak{b}(\mathcal{K}))$ . Hence  $\mathfrak{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable.

For (2), first observe that  $\mathfrak{V}(\mathfrak{b}(\mathcal{K}))$  is always not finitely axiomatisable if  $\mathcal{F}$  is infinite. When  $\mathcal{F}$  is finite, the class of total  $\mathcal{F}$ -algebras is a finitely axiomatised uH class (Example 2.2). Hence if  $\text{ISP}_r\mathcal{O}(\mathcal{K})$  is a uH class of total algebras, then we can assume without loss of generality that if  $\prod_{I/\mathcal{U}} \mathcal{T} \in \text{ISP}_r\mathcal{O}(\mathcal{K})$  then  $\mathcal{T}$  consists of total algebras. The claim now follows from the first statement.  $\dashv$

A further interesting consequence of the main results so far is the following.

**PROPOSITION 3.15.** *Let  $\mathcal{K}$  be a class of similar partial algebras. The variety  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable provided that there is an indexed set of partial algebras  $\mathcal{L} = \{\mathcal{A}_i \mid i \in I\}$  and an ultraproduct  $\prod_{I/\mathcal{U}} \mathcal{L}$  such that  $\mathbf{D}(\prod_{I/\mathcal{U}} \mathcal{L}) \subseteq \text{IDSP}_r\mathbf{O}(\mathcal{K})$  but for each  $i \in I$  we have:*

1. *there is  $p_i \in \mathcal{A}_i$  such that  $\mathcal{A}_i = \mathbf{D}_{p_i}(\mathcal{A}_i)$ ;*
2.  *$\mathcal{A}_i \notin \text{IDSP}_r\mathbf{O}(\mathcal{K})$ .*

*If  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is contained in a finitely axiomatisable subvariety of  $\mathbf{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$ , then the above condition holds if and only if  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable.*

**PROOF.** Let  $\mathcal{K}$  be a class of partial algebras of type  $\mathcal{F}$  for which a class  $\mathcal{L}$  exists satisfying the stated conditions. By Lemma 3.8,  $\mathfrak{b}(\mathcal{A}_i)$  is subdirectly irreducible, so by Theorem 3.3 and property (2) above, we have  $\mathfrak{b}(\mathcal{A}_i) \notin \mathbf{V}(\mathfrak{b}(\mathcal{K}))$ . Now we show that  $\prod_{I/\mathcal{U}} \mathfrak{b}(\mathcal{A}_i) \in \mathbf{V}(\mathfrak{b}(\mathcal{K}))$ .

Lemma 3.4 shows that  $\prod_{I/\mathcal{U}} \mathfrak{b}(\mathcal{A}_i) \cong \mathfrak{b}(\prod_{I/\mathcal{U}} \mathcal{A}_i)$ , while Corollary 3.11 and the condition on  $\prod_{I/\mathcal{U}} \mathcal{A}_i$  shows that  $\mathfrak{b}(\prod_{I/\mathcal{U}} \mathcal{A}_i) \in \mathbf{V}(\mathfrak{b}(\mathcal{K}))$  as required. This shows that the complement of  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is not closed under ultraproducts, and hence  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable.

For the reverse direction, assume that  $\mathcal{K}$  is a class of partial algebras of type  $\mathcal{F}$  such that  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is contained in a subvariety of  $\mathbf{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  axiomatised by a finite set  $\Phi$  of identities, and that  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  is not finitely axiomatisable. Let  $\text{Th}_{\text{id}}(\mathbf{V}(\mathfrak{b}(\mathcal{K})))$  denote the equational theory of  $\mathbf{V}(\mathfrak{b}(\mathcal{K}))$  (over some fixed countably infinite set of variables) and let  $I$  be the set of all finite subsets of  $\text{Th}_{\text{id}}(\mathbf{V}(\mathfrak{b}(\mathcal{K})))$  containing  $\Phi$ . For each  $\Sigma \in I$  there is an algebra  $\mathcal{B}_{\Sigma}$  satisfying  $\Sigma$  but not  $\text{Th}_{\text{id}}(\mathbf{V}(\mathfrak{b}(\mathcal{K})))$ . We can assume without loss of generality that  $\mathcal{B}_{\Sigma}$  is subdirectly irreducible because it is a subdirect product of its subdirectly irreducible quotients, and these all satisfy  $\Sigma$  (but cannot all satisfy  $\text{Th}_{\text{id}}(\mathbf{V}(\mathfrak{b}(\mathcal{K})))$ ).

Now each  $\Sigma \in I$  contains  $\Phi$ , and so the subdirectly irreducible algebra  $\mathcal{B}_{\Sigma}$  is a sink (with respect to  $\wedge$ ). Theorem 3.1 shows that  $\mathcal{B}_{\Sigma}$  is of the form  $\mathfrak{b}(\mathcal{A}_{\Sigma})$  for some partial algebra  $\mathcal{A}_{\Sigma}$ . By Lemma 3.9 there is  $p \in \mathcal{A}_{\Sigma}$  such that  $\mathcal{A}_{\Sigma} = \mathbf{D}_p(\mathcal{A}_{\Sigma})$ . We let  $\mathcal{L} := \{\mathcal{A}_i \mid i \in I\}$ . Now condition (1) is certainly satisfied, while condition (2) holds because of Theorem 3.3 and the fact that  $\mathcal{B}_{\Sigma} \notin \mathbf{V}(\mathfrak{b}(\mathcal{K}))$ . Finally, by the standard methods (see proof of [11, Corollary V.2.18] for example), there is an ultrafilter  $\mathcal{U}$  on the Boolean algebra of all subsets of  $I$  such that  $\prod_{I/\mathcal{U}} \mathcal{B}_i \in \mathbf{V}(\mathfrak{b}(\mathcal{K}))$ . But Lemma 3.4 shows that  $\prod_{I/\mathcal{U}} \mathcal{B}_i \cong \mathfrak{b}(\prod_{I/\mathcal{U}} \mathcal{A}_i)$  and then Corollary 3.11 shows that  $\mathbf{D}(\prod_{I/\mathcal{U}} \mathcal{A}_i) \subseteq \text{IDSP}_r\mathbf{O}(\mathcal{K})$ . ◻

**§4. Eliminating D.** When the operator  $\mathbf{D}$  is applied to a uH class  $\mathcal{K}$ , the resulting class  $\mathbf{D}(\mathcal{K})$  of partial algebras is often quite different to the original class; in particular if  $\mathcal{K}$  consists of total algebras then it is common that  $\mathbf{D}(\mathcal{K})$  is a class containing partial algebras that are not total. In this section we briefly examine the situation where the operator  $\mathbf{D}$  produces a subclass of the original uH class.

A quasi-identity of the form

$$f(x_0, \dots, x_{n-1}) \approx f(x_0, \dots, x_{n-1}) \ \& \ \left( \bigwedge_{0 \leq i \leq n-1} t_i(x_i, y_0^i, \dots, y_{m_i-1}^i) \approx z \right) \rightarrow s(f(x_0, \dots, x_{n-1}), x_0, \dots, x_{n-1}, y_0^0, \dots, y_{m_{n-1}}^{n-1}, z) \approx z$$

will be called a *division law* if the  $t_i$  are all explicit in their listed variables,  $f$  is an  $n$ -ary fundamental operation and  $s(u, \vec{x}, \vec{y}^0, \dots, \vec{y}^{n-1}, z)$  is explicit in  $u$  (but not necessarily the other variables). The conjunct of equalities to the left of the  $\rightarrow$  symbol will be called a *division sequence for  $f$* . We allow the  $n = 0$  case: the corresponding division law is  $f() \approx f() \rightarrow s(f(), z) \approx z$ .

**THEOREM 4.1.** *Let  $\mathcal{K}$  be a class of similar total algebras. The following are equivalent:*

1. *for every  $n$ -ary fundamental operation  $f$  and every division sequence  $f(\vec{x}) \approx f(\vec{x}) \ \& \ \left( \&_i t_i(x_i, \vec{y}^i) \approx z \right)$ , there is a term  $s(u, \vec{x}, \vec{y}^0, \dots, z)$  such that*

$$\mathcal{K} \models f(\vec{x}) \approx f(\vec{x}) \ \& \ \left( \&_i t_i(x_i, \vec{y}^i) \approx z \right) \rightarrow s(f(\vec{x}), \vec{x}, \vec{y}^0, \dots, z) \approx z;$$
2.  $\text{IDSP}_r\text{O}(\mathcal{K}) \subseteq \text{ISP}_r\text{O}(\mathcal{K})$ ;
3.  $\text{IDSP}_r\text{O}(\mathcal{K})$  consists of total algebras.

*More generally, when  $\mathcal{K}$  contains partial algebras, the implications (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (2) always hold.*

**PROOF.** (1)  $\Rightarrow$  (2). Let  $\mathcal{K}$  be any class of similar partial algebras, and consider  $A \in \text{SP}_r(\mathcal{K})$  and  $p \in A$ . We show that  $D_p(A)$  is a subalgebra of  $A$ . It suffices to show that if  $f$  is an  $n$ -ary operation and  $(a_0, \dots, a_n) \in \text{dom}(f^A) \cap (D_p(A))^n$ , then  $f(a_0, \dots, a_{n-1}) \in D_p(A)$ .

Now as  $a_i \rightsquigarrow p$  we obtain a division sequence of length  $n$  for  $p$ ; say

$$f(a_0, \dots, a_{n-1}) = f(a_0, \dots, a_{n-1}) \ \& \ \&_{0 \leq i \leq n-1} t_i(a_i, b_{i,0}, \dots, b_{i,m_i-1}) = p$$

for some terms  $t_i$  and elements  $b_j^k \in A$ . The corresponding division law shows that  $f(a_0, \dots, a_{n-1}) \rightsquigarrow p$ , as required.

(2)  $\Rightarrow$  (3). This is trivial when  $\mathcal{K}$  consists of total algebras.

(3)  $\Rightarrow$  (2). If  $A \in \text{SP}_r(\mathcal{K})$  and  $p \in A$  are such that  $B := D_p(A)$  is a total algebra, then  $B$  is a subalgebra of  $A$ .

(2)  $\Rightarrow$  (1). Let  $\mathcal{K}$  consist of total algebras. We prove the contrapositive. Assume there is a division sequence

$$f(x_0, \dots, x_{n-1}) \approx f(x_0, \dots, x_{n-1}) \ \& \ \&_{0 \leq i \leq n-1} t_i(x_i, y_{i,0}, \dots, y_{i,m_i-1}) \approx z$$

such that for every term  $s = s(u, \vec{x}, \vec{y}^0, \dots, z)$  (explicit in  $u$ ) we have a failure of the corresponding division law on some member  $A_s$  of  $\mathcal{K}$ . That is, there is  $p \in A_s$  such that

$$f(\vec{a}) = f(\vec{a}) \ \& \ \&_{0 \leq i \leq n-1} t_i(a_i, b_{i,0}, \dots, b_{i,m_i-1}) = p$$

but  $s(f(\vec{a}), \vec{b}^0, \dots, p) \neq p$ . Let  $A$  denote the direct product of all these  $A_s$  (in other words, the choices of the term  $s$  index the algebras), and define elements  $\underline{a}_0, \dots, \underline{a}_{n-1}, \underline{b}_0^0, \dots, \underline{b}_{m-1}^{n-1}, \underline{p}$  by setting the  $s^{\text{th}}$  entry to be the corresponding elements from  $A_s$  causing a failure of the division law for the term  $s$ . Let  $B$  be the subalgebra of  $A$  generated by these elements. We claim that  $D_{\underline{p}}(B)$  is a partial algebra.

Certainly  $\underline{a}_i \rightsquigarrow \underline{p}$  for each  $i$ , because the division sequence is fixed throughout the construction of the  $\underline{a}_i$  and  $\underline{b}_j^i$ . Similarly we have  $(\underline{a}_0, \dots, \underline{a}_{n-1}) \in \text{dom}(f^B)$  because

this holds on each coordinate. Now for each term  $s$  with  $s(u, x_0, \dots, y_{m_{n-1}-1}^{n-1}, z)$ , we have that on the  $s^{\text{th}}$ -coordinate

$$[s^{\mathbf{B}}(f^{\mathbf{B}}(\vec{a}), \underline{a}_0, \dots, \underline{b}_{n-1, m_{n-1}-1}, \underline{p})](s) \neq [\underline{p}](s).$$

Because  $\mathbf{B}$  is generated by  $\underline{a}_0, \dots, \underline{b}_{n-1, m_{n-1}-1}, \underline{p}$  we have that  $f^{\mathbf{B}}(\vec{a}) \not\sim \underline{p}$ . Hence  $D_{\underline{p}}(\mathbf{B})$  is not closed under  $f$  and hence is not a total algebra.  $\dashv$

The implication (2) $\Rightarrow$ (1) does not in general hold, since  $\text{IDSP}_r\text{O}(\mathcal{P}_{\mathcal{F}}) = \text{ISP}_r\text{O}(\mathcal{P}_{\mathcal{F}})$  holds trivially, while the only first order sentences satisfied by  $\mathcal{P}_{\mathcal{F}}$  are tautologies (in the logic of partial algebras).

EXAMPLE 4.2. If  $\mathcal{B}$  is a class of Brandt groupoids, then  $\text{IDSP}_r\text{O}(\mathcal{B}) \subseteq \text{ISP}_r\text{O}(\mathcal{B})$ .

Brandt groupoids are not total algebras, but we can deduce the containment  $\text{IDSP}_r\text{O}(\mathcal{B}) \subseteq \text{ISP}_r\text{O}(\mathcal{B})$  by showing that condition (1) of Theorem 4.1 holds. Let  $\mathbf{B}$  be a Brandt groupoid and  $p \in \mathbf{B}$ . If  $a \rightsquigarrow p$  in  $\mathbf{B}$  then certainly as  $a^{-1} \rightsquigarrow a$  we also have  $a^{-1} \rightsquigarrow p$ . Hence  $D_p(\mathbf{B})$  is closed under taking inverses. Now say that  $a, b \rightsquigarrow p$ . It is not hard to verify that the law  $xy \approx xy \rightarrow xyy^{-1} \approx x$  holds in  $\mathbf{B}$ . Hence, if  $ab$  is defined, then we also have  $ab \rightsquigarrow a \rightsquigarrow p$ , because  $(ab)b^{-1} = a$ . So for example, in terms of the division laws of Theorem 4.1, this shows that for a division sequence (in this case, for the binary partial operation of multiplication)

$$\Phi(x_0, x_1, \vec{y}_0, \vec{y}_1, z) := x_0x_1 \approx x_0x_1 \ \& \ t_0(x_0, y_{0,0}, \dots) \approx z \ \& \ t_0(x_1, y_{1,0}, \dots) \approx z$$

we may choose  $s(u, y_{0,0}, \dots, y_{1, m_1-1}, z)$  to be the term  $t_0(ux_1^{-1}, y_{0,0}, \dots, y_{0, m_0-1})$ . (The inversion case is similar.)

The class of Brandt groupoids actually satisfies the apparently stronger property that  $x \rightsquigarrow y$  implies  $y \rightsquigarrow x$ . Hence the class  $\text{D}(\mathcal{B}) =: \mathcal{B}_{\text{conn}}$  consists of the Brandt groupoids in which every element divides every other element (*connected* Brandt groupoids). This shows that  $\text{V}_{\text{s.i}}(\text{b}(\mathcal{B})) = \text{I}(\text{b}(\mathcal{B}_{\text{conn}}))$ .

EXAMPLE 4.3. Connected partial (di-)graph algebras.

A digraph is simply a binary relational structure (the vertices are the elements and the edge relation is the relation). A graph is a digraph satisfying the symmetry property  $x \sim y \rightarrow y \sim x$ . Digraphs can be conveniently thought of as partial algebras, by replacing the edge relation  $\sim$  by the first projection:  $\rho(x, y) := x$  if  $(x, y) \in \sim$ . The class of “digraph partial algebras” can be axiomatised by the property  $\rho(x, y) \approx \rho(x, y) \rightarrow \rho(x, y) \approx x$ . The flat extension of a partial digraph algebra is often called a *flat digraph algebra*.

Flat graph algebras are studied by Delić in [14] where a complete characterisation is obtained of which finite graphs have a flat graph algebra generating a finitely axiomatised variety. They also form the underlying construction used by Szekely [43] to give an example of a finite algebra whose variety has NP-complete finite membership problem (see Subsection 7.6 for further discussion). In [28] they are investigated from the perspective of natural dualities.

It is not hard to verify that if  $\mathbf{B}$  is a partial digraph algebra and  $p \in \mathbf{B}$ , then  $D_p(\mathbf{B})$  is the partial digraph algebra on the subgraph of  $\mathbf{B}$  consisting of all points from which  $p$  is reachable by a (finite, directed) path. In the case of graphs, this is of course just the component of  $p$ . So if  $\mathcal{GR}$  is the class of partial graph algebras, then  $\text{IDSP}_r\text{O}(\mathcal{GR})$  is the class of partial graph algebras of connected graphs (that is,

graphs with a single component). The following corollary is well known already, at least in the finitely generated case. A restricted version of it is the fundamental technique in [43] for example.

**COROLLARY 4.4.** *If  $\mathcal{K}$  is a class of graphs and  $\bar{\mathcal{K}}$  is the corresponding class of partial graph algebras, then (up to isomorphism)  $\mathbb{V}_{\text{s.i.}}(\mathfrak{b}(\bar{\mathcal{K}}))$  consists of the flat graph algebras of the connected members of the uH class generated by  $\mathcal{K}$ .*

With more terminology, a similar statement can be made for digraph algebras. The next example is central to the remainder of the paper.

**EXAMPLE 4.5.** If  $\mathcal{K}$  is a class of similar partial algebras for which there is a term  $p(x, y)$ , explicit in  $x$  and  $y$ , and such that  $\mathcal{K} \models p(x, y) \approx y$ , then  $\text{IDSP}_r\text{O}(\mathcal{K}) = \text{ISP}_r\text{O}(\mathcal{K})$ .

**PROOF.** Any partial algebra  $A$  satisfying  $p(x, y) \approx y$  has the property that the relation  $\rightsquigarrow$  is the universal relation: for any  $a, b \in A$  we have  $p^A(a, b) = b$  so that  $a \rightsquigarrow b$ . On such an algebra we have  $\text{D}(A) = \{A\}$ .  $\dashv$

Theorem 4.1 is not needed for this example, but it is clear that condition (1) does hold: for any possible division sequence we may choose the term  $s(u, \dots, z) := p(u, z)$ .

Of course, the term  $p$  in Example 4.5 is nothing more than the second projection, and every uH class is term-equivalent to one with a second projection operation. This will facilitate our translation of uH logic into equational logic.

**REMARK 4.6.** (Note added in proof.) The author has recently observed that the existence of a second projection binary term  $p(x, y)$  for the class  $\mathcal{K}$  as in Example 4.5 is equivalent to the operator  $\text{D}$  coinciding with the identity operator on each subclass of  $\text{ISP}_r\text{O}(\mathcal{K})$ . Also, every class of total algebras generating a congruence modular variety has such a term. The proofs of these claims will be given elsewhere (see Jackson and Trotta [23]).

**§5. The pointed semidiscriminator variety over a similarity type.** The flat extension of two term equivalent algebras may not be term equivalent, a fact that we now exploit heavily. We begin with an apparently unrelated construction, that mimics the notion of a discriminator variety (see [11] for example).

Let  $A$  be a (conventional) algebra and  $\infty \in A$ . If the element  $\infty$  is absorbing for all fundamental operations of  $A$ , then we say that the ternary function  $t : A^3 \rightarrow A$  given by

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \neq \infty, \\ \infty & \text{otherwise} \end{cases}$$

is a *pointed semidiscriminator function*. If there is a term  $d(x, y, z)$  representing a pointed semidiscriminator function on  $A$  with absorbing element  $\infty$ , then we say that  $d(x, y, z)$  is a *pointed semidiscriminator term* for  $A$ . A variety is a *pointed semidiscriminator variety*, if there is a ternary term representing a pointed semidiscriminator function on every subdirectly irreducible in the variety.

**EXAMPLE 5.1.** If  $\mathbf{G}$  is a group, then the term  $d(x, y, z) := (x \wedge y)^{-1}(x \wedge y)z$  is a pointed semidiscriminator term for the algebra  $\mathfrak{b}(\mathbf{G})$ . If  $\mathbf{R}$  is a ring, then

$(x \wedge y) - (x \wedge y) + z$  is a pointed semidiscriminator term for  $\mathfrak{b}(\mathbf{R})$  (cf. the second claim of Remark 4.6).

**LEMMA 5.2.** *An algebra  $\mathbf{A}$  has a pointed semidiscriminator term if and only if  $\mathbf{A}$  is term equivalent with  $\mathfrak{b}(\mathbf{B})$  for some partial algebra  $\mathbf{B}$  whose fundamental (partial) operations include a total binary operation  $\triangleright$  equal to the second projection.*

**PROOF.** Let  $\mathbf{A}$  be an algebra with a pointed semidiscriminator term  $d(x, y, z)$ . Let  $\wedge^{\mathbf{A}}$  and  $\triangleright^{\mathbf{A}}$  denote the term operations  $d^{\mathbf{A}}(x, y, y)$  and  $d^{\mathbf{A}}(x, x, y)$ . Then  $\wedge$  makes  $\mathbf{A}$  a flat semilattice operation and  $\mathbf{A}$  is term equivalent with the flat extension of the partial algebra on  $B := A \setminus \{\infty\}$  whose partial operations are the second projection  $\triangleright^{\mathbf{B}}$  along with the operations of  $\mathbf{A}$  restricted in domain and range to (tuples of) elements of  $B$ .

Conversely, if  $\mathbf{B}$  is a partial algebra whose operations include the second projection  $\triangleright$ , then the term  $(x \wedge y) \triangleright z$  is a pointed semidiscriminator term for  $\mathfrak{b}(\mathbf{B})$ .  $\dashv$

For a given partial algebra  $\mathbf{P} = \langle P; F \rangle$ , we define a new total operation  $a \triangleright b$  by  $a \triangleright b := b$  and let the partial algebra  $\mathbf{P}^{\triangleright} := \langle P; F \cup \{\triangleright\} \rangle$  be the result of adjoining the new operation  $\triangleright$ . Now define the *pointed semidiscriminator extension of  $\mathbf{P}$*  by  $\mathfrak{b}(\mathbf{P}^{\triangleright})$ ; we denote it by  $\text{ps}(\mathbf{P})$ . If there is a term  $t(x, y)$  of  $\mathbf{P}$  in the two variables  $x, y$  (where these variables explicitly appear) such that  $\mathbf{P} \models t(x, y) \approx y$ , then the pointed semidiscriminator extension of  $\mathbf{P}$  is term equivalent to  $\mathfrak{b}(\mathbf{P})$ . For example if  $\mathbf{P}$  is a group (in the language  $\{.,^{-1}\}$ ) then we may choose  $t(x, y) := x^{-1}xy$ , so pointed semidiscriminator extensions of groups are just flat extensions of groups (up to term equivalence). This particular choice of term gives rise to the pointed semidiscriminator term for flat algebras over groups given in Example 5.1. If no such term of  $\mathbf{P}$  exists, then  $\text{ps}(\mathbf{P})$  and  $\mathfrak{b}(\mathbf{P})$  will not be term equivalent except in some degenerate cases.

We will extend these notation as follows: if  $\mathcal{K}$  is a class of similar partial algebras, then  $\mathcal{K}^{\triangleright}$  and  $\text{ps}(\mathcal{K})$  denote respectively the members of  $\mathcal{K}$  augmented in type by the second projection operation  $\triangleright$  and the class of pointed semidiscriminator extensions of members of  $\mathcal{K}$ .

We can now state the following refinement of the results in previous sections.

**THEOREM 5.3.** *Let  $\mathcal{K}$  be a class of similar partial algebras. Up to isomorphism, the subdirectly irreducible members of  $\mathfrak{V}(\text{ps}(\mathcal{K}))$  are precisely the pointed semidiscriminator extensions of members of the *uH* class of  $\mathcal{K}$ :*

$$\mathfrak{I}(\text{ps}(\text{SP}_r\text{O}(\mathcal{K}))) = \mathfrak{V}_{\text{s.i.}}(\text{ps}(\mathcal{K})).$$

Furthermore,  $\mathfrak{V}_{\text{s.i.}}(\text{ps}(\mathcal{K}))$  is closed under taking subalgebras and its nontrivial members coincide with the simple members of  $\mathfrak{V}(\text{ps}(\mathcal{K}))$ . In other words,  $\mathfrak{V}(\text{ps}(\mathcal{K}))$  is hereditarily simple and semisimple.

**PROOF.** The final statements will follow from the first statement and the easily verified fact that a pointed semidiscriminator algebra is simple and its nontrivial subalgebras are pointed semidiscriminator algebras. Now we prove the first statement.

By Theorem 3.3 we have  $\mathfrak{V}_{\text{s.i.}}(\text{ps}(\mathcal{K})) = \mathfrak{V}_{\text{s.i.}}(\mathfrak{b}(\mathcal{K}^{\triangleright})) = \mathfrak{I}(\mathfrak{b}(\text{DSP}_r\text{O}(\mathcal{K}^{\triangleright})))$ . But  $\text{DSP}_r\text{O}(\mathcal{K}^{\triangleright}) = \text{SP}_r\text{O}(\mathcal{K}^{\triangleright})$  by Example 4.5 and it is easily seen that  $\mathfrak{ISP}_r\text{O}(\mathcal{K}^{\triangleright}) = \mathfrak{I}((\text{SP}_r\text{O}(\mathcal{K}))^{\triangleright})$ . Hence  $\mathfrak{V}_{\text{s.i.}}(\text{ps}(\mathcal{K})) = \mathfrak{I}(\mathfrak{b}((\text{SP}_r\text{O}(\mathcal{K}))^{\triangleright})) = \mathfrak{I}(\text{ps}(\text{SP}_r\text{O}(\mathcal{K})))$ .  $\dashv$

The variety generated by all pointed semidiscriminator extensions of partial algebras of some similarity type  $\mathcal{F}$  we call the *pointed semidiscriminator variety over the similarity type  $\mathcal{F}$*  and denote it by  $\text{PS}_{\mathcal{F}}$ .

The uH classes of a similarity type  $\mathcal{F}$  are ordered under inclusion and moreover form a lattice: the meet of two uH classes is their intersection (which is nonempty because of the empty algebra) while the join  $\mathcal{J} \vee \mathcal{K}$  of two uH classes  $\mathcal{J}$  and  $\mathcal{K}$  is the uH class  $\text{ISP}_r\text{O}(\mathcal{K} \cup \mathcal{J})$ . The uH classes of total algebras form a sublattice of this; see Example 2.2 for an axiomatisation of the class of all such structures (and restrict to uH classes containing the one element structure to get the sublattice of quasivarieties of total algebras). The following key fact is an easy corollary of Theorem 5.3.

**COROLLARY 5.4.** *Let  $\mathcal{F}$  be a fixed type of partial algebras. The mapping  $\mathcal{H} \mapsto \mathcal{V}(\text{ps}(\mathcal{H}))$  is a lattice isomorphism from the lattice of all uH classes of partial  $\mathcal{F}$ -algebras to the lattice of subvarieties of  $\text{PS}_{\mathcal{F}}$ .*

The remainder of this section is devoted to giving a simple axiomatisation for  $\text{PS}_{\mathcal{F}}$  and showing that the isomorphism just described translates most of the global properties of a uH class into corresponding properties of a variety.

We begin by showing that, when  $\mathcal{F}$  is finite,  $\text{PS}_{\mathcal{F}}$  is a finitely based subvariety of the variety generated by all sinks of type  $\mathcal{F} \cup \{\wedge\}$ . We define the set  $\Sigma_{\mathcal{F}}$  of identities in the operation symbols of  $\mathcal{F} \cup \{\wedge, \triangleright\}$  to be the union of the following sets.

$$\left. \begin{array}{l} (x \wedge y) \wedge z \approx x \wedge (y \wedge z), \quad x \wedge x \approx x, \\ x \wedge y \approx y \wedge x, \end{array} \right\} \Sigma_{\wedge}$$

$$\left. \begin{array}{l} (x \triangleright y) \triangleright z \approx x \triangleright (y \triangleright z), \quad x \triangleright x \approx x, \\ x \triangleright (y \triangleright z) \approx y \triangleright (x \triangleright z) \end{array} \right\} \Sigma_{\triangleright}$$

$$(1) \quad (x \wedge y) \triangleright y \approx x \wedge y, \quad (2) \quad (x \triangleright y) \wedge z \approx x \triangleright (y \wedge z) \quad \left. \right\} \Sigma_{\leq}$$

$$\left. \begin{array}{l} \text{for each } f \in \mathcal{F} \cup \{\triangleright, \wedge\} \text{ of arity } n \text{ and each } 1 \leq i \leq n \\ (1_{f,i}) \quad x \triangleright f(x_1, \dots, x_i, \dots, x_n) \approx f(x_1, \dots, x \triangleright x_i, \dots, x_n) \end{array} \right\} \Sigma_{\text{op}}$$

The identities  $\Sigma_{\wedge}$  show that models of  $\Sigma_{\mathcal{F}}$  are semilattices with respect to  $\wedge$ , while  $\Sigma_{\triangleright}$  shows that they are right normal bands with respect to  $\triangleright$ ; see [18] for example. We now adopt the convention that brackets are omitted from suitable places in terms involving associative binary operations, and that all operations take precedence over applications of  $\wedge$ ; so  $x \triangleright y \triangleright z \wedge w$  is the same as, say,  $(x \triangleright (y \triangleright z)) \wedge w$ .

The identities in  $\Sigma_{\text{op}}$  ensure that the other operations match the structure defined by  $\wedge$  and  $\triangleright$ . We note that there is redundancy in the definition of  $\Sigma_{\mathcal{F}}$ : modulo the semilattice laws, identities  $(1_{\wedge,1})$  and  $(1_{\wedge,2})$  of  $\Sigma_{\text{op}}$  are the same as  $\Sigma_{\leq}(2)$ , while  $(1_{\triangleright,1})$  and  $(1_{\triangleright,2})$  of  $\Sigma_{\text{op}}$  constitute the two 3-variable laws of  $\Sigma_{\triangleright}$ .

We now verify that  $\text{ps}(\mathbf{P}) \models \Sigma_{\mathcal{F}}$ , when  $\mathbf{P}$  is a partial algebra of type  $\mathcal{F}$ .

Associativity of  $\triangleright$  is easily verified on any  $\text{ps}(\mathbf{P})$ , so we can adopt the usual convention of omitting brackets. The following useful fact is easy; the proof is omitted.

LEMMA 5.5. *Let  $\mathbf{P}$  be a partial algebra of similarity type  $\mathcal{F}$ . Then for  $a_1, \dots, a_n, a$  in  $\text{ps}(\mathbf{P})$  we have*

$$a_1 \triangleright \cdots \triangleright a_n \triangleright a = \begin{cases} a & \text{if } a_1, \dots, a_n, a \neq \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Using this lemma it is now easy to verify the following fact.

LEMMA 5.6. *Let  $\mathbf{P}$  be a partial algebra of similarity type  $\mathcal{F}$ . Then  $\text{ps}(\mathbf{P}) \models \Sigma_{\mathcal{F}}$ .*

PROOF. As a sample, we prove satisfaction of

$$(1_{f,i}) \quad x \triangleright f(x_1, \dots, x_i, \dots, x_n) \approx f(x_1, \dots, x \triangleright x_i, \dots, x_n)$$

for  $f \in \mathcal{F}$  and leave the rest to the reader. Let  $\mathbf{P}$  be a partial algebra and  $f$  be a fundamental operation. Let  $\theta$  be a variable assignment into  $\text{ps}(\mathbf{P})$ . Both sides of  $(1_{f,i})$  contain the same variables, and as  $\infty$  is absorbing, we may assume without loss of generality that  $\theta$  assigns these variables values in  $\mathbf{P}$ . Hence  $\theta(x) \triangleright f(\theta(x_1), \dots, \theta(x_i), \dots, \theta(x_n)) = f(\theta(x_1), \dots, \theta(x_i), \dots, \theta(x_n)) = f(\theta(x_1), \dots, \theta(x) \triangleright \theta(x), \dots, \theta(x_n))$  (using the definition of  $\triangleright$  or the  $n = 1$  case of Lemma 5.5).  $\dashv$

This shows that  $\Sigma_{\mathcal{F}}$  defines a variety containing  $\text{PS}_{\mathcal{F}}$ .

LEMMA 5.7.  $\Sigma_{\mathcal{F}}$  defines a subvariety of the variety  $\mathbf{V}(\mathbf{b}(\mathcal{P}_{\mathcal{F} \cup \{\triangleright\}}))$  generated by all flat extensions of type  $\mathcal{F} \cup \{\triangleright\}$  partial algebras.

PROOF. We verify the identities from Theorem 3.1.

The  $\wedge$ -semilattice laws obviously hold. Now let  $t(x, z_1, \dots, z_n)$  and  $s(x, z_1, \dots, z_n)$  be two terms in  $\mathcal{F} \cup \{\triangleright\}$  and explicit in  $x$ . Then

$$\begin{aligned} t(x \wedge y, \overline{z}) &\approx t((x \wedge y) \triangleright (x \wedge y), \overline{z}) \quad (\text{by } \Sigma_{\triangleright}) \\ &\approx (x \wedge y) \triangleright t(x \wedge y, \overline{z}) \quad (\text{by repeated applications of } \Sigma_{\text{op}}) \end{aligned}$$

so that

$$\begin{aligned} t(x \wedge y, \overline{z}) \wedge s(x, \overline{z}) &\approx ((x \wedge y) \triangleright t(x \wedge y, \overline{z})) \wedge s(x, \overline{z}) \\ &\approx t(x \wedge y, \overline{z}) \wedge ((x \wedge y) \triangleright s(x, \overline{z})) \quad (\text{by } \Sigma_{\leq}(2)) \\ &\approx t(x \wedge y, \overline{z}) \wedge s((x \wedge y) \triangleright x, \overline{z}) \\ &\quad (\text{by repeated applications of } \Sigma_{\text{op}}) \\ &\approx t(x \wedge y, \overline{z}) \wedge s((x \wedge y), \overline{z}) \quad (\text{by } \Sigma_{\leq}(1)) \end{aligned}$$

and then by symmetry we get  $t(x \wedge y, \overline{z}) \wedge s(x, \overline{z}) \approx t(x \wedge y, \overline{z}) \wedge s((x \wedge y), \overline{z}) \approx t(x \wedge y, \overline{z}) \wedge s(y, \overline{z})$  as required.  $\dashv$

THEOREM 5.8. *The identities  $\Sigma_{\mathcal{F}}$  are a basis for the identities of  $\text{PS}_{\mathcal{F}}$ .*

PROOF. By Lemmas 5.6 and 5.7 it remains to show that every subdirectly irreducible sink satisfying  $\Sigma_{\mathcal{F}}$  is of the form  $\mathbf{b}(\mathbf{P}^{\triangleright})$  for some  $\mathcal{F}$ -partial algebra  $\mathbf{P}$ . By Lemma 5.7 and Theorem 3.1 we know that every subdirectly irreducible is of the form  $\mathbf{b}(\mathbf{Q})$  for some partial algebra of type  $\mathbf{Q}$  of type  $\mathcal{F} \cup \{\triangleright\}$ . To complete the proof it remains to show that if  $a, b \in \mathbf{Q}$  then  $a \triangleright b = b$ . By Theorem 3.3 there is  $p \in \mathbf{Q}$  such that  $a \rightsquigarrow p$  and  $b \rightsquigarrow p$  in  $\mathbf{Q}$ ; say  $\lambda(a) = p = \gamma(b)$  for some translations  $\lambda$  and  $\gamma$ . Hence in  $\mathbf{b}(\mathbf{Q})$  we have  $\lambda(a) \wedge \gamma(b) = p \neq \infty$ . So then  $p = \lambda(a) \wedge \gamma(b) = (a \triangleright \lambda(a)) \wedge (b \triangleright \gamma(b)) = a \triangleright (\lambda(a) \wedge (b \triangleright \gamma(b))) = a \triangleright b \triangleright$

$(\lambda(a) \wedge \gamma(b)) = a \triangleright b \triangleright p$ . But then  $a \triangleright b \neq \infty$  so  $a \triangleright b$  is well defined in  $\mathcal{Q}$  also. But  $b \wedge (a \triangleright b) = a \triangleright (b \wedge b) = a \triangleright b$  in  $\mathfrak{b}(\mathcal{Q})$  showing that  $a \triangleright b = b$  as required.  $\dashv$

We now describe how to translate uH formulæ in  $\mathcal{F}$  into identities in  $\mathcal{F} \cup \{\wedge, \triangleright\}$ . For a term  $t$  (set of terms  $T$ ) we let  $\text{var}(t)$  (or  $\text{var}(T)$ , respectively) denote the variables explicit in  $t$  (or in some member of  $T$ ). For a (quantifier free) formula  $\Phi$  we let  $x_\Phi$  denote a variable not appearing anywhere in  $\Phi$ . We isolate two cases:

1. Quasi-identities. Let  $\Phi$  be the quasi-identity  $\&_{0 \leq i \leq n-1} p_i \approx q_i \rightarrow p \approx q$ . Let

$$X_\Phi := \text{var}(p, q) \setminus \text{var}(p_0, q_0, \dots, p_{n-1}, q_{n-1}).$$

We define  $\Phi^\triangleright$  to be the identity

$$\left( \bigtriangleright_{0 \leq i \leq n-1} (p_i \wedge q_i) \right) \triangleright (p \wedge q) \triangleright x_\Phi \approx \left( \bigtriangleright_{0 \leq i \leq n-1} (p_i \wedge q_i) \right) \triangleright \left( \bigtriangleright_{x \in X_\Phi} x \right) \triangleright x_\Phi.$$

Note that when  $n = 0$  (that is,  $\Phi$  is an identity), then we obtain  $(p \wedge q) \triangleright x_\Phi \approx (\bigtriangleright_{x \in X_\Phi} x) \triangleright x_\Phi$ .

2. Disjuncts of negated atomic formulæ. Let  $\Phi$  be the formula  $\bigvee_{0 \leq i \leq n-1} p_i \not\approx q_i$ . We define  $\Phi^\triangleright$  to be the identity

$$\left( \bigtriangleright_{0 \leq i \leq n-1} (p_i \wedge q_i) \right) \triangleright x_\Phi \approx \left( \bigtriangleright_{0 \leq i \leq n-1} (p_i \wedge q_i) \right).$$

LEMMA 5.9. *Let  $\mathbf{P}$  be a partial algebra of similarity type  $\mathcal{F}$  and  $\Phi$  be a uH formula in  $\mathcal{F}$ . Then  $\mathbf{P} \models \Phi$  if and only if  $\text{ps}(\mathbf{P}) \models \Phi^\triangleright$ .*

PROOF. If  $\mathbf{P}$  is the empty algebra, then  $\text{ps}(\mathbf{P})$  is the one element algebra, and the result is trivial, since  $\mathbf{P}$  satisfies every universal sentence, while  $\text{ps} \mathbf{P}$  satisfies every identity. Now say that  $\mathbf{P}$  is nonempty. (1) First consider the case where  $\Phi = \&_{0 \leq i \leq n-1} p_i \approx q_i \rightarrow p \approx q$  is a quasi-identity. Say that  $\mathbf{P} \models \Phi$  and let  $\theta$  be a variable assignment for  $\Phi^\triangleright$  in  $\text{ps}(\mathbf{P})$ . Let  $\Phi_l^\triangleright$  and  $\Phi_r^\triangleright$  denote respectively, the left and right hand sides of the identity  $\Phi^\triangleright$ . As  $\text{var}(\Phi_l^\triangleright) = \text{var}(\Phi_r^\triangleright)$  we find that for  $\Phi^\triangleright$  to fail under  $\theta$ , all variables in  $\Phi^\triangleright$  must be assigned values in  $\mathbf{P}$ . Hence it suffices to assume that  $\theta$  is also a variable assignment into  $\mathbf{P}$ . Now as both sides of  $\Phi^\triangleright$  are of the form  $\bigtriangleright_{1 \leq i \leq n} (p_i \wedge q_i) \triangleright w$  for some term  $w$ , it suffices to assume that  $\theta(\bigtriangleright_{1 \leq i \leq n} (p_i \wedge q_i)) \neq \infty$ . Hence for each  $1 \leq i \leq n$  we have  $\theta(p_i)$  and  $\theta(q_i)$  are non- $\infty$  and equal. Hence  $\theta$  gives each expression  $p_i \approx q_i$  the value T in  $\mathbf{P}$ . Hence  $\theta(p)$  and  $\theta(q)$  are defined and equal, and then  $\theta(p \wedge q)$  is not equal to  $\infty$  in  $\text{ps}(\mathbf{P})$ . So by Lemma 5.5 we have  $\theta(\Phi_l^\triangleright) = \theta(x_\Phi) = \theta(\Phi_r^\triangleright)$  as required.

Now say that  $\mathbf{P} \not\models \Phi$ . So there is a variable assignment  $\theta$  giving  $\Phi$  the truth value F. By definition,  $\theta$  must give each  $p_i \approx q_i$  the value T and  $p \approx q$  the value F. We now extend  $\theta$  to a variable assignment into  $\text{ps}(\mathbf{P})$  by choosing  $\theta(x_\Phi)$  arbitrarily from  $\mathbf{P}$ . Now as  $\theta$  maps into  $\mathbf{P}$  and  $\theta(p_i) = \theta(q_i) \in \mathbf{P}$  we have that  $\theta(\Phi_r^\triangleright) = \theta(x_\Phi) \neq \infty$ . However as  $\theta(p \wedge q) = \infty$  (because  $\theta$  gives  $p \approx q$  the truth value F in  $\mathbf{P}$ ) we have  $\theta(\Phi_l^\triangleright) = \infty$ . Hence  $\Phi^\triangleright$  fails as required.

(2) Next we consider the case where  $\Phi$  is of the form  $\bigvee_{1 \leq i \leq n} p_i \not\approx q_i$ . If  $\mathbf{P} \models \Phi$  then every variable assignment into  $\mathbf{P}$  has the property that there is  $i$  such that either at least one of  $\theta(p_i)$  and  $\theta(q_i)$  are undefined or both are defined but unequal. In either case we have  $\theta(p_i \wedge q_i)$  in  $\text{ps}(\mathbf{P})$  equal to  $\infty$ . So for  $\theta$  mapping  $\text{var}(p_1, q_1, \dots)$

into  $\mathbf{P}$  we have  $\text{ps}(\mathbf{P}) \models \theta(\Phi^\triangleright)$ . However if  $\theta$  gives some variable of  $\Phi$  a value of  $\infty$ , then  $\theta$  gives both sides of  $\Phi^\triangleright$  the value  $\infty$ . So again,  $\text{ps}(\mathbf{P}) \models \theta(\Phi^\triangleright)$ .

Now if  $\Phi$  fails on  $\mathbf{P}$  under some assignment  $\theta$  then we can choose  $\theta(x_\Phi) = \infty$  we obtain a failing assignment for  $\Phi^\triangleright$  on  $\text{ps}(\mathbf{P})$ . This details are similar to the quasi-identity case and are left to the reader.  $\dashv$

**EXAMPLE 5.10.** Let  $\mathcal{S}$  be a partial algebra with a single partial binary operation  $\cdot$  (written as concatenation). The pointed semidiscriminator extension  $\text{ps}(\mathcal{S})$  is associative with respect to  $\cdot$  if and only if  $\mathcal{S} \models \{x(yz) \approx x(yz) \rightarrow (xy)z \approx x(yz), (xy)z \approx (xy)z \rightarrow (xy)z \approx x(yz)\}$ .

**PROOF.** Both directions are easily proved directly, but we here give an equational proof based on the translation just established. The pointed semidiscriminator translation of the two implications are  $((xy)z \wedge x(yz)) \triangleright w \approx (xy)z \triangleright w$  and  $((xy)z \wedge x(yz)) \triangleright w \approx x(yz) \triangleright w$ . Both of these obviously follow from associativity of  $\cdot$  on  $\text{ps}(\mathcal{S})$ . Conversely,

$$\begin{aligned} x(yz) &\approx x(yz) \triangleright x(yz) \\ &\approx (x(yz) \wedge (xy)z) \triangleright x(yz) \\ &\approx x(yz) \wedge (xy)z \quad (\text{by } \Sigma_{\leq}(1)) \\ &\approx \dots \approx (xy)z \quad (\text{by symmetry}). \end{aligned} \quad \dashv$$

Lemma 5.9 provides an alternative proof of one containment in the equality in Theorem 5.3. Indeed let  $\mathcal{K}$  be a class of similar partial algebras. If  $\mathbf{P} \notin \text{ISP}_r\text{O}(\mathcal{K})$  then there is a uH formula  $\Phi$  satisfied by  $\mathcal{K}$  and failing on  $\mathbf{P}$ . Hence  $\text{ps}(\mathcal{K}) \models \Phi^\triangleright$  while  $\text{ps}(\mathbf{P}) \not\models \Phi^\triangleright$ . So  $\text{ps}(\mathbf{P}) \notin \text{V}(\text{ps}(\mathcal{K}))$ .

The following corollary shows that the usual relation of syntactic consequence  $\vdash$  between universal Horn sentences transfers to the syntactic consequence relation  $\vdash$  for equational logic within  $\Sigma_{\mathcal{F}}$ . Both of these relations are complete (see [9, Theorem 2.14] for the completeness of  $\vdash$  for uH logic of partial algebras).

**COROLLARY 5.11.** *Let  $\mathcal{F}$  be a type of partial algebras and  $\Xi$  be a set of uH sentences in  $\mathcal{F}$ . If  $\Phi$  is a uH sentence in  $\mathcal{F}$ , then  $\Xi \vdash \Phi$  if and only if  $\Sigma_{\mathcal{F}} \cup \Xi^\triangleright \vdash \Phi^\triangleright$ .*

**PROOF.** We use Lemma 5.9 freely in this proof.

Say that  $\Xi \vdash \Phi$  and let  $\mathcal{A}$  be a partial algebra of type  $\mathcal{F}$  for which  $\text{ps}(\mathcal{A})$  satisfies  $\Xi^\triangleright$ . So  $\mathcal{A}$  satisfies  $\Xi$  and hence  $\Phi$  (by assumption). Hence  $\text{ps}(\mathcal{A})$  satisfies  $\Phi^\triangleright$ . This shows that any model of  $\Sigma_{\mathcal{F}} \cup \Xi^\triangleright$  (as a subdirect product of pointed semidiscriminator algebras satisfying  $\Xi^\triangleright$ ) satisfies  $\Phi^\triangleright$ . By completeness of equational logic, we have  $\Sigma_{\mathcal{F}} \cup \Xi^\triangleright \vdash \Phi^\triangleright$ .

Conversely, if  $\Xi \not\vdash \Phi$ , then the completeness of  $\vdash$  for uH logic shows that there is a partial algebra  $\mathcal{A}$  satisfying  $\Xi$  but not  $\Phi$ . Then  $\text{ps}(\mathcal{A})$  is a model of  $\Sigma_{\mathcal{F}} \cup \Xi^\triangleright$  not satisfying  $\Phi^\triangleright$ , so that  $\Sigma_{\mathcal{F}} \cup \Xi^\triangleright \not\vdash \Phi^\triangleright$ .  $\dashv$

In the following theorem, a variety or quasivariety is said to be *finitely generated* if it is generated by a finite set of finite algebras or partial algebras.

**THEOREM 5.12.** *Let  $\mathcal{K}$  be a class of partial algebras of some finite similarity type  $\mathcal{F}$ . The following implications hold:*

ISP <sub>r</sub> O( $\mathcal{K}$ ) is:		HSP(ps( $\mathcal{K}$ )) is:
<i>finitely generated</i>	$\Leftrightarrow$	<i>finitely generated</i>
<i>generated by its finite members</i>	$\Leftrightarrow$	<i>generated by its finite members</i>
<i>locally finite</i>	$\Leftrightarrow$	<i>locally finite</i>
<i>finitely axiomatised</i>	$\Leftrightarrow$	<i>finitely axiomatised</i>
<i>decidable uH theory</i>	$\Leftarrow$	<i>decidable equational theory</i>

If  $\mathcal{F}$  is infinite, all implications continue to hold provided that the local finiteness of ISP<sub>r</sub>O( $\mathcal{K}$ ) is replaced by regular local finiteness, and the finite axiomatisability of HSP(ps( $\mathcal{K}$ )) is replaced by finite axiomatisability within PS $_{\mathcal{F}}$ .

PROOF. First observe that a partial algebra  $\mathbf{P}$  with more than one element is  $n$ -generated if and only if the algebra ps( $\mathbf{P}$ ) is  $n$ -generated (of course, a one element partial algebra may have pointed semidiscriminator extension that is two generated but not one-generated and similarly the empty partial algebra is 0-generated, but its pointed semidiscriminator extension may be either 0- or 1-generated, depending on whether there are any nullaries in the type).

Rows 1 and 2 are easy applications of the Theorem 5.3.

**Local finiteness.** Put  $\mathcal{V} := \text{HSP}(\text{ps}(\mathcal{K}))$ . We have ISP<sub>r</sub>O( $\mathcal{K}$ ) is regularly locally finite if and only if  $\text{l}(\text{ps}(\text{SP}_r\text{O}(\mathcal{K}))) = \mathcal{V}_{s,i}$  is regularly locally finite if and only if  $\mathcal{V}$  is locally finite (the final rightward implication is given by [8, Theorem 3.7.3]). When  $\mathcal{F}$  is finite, Proposition 2.6 shows that regular local finiteness of ISP<sub>r</sub>O( $\mathcal{K}$ ) coincides with local finiteness.

**Finite axiomatisability.** Let  $\Xi$  be an axiomatisation of the uH theory of  $\mathcal{K}$ . By Lemma 5.9,  $\Sigma_{\mathcal{F}} \cup \Xi^{\triangleright}$  is an axiomatisation of the equational theory of  $\text{V}(\text{ps}(\mathcal{K}))$ . Hence if  $\Xi$  and  $\mathcal{F}$  are finite, then  $\text{V}(\text{ps}(\mathcal{K}))$  has a finite equational basis, while if  $\Xi$  is finite but  $\mathcal{F}$  infinite, then  $\text{V}(\text{ps}(\mathcal{K}))$  has a finite equational basis within PS $_{\mathcal{F}}$ . Conversely, if  $\Sigma$  is a finite equational basis for the equational theory of  $\text{V}(\text{ps}(\mathcal{K}))$  (within PS $_{\mathcal{F}}$ ), then the compactness theorem for equational logic shows that there is a finite subset  $\Gamma$  of  $\Sigma_{\mathcal{F}} \cup \Xi^{\triangleright}$  (of  $\Xi^{\triangleright}$ , respectively) that is an equational basis for  $\text{V}(\text{ps}(\mathcal{K}))$ . Let  $\Xi_1 := \{\Phi \in \Xi \mid \Phi^{\triangleright} \in \Gamma\}$ . Then  $\Xi_1$  is a finite basis for the uH theory of  $\mathcal{K}$ . (An alternative proof is to use the second part of Proposition 3.15.)

**Decidability of equational theory.** The contrapositive of this statement is a direct corollary of Lemma 5.9.  $\dashv$

**§6. Congruences.** In this section we show that for any similarity type  $\mathcal{F}$ , the variety PS $_{\mathcal{F}}$  has definable principal congruences. Here the pointed semidiscriminator term  $d(x, y, z) = (x \wedge y) \triangleright z$  plays a central role.

Let  $A$  be an algebra and  $F_x$  be a set of terms (of the appropriate language) each involving (but necessarily explicit in) the variable  $x$ . Following the notation of [4], we write

$$\{c, d\} \mathfrak{q}_{F_x}^1 \{a, b\}$$

to denote the fact that there exist  $f(x, x_1, \dots, x_n) \in F_x$  and  $e_1, \dots, e_n \in A$  such that

$$\{f^A(c, e_1, \dots, e_n), f^A(d, e_1, \dots, e_n)\} = \{a, b\}.$$

The notation

$$\{c, d\} \mathfrak{q}_{F_x}^m \{a, b\}$$

means that there are  $a_0, a_1, \dots, a_m \in A$  with  $a = a_0$ ,  $b = a_m$  and with  $\{c, d\} \mathfrak{q}_{F_x}^1 \{a_i, a_{i+1}\}$  for each  $i = 0, \dots, m-1$ . In this notation, Maltsev's Lemma states

that  $(a, b) \in \text{Cg}(c, d)$  if and only if there is an  $n$  and a finite set of terms  $F_x$  with distinguished variable  $x$  (depending on  $\{a, b, c, d\}$ ) for which  $\{c, d\} \varrho_{F_x}^n \{a, b\}$ . A variety  $\mathcal{V}$  has *definable principal congruences* if there is a first order formula  $\Phi(u, v, x, y)$  such that for all  $A \in \mathcal{V}$  and  $a, b, c, d \in A$  we have  $(a, b) \in \text{Cg}^A(c, d)$  if and only if  $A \models \Phi(a, b, c, d)$ . This is also known to be equivalent to the existence of  $n \in \mathbb{N}$  and finite set  $F_x$  of terms (with distinguished variable  $x$ ) such that for any  $A \in \mathcal{V}$  we have  $(a, b) \in \text{Cg}^A(c, d)$  if and only if  $\{c, d\} \varrho_{F_x}^n \{a, b\}$  (see [11, Exercise V.3.5] for example).

**PROPOSITION 6.1.** *The pointed semidiscriminator variety  $\text{PS}_{\mathcal{F}}$  over any similarity type  $\mathcal{F}$  has definable principal congruences. Specifically, for any algebra  $A \in \text{PS}_{\mathcal{F}}$  and  $a, b, c, d \in A$  we have  $(a, b) \in \text{Cg}^A(c, d)$  if and only if  $\{c, d\} \varrho_{\{(x \wedge x_1) \triangleright x_2\}}^2 \{a, b\}$ .*

The proof of this proposition covers the rest of this section.

First note that one direction is trivial: if  $\{c, d\} \varrho_{\{(x \wedge x_1) \triangleright x_2\}}^2 \{a, b\}$ , then by Maltsev's Lemma we also have  $(a, b) \in \text{Cg}^A(c, d)$ .

**LEMMA 6.2.** *The term  $(x_1 \wedge x) \triangleright x_2$  determines principal congruences. That is, for  $A \in \text{PS}_{\mathcal{F}}$  and  $a, b, c, d \in A$  with  $(a, b) \in \text{Cg}^A(c, d)$  there is  $n \in \mathbb{N}$  depending on  $a, b, c, d$  such that  $\{c, d\} \varrho_{\{(x \wedge x_1) \triangleright x_2\}}^n \{a, b\}$ .*

**PROOF.** Let  $t(x, z_1, \dots, z_m)$  be a term with a single occurrence of the variable  $x$ , and let  $F = F(x, z_1, \dots, z_m)$  be the free algebra in  $\text{PS}_{\mathcal{F}}$  freely generated by  $x, z_1, \dots, z_m$ . By [12, §2, §3] it suffices to show that

$$\{x, y\} \varrho_{\{(x \wedge x_1) \triangleright x_2\}}^n \{t^F(x, z_1, \dots, z_m), t^F(y, z_1, \dots, z_m)\}$$

for some  $n$ . Now, induction on the number of operation symbols in a term easily shows that for any term  $s(x, z_1, \dots)$  with a single occurrence of the variable  $x$ , the identity  $z \triangleright s(x, z_1, \dots) \approx s(z \triangleright x, z_1, \dots)$  is satisfied by  $\text{PS}_{\mathcal{F}}$  (using  $\Sigma_{\text{op}}$ ). Hence in  $F$  we have

$$\begin{aligned} t^F(x, z_1, \dots, z_m) &= t^F((x \wedge x) \triangleright x, z_1, \dots, z_m) \\ &= (x \wedge x) \triangleright t^F(x, z_1, \dots, z_m), \\ (x \wedge y) \triangleright t^F(x, z_1, \dots, z_m) &= t^F((x \wedge y) \triangleright x, z_1, \dots, z_m) \\ &= t^F((x \wedge y) \triangleright y, z_1, \dots, z_m) \\ &= (y \wedge x) \triangleright t^F(y, z_1, \dots, z_m), \\ (y \wedge y) \triangleright t^F(y, z_1, \dots, z_m) &= y \triangleright t^F(y, z_1, \dots, z_m) \\ &= t^F(y \triangleright y, z_1, \dots, z_m) \\ &= t^F(y, z_1, \dots, z_m), \end{aligned}$$

as required. ⊔

We now prove Proposition 6.1.

**PROOF.** Let  $A \in \text{PS}_{\mathcal{F}}$  and say that  $(a, b) \in \text{Cg}^A(c, d)$ . By Lemma 6.2 and the Maltsev's Lemma, there are  $a_0, a_1, \dots, a_n \in A$  with  $a = a_0$ ,  $b = a_n$  and translations  $\lambda_i(x) := (x \wedge e_i) \triangleright f_i$  in  $A$  such that for each  $i = 0, 1, \dots, n-1$  we have  $\{\lambda_i(c), \lambda_i(d)\} = \{a_i, a_{i+1}\}$ . We can assume that the number  $n$  is chosen to be

minimal. For each  $i = 0, 1, \dots, n-1$ , let  $g_i, h_i \in \{c, d\}$  be such that  $\lambda_i(g_i) = a_i$  and  $\lambda_i(h_i) = a_{i+1}$ .

For  $i = 1, 2, \dots, n-2$  let  $\lambda'_i(x) := (x \wedge ((c \wedge d) \triangleright e_i)) \triangleright f_i$ , let  $\lambda'_0(x) := (x \wedge (e_0 \wedge g_0)) \triangleright f_0$  and  $\lambda'_{n-1}(x) := (x \wedge (e_{n-1} \wedge h_{n-1})) \triangleright f_0$ . These are all translations built from the term  $(x \wedge x_1) \triangleright x_2$ .

When  $1 \leq i \leq n-2$ , the laws  $\Sigma_{\mathcal{F}}$  are easily used to show that  $\lambda'_i(x) = (c \wedge d) \triangleright \lambda_i(x)$ , and that  $(c \wedge d) \triangleright \lambda_i(c) = (c \wedge d) \triangleright \lambda_i(d)$ . So, for  $i \in \{1, \dots, n-2\}$  we have  $\lambda'_i(c) = \lambda'_i(d) = (c \wedge d) \triangleright a_i = (c \wedge d) \triangleright a_{i+1}$ . Also,  $\{\lambda'_0(c), \lambda'_0(d)\} = \{a_0, (c \wedge d) \triangleright a_1\}$ , since  $\lambda'_0(g_0) = \lambda_0(g_0 \wedge g_0) = a_0$ , while  $\lambda'_0(h_0) = \lambda_0(c \wedge d) = \lambda_0((c \wedge d) \triangleright h_0) = (c \wedge d) \triangleright \lambda_0(h_0) = (c \wedge d) \triangleright \lambda_1(g_1) = (c \wedge d) \triangleright a_1$ . Similarly,  $\{\lambda'_{n-1}(c), \lambda'_{n-1}(d)\} = \{(c \wedge d) \triangleright a_{n-1}, a_n\}$ . By minimality, we have  $n \leq 2$ .  $\dashv$

**§7. Examples.** In this final section we give some corollaries and examples that deserve extra discussion.

**7.1. General Structures.** By a partial structure  $\langle M; F, R \rangle$ , we mean a set  $M$  endowed with some partial operations (listed in  $F$ ) and some finitary relations (listed in  $R$ ). We now observe how our translation can also be made meaningful for partial structures. The basic idea is already in Example 4.3.

UH formulæ are defined for partial structures in the same way as for partial algebras except that we include expressions of the form  $(t_1(\vec{x}), \dots, t_n(\vec{x})) \in r$  (for an  $n$ -ary relation symbol  $r$  and terms  $t_i(\vec{x})$  in the partial operation symbols of  $F$ ) as atomic expressions. For satisfaction of uH formulæ we say that a truth assignment  $\theta$  into a partial structure  $\mathbf{P}$  gives an atomic expression  $(t_1(\vec{x}), \dots, t_n(\vec{x})) \in r$  the value  $\top$  if each  $t_i^{\mathbf{P}}(\theta(\vec{x}))$  is defined and  $(t_0^{\mathbf{P}}(\theta(x)), \dots, t_n^{\mathbf{P}}(\theta(x)))$  is in  $r^{\mathbf{P}}$ ; otherwise  $(t_1(\vec{x}), \dots, t_n(\vec{x})) \in r$  is assigned  $\text{F}$ .

Let  $r$  be an  $n$ -ary relation on a set  $M$ . Then we let  $\bar{r} : M^n \rightarrow M$  denote the first projection partial operation with domain  $r$ . For a similarity type  $\mathcal{F}$  we let  $\bar{\mathcal{F}}$  consist of the (partial) operation symbols in  $\mathcal{F}$  along with a new (partial) operation symbol  $\bar{r}$  of arity  $n$  for each  $n$ -ary relation symbol  $r$  of  $\mathcal{F}$ . For each partial  $\mathcal{F}$ -structure  $\mathbf{M} := \langle M; F, R \rangle$ , we let  $\langle M; F, R \rangle^{\bar{\mathcal{F}}}$  denote the partial  $\bar{\mathcal{F}}$ -algebra  $\langle M; F \cup \{\bar{r} : r \in R\} \rangle$ . It is trivial to verify that there is a category theoretic equivalence between the category of all partial  $\mathcal{F}$ -structures and of all partial  $\bar{\mathcal{F}}$ -algebras satisfying  $\{\bar{r}(x_1, \dots, x_n) \approx \bar{r}(x_1, \dots, x_n) \rightarrow \bar{r}(x_1, \dots, x_n) \approx x_1 : r \in R\}$ . One may translate atomic expressions (and then arbitrary uH formulæ) in  $F, R$  into atomic expressions in  $F \cup \bar{R}$  by replacing  $(t_1(\vec{x}), \dots, t_n(\vec{x})) \in r$  by  $\bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x})) \approx \bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x}))$ .

In the other direction we may translate an atomic expression in  $F \cup \bar{R}$  into a conjunct of atomic expressions in  $F, R$ . Indeed, if  $s(\vec{x}) \approx t(\vec{x})$  is an atomic formula in  $F \cup \bar{R}$ . Say that  $\bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x}))$  is a subterm of  $t(\vec{x})$  consisting of an application of a relational operation  $\bar{r}$  of maximal height in  $t$  (so each  $t_i(\vec{x})$  are terms in  $F$ ), then we can replace  $\bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x}))$  in  $t(\vec{x})$  by  $t_1(\vec{x})$  (denote the resulting expression by  $t'(\vec{x})$ ) and replace by  $s(\vec{x}) \approx t(\vec{x})$  by  $(s(\vec{x}) \approx t'(\vec{x})) \ \& \ \bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x})) \approx \bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x}))$ . Certainly, the two formulæ have the same truth value in  $\mathbf{M}^{\bar{\mathcal{F}}}$ . Continuing in this way we find that each atomic expression is equivalent to a conjunct of atomic expressions, each of whose atomic subformulæ is either an atomic formulæ in  $F$  or of the form  $\bar{r}(t_1(\vec{x}), \dots, t_n(\vec{x})) \approx$

$\bar{r}(t_1(\bar{x}), \dots, t_n(\bar{x}))$ . A formula  $\& \Phi_i \rightarrow \Phi$  or  $\bigvee \neg \Phi_i$ , where each  $\Phi$  is of this form, is easily seen to be equivalent to a finite set of uH formulæ in the similarity type  $F$ .

In this way, one obtains versions of Theorem 5.3 and 5.12, where  $\text{ps}(\mathcal{K})$  is replaced by  $\text{ps}(\mathcal{K}^\pi)$  and “partial algebras” is replaced by “partial structures”.

**7.2. The finite q-basis problem.** (Throughout this subsection, and those following, we assume finite similarity type.) The *finite identity basis problem* for finite algebras (often called *Tarski’s finite basis problem*) asks if a given finite algebra has a finite basis for its identities. The *finite q-basis problem* is the same but refers to the quasi-identities of an algebra. We may likewise define the *finite uH-basis problem*, however it is easy to see that a uH class is finitely axiomatisable if and only if the quasivariety it generates is finitely axiomatisable (because the latter class differs from the former by at most the one element structure). We note that a finite algebra has a finite basis for its quasi-identities (identities) if and only if the quasivariety (variety, respectively) it generates is strictly elementary; that is, if it is the class of models of some finite set of first order sentences.

The finite identity basis problem was shown to be undecidable by McKenzie in [34], but the possible undecidability of the finite q-basis problem remains one of the most well known open problems in universal algebra. Theorem 5.12 shows that the finite q-basis problem for finite partial algebras of a given type  $\mathcal{F}$  is equivalent to the finite identity basis problem for the variety  $\text{PS}_{\mathcal{F}}$ . Of course, the finite q-basis problem for total algebras of type  $\mathcal{F}$  corresponds to the finite identity basis problem within a (finitely based) subvariety of  $\text{PS}_{\mathcal{F}}$ .

The variety  $\text{PS}_{\mathcal{F}}$  appears to have many properties usually associated with well behaved varieties: it has definable principal congruences, is semisimple and hereditarily simple, for example. This suggests that the finite q-basis property might be slightly better behaved than the finite identity basis property. On the other hand, the variety  $\text{PS}_{\mathcal{F}}$  is generated by flat algebras, which also form the basic framework of McKenzie’s proof of the undecidability of the finite identity basis problem (McKenzie’s constructions are not sinks, since the bottom element is not absorbing in all operations).

We mention that results of Baker and Wang [5] show that a finite member of  $\text{PS}_{\mathcal{F}}$  has a finite basis for its identities if and only if the subdirectly irreducibles in its variety form a strictly elementary class; however in the case of pointed semidiscriminator varieties this can be deduced directly from Theorem 5.3.

**7.3. Inherent non-finitely axiomatisability.** An algebra is said to be *inherently non-finitely based* (abbreviated to INFB) if every locally finite variety containing it is not finitely axiomatisable. Likewise, a (possibly partial) algebra is *inherently non-finitely q-based* (INFQB) if it is not contained in a finitely axiomatisable locally finite quasivariety. Obviously an INFQB algebra is also INFB. INFQB algebras seem to be quite elusive: the only examples known are due to Lawrence and Willard [29]; these examples are unary semigroups. It is clear that a pointed semidiscriminator extension  $\text{ps}(\mathbf{P})$  of a finite partial algebra is INFB if and only if  $\mathbf{P}$  is INFQB.<sup>6</sup>

The pointed semidiscriminator varieties offer a possibly new approach to finding new inherently non-finitely q-based algebras and partial algebras. As an example

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<sup>6</sup>If  $\mathbf{P}$  is INFQB then Theorem 5.12 only shows that  $\text{ps}(\mathbf{P})$  is INFB within the pointed semidiscriminator variety over the type of  $\mathbf{P}$ . However it is easy to see that an algebra is INFB within some finitely based variety if and only if it is INFB amongst all algebras of its type.

we observe that the partial algebra  $\mathbf{M}$  on  $\{1, 2, h, c, d\}$  with a single binary partial operation given by  $1 \cdot c = c$ ,  $h \cdot c = d$  and  $2 \cdot d = d$  is inherently non-finitely  $q$ -based. Indeed [32, Lemma 6.2] shows that  $b(\mathbf{M})$  is inherently non-finitely based (as does Proposition 3.2 above), and the method of [32] (but not Proposition 3.2) is trivially seen to hold for  $ps(\mathbf{M})$  as well. (We note that Lemma 6.2 of [32] involves an algebra  $A$  of which  $b(\mathbf{M})$  is a subreduct, however the extra elements and operations become degenerate in this lemma, so the proof is essentially a proof that  $b(\mathbf{M})$  is inherently non-finitely based.)

We also observe that the algorithmic problem of deciding when a finite partial algebra is inherently non-finitely  $q$ -based is equivalent to the problem of finding when a finite pointed semidiscriminator algebra is inherently non-finitely based. For general algebras, this latter property is also known to be undecidable [34].

**7.4. The Eilenberg-Schützenberger problem.** In 1976, Eilenberg and Schützenberger [16] asked whether or not the following condition is sufficient for a finite algebra  $A$  of finite type to have a finite basis for its identities (it is certainly necessary): there is a finite set of identities  $\Sigma$  such that for every *finite* algebra  $B$ , we have  $B \in V(A)$  if and only if  $B \models \Sigma$ . We call this the *Eilenberg-Schützenberger identity basis question*. This question has been answered in the positive for semigroups by Sapir [40], and has recently gained renewed interest; see the [7, Problem, 13]. The same question may be asked for the quasivariety of a (partial or total) algebra  $P$  (where now,  $\Sigma$  is some set of quasi-identities), which we could call the *Eilenberg-Schützenberger quasi-identity basis question*. (As in Section 7.2, the corresponding question for universal Horn sentences is equivalent to the quasi-identity version.) The methods of Section 5 easily show that a partial algebra  $P$  gives a negative answer to the Eilenberg-Schützenberger quasi-identity basis question if and only if  $ps(P)$  gives a negative answer to the Eilenberg-Schützenberger identity basis question.

**7.5. Q-universality.** A lattice  $M$  is *Q-universal* if the lattice  $L_{\mathcal{F}}$  of all quasivarieties (of algebras) of any finite similarity type  $\mathcal{F}$  is a factor of  $M$ . A quasivariety is *Q-universal* if its lattice of subquasivarieties is Q-universal. Q-universality was introduced by Mark Sapir [39] who showed that there is a finite semigroup whose quasivariety is Q-universal. Many other examples are now known; see [1]. Theorem 5.12 shows that the pointed semidiscriminator extension of any such algebra generates a variety whose lattice of *subvarieties* is Q-universal.

We can likewise say that a lattice  $M$  is *V-universal* if any lattice of varieties (in any finite similarity type) is a factor of  $M$ . A variety is V-universal if its lattice of subvarieties is. Now the lattice of subvarieties of a variety  $V$  is a homomorphic image of the lattice of subquasivarieties of  $V$  under the map that takes a quasivariety to the variety it generates. Hence, using the class operator identity  $HS = HS$ , a Q-universal lattice is V-universal. Conversely, Corollary 5.4 shows that any V-universal lattice is Q-universal, so the concepts of Q-universality and V-universality coincide.

One might also extend the definition of Q-universality to include the lattice of quasivarieties of partial structures of finite type. However the above comments show that this extended notion coincides with the conventional algebra version.

**EXAMPLE 7.1.** Let  $\mathbf{2}$  denote the 2 element graph  $\langle \{a, b\}; \{(a, b), (b, a), (a, a)\} \rangle$ . Then (using the notation of Subsection 7.1) the lattice of subvarieties of  $V(ps(\mathbf{2}^n))$  is Q-universal and V-universal.

This follows from the above comments and [2, Theorem 1.2], where it is shown that  $\mathbf{2}$  generates a  $\mathbf{Q}$ -universal quasivariety. We observe that every two element algebra has a lattice of subvarieties with precisely two elements.

**7.6. Membership problems.** There has been recent interest in the complexity of the problem of deciding membership of finite algebras in the variety of some fixed finite algebra  $\mathcal{A}$  (the *finite membership problem* for  $\mathcal{V}(\mathcal{A})$ ). Szekely [43] found an algebra  $\mathcal{A}$  for which this problem is NP-complete, while the author and McKenzie [20] found a 55 element semigroup with this property. (As observed in [20], the Szekely algebra can be chosen to have just 6 elements.)

Szekely's algebra is the sink  $\mathfrak{b}(\mathbf{C}_3^\pi)$ , where  $\mathbf{C}_3$  is a finite graph whose uH class is known to generate the uH class of all 3 colourable graphs. The NP-hardness of the finite membership problem for  $\mathcal{V}(\mathfrak{b}(\mathbf{C}_3^\pi))$  follows from Corollary 4.4 and the fact that graph 3-colourability is a known NP-complete problem; see [17] for example. (Of course, Corollary 4.4 is itself a generalisation of Szekely's methods.) Our general results yield many new examples. We refer the reader to [17] for the definition of Turing reduction and Karp (or many-one) reduction.

**PROPOSITION 7.2.** *Let  $\mathcal{K}$  be a finite set of partial algebras of finite similarity type  $\mathcal{F}$ . The finite membership problem for  $\text{ISP}_r\mathcal{O}(\mathcal{K})$  is polynomially equivalent to the finite membership problem for  $\mathcal{V}(\text{ps}(\mathcal{K}))$  (via polynomial time Turing reductions).*

**PROOF.** Theorem 5.3 shows that the pointed semidiscriminator extension construction yields an almost trivial reduction from the finite membership problem  $\text{ISP}_r\mathcal{O}(\mathcal{K})$  to the finite membership problem for  $\mathcal{V}(\text{ps}(\mathcal{K}))$  (this is in fact a Karp reduction).

In the other direction, we first observe that there is a polynomial time Turing reduction from the finite membership problem in any variety  $\mathcal{V}$  to the finite membership problem in  $\mathcal{V}_{\text{s.i.}}$ . (This basically follows the general method of Szekely [43].) Consider a finite algebra  $\mathcal{A}$ . Using the main result of Demel [15], we can construct in polynomial time a sequence of at most  $|\mathcal{A}| - 1$  subdirectly irreducible quotients  $\mathcal{A}_1, \dots, \mathcal{A}_{n-1}$  of  $\mathcal{A}$  with the property that  $\mathcal{A}$  subdirectly embeds into  $\prod_{1 \leq i \leq n} \mathcal{A}_i$  (the precise complexity here depends on the similarity type of  $\mathcal{A}$ ). Basic universal algebra shows that  $\mathcal{A} \in \mathcal{V}$  if and only if all of the quotients  $\mathcal{A}_i$  are in  $\mathcal{V}_{\text{s.i.}}$ ; this gives the claimed polynomial Turing reduction.

It remains to show that there is a polynomial time Turing reduction from the finite membership problem for  $\mathcal{V}_{\text{s.i.}}(\text{ps}(\mathcal{K}))$  to the finite membership problem for  $\text{ISP}_r\mathcal{O}(\mathcal{K})$ . It is easy to check (in polynomial time) if a given algebra  $\mathcal{S}$  in the type  $\mathcal{F} \cup \{\wedge, \triangleright\}$  is a pointed semidiscriminator algebra; if not, then we can return false. Otherwise, we can construct the partial algebra  $\mathcal{P}$  for which  $\text{ps}(\mathcal{P}) \cong \mathcal{S}$ . We now have  $\mathcal{S} \in \mathcal{V}_{\text{s.i.}}(\text{ps}(\mathcal{K}))$  if and only if  $\mathcal{P} \in \text{ISP}_r\mathcal{O}(\mathcal{K})$ , which yields the desired Turing reduction.  $\dashv$

Recent results of M. Kozik [27] have shown that the finite membership problem for a finitely generated variety can be EXPSPACE-hard. However, in the particular case of  $\mathfrak{b}(\mathbf{C}_3^\pi)$ , Szekely showed that the finite membership problem is NP-complete. Finitely generated pointed semidiscriminator varieties also have NP finite membership problem (so that NP-complete is an upper bound). Indeed Proposition 7.2

describes a polynomial time Turing reduction from such a variety to a finitely generated uH class, which necessarily has NP finite membership problem; see Bergman and Slutzki [6].

**7.7. Groups.** The pointed semidiscriminator extensions of groups are particularly interesting.

For  $\mathcal{F} \in \{\{\cdot\}, \{\cdot,^{-1}\}, \{\cdot,^{-1}, 1\}\}$ , let  $\mathcal{G}_{\mathcal{F}}$  denote the class of all groups in the signature  $\mathcal{F}$ . For a group  $\mathbf{G}$  from  $\mathcal{G}_{\{\cdot,^{-1}\}}$  or  $\mathcal{G}_{\{\cdot,^{-1}, 1\}}$  we have  $x \triangleright y = x^{-1}xy$  and hence  $\mathfrak{b}(\mathbf{G})$  and  $\text{ps}(\mathbf{G})$  are term equivalent. The flat extension of a group in  $\mathcal{G}_{\{\cdot\}}$  is not a pointed semidiscriminator algebra unless  $\triangleright$  is a term function (such as when  $\mathbf{G}$  has finite exponent).

Ol'shanskii proved that a finite group has a finite basis of quasi-identities if and only if its Sylow subgroups are abelian [38]. More precisely, Ol'shanskii shows that if  $\mathbf{G}$  is a finite group with a non-abelian Sylow subgroup, then for each pair of positive integers  $n, k$  there is a group  $\mathbf{C}_{n,k}$  such that all  $k$ -generated subgroups of  $\mathbf{C}_{n,k}$  lie in the quasivariety of  $\mathbf{G}$ , but  $\mathbf{C}_{n,k}$  does not lie in the quasivariety generated by any set  $\mathcal{K}$  of finite algebras whose cardinality is bounded by  $n$ . This statement also holds if  $\mathcal{K}$  contains partial algebras: Ol'shanskii's proof continues to hold in this more general setting, but alternatively, one may easily verify that a finite total algebra lies in the quasivariety of a finite set of finite partial algebras if and only if it lies in the quasivariety generated by the total subalgebras of the partial algebras. Combined with our observations above we get the following result.

**THEOREM 7.3.** *Let  $\mathbf{G}$  be a finite group in one of the usual signatures  $\mathcal{F} = \{\cdot,^{-1}, 1\}$ ,  $\{\cdot,^{-1}\}$  or  $\{\cdot\}$ . The following are equivalent:*

1. *all Sylow subgroups of  $\mathbf{G}$  are abelian;*
2.  *$\mathbf{G}$  has a finite basis of quasi-identities;*
3. *there is a finitely generated, finitely based quasivariety of (possibly partial) algebras that contains  $\mathbf{G}$ ;*
4.  *$\mathfrak{b}(\mathbf{G})$  has a finite basis of identities;*
5. *there is a finitely generated, finitely based subvariety of  $\text{PS}_{\mathcal{F}}$  containing  $\text{ps}(\mathbf{G})$ ;*
6. *there is a finitely generated, finitely based subvariety of  $\mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$  containing  $\mathfrak{b}(\mathbf{G})$ .*

**PROOF.** The equivalence of (1)–(5) all follow from the above discussion (in particular, see Subsection 7.2), while (5) $\Rightarrow$ (6) is trivial. Now say that (1) fails; we show that (6) fails. Consider a variety  $\mathcal{V} \subseteq \mathcal{V}(\mathfrak{b}(\mathcal{P}_{\mathcal{F}}))$ , generated by a finite set of finite algebras  $\mathcal{K}$  and containing  $\mathfrak{b}(\mathbf{G})$ ; we need to show that  $\mathcal{V}$  is not finitely axiomatisable (proving that (6) fails). As  $\mathcal{V}$  can be generated by the subdirectly irreducible quotients of the members of  $\mathcal{K}$ , we may (by Theorem 3.1) assume without loss of generality that  $\mathcal{K}$  consists of sinks. Hence there is a finite set  $\mathcal{L}$  of finite partial algebras such that  $\mathcal{K} = \mathfrak{b}(\mathcal{L})$ ; also as  $\mathfrak{b}(\mathbf{G}) \in \mathcal{V}$ , we can assume that  $\mathbf{G} \in \mathcal{L}$ . Now we use Corollary 3.14(1) (Proposition 3.15 works similarly). Let  $n$  be larger than the maximal cardinality of any member of  $\mathcal{L}$ . We let the family of total algebras  $\mathcal{T}$  consist of Ol'shanskii's groups  $\{\mathbf{C}_{n,k} \mid k \in \mathbb{N}\}$ . As explained above,  $\mathbf{C}_{n,k}$  is not contained in  $\text{ISP}_r\text{O}(\mathcal{L})$ , but the  $k$ -generated subgroups of  $\mathbf{C}_{n,k}$  are contained in  $\text{ISP}_r\text{O}(\mathbf{G})$ ; equivalently,  $\mathbf{C}_{n,k}$  satisfies all  $k$ -variable quasi-identities of  $\mathbf{G}$ . Hence any non-principal ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  has the property that the ultraproduct  $\prod_{\mathbb{N}/\mathcal{U}} \mathcal{T}$

satisfies all quasi-identities of  $\mathbf{G}$  and therefore is contained in  $\text{ISP}_r(\mathbf{G}) \subseteq \text{ISP}_r\mathbf{O}(\mathcal{L})$ . Corollary 3.14(1) now shows that  $\mathcal{V}$  is not finitely axiomatisable.  $\dashv$

**REMARK 7.4.** The equivalence of properties (2) and (4) holds for any partial algebra of finite type on which the (total) projection function  $p(x, y) := y$  is represented by a term function. For example, Remark 4.6 implies that this equivalence holds for any total algebra generating a congruence modular variety.

Theorem 7.3 demonstrates an amusing—if perhaps somewhat artificial—switch between the axiomatisability properties of finite groups and their flat extensions. Theorem 7.3 shows that a finite group is finitely based with respect to quasi-identities if and only if its flat extension is finitely based with respect to identities. And similarly (although somewhat vacuously), a finite group is finitely based with respect to identities if and only if its flat extension is finitely based with respect to quasi-identities: this is because Oates and Powell [37] showed that a finite group always has a finite basis of identities, while Jezek, Maroti and McKenzie [24] showed the same for the quasi-identities of flat extensions of finite groups.

In [44] a finite algebra is said to be *strongly non-finitely based* (abbreviated to SNFB) if every finitely generated variety containing it is without a finite identity basis. Clearly every INFB algebra is SNFB, but is an open problem as to whether or not there is a SNFB finite algebra that is not also INFB (see [44] and [19]). A near example has recently been found by Kad'ourek [25] who has shown that the six element Brandt inverse semigroup  $\mathbf{B}_2^1$  is SNFB amongst certain varieties of inverse semigroups, while Sapir [41] has shown that no inverse semigroup is INFB. Theorem 7.3 supplies a second example: the flat extension of a finite group  $\mathbf{G}$  with a non-abelian Sylow subgroup is SNFB within the variety  $\mathbf{V}(\mathbf{b}(\mathcal{P}_{\mathcal{F}}))$  (where  $\mathcal{F}$  is the type of  $\mathbf{G}$ ). However  $\mathbf{b}(\mathbf{G})$  is not INFB because the variety  $\mathbf{V}(\mathbf{G})$  generated by  $\mathbf{G}$  is finitely axiomatisable by the Oates Powell Theorem [37], and hence also finitely axiomatisable as a quasivariety. Because  $\mathbf{V}(\mathbf{G})$  is locally finite and of finite exponent, the projection operation  $\triangleright$  is a term function and Theorem 5.12 shows that  $\mathbf{V}(\mathbf{b}(\mathbf{V}(\mathbf{G})))$  is a finitely axiomatisable locally finite variety containing  $\mathbf{b}(\mathbf{G})$ . This result was announced in [19].

**7.8. Naturally semilattice ordered inverse semigroups.** Flat extensions of groups also generate a quite natural variety, which we examine in more detail here. Recall that an *inverse semigroup* in the signature  $\{\cdot, {}^{-1}\}$  is a semigroup with a unary operation  ${}^{-1}$  satisfying the additional laws  $x^{-1}xy^{-1}y \approx y^{-1}yx^{-1}x$ ,  $xx^{-1}x \approx x$  and  $(x^{-1})^{-1} \approx x$ ; the law  $(xy)^{-1} \approx y^{-1}x^{-1}$  is a consequence. There is a *natural order* on any inverse semigroup which is given by  $x \leq y$  if  $x = yx^{-1}x$  (there are many equivalent definitions; see [18, §5.2] for example). The natural order is stable under left and right multiplication. In many places where inverse semigroups arise (for example, the local automorphism semigroup of an algebra), this partial order is a meet semilattice order. Accordingly, several authors have considered inverse semigroups with an additional operation  $\wedge$  whose order agrees with the natural order (see Leech [30] for example). As is shown in [30] or [21], the class is a variety defined by the inverse semigroup axioms for  $\cdot$  and  ${}^{-1}$ , the semilattice axioms for  $\wedge$ , left and right distributivity of  $\cdot$  over  $\wedge$  and the following law:

- $x \wedge y = x(x \wedge y)^{-1}(x \wedge y)$

which ensures that the natural inverse semigroup order agrees with the  $\wedge$ -order. We call this variety, the *naturally semilattice ordered inverse semigroups* [21], or *inverse algebras* [30]. A *Clifford semigroup* is an inverse semigroup whose idempotent elements are central (that is, commute with all elements). A *naturally semilattice ordered Clifford semigroup*, or *Clifford inverse algebra* is an inverse algebra whose underlying inverse semigroup is a Clifford semigroup. (Note that ‘‘Clifford algebra’’ has a different meaning).

**THEOREM 7.5.** *The variety of naturally semilattice ordered Clifford semigroups is  $\mathbb{V}(\mathfrak{b}(\mathcal{G}_{\{\cdot, -1\}}))$ , while the variety of semilattice ordered Clifford monoids is  $\mathbb{V}(\mathfrak{b}(\mathcal{G}_{\{\cdot, -1, 1\}}))$  (term equivalent using  $x \triangleright y := xy^{-1}y$  to  $\mathbb{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1\}}))$  and  $\mathbb{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1, 1\}}))$  respectively).*

**PROOF.** We prove the semigroup case and leave the very similar monoid case to the reader. Let  $\mathcal{C}$  denote the class of naturally semilattice ordered Clifford semigroups, and let  $\mathcal{S} = \langle \mathcal{S}; \cdot, \wedge, -1 \rangle \in \mathcal{C}$ . We show that  $\mathcal{S} \models \Sigma_{\{\cdot, -1\}}$  (see Section 5), where  $x \triangleright y$  is the term operation  $yx^{-1}x$ .

In addition to the inverse algebra axioms listed above,  $\mathcal{S}$  must satisfy the law  $x^{-1}xy \approx yx^{-1}x$  (centrality of idempotents). We use these laws freely in the deductions to follow and give precedence of  $-1$  over  $\cdot$  over  $\triangleright$  over  $\wedge$  (so that  $xz \triangleright xy^{-1} \wedge z$  abbreviates  $((xz) \triangleright (xy^{-1})) \wedge z$  for example).

The laws in  $\Sigma_{\wedge}$  are simply the semilattice axioms. For  $(x \triangleright y) \triangleright z \approx x \triangleright (y \triangleright z)$  note that both sides reduce to  $zx^{-1}xy^{-1}y$ , and then the centrality of  $x^{-1}x$  and  $y^{-1}y$  also give  $x \triangleright y \triangleright z \approx y \triangleright x \triangleright z$ . The law  $x \triangleright x \approx x$  is one of the inverse semigroup axioms. Law (1) of  $\Sigma_{\leq}$  comes straight from the definition of the natural order. Law (2) of  $\Sigma_{\leq}$  is  $xa^{-1}a \wedge y \approx (x \wedge y)a^{-1}a$ , which holds because  $(x \wedge y)a^{-1}a \approx xa^{-1}a \wedge ya^{-1}a \approx xa^{-1}aa^{-1}a \wedge ya^{-1}a \approx (xa^{-1}a \wedge y)a^{-1}a$ . Finally, laws  $(1_{\{\cdot\}})$  and  $(1_{\{-1\}})$  of  $\Sigma_{\text{op}}$  come directly from the centrality law  $x^{-1}xy \approx yx^{-1}x$ .

Thus  $\mathcal{C} \models \Sigma_{\{\cdot, -1\}}$  and is a subvariety of the pointed semidiscriminator variety over the similarity type  $\langle 2, 1 \rangle$ . Therefore  $\mathcal{C}$  is equal to  $\mathbb{V}(\text{ps}(\mathcal{Q}))$  for some uH class  $\mathcal{Q}$  of partial algebras. Now we verify three further identities; namely  $x^{-1} \triangleright z \approx x \triangleright z$ ,  $xy \triangleright z \approx x \triangleright y \triangleright z$  and  $(xy^{-1}y \wedge x) \triangleright z \approx x \triangleright y \triangleright z$ .

The first holds as

$$x^{-1} \triangleright z = zx^{-1} \approx zx[x^{-1}x]x^{-1} \approx z[xx^{-1}]x^{-1}x \approx zx^{-1}xx^{-1}x \approx zx^{-1}x = x \triangleright z$$

by using the centrality of  $x^{-1}x$  and  $xx^{-1}$  (the square brackets indicate which of these is to be commuted next). The second holds because  $xy \triangleright z = z(xy)^{-1}xy \approx zy^{-1}[x^{-1}x]y \approx zy^{-1}yx^{-1}x = x \triangleright (y \triangleright z)$ . The third holds because  $(xy^{-1}y \wedge x) \triangleright z \approx ((y \triangleright x) \wedge x) \triangleright z \approx (y \triangleright (x \wedge x)) \triangleright z \approx x \triangleright y \triangleright z$  (using  $\Sigma_{\leq}(2)$  for the second equality).

The three identities are the pointed semidiscriminator translations of the atomic formulæ  $xy \approx xy$ ,  $x^{-1} \approx x^{-1}$  and  $xy^{-1}y \approx x$ . By Lemma 5.9, the first and second of the identities guarantee that  $\mathcal{Q}$  consists of total algebras. But then, the members of  $\mathcal{Q}$  are subalgebras of  $\{\cdot, -1\}$ -reducts of subdirectly irreducible naturally semilattice ordered Clifford semigroups and so it follows that  $\mathcal{Q}$  itself consists of Clifford semigroups. However any Clifford semigroup satisfying the third identity  $xy^{-1}y \approx x$  is a group. Hence  $\mathcal{Q} \subseteq \mathcal{G}_{\{\cdot, -1\}}$  is a uH class of groups. This shows that

$\mathcal{C} \subseteq \mathbf{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1\}}))$ . The reverse containment  $\mathcal{C} \supseteq \mathbf{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1\}}))$  follows because  $\text{ps}(\mathcal{G}_{\{\cdot, -1\}})$  consists of naturally semilattice ordered Clifford semigroups.  $\dashv$

The following corollary of Theorems 5.12 and 7.5 was first proved in [22].

**COROLLARY 7.6.** *The variety of naturally semilattice ordered Clifford semigroups has undecidable equational theory.*

**PROOF.** This follows immediately from Theorem 5.12, because the uniform word problem for the class of groups is undecidable (and this is equivalent in a variety to the undecidability of the quasi-equational theory; see [26, Connection 2.3]).  $\dashv$

Both  $\mathbf{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1\}}))$  and  $\mathbf{V}(\text{ps}(\mathcal{G}_{\{\cdot, -1, 1\}}))$  are finitely based, however  $\mathbf{V}(\text{ps}(\mathcal{G}_{\{\cdot\}}))$  is not. Indeed, the quasivariety generated by  $\mathcal{G}_{\{\cdot\}}$  consists of the class of all semigroups embeddable in a group. This quasivariety was studied by Maltsev who gave an infinite system of quasi-identities for it, and showed that no finite set was sufficient (see [13] for details).

More generally, the pointed semidiscriminator extensions of arbitrary semigroups are a semilattice ordered variety of “1-stacks” in the sense of Schein [42]. We omit the details of this claim, but comment that the symbol  $\triangleright$  is borrowed from that paper. In the monoid case we arrive at a variety that is term equivalent to a subvariety of the “agreeable monoids” in the sense of [21] (this fact underlies the methods in [22, Section 6]).

**7.9. Brandt groupoids.** The flat extension of a Brandt groupoid (see Example 4.2) is also a naturally semilattice ordered inverse semigroup. We use  $\mathcal{B}^\wedge$  to abbreviate  $\mathbf{V}(\mathfrak{b}(\mathcal{B}))$ , the variety generated by flat extensions of Brandt groupoids. The naturally semilattice ordered Clifford semigroups  $\mathcal{C}$  are a subvariety, but are closely related: the equational theory of this second variety interprets in  $\mathcal{B}^\wedge$  as follows.

**LEMMA 7.7.** *Let  $s(x_0, \dots, x_{n-1}) \approx t(x_0, \dots, x_{n-1})$  be an identity in the language  $\{\cdot, \wedge, {}^{-1}\}$  and let  $x$  be a new variable. The identity  $s(x_0, \dots, x_{n-1}) \approx t(x_0, \dots, x_{n-1})$  holds in  $\mathcal{C}$  if and only if*

$$s(xx^{-1}x_0xx^{-1}, \dots, xx^{-1}x_{n-1}xx^{-1}) \approx t(xx^{-1}x_0xx^{-1}, \dots, xx^{-1}x_{n-1}xx^{-1})$$

holds in  $\mathcal{B}^\wedge$ .

**PROOF.** Let us say that  $s(x_0, \dots, x_{n-1}) \approx t(x_0, \dots, x_{n-1})$  holds in  $\mathcal{C}$ , and consider an assignment  $\nu$  of the variables  $x, x_0, \dots, x_{n-1}$  into the flat extension of a Brandt groupoid  $\mathfrak{b}(\mathcal{B})$ . The flat extension of the one element group is in  $\mathcal{C}$  (it is term equivalent to the two element semilattice) and hence satisfies  $s(x_0, \dots, x_{n-1}) \approx t(x_0, \dots, x_{n-1})$ . It is easy to see that this implies that the terms  $s(x_0, \dots, x_{n-1})$  and  $t(x_0, \dots, x_{n-1})$  are explicit in the same variables; we assume that they are explicit in all of  $x_0, \dots, x_{n-1}$ . This shows that if  $\nu(xx^{-1}x_i xx^{-1}) = \infty$  for some  $i$ , then both sides of  $s(\dots, xx^{-1}x_i xx^{-1}, \dots) \approx t(\dots, xx^{-1}x_i xx^{-1}, \dots)$  take the value  $\infty$ . Now assume that  $\nu(xx^{-1}x_i xx^{-1}) \in \mathcal{B}$  for each  $i$ . This implies that there is a (total) subgroup  $\mathbf{G}$  of  $\mathcal{B}$  such that  $\nu(x_i) \in \mathbf{G}$  for all  $i$ , and  $\nu(xx^{-1})$  is the identity element for  $\mathbf{G}$ . So  $\nu$  maps into the subalgebra  $\mathfrak{b}(\mathbf{G})$  of  $\mathfrak{b}(\mathcal{B})$ . As  $\mathfrak{b}(\mathbf{G})$  is in  $\mathcal{C}$ , both sides of the identity  $s(\dots, xx^{-1}x_i xx^{-1}, \dots) \approx t(\dots, xx^{-1}x_i xx^{-1}, \dots)$  take the same value under  $\nu$ . Hence this identity is satisfied by  $\mathfrak{b}(\mathcal{B})$  and hence by  $\mathcal{B}^\wedge$ .

For the other direction, assume that

$$\mathcal{B}^\wedge \models s(\dots, xx^{-1}x_i xx^{-1}, \dots) \approx t(\dots, xx^{-1}x_i xx^{-1}, \dots).$$

Hence  $\mathcal{C} \models s(\dots, xx^{-1}x_i xx^{-1}, \dots) \approx t(\dots, xx^{-1}x_i xx^{-1}, \dots)$ . Now let  $v$  be any assignment of  $x_0, \dots, x_{n-1}$  into the flat extension of a group  $\mathbf{G}$ . Extend  $v$  to  $x$  by setting  $v(x) \neq \infty$  arbitrarily. Then it is easy to see that  $v(s(\dots, x_i, \dots)) = v(s(\dots, xx^{-1}x_i xx^{-1}, \dots)) = v(t(\dots, xx^{-1}x_i xx^{-1}, \dots)) = v(t(\dots, x_i, \dots))$  so that  $s(\dots, x_i, \dots) \approx t(\dots, x_i, \dots)$  is satisfied by  $\mathfrak{b}(\mathbf{G})$ , whence  $\mathcal{C}$ .  $\dashv$

The following extends Corollary 7.6.

**COROLLARY 7.8.** *The variety generated by the naturally semilattice ordered Brandt semigroups has undecidable equational theory.*

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