

DUALISABILITY OF FINITE SEMIGROUPS

MARCEL JACKSON

*Department of Mathematics, La Trobe University 3086, Australia
m.g.jackson@latrobe.edu.au*

Revised 25 July 2002

Communicated by R. McKenzie

We describe the inherently non-dualisable finite algebras from some semigroup related classes. The classes for which this problem is solved include the variety of bands, the pseudovariety of aperiodic monoids, commutative monoids, and (assuming a reasonable conjecture in the literature) the varieties of all finite monoids and finite inverse semigroups. The first example of an inherently non-dualisable entropic algebra is also presented.

Keywords: Natural duality; finite semigroup; quasivariety.

Mathematics Subject Classification: Primary 08C15, 20M30

The general notion of a duality for algebras was introduced by Davey and Werner [9] as a unification of the techniques in various particular dualities such as the Stone duality for Boolean algebras, the Pontryagin duality for abelian groups, the duality of Arens and Kaplansky for certain classes of rings and the Priestley duality for distributive lattices. While the reader is directed to [1] for a full treatment of the theory of dualities and their application, the basic definition of a duality can be given as follows.

Let \mathbf{M} be a finite algebra and $\underline{\mathbf{M}}$ be a topological structure $\langle M, H, R, \mathcal{T} \rangle$ where H is set of an algebraic partial operations on \mathbf{M} (that is, homomorphisms $f : \mathbf{T} \rightarrow \mathbf{M}$ where \mathbf{T} is a subalgebra of \mathbf{M}^n for some $n \in \mathbb{N}$), R is a set of algebraic relations on \mathbf{M} (that is, a set of subalgebras of finite powers of \mathbf{M}) and \mathcal{T} is the discrete topology. We call $\underline{\mathbf{M}}$ an *alter ego* of \mathbf{M} . For an algebra \mathbf{A} in the quasivariety $\mathbb{ISP}(\mathbf{S})$, we may define the *dual* $D(\mathbf{A})$ to be the topological structure of all homomorphisms from \mathbf{A} into \mathbf{M} regarded as a substructure of $\underline{\mathbf{M}}^A$. Now we define $E(D(\mathbf{A}))$ (the *dual* of $D(\mathbf{A})$) to be the algebra of all continuous homomorphisms from $D(\mathbf{A})$ into $\underline{\mathbf{M}}$, considered as a substructure of \mathbf{M}^{D_A} , where D_A denotes the universe of $D(\mathbf{A})$.

There is a natural mapping $e_{\mathbf{A}} : \mathbf{A} \rightarrow E(D(\mathbf{A}))$ of \mathbf{A} into the algebra $E(D(\mathbf{A}))$ that associates with each element a of A the *evaluation* map given by $e_{\mathbf{A}}(a)(x) = x(a)$. It is routinely verified that this map constitutes an embedding of \mathbf{A} into $E(D(\mathbf{A}))$. If $e_{\mathbf{A}}$ is also a surjection we say that $\underline{\mathbf{M}}$ yields a duality on \mathbf{A} . The alter ego $\underline{\mathbf{M}}$ is said to yield a duality on $\mathbb{ISP}(\mathbf{M})$ if it yields a duality on every member of $\mathbb{ISP}(\mathbf{M})$. In this case we say that \mathbf{M} is *dualisable*. Any finite generator for the quasivariety of a finite dualisable algebra is also dualisable [10, 27] and thus we may also speak of dualisable quasivarieties.

If a finite algebra admits a duality, then the alter ego of all (finitary) algebraic relations on \mathbf{M} will dualise $\mathbb{ISP}(\mathbf{M})$; however usually a much simpler alter ego is possible. At the extreme end is the well-known Stone duality for Boolean algebras; here we take as the alter ego the discrete topological space on $\{0, 1\}$ with no extra partial operations or relations. On the other hand for some finite algebras, even the alter ego of all finitary algebraic relations fails to yield a duality on every member of the quasivariety. Such an algebra (or the quasivariety generated by such an algebra) is said to be *non-dualisable* (abbreviated to ND). Some classes containing both dualisable and non-dualisable finite algebras but for which a complete description of the finite dualisable members is known are: the class of all two element algebras [1]; the class of all three element unary algebras [3]; the variety of commutative rings with identity [4]; the variety of pseudo-complemented semilattices [8]; the class of graph algebras [5] and of flat graph algebras [16]. For numerous other examples the reader is again directed to [1].

There has been some progress toward a description of dualisability for finite semigroups, in particular within the classes of groups and bands (idempotent semigroups). In the positive direction, Davey and Knox have given simple dualities for the quasivarieties of rectangular bands [7] (see also [1]), left normal, right normal and normal bands [6], while Quackenbush and Szabó have shown that every finite group with cyclic Sylow subgroups [24] is dualisable. In the negative direction, Quackenbush and Szabó have shown that finite groups with non-abelian Sylow subgroups are not dualisable [23] while Hobby has constructed an infinite collection of non-dualisable bands [14]. More recently, the techniques for establishing non-dualisability have been refined and simplified and this enables us to give some substantial extensions of the known situation for semigroup-related algebras.

Our first result is to show that a finite band is contained in the quasivariety of a finite dualisable algebra if and only if it is a normal band. We then establish the non-dualisability of some small semigroups including a commutative example. We finish with an examination of the dualisability of inverse semigroups and of monoids. Here we show that, modulo an (increasingly reasonable) conjecture from [23], a finite inverse semigroup or a finite monoid is contained in the quasivariety of a finite dualisable algebra if and only if it is a finite semilattice of groups with abelian Sylow subgroups. This shows, for example, that the dualisable aperiodic monoids are exactly the semilattice monoids.

1. Preliminaries

An algebra \mathbf{M} is said to be *inherently non-dualisable* (or IND) if whenever \mathbf{V} is a quasivariety containing \mathbf{M} and \mathbf{V} is generated by a finite algebra, then \mathbf{V} is non-dualisable. Pitkethly has observed that since the dualisability of a quasivariety is independent of the finite generating algebra, and since a non-empty power of an algebra generates the same quasivariety as the original algebra, a finite algebra is IND if and only if it does not embed into any finite dualisable algebra [20].

The properties of non-dualisability and inherent non-dualisability are known to be distinct. For example, it is known that every finite unary algebra is *not* IND [2] although (as long as at least two unary operations are present) can be embedded into a dualisable or a non-dualisable unary algebra of the same type [21]. On the other hand, the 2-element implication algebra is known to be IND. This last example is of interest since by adjoining the nullary operation 1 to the signature, the structure becomes term equivalent to the 2-element Boolean algebra which certainly is dualisable. Furthermore this example generates a minimal IND quasivariety in the sense that all subquasivarieties are dualisable (because the only proper subquasivariety is the trivial quasivariety). The following two lemmas are very useful in proving the non-dualisability of algebras (here and elsewhere if \mathbf{A} is a subalgebra of a direct product of algebras, we use π_s to denote the projection onto the s th coordinate).

Lemma 1. [3] *Let \mathbf{M} be a finite algebra. Assume there exists an infinite set S , a subalgebra \mathbf{A} of \mathbf{M}^S , an infinite subset A_0 of A and a number n such that*

- (i) *for each homomorphism $\phi : \mathbf{A} \rightarrow \mathbf{M}$ there is a unique block of $\ker(\phi|_{A_0})$ with more than n elements,*
- (ii) *the element $g \in M^S$ defined by $g(s) := a_s(s)$ is not contained in A , where a_s is any element of the infinite block of $\ker(\pi_s|_{A_0})$.*

Then \mathbf{M} is not dualisable.

Lemma 2. [5] *Let \mathbf{M} be a finite algebra. Assume there exists an infinite set S , a subalgebra \mathbf{A} of \mathbf{M}^S , an infinite subset A_0 of A and a function $u : \mathbb{N} \rightarrow \mathbb{N}$ such that*

- (i) *for each congruence θ on \mathbf{A} with finite index n there is a unique block of $\theta|_{A_0}$ with more than $u(n)$ elements,*
- (ii) *the element $g \in M^S$ defined by $g(s) := a_s(s)$ is not contained in A , where a_s is any element of the infinite block of $\ker(\pi_s|_{A_0})$.*

Then \mathbf{M} is IND.

The rough idea behind these lemmas is that while the element g is not actually contained in \mathbf{A} , it is “close enough” to \mathbf{A} to guarantee that for any alter ego of \mathbf{M} , there is an element of $E(D(\mathbf{A}))$ that is “missed” by the natural map $e_{\mathbf{A}} : \mathbf{A} \rightarrow E(D(\mathbf{A}))$. The element g in these lemmas is known as the *ghost element* and this type of approach for showing non-dualisability is known as the *ghost element method*.

The definition of dualisability concerns only finitary algebraic relations and partial operations, however it is known that by allowing relations and partial operations of all possible arities a “duality” is always achieved. For a fixed cardinal κ , we will say that a finite algebra \mathbf{M} is κ -dualisable if there is an alter ego $\widetilde{\mathbf{M}}$ dualising $\mathbb{ISP}(\mathbf{M})$ for which the maximal arity of the algebraic relations, operations and partial operations is at most κ . In the known cases, non-dualisable algebras have turned out to be non- κ -dualisable for every κ . The non- κ -dualisability of a finite algebra can be established using Lemma 1 or Lemma 2 if the set S in these lemmas can be chosen to have $|S| = \kappa$. We will restrict ourselves to considering $\kappa = \omega$, however obvious variations of the proofs can be performed for larger cardinals.

If a, a_1, a_2, \dots, a_n are elements of a finite set S and N_1, N_2, \dots, N_n are disjoint subsets of \mathbb{N} then we will use $a_{N_1 N_2 \dots N_n}^{a_1 a_2 \dots a_n}$ to denote the element of $S^{\mathbb{N}}$ defined by

$$a_{N_1 N_2 \dots N_n}^{a_1 a_2 \dots a_n}(k) = \begin{cases} a_j & \text{if } k \in N_j \\ a & \text{otherwise.} \end{cases}$$

When using this notation we will adopt the convention that brackets may be omitted from singleton N_i . For example, when N is the set $\{3, 4\}$ then a_{1N}^{bc} is the element $(b, a, c, c, a, a, \dots)$. To avoid confusion, we will also denote the constant map $(a, a, a, \dots) \in S^{\mathbb{N}}$ by \underline{a} .

We complete this section by recalling a few elementary notions from the theory of semigroups. Our definitions are completely standard and the reader familiar with the basics of semigroup theory can freely skip to the next section. Other readers should note that the following concepts are introduced in a fashion very much tailored to the needs of this paper. For a complete and balanced introduction to the topic, the reader is directed to a book such as [15].

If \mathbf{S} is a finite semigroup then there are smallest numbers $i, p \in \mathbb{N}$ such that $\mathbf{S} \models x^i \approx x^{i+p}$. These numbers are called the *index* and *period* of \mathbf{S} . The index and period of an element $a \in S$ is the index and period of the subsemigroup of \mathbf{S} that is generated by a . We will call an element of a semigroup \mathbf{S} that lies in a subgroup of \mathbf{S} a *group element*; for finite semigroups, the group elements are exactly the elements of index 1. A *completely regular semigroup* is a semigroup in which every element is a group element, or equivalently a semigroup that is a union of groups. Clearly a group is an example of a completely regular semigroup, as is a band (a semigroup in which every element is idempotent). A particular kind of band that will be important below is the variety of rectangular bands. These are defined within semigroups by the laws $xyx \approx x$ and $xx \approx x$. A typical rectangular band can be constructed by taking two sets, I and J say, and giving the cartesian product $I \times J$ the operation $(i_1, j_1) \cdot (i_2, j_2) = (i_1, j_2)$ (up to isomorphism, all rectangular bands arise in this fashion). Equivalently, a rectangular band is a direct product of a left zero semigroup with a right zero semigroup (as defined by the laws $xy \approx x$ and $xy \approx y$ respectively).

If K is a class of semigroups, then we say that a semigroup \mathbf{S} is a band of K -semigroups if there is a band $\mathbf{I} := \langle I, \cdot \rangle$ and a family $\{\mathbf{S}_i : i \in I\}$ of semigroups from K such that $S = \dot{\cup}_{i \in I} S_i$ and for $s, t \in S_i, s' \in S_j$, the product $s \cdot t$ in \mathbf{S} is equal to the same product in \mathbf{S}_i while the product $s \cdot s'$ is contained in $S_{i,j}$. The case of rectangular bands of groups turns out to be of fundamental importance in the theory of semigroups and these structures are given the name *completely simple semigroups*. The construction above actually ensures that the subgroups of a completely simple semigroup are all isomorphic (though this is not obvious from our presentation of the idea). The following theorem due to Clifford demonstrates one of the reasons for the importance of completely simple semigroups in the theory of semigroups.

Theorem 3. *Every completely regular semigroup is a semilattice of completely simple semigroups.*

Restricted to the class of bands, this theorem implies that every band is a semilattice of rectangular bands and hence every non-commutative band must contain a non-trivial left zero or right zero subsemigroup.

2. Dualisability of Finite Bands

As noted above, an infinite collection of non-dualisable bands (the smallest having seven elements) was found by Hobby [14]. These examples are constructed using a variation of Lemma 1 and some interesting techniques involving hypergraphs. Denote by \mathbf{L} the two element left zero semigroup and by \mathbf{R} the two element right zero semigroup. By \mathbf{L}^1 and \mathbf{R}^1 we will mean the semigroups obtained from \mathbf{L} and \mathbf{R} by adjoining an identity element 1. The bands \mathbf{L}^1 and \mathbf{R}^1 are natural examples to consider and their possible dualisability is posed as an open question in [14]. We now use Lemma 2 to show that they are IND and then use this fact to describe all IND finite bands.

Proposition 4. \mathbf{L}^1 is IND.

Proof. Let \mathbf{M} be a finite semigroup with elements a, b, c with a and b forming a left zero subsemigroup of \mathbf{M} , $bx \neq a$ for all $x \in M$ and $ca = a, cb = b$. Clearly \mathbf{L}^1 is such a semigroup. We will show that \mathbf{M} is IND.

Let \mathbf{A} be the subsemigroup of $\mathbf{M}^{\mathbb{N}}$ generated by all elements h for which $h(i) = b$ for some $i \in \mathbb{N}$ and let A_0 be the set $\{a_i^b : i \in \mathbb{N}\} \subset A$. Note that since $bx \neq a$ for all $x \in M$, we have that the constant map \underline{a} is not contained in the semigroup \mathbf{A} . Indeed, if $h := h_1 h_2 \dots h_n$ is a product of generators in \mathbf{A} then there is an i such that $h_1(i) = b$. But then $h(i) = b h_2(i) h_3(i) \dots h_n(i) \neq a$ as required. The element \underline{a} will be the ghost element. Note that if \mathbf{M} is a monoid, we may also ask that \mathbf{A} contain the constant map $\underline{1}$.

We aim to use Lemma 2. Choose $n = 1$ and let θ be any congruence on \mathbf{A} such that $a_i^b \theta a_j^b$ and $a_k^b \theta a_\ell^b$, with i, j, k, ℓ pairwise unequal. We show that $a_j^b \theta a_k^b$

and therefore there can be at most one class of $\theta|_{A_0}$ that contains more than one element of A_0 . Now $a_{jk}^{cb} \cdot a_j^b = a_{jk}^{bb}$ and $a_{jk}^{cb} \cdot a_i^b = a_k^b$ and therefore $a_{jk}^{bb} \theta a_k^b$. Then by symmetry we have $a_j^b \theta a_{kj}^{bb} = a_{jk}^{bb} \theta a_k^b$ as required.

To complete the proof it will suffice to show that the element g defined in Lemma 2 is equal to $\underline{a} \notin A$. Let π_i be a projection map. Clearly $\pi_i(a_j^b) = a$ whenever $j \neq i$. Hence we have $g(i) = a$ for all $i \in \mathbb{N}$ and so $g = \underline{a}$. □

Since \mathbf{R}^1 is obtained by reversing the multiplication in \mathbf{L}^1 , the following corollary holds.

Corollary 5. \mathbf{R}^1 is IND.

To complete our description of the inherently non-dualisable finite bands we need some elementary lemmas concerning quasivarieties of bands. Recall that a *normal band* is a band satisfying the identity $xyxzx \approx xzxyx$ (or equivalently, $xyzw \approx xzyw$).

Lemma 6. If \mathbf{B} is a non-normal band then \mathbf{B} contains a subsemigroup isomorphic to \mathbf{L}^1 or \mathbf{R}^1 .

Proof. By definition, there must be elements $e, f, g \in B$ such that $efege \neq egefe$ in \mathbf{B} . Consider the submonoid $e\mathbf{B}e$ of \mathbf{B} consisting of all elements of the form ebe ($b \in B$). This band monoid is not commutative since $efeege = efege \neq egefe = egeefe$. Hence it contains a left zero or right zero subsemigroup. Therefore it contains \mathbf{L}^1 or \mathbf{R}^1 as a subsemigroup. □

Theorem 7. A band is IND if and only if it is not a normal band.

Proof. To prove this, it will suffice to show that the quasivariety of all normal bands is dualisable. This follows from existing results as follows. Firstly, in [7] it is shown that every finite rectangular band is dualisable. In particular, it is shown that the rectangular band $\mathbf{L} \times \mathbf{R}$ is dualisable. Secondly, the variety $\mathcal{R}e$ of rectangular bands satisfies the identity $xyx \approx x$ and therefore the term $t(x, y) := xyx$ makes $\mathcal{R}e$ strongly irregular in the sense of [6] (since $\mathcal{R}e \models t(x, y) \approx x$). A central result of [6] now implies that the semigroup \mathbf{N} obtained from $\mathbf{L} \times \mathbf{R}$ by adjoining a zero element is also dualisable. To complete the proof we show that \mathbf{N} generates the quasivariety of all normal bands. (The same arguments also apply to the quasivarieties of all left normal and right normal bands by using \mathbf{L} and \mathbf{R} instead of $\mathbf{L} \times \mathbf{R}$ respectively.)

First note that the only subdirectly irreducible semigroups in the quasivariety of all normal bands are \mathbf{L}^0 (the two element left zero semigroup with adjoined 0), \mathbf{L} , their right handed duals, the two element semilattice and the one-element semigroup [11]. These are all subalgebras of \mathbf{N} and hence the quasivariety generated by \mathbf{N} contains all subdirectly irreducible normal bands. Since every normal band is a subdirect product (and so certainly a subalgebra of a direct product) of subdirectly

irreducible bands it follows that every normal band is contained in $\mathbb{ISP}(\mathbf{N})$, as required. \square

An interesting feature of the above result is the following.

Corollary 8. *The semigroups \mathbf{L}^1 and \mathbf{R}^1 generate minimal IND quasivarieties.*

In fact Lemma 6 shows that the quasivariety $\mathbb{ISP}(\mathbf{L}^1)$ is a “just-non-left-normal” quasivariety of bands in the sense that every proper subquasivariety of $\mathbb{ISP}(\mathbf{L}^1)$ is a quasivariety of left normal bands. By [12] there are exactly five quasivarieties of left normal bands and hence the lattice of subquasivarieties of $\mathbb{ISP}(\mathbf{L}^1)$ has exactly six elements (this last statement also holds for $\mathbb{ISP}(\mathbf{R}^1)$ of course).

The following corollary follows immediately from Theorem 7 and the results of [13] or [17, 25, 26] (respectively).

Corollary 9. *The following are equivalent for a finite band \mathbf{B} :*

- (i) \mathbf{B} is a normal band;
- (ii) \mathbf{B} is not IND;
- (iii) \mathbf{B} generates a residually finite variety;
- (iv) \mathbf{B} generates a variety with only finitely many subquasivarieties;
- (v) \mathbf{B} generates a variety with only countably many subquasivarieties.

As noted in the proof of Proposition 4, \mathbf{L}^1 is also IND as a monoid (of type $\langle 2, 0 \rangle$). Therefore any band monoid containing a left zero or right zero subsemigroup contains \mathbf{L}^1 or \mathbf{R}^1 as a submonoid. This gives the following result.

Corollary 10. *A band monoid (of type $\langle 2, 0 \rangle$) is dualisable if and only if it is a semilattice.*

3. Other Small Finite Semigroups

Recall that an algebra \mathbf{A} is said to be *entropic* if all its operations are homomorphisms from a power of \mathbf{A} into \mathbf{A} . A semigroup is entropic if and only if it satisfies $wx \cdot yz \approx wy \cdot xz$. It follows, for example, that the entropic bands are exactly the normal bands. For some classes the property of being entropic is known to be equivalent to being dualisable (apart from the example of bands, this is also true for graph algebras [5] and flat graph algebras [16]) or at least implies it (for example, the entropic groups are exactly the abelian groups) and in [16] it is “speculated” that entropicity always implies dualisability. A counterexample to this suspicion is given in [22]; it is a five element unary algebra. We now prove another lemma which enables us to construct a three element (commutative) semigroup counterexample. Note that every two element entropic algebra is dualisable [1] while the result of [3] shows that every three element entropic unary algebra is dualisable.

Lemma 11. *Let \mathbf{M} be a finite semigroup with 0. Say that \mathbf{M} contains elements a, e, f with $ea = a, af = a, e^2 = e, f^2 = f, a \neq a^2, (\exists n \in \mathbb{N}) a^n = 0$ and that*

for every non-idempotent element x we have $yz = x$ implies that either $y = x$ and $z^2 = z$ or $z = x$ and $y^2 = y$. Then \mathbf{M} is ND.

Proof. Let \mathbf{M} be as in the statement of the lemma and let \mathbf{A} be the subsemigroup of $\mathbf{M}^{\mathbb{N}}$ on the set $\{h \in M^{\mathbb{N}} \mid (\exists i \in \mathbb{N}) h(i) = 0\}$. By the definition of a zero element and since $a \neq 0$ (because $a^2 \neq a$), the constant map \underline{a} is not contained in \mathbf{A} . Let A_0 be the set $\{a_i^0 : i \in \mathbb{N}\}$ and let ϕ be a homomorphism of \mathbf{A} into \mathbf{S} . We show that there is at most one block of $\ker(\phi \upharpoonright_{A_0})$ with size more than one. Lemma 1 then implies that \mathbf{S} is non-dualisable. For the remainder of the proof we fix n such that $a^n = 0$.

Firstly, if $\phi(a_i^0)$ is idempotent for every i then we have $\phi(a_i^0) = \phi(a_i^0)^2 = \phi(a_i^0)^n = \phi(\underline{0}) = \phi(a_i^0)$, showing that $\ker(\phi \upharpoonright_{A_0})$ has exactly one block.

Now say that $\phi(a_i^0)$ is not idempotent. Therefore we have $\phi(a_i^0) = \phi(e_{ij}^{0a} \cdot a_{ij}^{0f}) = \phi(e_{ij}^{0a})\phi(a_{ij}^{0f})$. By the hypotheses of the lemma this implies that either $\phi(e_{ij}^{0a})$ is an idempotent left identity of $\phi(a_i^0)$ and $\phi(a_{ij}^{0f}) = \phi(a_i^0)$ or $\phi(a_{ij}^{0f})$ is an idempotent right identity of $\phi(a_i^0)$ and $\phi(e_{ij}^{0a}) = \phi(a_i^0)$. We now look at two cases.

Firstly assume that for all $j \in \mathbb{N} \setminus \{i\}$ we have $\phi(e_{ij}^{0a})$ is an idempotent left identity for $\phi(a_i^0)$ and that $\phi(a_{ij}^{0f}) = \phi(a_i^0)$. Then for any $k \neq i$ we have $\phi(e_{ik}^{0a})\phi(a_k^0) = \phi(e_{ik}^{0a})^n\phi(a_k^0) = \phi((e_{ik}^{0a})^n \cdot a_k^0) = \phi((e_{ik}^{00}) \cdot a_k^0) = \phi(a_{ik}^{00}) = \phi(e_{ik}^{00} \cdot a_i^0) = \phi(e_{ik}^{0a})^n\phi(a_i^0) = \phi(a_i^0)$. Since $\phi(a_i^0)$ is not an idempotent and $\phi(e_{ik}^{0a})$ is an idempotent, the conditions assumed in the theorem imply that $\phi(a_k^0) = \phi(a_i^0)$. That is, there is exactly one block in $\ker(\phi \upharpoonright_{A_0})$.

If the previous assumption is not true then there must be a number j such that $\phi(a_{ij}^{0f})$ is an idempotent right identity of $\phi(a_i^0)$ (and $\phi(e_{ij}^{0a}) = \phi(a_i^0)$). Now note that $\phi(a_{ij}^{0f})^n = \phi((a_{ij}^{0f})^n) = \phi(0_j^f)$ and so $\phi(a_i^0) = \phi(a_i^0)\phi(a_{ij}^{0f}) = \phi(a_i^0)\phi(0_j^f) = \phi(0_j^a)$. Let k be any positive integer other than i or j . Then $\phi(a_k^0)\phi(0_j^f) = \phi(0_j^a) = \phi(a_i^0)$. Since $\phi(0_j^f)$ is idempotent and $\phi(a_i^0)$ is not, we have $\phi(a_k^0) = \phi(a_i^0)$. Therefore there is exactly one block of size more than one in $\ker(\phi \upharpoonright_{A_0})$ (and possibly one block of size one: the block containing $\phi(a_j^0)$). Since the ghost element g in Lemma 1 is easily seen to be the element \underline{a} and since $\underline{a} \notin A$, we have shown that \mathbf{S} is non-dualisable. □

While the conditions of this theorem seem to be rather technical, we immediately get some very basic examples. For example, the following two semigroups can be shown to be ND using the theorem:

\cdot	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

\cdot	0	a	e	f
0	0	0	0	0
a	0	0	0	a
e	0	a	e	0
f	0	0	0	f

In the first case we take $e = f = 1$; in the second take all elements as in the statement of the theorem. In general we will denote a one-generated monoid with

index i and period p by $\mathbf{C}_{i,p}^1$. By $\mathbf{C}_{i,p}$ and $\mathbf{C}_{i,p}^0$ we will denote respectively the one generated semigroup and one generated semigroup with adjoined 0, each with index i and period p . The first example above is $\mathbf{C}_{2,1}^1$ while $\mathbf{C}_{1,n}$ is the cyclic group of order n . Note that $\mathbf{C}_{1,1}^1$ and $\mathbf{C}_{1,1}^0$ are both isomorphic to the two element semilattice.

The non-dualisability of $\mathbf{C}_{2,1}^1$ is of interest since it is commutative and therefore provides the desired example of a non-dualisable entropic semigroup. The second example is not entropic since $eeaf = a \neq 0 = eae f$. We note Lemma 11 only shows that $\mathbf{C}_{2,1}^1$ is ND and not that it is IND. Furthermore, the only other known ND entropic algebra is not inherently non-dualisable. The question now arises as to whether every finite entropic algebra is not inherently non-dualisable. We return to this problem in Sec. 5 by showing that $\mathbf{C}_{2,1}^1$ is in fact IND.

Another interesting example is the Brandt semigroup \mathbf{B}_2 as given by the following table.

·	0	a	b	e	f
0	0	0	0	0	0
a	0	0	e	0	a
b	0	f	0	b	0
e	0	a	0	e	0
f	0	0	b	0	f

Note that \mathbf{B}_2 has exactly two non-idempotent elements: a and b . However a and b both have unique left and right identity elements, and whenever $xy = a$ for some x, y we have either $x = a$ and $y = f$ (the right identity element) or $x = e$ (the left identity element) and $y = a$. A “dual” statement holds for b (with e and f interchanged). Therefore Lemma 11 implies that \mathbf{B}_2 is ND. Note that the second of the two examples presented before \mathbf{B}_2 is in fact a subsemigroup of \mathbf{B}_2 .

In general if κ is a cardinal and S is a set with $|S| = \kappa$ then we denote by \mathbf{B}_κ the semigroup on the set $S \times S \cup \{0\}$ with the multiplication $(i, j) \cdot (j, k) = (i, k)$ and all other products equalling 0. In the Cayley table above, we may choose $S = \{1, 2\}$ and let a, b, e, f denote the elements $(1, 2), (2, 1), (1, 1), (2, 2)$ respectively. The semigroups \mathbf{B}_κ are also known as *Brandt semigroups* and play a critical role in many aspects of semigroup theory, particularly inverse semigroup theory.

4. Inverse Semigroups

Recall that a *regular semigroup* is a semigroup satisfying $(\forall x)(\exists y) xyx \approx x$. As semigroups, the class of *inverse semigroups* can be defined as regular semigroups in which idempotents commute. Equivalently inverse semigroups can be defined as the variety of semigroups with an adjoined unary operation $^{-1}$ and satisfying $\{xx^{-1}x \approx x, xx^{-1}yy^{-1} \approx yy^{-1}xx^{-1}, (x^{-1})^{-1} \approx x\}$. The semigroup \mathbf{B}_2 is an example of an inverse semigroup and it can be easily verified that our proof of its non-dualisability as a semigroup can be also performed in the language of inverse semigroups. In

particular if in Lemma 11 we start with \mathbf{M} being an inverse semigroup, then the semigroup \mathbf{A} in the proof can be seen to be closed under the taking of inverses (since 0 is its own inverse) and so all remaining arguments hold. We now show that \mathbf{B}_2 is in fact IND and furthermore that every finite inverse semigroup containing a non-group element (that is, an element that is not contained in a subgroup) is IND. This means that a dualisable finite inverse semigroup is necessarily completely regular. This statement will be true in either the semigroup signature or in the inverse semigroup signature. Unfortunately our method does not extend to show that the three and four element examples of the last section are IND.

We note that a completely regular inverse semigroup is necessarily a semilattice of groups (rather than just a semilattice of completely simple semigroups as is implied by Theorem 3) and is also given the name *Clifford semigroup*. We will call a quasivariety (or an inverse semigroup generating such a quasivariety) that contains a non-Clifford inverse semigroup a *non-Clifford* quasivariety (or inverse semigroup). We now show that all finite non-Clifford inverse semigroups are IND.

Theorem 12. *If \mathbf{S} is a finite non-Clifford inverse semigroup then \mathbf{S} is IND.*

Proof. Let \mathbf{S} be a finite non-Clifford inverse semigroup. Evidently, S contains an element x such that for $n \geq 2$, $x^n \neq x$. Let i be maximal such that for $(\forall n \in \mathbb{N}) x^{i+n} \neq x^i$. Note that x^i is a non-group element but all higher powers of x^i lie in the same subgroup, that is, the index of x^i is 2. We will denote this element of index two in \mathbf{S} by a . Our proof will be concentrated on the inverse subsemigroup of \mathbf{S} generated by a (note that while we use the inverse operation to generate the semigroup, the resulting structure is still a subsemigroup of \mathbf{S} and so our result does not depend on the explicit presence of the inverse operation). This structure will be called \mathbf{M} .

The general structure of \mathbf{M} is quite easy to describe. Firstly, we have an element a^{-1} which we will denote by b . Secondly, we have the idempotent elements ab and ba which we denote by c and d respectively. Since a^2 lies in subgroup \mathbf{G} of \mathbf{M} we have $b^2 = (a^{-1})^2$ which is the inverse of a^2 in \mathbf{G} . Hence we have $a^2, b^2, cd, dc \in G$. Let p be the exponent of \mathbf{G} . Since G is finite we have that \mathbf{G} is generated by a^{p+1} (since for example, $b^2 = a^{-2} = (a^{p+1})^{p-2}$). It follows that the set of elements of \mathbf{M} is $\{a, b, c, d, a^2, a^3, \dots, a^{p+1}\}$. Since the idempotents of an inverse semigroup form a subsemigroup, the elements cd and dc of G must be idempotent and therefore equal the identity element of \mathbf{G} . We denote this element by e (in fact e is exactly the element a^p for $p > 1$ or a^2 for $p = 1$).

We now show that \mathbf{M} is IND. We begin by taking \mathbf{A} to be the subsemigroup of $\mathbf{M}^{\mathbb{N}}$ on the set $\{h \in M^{\mathbb{N}} | (\exists i \in \mathbb{N}) h(i) \in G\} \cup \{c_i^d : i \in \mathbb{N}\}$. It is routinely verified that this forms an inverse semigroup and therefore also a semigroup (the crucial ideas are: that \mathbf{G} is an ideal of \mathbf{M} and so is closed under products and inverses; that both c and d are self inverse; and that for $i \neq j$, $(c_i^d \cdot c_j^d)(i) \in G$). We now choose A_0 to be the set $\{c_i^d | i \in \mathbb{N}\}$. Note that the element \underline{c} is not contained in A ; this will be our ghost element.

Let θ be a congruence such that $c_i^d \theta c_j^d$ and $c_k^d \theta c_\ell^d$ for four distinct positive integers i, j, k, ℓ . We show that $c_i^d \theta c_k^d$ hence showing that $\theta|_{A_0}$ has at most one block of size more than 1. Firstly note that $(c_i^d)^2 = c_i^d$ and so $c_i^d \theta c_i^d c_j^d = c_{ij}^{ee}$. By symmetry $c_i^d \theta c_j^d \theta c_{ij}^{ee}$ and $c_k^d \theta c_\ell^d \theta c_{k\ell}^{ee}$ also. Now note that $c_i^e c_i^d = c_i^e$ while $c_i^e c_{ij}^{ee} = c_{ij}^{ee} \theta c_i^d$. That is, $c_i^d \theta c_i^e \theta c_j^e \theta c_j^d$ and similarly $c_k^d \theta c_k^e \theta c_\ell^e \theta c_\ell^d$.

Now note that $acb = aabb = a^2 \cdot a^{-2} = e$ and $adb = abab = ab = c$. Therefore $c_{ijk}^{eee} = c_{ik}^{ae} c_j^d c_{ik}^{be} \theta c_{ik}^{ae} c_i^d c_{ik}^{be} = c_k^e \theta c_k^d \theta c_\ell^d$ and then by symmetry we have that $c_{ijk}^{eee} \theta c_{ij\ell}^{eee} \theta c_k^e \theta c_\ell^e \theta c_k^d \theta c_\ell^d$. Since these are idempotents, $c_k^d \theta c_{ijk}^{eee} \theta c_{ij\ell}^{eee} \theta c_{ij\ell}^{eee} = c_{ijk\ell}^{eeee}$. Hence by symmetry, $c_k^d \theta c_{ijk\ell}^{eeee} \theta c_i^d$ as required. It is easily established that \underline{c} is the required ghost element, and as discussed above, $\underline{c} \notin A$. Hence \mathbf{M} is IND and since \mathbf{M} is a subsemigroup (and a sub-inverse semigroup) of \mathbf{S} it follows that \mathbf{S} is also IND. \square

In [23] it is shown that a group with a non-abelian Sylow subgroup is non-dualisable. This paper can be decomposed into two sections. The first is establishing certain structural characteristics of groups with a non-abelian Sylow subgroup; roughly speaking this corresponds to the first two paragraphs of our proof of Theorem 12 (though is more difficult). Here it is shown that every finite group \mathbf{H} with a non-abelian Sylow subgroup contains a subgroup \mathbf{G} of a particular form (corresponding to the inverse semigroup \mathbf{M} in the proof of Theorem 12). The second part is constructing a subgroup \mathbf{D} of an infinite power of \mathbf{G} which has a subset $V := \{v_{0,1}, v_{0,2}, \dots\}$ (using the notation of [23]) such that any homomorphism ϕ from \mathbf{D} into \mathbf{H} has the property that there is a number k for which for all $i, j \geq k$, $\phi(v_{0,i}) = \phi(v_{0,j})$. The existence of the number k in this proof depends only on the fact that the group \mathbf{H} is finite and not that it actually is a group. By itself, this is not enough to guarantee non-dualisability and further arguments are required. At first these arguments appear to depend on the fact that \mathbf{H} is a group. In particular in [23, Definition 3.3], a number M is chosen to be the maximum of the number of pairs satisfying a particular condition on the commutators of \mathbf{H} . The crucial argument (in [23, Lemma 3.4]) is a proof by contradiction of the maximality of M . If instead we take M to equal $|H|$ (which does not depend on group theoretic concepts) then a contradiction is still obtained; indeed the arguments would otherwise show that the map $\mu : \mathbf{D} \rightarrow \mathbf{H}$ under consideration has a kernel of index greater than $|H|$. All remaining arguments continue to hold and are independent of anything other than the size of \mathbf{H} .

This suffices to show that \mathbf{H} is IND since if \mathbf{A} is an arbitrary finite algebra (of appropriate type) whose quasivariety contains \mathbf{H} , then \mathbf{D} is also in the quasivariety of \mathbf{A} . All the arguments previously used for \mathbf{H} now hold for \mathbf{A} also, since A is finite.

These arguments are all true in the type $\langle 2, 1, 0 \rangle$ but also in the other conventional types used when considering finite groups; for example as inverse semigroups (of type $\langle 2, 1 \rangle$), as semigroups (of type $\langle 2 \rangle$) or as monoids (of type $\langle 2, 0 \rangle$).

Summarizing, we have the following.

Theorem 13. [23] *A finite group with a non-abelian Sylow subgroup is IND (in any of the types $\langle 2, 1, 0 \rangle$, $\langle 2, 1 \rangle$, $\langle 2, 0 \rangle$, $\langle 2 \rangle$).*

It is conjectured in [23] that a finite group with abelian Sylow subgroups is dualisable and in [24] this is proved in the case when all Sylow subgroups are cyclic. We call this the *Quackenbush-Szabó conjecture*. Note that a finite group in the type $\langle 2, 1, 0 \rangle$ generates a group variety that is term equivalent to the variety of type $\langle 2, 1 \rangle$ that it generates as an inverse semigroup and to the variety it generates as a semigroup of type $\langle 2 \rangle$. This means that the Quackenbush-Szabó conjecture for groups in any of these types coincide.

Theorem 14. *If the Quackenbush-Szabó conjecture is true, then a finite inverse semigroup (in the type $\langle 2, 1 \rangle$ or $\langle 2 \rangle$) is contained in a quasivariety generated by a finitely dualisable algebra if and only if it is a semilattice of groups with abelian Sylow subgroups.*

Proof. It is known [18] that a finitely generated variety \mathcal{V} of groups with abelian Sylow subgroups contains only finitely many subdirectly irreducible groups, all finite (in fact the converse is also true). Hence this variety can be generated by the direct product of all these subdirectly irreducibles; say \mathbf{G} . Every group in \mathcal{V} is a subdirect product of subdirectly irreducible groups in \mathcal{V} and therefore a subdirect product of subgroups of \mathbf{G} . Hence \mathbf{G} generates a quasivariety that coincides with \mathcal{V} . By the Quackenbush-Szabó conjecture, this quasivariety is dualisable. Now adjoin a zero element onto the group \mathbf{G} , giving a semigroup, say \mathbf{G}^0 . This generates the variety of all semilattices of groups in \mathcal{V} . However the only subdirectly irreducibles in such a variety are semigroups of the form \mathbf{H} and \mathbf{H}^0 , where \mathbf{H} is a subdirectly irreducible group. All these semigroups are subsemigroups of \mathbf{G}^0 and hence again, \mathbf{G}^0 generates a quasivariety that coincides with the variety of all semilattices of groups in \mathcal{V} . Every group is strongly irregular and hence by [6], \mathbf{G}^0 generates a dualisable quasivariety. Hence if \mathbf{H} is a semilattice of groups with abelian Sylow subgroups, then \mathbf{H} is contained in a dualisable quasivariety (generated by a finite algebra) and therefore is not IND.

In the other direction, simply note that if a finite inverse semigroup contains a subgroup with non-abelian Sylow subgroup or if it contains a non-group element, then the inverse semigroup is IND by Theorem 12 or Theorem 13. \square

Theorem 15. *If the Quackenbush-Szabó conjecture is true, then a finite inverse semigroup is IND if and only if it generates a non-residually finite variety.*

Proof. It is well-known that a finite inverse semigroup containing a non-group element generates a variety containing the semigroup \mathbf{B}_κ for every cardinal κ ; see [19] for example. Each algebra \mathbf{B}_κ is congruence free and so it follows that a finite inverse semigroup containing a non-group element generates a residually large variety. Therefore a residually finite (and finitely generated) variety of inverse semigroups is generated by a Clifford semigroup. The subgroups of a Clifford semigroup generating a residually finite variety must also generate residually finite varieties and so by [18], they must have abelian Sylow subgroups. Conversely, a Clifford semigroup

of groups with abelian Sylow subgroups generates a residually finite variety by [13] or [17] (as discussed above, the subdirectly irreducible algebras in such a variety are either subdirectly groups or semigroups of the form \mathbf{G}^0 where \mathbf{G} is a subdirectly irreducible group). The result now follows by Theorem 14. \square

It is natural to ask whether “IND” can be replaced by “ND” in the statement of these theorems. There is some evidence to suggest that this is unlikely. In particular we note that if p is a prime then the (dualisable) quasivariety generated by the semigroup $\mathbf{C}_{1,p}^0$ has uncountably many subquasivarieties [26]. On the other hand there do not appear to be any known examples of ND but not IND finite algebras outside of the unary algebras (where there are no IND algebras).

5. Monoids

Typically the presence of an identity element in a semigroup results in the semigroup generating a quasivariety with many subquasivarieties and a variety with many subdirectly irreducibles. This makes testing for dualisability much easier, since on the available evidence, these kind of properties are associated with non-dualisability. Indeed we now give a complete description (modulo the Quackenbush-Szabó conjecture) of the IND monoids (considered in either the type $\langle 2, 0 \rangle$ or as semigroups).

Theorem 16. *If the Quackenbush-Szabó conjecture is true, then a finite monoid is contained in a dualisable quasivariety generated by a finite algebra if and only if it is a Clifford semigroup of groups with abelian Sylow subgroups.*

Most of the rest of this section will be devoted to proving this result. Note that sufficiency follows from Theorem 14. In fact this theorem implies that we need to show that a monoid that is not a Clifford semigroup is IND. There are two cases. The first case corresponds to the situation when the monoid is a completely regular, but non-Clifford semigroup. The second case will correspond to when the monoid contains a non-group element.

In the first case, let \mathbf{S} be a completely regular, but non-Clifford monoid. By Theorem 3, \mathbf{S} is isomorphic to a semilattice of completely simple semigroups and the assumption that \mathbf{S} is non-Clifford implies that at least one of the completely simple subsemigroups must not be a group. A completely simple semigroup other than a group contains a subsemigroup that is either a left zero semigroup or a right zero semigroup (for the reader familiar with Green’s relations, note that \mathcal{L} - or \mathcal{R} -related idempotents form a right or left zero subsemigroup). In the presence of an identity element it then follows that \mathbf{S} contains either \mathbf{L}^1 or \mathbf{R}^1 as a submonoid. By Proposition 4 and Corollary 5, \mathbf{S} is IND.

It now remains to prove the second case.

Proposition 17. *Let \mathbf{S} be a finite monoid containing a non-group element. Then \mathbf{S} is IND (as a semigroup or as a monoid).*

Proof. For the sake of notational convenience, for any element x of a semigroup of period p and index i , we will let x^0 denote the element x^{ip} . This element is the unique idempotent power of the element x since $x^{ip+ip} = x^{ip}$. By choosing a suitable non-group element (as in Theorem 12), we can find a submonoid of \mathbf{S} isomorphic to one of the form $\{1, a, a^2, \dots, a^{1+p}\}$ endowed with the multiplication implied by setting $a^{2+p} = a^2$. We denote this structure by \mathbf{M} and the set $\{a^2, \dots, a^{1+p}\}$ by G . We will also denote the particular elements a^{1+p} and a^0 of \mathbf{M} by b and e respectively. Note that e is the only idempotent in \mathbf{M} other than 1 and that \mathbf{M} is isomorphic to the monoid $\mathbf{C}_{2,p}^1$.

We aim to apply Lemma 2. Let $A_1 := \{h \in M^{\mathbb{N}} \mid (\exists i \in \mathbb{N}) h(i) \in G\}$, $A_2 := \{1_N^a \mid N \subset \mathbb{N} \text{ and } N \text{ finite}\}$, $A_0 := \{a_i^b \mid i \in \mathbb{N}\} \subset A_1$, and $A := A_1 \cup A_2$. Note that products between elements of A_1 and other elements in A in $\mathbf{M}^{\mathbb{N}}$ are equal to elements of A_1 while products between elements of A_2 either lie in A_2 or A_1 . Hence A is a subuniverse of $\mathbf{M}^{\mathbb{N}}$; the corresponding submonoid will be denoted by \mathbf{A} . Note that the element \underline{a} is not contained in A and this will be our ghost element.

The remaining part of the proof will depend heavily on the following observation: if \mathbf{B} is an n element monoid in the variety generated by \mathbf{M} and $x := x_1 x_2 \dots x_n$ is a product of length n in \mathbf{B} , then there is an $i \leq n$ such that $x_i^0 x = x$. Indeed, assume that none of x_1, \dots, x_n lie in a subgroup of \mathbf{B} . Now B has only n elements and at least one of these must be an idempotent. At most $n - 1$ of the n elements of \mathbf{B} can be non-group elements since an idempotent is certainly a group element. It follows that there are distinct $i, j \leq n$ such that $x_i = x_j$. The product $g := x_i x_j = x_i^2$ lies in a subgroup of \mathbf{B} (considered as a semigroup) since the square of any element in \mathbf{B} lies in a subgroup (recall that $\mathbf{M} \models x^2 \approx x^{2+p}$). But then $g^0 x = g^0 g x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_{j+1} \dots x_m = g x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_{j+1} \dots x_m = x$. The case when one of the x_i lies in a subgroup is almost identical.

Now let θ be a congruence of \mathbf{A} of index n ; obviously we may assume that $n > 1$. We are first going to show that there are at most $n - 1$ distinct choices of $j \in \mathbb{N}$ for which $1_j^e a_i^b$ is not equivalent to a_i^b modulo θ . Indeed, if there are $n - 1$ such elements exist, say j_1, \dots, j_{n-1} then by taking $M = \mathbb{N} \setminus \{i, j_1, \dots, j_{n-1}\}$ we have $1_{i,M}^{b,a} 1_{j_1}^a \dots 1_{j_{n-1}}^a = a_i^b$. The left hand side of this equality is a product of length n and hence one of the elements involved in the product must have its idempotent power equivalent modulo θ to a left identity of a_i^b . By assumption, this must be the element $1_{i,M}^{b,a}$; that is, $1_{M \cup \{i\}}^e / \theta$ is a left identity for a_i^b / θ . But then for all $j \in M \cup \{i\}$ we have $a_i^b \theta 1_{M \cup \{i\}}^e a_i^b = 1_j^e 1_{M \cup \{i\}}^e a_i^b \theta 1_j^e a_i^b$ as required.

Now let i_1, \dots, i_n and j_1, \dots, j_n be pairwise distinct positive integers such that $a_{i_k}^b \theta a_{i_\ell}^b$ and $a_{j_k}^b \theta a_{j_\ell}^b$ for each $k, \ell \leq n$. Then at least one element $1_{i_k}^e$ is such that $a_{j_1}^b \theta 1_{i_k}^e a_{j_1}^b = a_{i_k j_1}^{b,b}$. But there is an ℓ such that $a_{i_k}^b \theta 1_{j_\ell}^e a_{i_k}^b = a_{i_k j_\ell}^{b,b}$. Then

$$a_{j_1}^b \theta a_{i_k j_1}^{b,b} = 1_{i_k}^e a_{j_1}^b \theta 1_{i_k}^e a_{j_\ell}^b = a_{i_k j_\ell}^{b,b} \theta a_{i_k}^b.$$

It follows that there is at most one congruence class of θ with more than $n - 1$ elements. By Lemma 2 (with $u(n) := n - 1$ for $n > 1$ and $u(1) := 1$) it follows that \mathbf{M} is IND. □

This also completes the proof of Theorem 16. □

We note that this proof appears to be the first direct application of Lemma 2 in which the function u is not the constant function $\underline{1}$.

The results of [13] and [17] show that the class of monoids described in Theorem 16 is exactly the class of monoids that generate residually finite varieties.

Within any restricted class of monoids, whose members contain no subgroups with non-abelian Sylow subgroups, the Quackenbush-Szabó conjecture is no longer required. For example, since the quasivariety of semilattices is dualisable (see [1] for example) and since there can be no left zero or right zero subsemigroups of a commutative semigroup, the following two corollaries can be easily extracted from Theorem 16 (recall that an aperiodic semigroup is a semigroup with only trivial subgroups).

Corollary 18. *Let \mathbf{M} be a finite aperiodic monoid. The following are equivalent:*

- (i) \mathbf{M} is dualisable;
- (ii) \mathbf{M} is not IND;
- (iii) \mathbf{M} is a semilattice;
- (iv) \mathbf{M} generates a variety not containing any of the monoids $\mathbf{C}_{2,1}^1$, \mathbf{L}^1 and \mathbf{R}^1 ;
- (v) \mathbf{M} generates a residually finite variety.

Corollary 19. *Let \mathbf{M} be a finite commutative monoid. The following are equivalent:*

- (i) \mathbf{M} is not IND;
- (ii) \mathbf{M} is a Clifford semigroup;
- (iii) For each $n \in \mathbb{N}$, the semigroup $\mathbf{C}_{2,n}^1$ is not a subalgebra of \mathbf{M} ;
- (iv) \mathbf{M} generates a variety not containing $\mathbf{C}_{2,1}^1$;
- (v) \mathbf{M} generates a residually finite variety.

We note that $\mathbf{C}_{2,1}^1$ (shown in Sec. 3 to be ND) can now be seen to be an IND entropic algebra; the only other known example of a ND entropic algebra (in [22]) is not IND. In fact $\mathbf{C}_{2,1}^1$ provides an infinite collection of entropic IND algebras. Recall that a *binar* (also known as a groupoid) is an algebra with a single binary operation.

Proposition 20. *If \mathbf{A} is a finite entropic binar, then there is a finite IND entropic binar \mathbf{B} containing \mathbf{A} as a subalgebra.*

Proof. First adjoin a new element ∞ to the universe of \mathbf{A} and extend the multiplication by setting $\infty \cdot x = x \cdot \infty = \infty$ for all $x \in A \cup \{\infty\}$. This new binar is still entropic (for example one can show that the law $(wx)(yz) \approx (wy)(xz)$ is still satisfied). Now consider the direct product of this algebra with $\mathbf{C}_{2,1}^1$ and denote the resulting structure by \mathbf{B} . Now \mathbf{B} is still entropic and the maps $x \mapsto (\infty, x)$

and $x \mapsto (x, 0)$ are embeddings of $\mathbf{C}_{2,1}^1$ and \mathbf{A} into \mathbf{B} . Hence \mathbf{B} is an IND entropic algebra with a subalgebra isomorphic to \mathbf{A} . \square

References

1. D. Clark and B. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, 1998.
2. D. Clark, B. Davey and J. Pitkethly, Binary homomorphisms and natural dualities, *J. Pure Appl. Algebra* **169** (2002), 1–28.
3. D. Clark, B. Davey and J. Pitkethly, Dualisability of three-element unary algebras, *Int. J. Algebra Comput.* **13** (2003), 361–391.
4. D. Clark, P. Idziak, L. Sabourin, C. Szabó and R. Willard, Natural dualities for quasivarieties generated by a finite commutative ring, *Algebra Universalis* **46** (2001), 285–320.
5. B. Davey, P. Idziak, W. Lampe and G. McNulty, Dualizability and graph algebras, *Discrete Math.* **214** (2000), 145–172.
6. B. Davey and B. Knox, Regularising natural dualities, *Acta Math. Univ. Comenianae* **68** (1999), 295–318.
7. B. Davey and B. Knox, From rectangular bands to k -primal algebras, *Semigroup Forum* **64** (2002), 29–54.
8. B. Davey and J. Pitkethly, Dualisability of p -semilattices, *Algebra Universalis* **45** (2001), 149–153.
9. B. A. Davey and H. Werner, Dualities and equivalences for varieties of algebras, Contributions to lattice theory (Szeged, 1980), eds. A. P. Huhn and E. T. Schmidt, Colloquia Mathematica Societatis János Bolyai 33, North-Holland, Amsterdam (1983), 101–275.
10. B. Davey and R. Willard, The dualisability of a quasivariety is independent of the generating algebra, *Algebra Universalis* **45** (2001), 103–106.
11. J. Gerhard, Subdirectly irreducible idempotent semigroups, *Pacific J. Math.* **39** (1971), 669–676.
12. J. Gerhard and A. Shafaat, Semivarieties of idempotent semigroups, *Proc. London Math. Soc.* **22** (1971), 667–680.
13. È. A. Golubov and M. V. Sapir, Varieties of finitely approximable semigroups, *Dokl. Akad. Nauk SSSR* **247**(5) (1979), 1037–1041 [Russian; English translation in *Soviet Math. Dokl.* **20** (1979), 828–832].
14. D. Hobby, Non-dualizable semigroups, *Bull. Austral. Math. Soc.* **65** (2002), 491–502.
15. J. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, Oxford, 1995.
16. W. A. Lampe, G. F. McNulty and R. Willard, Full duality among graph algebras and flat graph algebras, *Algebra Universalis* **45** (2001), 311–334.
17. R. McKenzie, Residually small varieties of semigroups, *Algebra Universalis* **13** (1981), 171–201.
18. A. Ju. Ol’sanskiĭ, Varieties of finitely approximable groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **33** (1969), 915–927 [Russian; English translation in *Math. USSR Izv.* **3** (1969), 867–877].
19. M. Petrich, *Inverse Semigroups*, Wiley Interscience, New York, 1984.
20. J. Pitkethly, Dualisability: unary algebras and beyond, Ph.D. Thesis, La Trobe University, Melbourne, 2001.
21. J. Pitkethly, Inherent dualisability, *Discrete Math.* **269** (2003), 219–237.
22. J. Pitkethly and B. Davey, A non-dualisable entropic algebra, *Algebra Universalis* **47** (2002), 51–54.

23. R. Quackenbush and C. S. Szabó, Nilpotent groups are not dualizable, *J. Austral. Math. Soc.* **A72** (2002), 173–179.
24. R. Quackenbush and C. S. Szabó, Strong duality for metacyclic groups, *J. Austral. Math. Soc.* **73** (2002), 377–392.
25. M. Sapir, Varieties with a finite number of subquasivarieties, *Sibirsk. Mat. Zh.* **22** (1981), 168–187 [Russian; English translation in *Siberian Math. J.* **22** (1981), 934–949].
26. M. Sapir, Varieties with a countable number of subquasivarieties, *Sibirsk. Mat. Zh.* **25** (1984), 148–163 [Russian; English translation in *Siberian Math. J.* **25** (1984), 461–473].
27. M. J. Saramago, Some remarks on dualisability and endodualisability, *Algebra Universalis* **43** (2000), 197–212.