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Semilattice Pseudo-complements on Semigroups#

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ABSTRACT

The notion of a pseudo-complement is extended to a broad class of semigroups, which we say are semilattice pseudo-complemented (*SP*). We examine the relationship between these and some other classes of unary semigroups, namely semigroups with closure and interior operations, and we also consider the concepts in a ring theoretic setting. Under some natural strengthenings of the defining axioms, some fundamental congruences are described, extending cases from the theory of inverse semigroups. The class of *SP*-semilattices coincides with a natural class of ordered structures related to topological spaces but differs slightly from the standard definition of a pseudo-complemented semilattice. These *SP*-semilattices arise naturally in the general theory and are given particular attention.

Key Words: Pseudo-complement; Semigroup; *p*-Semilattice; Ring.

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1. INTRODUCTION

Let S be a semigroup with distinguished subsemilattice L given its usual ordering: $e \leq f$ if and only if $e = ef$, for all $e, f \in L$.

The authors considered the case in which $C(a) := \min\{e \in L \mid ae = a\}$ exists for all $a \in S$ (Jackson and Stokes, 2001). These semigroups can be characterised equationally using C as an additional unary operation, the resulting structures called *right closure semigroups* (*RC-semigroups*). Similarly, if $I(a) := \max\{e \in L \mid ae = e\}$ exists for all $a \in S$, one obtains *interior semigroups* (*I-semigroups*); see Kelarev and Stokes (1999). Again, an equational characterisation in terms of I is available. Indeed in Jackson and Stokes (2001) and Kelarev and Stokes (1999), the equational characterisations were the beginning point and the existence of L with the stated properties was subsequently deduced.

A standard example of an RC-semigroup is that of all partial maps on a set X equipped with a closure operator, with $C(f)$ the identity map restricted to the closure of the domain of f; here L is all restrictions of the identity to closed subsets of X. In the I-semigroup case, instead assume X has a topology and let $I(f)$ be the restriction of f to the interior of the set of values in X which it fixes; this time L is all restrictions of the identity to open sets, a subsemigroup since finite intersections of open sets are open. (Each of these examples generalises to the semigroup of all relations on X.)

On similar lines, one could ask for the restriction $P(f)$ of the identity to the interior of the complement of the domain of a partial map f on X. This time, the relevant definition is $P(f) = \max\{e \in L \mid fe = 0\}$, where L is as for the I-semigroup example and 0 is the empty partial map; note that $P(0)$ is the top element in L. Evidently, $P(f)$ is like a semilattice-valued pseudo-complement of f.

Accordingly, the following general definition applies. Suppose S has a distinguished element 0 (which will turn out to be a zero for a subsemilattice L of S). Call S a *semilattice pseudo-complemented semigroup* (SP-semigroup) (relative to L) if $P(a) := \max\{e \in L \mid ae = 0\}$ exists for all $a \in S$. In this case, P is a (*right*) *semilattice pseudo-complement* on S. Left-sided versions are defined analogously.

For T a semigroup having a unary operation Q, let

$$Q^*(T) := \{e_1 e_2 \cdots e_k \mid k \in \mathbb{N}, e_i \in Q(T)\},$$

a subsemigroup of T. Note that if Q maps into a subsemilattice of T, then $Q^*(T)$ is also a subsemilattice of T. In particular, if S is an SP-semigroup relative to L, then it is also an SP-semigroup relative to $P^*(S)$.

Proposition 1.1. *Suppose S is a semigroup with distinguished subsemilattice L and element 0 and P is a unary operation on S having image in L. The following are equivalent*:

(i) *S is an SP-semigroup relative to $P^*(S)$, with $P(a) = \max\{e \in P^*(S) \mid ae = 0\}$*;
(ii) *For all $a, b \in S$, $aP(a) = 0$ and $P(a)P(b) = P(aP(b))P(b)$.*

In this case, $0 = P^2(0)$ is a zero for $P^*(S)$ and a right zero for S and $P(0)$ is the top element of the subsemilattice $P^*(S)$.

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Proof. Let $a, b \in S$. Assuming (i), $aP(a) = 0$ and $0P(a) = aP(a)P(a) = aP(a) = 0$, so 0 is a left zero for elements of $P(S)$. Hence $0 = 0P(b) = (aP(a))P(b) = aP(b)P(a)$, so $P(aP(b)) \geq P(a)$, and therefore $P(aP(b))P(b) \geq P(a)P(b)$. Conversely, $0 = (aP(b))P(aP(b)) = a(P(b)P(aP(b)))$, so $P(a) \geq P(b)P(aP(b))$ and so $P(a)P(b) \geq P(b)P(aP(b))$.

Conversely, suppose P satisfies (ii). Note that $P(0)P(a) = P(aP(a))P(a) = P(a)P(a) = P(a)$, so $P(0)$ is the top element of $P^*(S)$. Now $aP(a) = 0$, and if $ae = 0$ for some $e \in P^*(S)$, then writing $e := P(b_1)P(b_2)\cdots P(b_k)$ for some $b_i \in S$,

$$\begin{aligned}P(a)e &= P(a)P(b_1)P(b_2)\cdots P(b_k)\\ &= P(aP(b_1)P(b_2)\cdots P(b_k))P(b_1)P(b_2)\cdots P(b_k)\\ &= P(ae)e = P(0)e = e,\end{aligned}$$

and so $P(a) = \max\{e \in P^*(S) \mid ae = 0\}$.

To show that 0 is a right zero for S note that, since $P(0)$ is the top element of $P^*(S)$, we have that $0 = P(0)P^2(0) = P^2(0)$ is contained in $P^*(S)$ and is a zero element for $P^*(S)$. Then $a0 = a(P(a)0) = 00 = 0$. □

Note that we cannot weaken the above result by replacing $P^*(S)$ by L since L may be too big. It is now straightforward to give an equational characterisation of *SP*-semigroups.

Proposition 1.2. *Let S be a semigroup equipped with additional unary operation P and nullary* 0*. Then P is an SP-semigroup relative to some subsemilattice L if and only if the following laws are satisfied*:

(i) $aP(a) = 0$;
(ii) $P(0)P(a) = P(a)$;
(iii) $P(a)P(b) = P(b)P(a)$; *and*
(iv) $P(a)P(b) = P(aP(b))P(b)$

Proof. The necessity is immediate by Proposition 1.1. Now assume the laws are satisfied. For $a \in S$, by (i), (ii) and (iv), $P(a)^2 = P(aP(a))P(a) = P(0)P(a) = P(a)$, and by (iii), $P(S)$ generates a semilattice, namely $P^*(S)$, which trivially P maps into. Hence, by Proposition 1.1, S is an *SP*-semigroup relative to $L = P^*(S)$. □

Thus we are at liberty to use the laws in the above proposition to define *SP*-semigroups. Note that different choices of L can give rise to the same operation P; the salient aspect is the subsemilattice generated by $P(S)$. Consequently, we generally take the equational description to be the definition. Note that under this approach, there is no need even to specify the existence of the distinguished element 0 in advance: law (i) can be captured without mention of 0 by the identity $aP(a) = bP(b)$.

In most of our examples, the element 0 will be a two-sided zero element, although this need not be the case in general: consider the two element right zero semigroup on $\{0, 1\}$ with $P(1) = P(0) = 0$, for instance.

Some useful facts are summarised in the next result.

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Proposition 1.3. *In the SP-semigroup S, the following hold for all $a, b \in S$:*

(i) $P(a) \leq P^3(a)$;
(ii) $P^4(a) = P^2(a)$;
(iii) $P(a)P^4(a) = 0$;
(iv) $P(ab)P(b) = P(b)$ *and* $P^2(ab)P^2(b) = P^2(ab)$.

Proof. (i) holds since

$$P^3(a)P(a) = P(P^2(a)P(a))P(a) = P(0)P(a) = P(a).$$

This implies that $P^2(a) \geq P^4(a)$. Since $P^3(a)P^2(a) = 0$, it follows that $P^2(a) \leq P^4(a)$, giving (ii); (iii) follows immediately.

For the equalities in (iv), note that for all $a, b \in S$,

$$P(ab)P(b) = P(abP(b))P(b) = P(a0)P(b) = P(0)P(b) = P(b),$$

showing that $P(b) \leq P(ab)$ and therefore $P^2(ab) \leq P^2(b)$. □

In the commutative case, a nice replacement rule holds.

Proposition 1.4. *Let S be a commutative SP-semigroup. Then for all $a \in S$ and every derived unary operation f on S, $f(a)P(a) = f(0)P(a)$.*

Proof. The result is proved by induction on the depth of nesting of f as a derived operation.

In the depth zero case, $f(x) = 0$ or x, but the former case obviously satisfies the condition and as for the second, $aP(a) = 0 = 0P(a)$. Assuming the condition is satisfied up to depth n, suppose $f(x)$ has depth $n + 1$. Now either $f(x) = g(x)h(x)$, where $g(x), h(x)$ both have depth at most n, or else $f(x) = P(g(x))$ where $g(x)$ has depth n. In the former case, $f(a)P(a) = g(a)h(a)P(a) = g(0)h(0)P(a) = f(0)P(a)$ by commutativity and the inductive assumption, while in the latter case,

$$\begin{aligned} f(a)P(a) &= P(g(a))P(a) = P(g(a)P(a))P(a) = P(g(0)P(a))P(a) \\ &= P(g(0))P(a) = f(0)P(a) \end{aligned}$$

as required. □

In the next section, some constructions for producing examples are considered. In Secs. 3–6, some connections between *SP*-semigroups and other unary semigroups are examined, and some congruence theory from these classes is extended to the variety of SP-semigroups; these ideas are further developed for rings in Sec. 7. In Sec. 8, the variety of *SP*-semilattices, which appears to play a central role in the theory of *SP*-semigroups, is considered.

2. SOME CONSTRUCTIONS

As discussed above, one of the main motivating examples of an *SP*-semigroup is as follows. Let X be a set equipped with a topological interior operator I. Let $S := \mathscr{P}_X$ be the semigroup (under composition) of all partial maps $X \to X$ and let $P(a)$ be the restriction of the identity map on X to $I(\mathrm{dom}(a)')$, for all $a \in S$. Letting L be all restrictions of the identity map on X to open subsets of X, it is clear that L is a subsemilattice of S since function composition in L corresponds to intersection of subsets. Moreover, for any $f \in S$, there is a largest open subset contained in the complement of the domain of f, and the restriction of the identity to that subset is evidently the largest $g \in L$ (under the usual semilattice partial order) for which $fg = \varnothing$ (the empty map). Hence S can be made into an *SP*-semigroup with $P(S) = P^*(S) = L$. A similar construction on the semigroup $\mathscr{B}_X$ of all binary relations (viewed as multivalued partial functions) yields a second example.

It is not possible to weaken this by replacing ''open'' by ''complement of closed'' for an arbitrary closure operator (that is, a closure operator not necessarily distributing over unions). For instance, letting $X := \{a, b, c\}$, with ''open'' subsets $\varnothing, \{a, b\}, \{a, c\}, \{b, c\}$ and X, clearly the open sets are closed under union (but not intersection). However, in the semigroup $\mathscr{P}_X$, although there is a largest restriction of the identity with domain an open subset in the complement of the domain of $f = \{(a, a), (b, b)\}$, namely $P(f) = \varnothing$, this is not the same as the largest element in the semilattice generated by such maps, the latter map being $1_c = \{(c, c)\}$. (Part (iv) of Proposition 1.2 breaks down: defining P as for $P(f)$ above, $P(f)P(1_a) = \varnothing \neq P(fP(1_a))P(1_a) = P(1_b)P(1_a) = 1_{ac}1_{bc} = 1_c$.) However, it is possible to weaken the notion of a topological space and still have the basic construction work; see below.

Examples exist for which $P(S) \neq P^*(S)$; for instance, let $X = \{a, b, c\}$ and $S = 2^X \backslash \{\{b, c\}\}$ which is a semilattice under $\cap$ and take $L = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then S is a *SP*-semilattice relative to L and $P(S) = \{\varnothing, X, \{a, b\}, \{a, c\}\} \neq L = P^*(S)$.

Define a *weak topological space* to be any set X together with a family of ''open'' subsets $\mathscr{T}$ closed under finite intersections, arbitrary unions and containing the empty set. This differs from the usual definition of a topological space only in that X is not necessarily contained in $\mathscr{T}$. The concept of an open basis and subbasis for such a topology are seen to extend trivially to this situation. Note that every weak topological space has a maximal open set, namely the union of all its open sets. Note also that the earlier definition of *SP*-semigroup structure on $\mathscr{P}_X$ still works if X is only a weak topological space, if the interior operator I on the subset T of X is defined to be the union of all open subsets in T.

A wide class of *SP*-semigroups, many commutative, arise as follows. Let $S_i, i \in X$ be a collection of monoids with zero, with X a weak topological space. On the Cartesian product semigroup $S := \prod_{i \in X} S_i$, define $P((a_i))$ to be (b_i), where $b_i = 1$ for i in the interior of the set $\{i \in X \mid a_i = 0\}$ and $b_i = 0$ for all other i. Obviously elements of the form $P(s), s \in S$ are exactly the ''characteristic functions'' of open sets in X, and form a subsemilattice $P(S)$ of S. It is also evident that $P(s) = \max\{e \in P(S) \mid se = 0\}$, and so P makes S into an *SP*-semigroup, commutative if all the S_i are.

For any given weak topological space $(X, \mathscr{T})$ it is also possible to construct an SP-semilattice S by taking S to be the semilattice 2^X of all subsets of X under intersection and, for each $A \subseteq X$, letting $P(A)$ be the interior of the complement of A in X. Under this construction, $P(S)$ is the semilattice of open subsets of X. If $X \in \mathscr{T}$ (in which case X is a standard topological space), then the element $P(\varnothing)$ is the identity element, which is not in general the case for SP-semilattices. As now shown, these types of examples are generic amongst SP-semilattices.

Proposition 2.1. *Every SP-semilattice S is embeddable in the semilattice of subsets of S equipped with a weak topology such that $P(a)$ is the largest open subset in the complement of a.*

Proof. Let θ be an embedding of S into the semilattice of subsets of some set X with $\theta(0) = \varnothing$ (for example take the map $\theta : S \to 2^{S\backslash\{0\}}$ given by $\theta(x) := \{s \in S\backslash\{0\} \mid s \leq x\}$). For all $a \in S$, let $X_a := \theta(P(a))$, and let all $\{X_a \mid a \in S\}$ be an open subbase for a weak topology on X. Let P also denote the semilattice pseudo-complement operation induced on 2^X as in the example. It must be shown that $\theta(P(a)) = P(\theta(a))$.

First, $\theta(a)\theta(P(a)) = \theta(aP(a)) = \theta(0) = \varnothing$, so $\theta(P(a)) \subseteq P(\theta(a))$. Now $P(\theta(a))$ is the largest open subset of X in the complement of $\theta(a)$ and therefore is the union of all open base sets it contains. If the typical open base set $\theta(P(b_1)) \cap \theta(P(b_2)) \cap \cdots \cap \theta(P(b_n))$ is contained in $P(\theta(a))$, then $\theta(P(b_1)) \cap \theta(P(b_2)) \cap \cdots \cap \theta(P(b_n)) \cap \theta(a) = \varnothing$, so $\theta(aP(b_1)P(b_2)\cdots P(b_n)) = \varnothing$, and so $aP(b_1)P(b_2)\cdots P(b_n) = 0$, which means $P(b_1)P(b_2)\cdots P(b_n) \leq P(a)$, and so $\theta(P(b_1)) \cap \theta(P(b_2)) \cap \cdots \cap \theta(P(b_n)) = \theta(P(b_1)P(b_2)\cdots P(b_n)) \subseteq \theta(P(a))$; hence $P(\theta(a)) \subseteq \theta(P(a))$ and so the two sets are equal. Hence θ respects P also. □

Note that if $P(0)$ is the identity element in an SP-semilattice, then the open subbasis constructed in this proof contains the entire set, and the weak topology is in fact a standard topology. Hence the following

Corollary 2.2. *Every SP-semilattice satisfying $xP(0) = x$ is embeddable in the semilattice of subsets of S equipped with a topology such that $P(a)$ is the largest open subset in the complement of a.*

At this point it is worth noting the distinction between SP-semigroups and another similar structure in the literature, the *pseudo-complemented semilattice* (*PC-semilattice*), which is a semilattice S in which for each $e \in S$ there exists a maximum element e^* of S for which $ee^* = 0$. Evidently PC-semilattices are precisely SP-semigroups S relative to S. The variety of PC-semilattices can be described within the variety of SP-semilattices by adjoining the identities $P^2(xy) = P^2(x)P^2(y)$ and $xP^2(x) = x$ (see for example Lemma 3.5 below).

PC-semilattices were introduced by Frink (1962), generalising concepts in lattice theory, and have been investigated by a number of authors. Proposition 2.1 and Corollary 2.2 as well as other examples above indicate that SP-semigroups are a natural extension of PC-semilattices. By definition, the subsemilattice $P^*(S)$ of an SP-semigroup S is a PC-semilattice, and so aspects of the theory of PC-semilattices can be transferred to it. Two useful and elementary consequences of facts established in Frink (1962) are given in the following proposition.

Proposition 2.3. *Let S be an SP-semigroup relative to a subsemilattice L.*

(i) *The subset $P(P(S))$ is a subsemilattice of S which is a Boolean algebra, with meet operation given by the multiplication of S, complement operation given by P and join operation given by* $P^2(x) \vee P^2(y) := P(P^3(x)P^3(y))$.
(ii) *If* $e, f \in L$ *then* $P^2(e)P^2(f) = P^2(ef)$ *and* $eP^2(e) = e$.

Some of the laws in Proposition 1.3 can also be derived as consequences of $P^*(S)$ being a *PC*-semilattice.

The class of *PC*-semilattices has previously been extended to semigroups (Blyth, 1965) and even to groupoids (Nirmula Kumari Amma, 1978), though the approach in these papers is quite different to the construction used above. For example, in Blyth (1965) one requires a partially ordered semigroup $S := \langle S, \cdot, \leq \rangle$ with multiplicative zero element 0 whose multiplication respects $\leq$ (in the sense that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$) and such that for each a there is a unique element a^* that is the maximum with respect to $\leq$ such that $a^*a = aa^* = 0$. Both Blyth (1965) and Nirmula Kumari Amma (1978) obtain various generalisations of results in Frink (1962).

3. CLOSURE SEMIGROUPS FROM *SP*-SEMIGROUPS

Recall from Jackson and Stokes (2001) that a semigroup S with unary operation C is a *right closure semigroup* (*RC-semigroup*) if it satisfies the identities

$$xC(x) = x, \quad C(x)C(y) = C(y)C(x), \quad C(C(x)) = C(x), \quad C(xy) = C(xy)C(y),$$

or equivalently, S is an *RC*-semigroup if there is a subsemilattice L such that for all $x \in S$, $C(x) := \min\{e \in L \mid xe = x\}$ exists. For commutative examples, we use the term "*C*-semigroup."

Any finite *SP*-semigroup S satisfying $xP(0) = x$ admits an *RC*-semigroup closure operation in the following way. For every x there is $e \in P^*(S)$ such that $xe = x$ (taking $e = P(0)$) and if $e, f \in P^*(S)$ are such that $xe = x$ and $xf = x$ then $xef = x$. Thus there is a minimum element e of $P^*(S)$ such that $xe = x$. This argument fails in the infinite case; and even in the finite case, the closure operation is not necessarily term-definable using the semilattice pseudo-complement. In many cases however, a closure operation does arise in a natural and term definable way. Proposition 2.3(ii) shows that P^2 is a homomorphism that acts as some kind of closure operation on L. In fact it shows that on the subsemilattice L, P^2 is a right closure operation. Accordingly, we will say that an *SP*-semigroup S is *closable* if $aP^2(a) = a$ for all $a \in S$.

We now show how every closable *SP*-semigroup can be turned into an *RC*-semigroup. First note that since $P^2(S)$ is a subsemilattice of an *SP*-semigroup S, it can be given a closure operation C, the identity map on $P^2(S)$ for example.

Proposition 3.1. *Let S be a closable SP-semigroup and let C be a closure operation on $P^2(S)$. Then $P = P^3$ and by defining $C'(x) := C(P^2(x))$ for all $x \in S$, S becomes an RC semigroup. Furthermore C', when restricted to $P^2(S)$, coincides with the existing closure C on $P^2(S)$.*

Proof. First note that the identity $xP^2(x) = x$ implies that $P(x) = P^3(x)$. This is because $xP^3(x) = xP^2(x)P^3(x) = x0 = 0$ and so $P^3(x) \leq P(x)$, whence equality follows by Proposition 1.3(i). Now it suffices to show that $C(P^2(x))$ is the minimum element e of $C'(S) = C(P^2(S))$ for which $xe = x$.

Let $e \in C(P^2(S))$ be such that $xe = x$. Note that since C is a closure on $P^2(S)$, and $P^3 = P$, it must be that $e = P^2(e)$. Now $xP(e) = xeP(e) = x0 = 0$ and so $P(e) \leq P(x)$. Therefore $P^2(x) \leq P^2(e) = e$ and so $C(P^2(x)) \leq C(e) = e$. Finally for $e \in P^2(S)$, $C'(e) = C(P^2(e)) = C(e)$ since $P^2(e) = e$ in S. □

The case when the closure on $P^2(S)$ is exactly the identity operation is of particular interest.

Corollary 3.2. *If S is a closable SP-semigroup then letting $C = P^2$ makes S an RC-semigroup for which $C(S)$ is a Boolean algebra.*

Proof. That P^2 is a right closure is immediate from Proposition 3.1 and that $P^2(S)$ is a Boolean algebra follows from Proposition 2.3(i). □

In fact the definition of C' in Proposition 3.1 is in some sense unique, as is shown next.

Theorem 3.3. *Let S be a closable SP-semigroup and an RC-semigroup for which $C(S) \subseteq P(S)$. Then $C = CP^2$, that is, the closure is determined by its action on $P(S)$.*

Proof. Now $xP(C(x)) = xC(x)P(C(x)) = x0 = 0$ and so $P(C(x)) \leq P(x)$. Therefore $P^2(C(x)) \geq P^2(x)$. However, $C(x) = P(y)$ for some y and so it follows that $P(P(C(x))) = P(P(P(y))) = P(y) = C(x)$ (since $P(x) = P^3(x)$ on a closable SP-semigroup). That is, $C(x) \geq P(P(x))$. But $x = xP(P(x)) = xP(P(x))C(P(P(x))) = xC(P(P(x)))$ and so $C(P(P(x))) \geq C(x)$. Therefore $C(x) = C(P(P(x)))$, as required. □

The converse of Corollary 3.2 is a simple corollary.

Corollary 3.4. *If S is a closable SP-semigroup that is also an RC-semigroup with $C(S) = P^2(S)$, then $C = P^2$.*

It is well known (Frink, 1962) that PC-semilattices (which satisfy $xP^2(x) = x$ and $P^2(xy) = P^2(x)P^2(y)$) are closable SP-semigroups.

Lemma 3.5. *The class of PC-semilattices coincides with the variety of closable SP-semilattices satisfying $P^2(xy) = P^2(x)P^2(y)$.*

Proof. It suffices to show that if S is a closable SP-semilattice satisfying $P^2(xy) = P^2(x)P^2(y)$, then $xy = 0 \Rightarrow y \leq P(x)$ holds. Let $a, b \in S$ be such that $ab = 0$. Then

$$P(a)b = P(a)bP^2(b) = P(aP^2(b))b = P(P^2(aP^2(b)))b$$
$$= P(P^2(a)P^2(b))b = P(P^2(ab))b = P(0)b = b$$

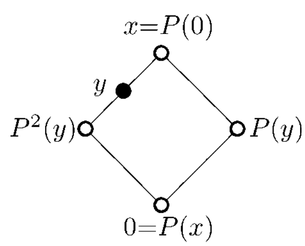
as required. □

An SP-semigroup S such that for all $x \in S$ and $e \in P^*(S)$ there exists $f \in P^*(S)$ such that $ex = xf$ is called *translucent*. Corollary 3.2 shows that a closable and translucent SP-semigroup gives rise to an RC-semigroup that is translucent in the sense of Jackson and Stokes (2001). There it is shown that translucency for RC-semigroups is equivalent to the identity $C(x)y = yC(C(x)y)$. Thus the following result holds.

Lemma 3.6. *A closable SP-semigroup is translucent if and only if it satisfies the identity* $P^2(x)y = yP^2(P^2(x)y)$.

In contrast with the situation for RC-semigroups, the following example demonstrates that the identity in the previous lemma is not equivalent to translucency.

Example 3.7. The SP-semilattice given by the following Hasse diagram (the elements of $P^*(S)$ are given by hollow vertices) is translucent but does not satisfy $P^2(x)y = yP^2(P^2(x)y)$.



Adjoining the law $C(x)y = yC(xy)$ to the definition of an RC-semigroup allows numerous aspects of inverse semigroup theory to be extended to RC-semigroups (Jackson and Stokes, 2001). An RC-semigroup satisfying this law is called *twisted*; every twisted RC-semigroup is translucent. Proposition 2.3(ii) shows that for a closable SP-semigroup S, the closure P^2 on $P^*(S)$ in fact makes $P^*(S)$ a twisted C-semilattice. Another example of a twisted RC-semigroup is any inverse semigroup, where $C(x)$ is defined by $x'x$. It is not hard to verify that the SP-semigroup $\mathscr{P}_X$, with $P(f)$ the identity restricted to the complement of the domain of $f \in \mathscr{P}_X$, is closable, and indeed twisted as an RC-semigroup. Moreover it satisfies the additional identities $P(x)y = yP(xy)$ and $xP^2(y) = xP(xP(y))$. These two laws are in general sufficient for twistedness of P^2, for if they hold, we have $P(P(a))b = bP(P(a)b) = bP(bP(ab)) = bP(P(ab))$ for any $a, b \in S$.

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4. *SP*-SEMIGROUPS DEFINED ON CLOSURE SEMIGROUPS

The ability to construct closures from closable *SP*-semigroups raises the question of whether it is possible to construct semilattice pseudo-complements on an *RC*-semigroup. With a few necessary restrictions, the answer is "yes." First it must at least be assumed that the semilattice of closed elements is an *SP*-semilattice. It also seems necessary to assume that S is *normal* in the sense of Jackson and Stokes (2003), that is, for all $a \in S$ and $\alpha \in C(S)$, $C(a)\alpha = C(a\alpha)$ (necessarily equal to $C(a\alpha)\alpha$). The property of normality is implied by the twisted property and is possessed by many *RC*-semigroups, for instance any inverse semigroup with closure given by $C(x) := x^{-1}x$ (which is twisted) or the *RC*-semigroup $\mathscr{B}_X$ of all relations on a set X (which is in general not twisted).

Proposition 4.1. *Let S be a normal RC-semigroup such that $C(S)$ is an SP-semilattice, and suppose S has a closed right zero* 0. *Then setting $P(a) = P(C(a))$ makes S into an SP-semigroup extending the SP-semigroup structure on $C(S)$. If $P^2 = C$ on S then S is a closable SP-semigroup and so $P = P^3$.*

Proof. For all $\alpha \in C(S)$ there is a largest $\beta \in P^*(C(S))$ for which $\alpha\beta = 0$. Note that 0 must be the zero of $C(S)$. Define $P(a) = P(C(a))$ for all $a \in S$ (which is clearly well defined on $C(S)$), so that $P(a) \in C(S)$, and therefore $P^*(S) \subseteq C(S)$. Then $aP(a) = aC(a)P(C(a)) = a0 = 0$, and if also $e \in P^*(S)$ is such that $ae = 0$, then $C(a)e = C(a)C(e) = C(aC(e)) = C(ae) = C(0) = 0$, so $e \leq P(C(a)) = P(a)$ in $C(S)$. Hence $P(a)$ is the largest α in the semilattice $P^*(C(S))$ for which $a\alpha = 0$, and so S is an *SP*-semigroup relative to $L = P^*(C(S))$.

The final statement is immediate from the definition of closable and from Proposition 3.1. □

The converse of this result is also true.

Theorem 4.2. *Let S be an SP-semigroup that admits a normal right closure C satisfying the identity $C(P(x)) = P(x)$. Then $P(x) = P(C(x))$ holds.*

Proof. First note that by Proposition 1.1, $0 = P^2(0)$ and so $C(0) = C(P^2(0)) = P^2(0) = 0$.

Now $xP(C(x)) = xC(x)P(C(x)) = x0 = 0$ and so $P(C(x)) \leq P(x)$. However,

$$\begin{aligned} P(C(x))P(x) &= P(C(x)P(x))P(x) = P(C(x)C(P(x)))P(x) \\ &= P(C(xC(P(x))))P(x)\,(\text{by normality}) \\ &= P(C(xP(x)))P(x) = P(C(0))P(x) = P(0)P(x) = P(x) \end{aligned}$$

and so $P(x) \leq P(C(x))$. Hence $P(C(x)) = P(x)$. □

In general a normal *RC*-semigroup may contain many idempotents that are not closed, and if the assumption that $P(x) = C(P(x))$ is dropped then the theorem no longer holds. For example if S is a monoid with zero, define $C(x) = 1$ for

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all x and let $P(x) = 0$ if $x \neq 0$ and $P(0) = 1$. This is easily seen to be a twisted RC-semigroup and hence normal (in fact it is even a two-sided closure semigroup), and is also an SP-semigroup. However $P(C(0)) = P(1) = 0$ while $P(0) = 1$.

On the other hand, for *full* twisted RC-semigroups (that is, twisted RC-semigroups where every idempotent is closed), the assumption $C(P(x)) = P(x)$ in Theorem 4.2 automatically holds (since $P(x)$ is an idempotent). The class of full twisted RC-semigroups exactly coincides with the right dual version of the class of *weakly left ample semigroups*, (see Gomes and Gould, 1999 for example) and includes the class of all inverse semigroups and all right type-A semigroups (in the sense of Fountain, 1991). The "closure" for weakly right ample semigroups is usually denoted by x^+.

Corollary 4.3. *If an inverse semigroup (or more generally, a weakly right ample semigroup) S is an SP-semigroup then $P(x) = P(x'x)$ $(P(x) = P(x^+)$, respectively). That is, the P operation on S is completely determined by the SP-subsemilattice $E(S)$. Conversely if S is an inverse semigroup (or right type A-semigroup) and $E(S)$ is an SP-semilattice containing a right 0 for S, then S is an SP-semigroup.*

Of course, any semigroup with zero element 0 can be made into a "trivial" SP-semigroup by letting $P(x) = 0$ for all x, but it can now be seen that any inverse semigroup with zero admits as many different semilattice pseudo-complement operations as does its semilattice of idempotents. For example on a three element chain E, there are four subsemilattices L containing 0, and each of these gives rise to a semilattice pseudo-complement on E (in fact, as is easily seen, any semigroup with 0 whose idempotents form a complete totally ordered subsemilattice admits P operations in this way); thus an inverse semigroup S with zero element and with $E(S) = E$ admits exactly four possible semilattice pseudo-complementations. This shows that the class of SP-semilattices has particular interest, and we return to it in a later section.

5. INTERIOR SEMIGROUPS FROM *SP*-SEMIGROUPS

Recall from Kelarev and Stokes (1999) that a semigroup S with unary operation I is an *interior semigroup* (*I-semigroup*) if it satisfies the identities

$$I(x)I(y) = I(y)I(x), \quad xI(x) = I(x), \quad I(I(x)) = I(x), \quad I(xy)I(y) = I(x)I(y),$$

or equivalently, S is an I-semigroup if there is a subsemilattice L such that for each $x \in S$, $I(x) := \max\{e \in L \mid xe = e\}$ exists. Call an SP-semigroup satisfying $xP^2(x) = P^2(x)$ *openable*. An example of an openable SP-semigroup is the SP-semilattice in Example 3.7.

First we show that every openable SP-semigroup satisfies the identity $P^2(xy) = P^2(x)P^2(y)$ and hence P^2 is a homomorphism onto $P(S)$.

Lemma 5.1. *Let S be an openable SP-semigroup. Then S satisfies the following identities*:

(i) $P(x) = P^3(x)$;
(ii) $P(x) = P(x^2)$;
(iii) $P(xy) = P(yx)$.

Proof. (i) Now $P(x)P^3(x) = P(x)P^2(P(x)) = P^3(x)$. But $P(x) \leq P^3(x)$ by Proposition 1.3(i).

(ii) First, $P^2(xx) \leq P^2(x)$ by Proposition 1.3(iv). But by Proposition 1.3(ii),

$$P^2(xx)P^2(x) = P^2(xxP^2(x))P^2(x) = P^2(xP^2(x))P^2(x) = P^4(x)P^2(x) = P^2(x),$$

giving $P^2(x) = P^2(xx)$. By part (i), it follows that $P(x) = P(xx)$ as required.

(iii) By (ii) and Proposition 1.3(iv),

$$\begin{aligned} P^2(xy) &= P^2(xyxy) = P^2(xyxy)P^2(yxy) = P^2(xy)P^2(yxy) \\ &= P^2(xy)P^2(yxy)P^2(y) = P^2(xy)P^2(yxyP^2(y))P^2(y) \\ &= P^2(xy)P^2(yxP^2(y))P^2(y) = P^2(xy)P^2(yx)P^2(y) = P^2(xy)P^2(yx) \end{aligned}$$

and then the result follows by symmetry. □

Note that part (i) shows that in an openable SP-semigroup S, $P^2(S) = P(S)$.

Theorem 5.2. *Let S be an openable SP-semigroup. Then S satisfies $P^2(xy) = P^2(x)P^2(y)$ and as a semigroup, S is a semilattice of semigroups with right zero element.*

Proof. By Lemma 5.1 and Proposition 1.3(iv),

$$\begin{aligned} P^2(xy) &= P^2(xy)P^2(y) = P^2(yx)P^2(y)P^2(x) = P^2(yxP^2(x))P^2(x)P^2(y) \\ &= P^2(yP^2(x))P^2(x)P^2(y) = P^2(yP^2(y)P^2(x))P^2(x)P^2(y) \\ &= P^2(P^2(x)P^2(y))P^2(x)P^2(y) = P^2(x)P^2(y). \end{aligned}$$

Now define an equivalence ρ by $(a, b) \in \rho$ if $P^2(a) = P^2(b)$. By Lemma 5.1(i) each equivalence class contains exactly one element of $P(S)$. For each $e \in P(S)$ let S_e denote the equivalence class of e modulo ρ. Now let $e, f \in P(S)$, with $a \in S_e$ and $b \in S_f$. Then $P^2(ab) = P^2(a)P^2(b) = ef$ so that $ab \in S_{ef}$. Therefore each equivalence class forms a subalgebra of S. It follows that as a semigroup, S is a (Boolean) semilattice of semigroups S_e, each with a right zero element (the elements in $P(S)$). □

Proposition 5.3. *Let S be an openable SP-semigroup. Then an interior operation I on $P(S)$ extends to an interior operation I' on S by defining $I'(x) := I(P^2(x))$. Conversely, if S admits an I-operation and $I(S) \subseteq P(S)$ then $I(x) = I(P^2(x))$.*

Proof. First, let $e \in I(P(S))$ be such that $xe = e$. Then $eP(x) = xeP(x) = xP(x)e = 0e = 0$. So $P(x) \leq P(e)$ and therefore $e = P^2(e) \leq P^2(x)$. It now follows that $I(P^2(x)) \geq I(e) = e$, so $I(P^2(x))$ is the maximum element f of $I(P(S))$ for which $xf = f$. Hence extending I to all of S by $I(x) = I(P^2(x))$ makes S an I-semigroup.

Conversely, if S admits an I-operation for which $I(S) \subseteq P(S)$, then $I(x)P(x) = xP(x)I(x) = 0$ so $P(x) \leq P(I(x))$ and therefore $I(x) = P^2(I(x)) \leq P^2(x)$. Since $I(I(x)) = I(x)$, it follows that $I(x) \leq I(P^2(x))$. But then

$$xI(P^2(x)) = xP^2(x)I(P^2(x)) = P^2(x)I(P^2(x)) = I(P^2(x))$$

and so $I(P^2(x)) \leq I(x)$. Hence $I(P^2(x)) = I(x)$. □

Of course $P(S)$ always admits the interior operation given by $I(x) = x$.

Corollary 5.4. *Let S be an openable SP-semigroup. Then S is an I-semigroup with* $I(x) := P^2(x)$. *Conversely, if S is an I-semigroup with* $I(S) = P(S)$, *then* $I(x) = P^2(x)$.

As an example of an openable SP-semigroup with reasonably rich structure, take a non-empty set X with subset $Y \subseteq X$ and let $\mathscr{S}(X)$ be the semigroup $\{s \in \mathscr{P}(X) \mid x \in Y \cap \mathrm{dom}(s) \Rightarrow s(x) = x\}$ under composition with pseudo-complement given by defining $P(s)$ to be the identity map restricted to the intersection of Y with the complement of the domain of s. This makes $\mathscr{S}(X)$ an SP-semigroup and it is routine to check that it satisfies the identity $xP^2(x) = P^2(x)$.

Unlike the case of RC-semigroups, if S is an I-semigroup with $I(S)$ an SP-semilattice then it does not in general seem possible to extend the P-operation to all of S. For example $S := \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2,3\}\}$ is an I-semilattice under $\cap$ if we let $I(S) := \{\varnothing, \{2\}, \{3\}, \{1,2,3\}\}$. However, while the subsemilattice on $I(S)$ is a PC-semilattice (and so certainly an SP-semilattice), there is no maximal element A of $I(S)$ for which $\{1\} \cap A = \varnothing$.

6. CONGRUENCES ON *SP*-SEMIGROUPS

The close relationship between closable SP-semigroups and RC-semigroups enables some of the results of Jackson and Stokes (2001) to be transferred into the pseudo-complement environment. In Jackson and Stokes (2001) it is shown that if S is an RC-semigroup satisfying the identity $C(C(x)y) = C(xy)$ and ρ is a congruence on $C(S)$ satisfying for all $x \in S$, $C(a)\rho C(b) \Rightarrow C(ax)\rho C(bx)$, then the relation

$$\rho_{\max} = \{(a,b) \mid C(a)\rho C(b) \text{ and } C(ea)\rho C(eb) \text{ for all } e \in C(S)\}$$

is the largest congruence on S which equals ρ when restricted to $C(S)$. This congruence generalises an analogous congruence definable on any inverse semigroup. (Every inverse semigroup admits the structure of an RC-semigroup for which this congruence and the analogous inverse semigroup congruence coincide; see Howie, 1995 for an account of the inverse semigroup theory version.) By replacing C everywhere

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in the definition of this congruence by P^2, it is not difficult to show that an analogous result holds for closable SP-semigroups satisfying $P^2(P^2(x)y) = P^2(xy)$. In fact the condition that S be closable and satisfy $P^2(P^2(x)y) = P^2(xy)$ may be replaced with the weaker restriction that S satisfy $P(x) = P^3(x)$ and $P^2(P^2(x)y) = P^2(xy)$. The following lemma simplifies this collection of laws.

Lemma 6.1. *The variety of SP-semigroups given by* $\{P(x) = P^3(x),\ P^2(P^2(x)y) = P^2(xy)\}$ *is the same as the variety of SP-semigroups given by the single identity* $P(xy) = P(P^2(x)y)$.

Proof. First, using $\{P(x) = P^3(x), P^2(P^2(x)y) = P^2(xy)\}$, it follows that $P(xy) = P(P^2(xy)) = P(P^2(P^2(x)y)) = P(P^2(x)y)$. Secondly, using the law $P(xy) = P(P^2(x)y)$, it follows that

$$\begin{aligned} P(x) &= P(x)P(0) = P(xP(0))P(0) \\ &= P(P^2(x)P(0))P(0) = P(P^2(x))P(0) = P^3(x) \end{aligned}$$

and $P(P(xy)) = P(P(P^2(x)y)) = P^2(P^2(x)y)$. □

Of course Proposition 2.3 shows that the identity $P(x) = P^3(x)$ makes $P(S) = P^2(S) = P^*(S)$ into a Boolean algebra.

For an SP-semigroup S satisfying $P(xy) = P(P^2(x)y)$, define a congruence ρ on the subsemilattice $P(S)$ to be *normal* if, for all $x \in S$, $P^2(a)\rho P^2(b)$ implies $P^2(ax)\rho P^2(bx)$. Our goal is to extend the congruence ρ to S. If this is to happen then certainly $P^2(a)\rho P^2(b)$ implies that $P^2(a)x\rho P^2(b)x$ for all $x \in S$, and so $P^2(ax) = P^2(P^2(a)x)\rho P^2(P^2(b)x) = P^2(bx)$. Equivalently (by applying P to both sides and using $P(x) = P^3(x)$), define normality for congruences ρ on $P(S)$ as $P(a)\rho P(b)$ implies $P(ax)\rho P(bx)$. The first definition however is in line with the following more general result. The proof is routine and omitted (it is given in Jackson and Stokes (2001) in the language of RC-semigroups, but only the properties given below are used).

Proposition 6.2. *Let S be a semigroup with unary operation* $U : S \to S$ *satisfying the identities*

$$\{U^2(x) = U(x),\ U(U(x)y) = U(xy),\ U(U(x)U(y)) = U(x)U(y)\}$$

(making the set $U(S) := \{U(x) \mid x \in S\}$ *a subsemigroup of S which is fixed by U). If* ρ *is a congruence on* $U(S)$ *satisfying the implication* $U(a)\rho U(b) \Rightarrow U(ax)\rho U(bx)$, *then* ρ *may be extended to a congruence on S respecting U and moreover the relation*

$$\rho_{\max} = \{(a, b) \mid U(a)\rho U(b) \text{ and } U(ea)\rho U(eb) \text{ for all } e \in U(S)\}$$

is the maximum congruence on S closed under U and coinciding with ρ *on the subsemilattice* $U(S)$.

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Next, Proposition 6.2 is used to establish a version of $\rho_{\max}$ on SP-semigroups satisfying $P(P^2(x)y) = P(xy)$. This will be immediate once the following elementary lemma is noted, the proof of which is clear.

Lemma 6.3. *Let S be an SP-semigroup satisfying $P(x) = P^3(x)$ and let θ be a semigroup congruence on S which is stable under the derived unary operation P^2 and is stable under the P-operation when restricted to $P(S) = P^2(S)$. Then θ is closed under the P-operation everywhere on S.*

This shows, for example, that an RC-semigroup congruence on a closable SP-semigroup that is closed under pseudo-complementation on $P(S)$ is an SP-semigroup congruence.

Lemma 6.1, Proposition 6.2 and Lemma 6.3 now establish the desired SP-semigroup congruence.

Proposition 6.4. *Let S be an SP-semigroup satisfying the identity $P(P^2(x)y) = P(xy)$. If ρ is a normal congruence on $P(S)$ then the relation*

$$\rho_{\max} = \{(a,b) \mid P(a)\rho P(b) \text{ and } P(ea)\rho P(eb)\ \forall e \in P(S)\}$$

is the maximum congruence on S whose restriction to $P(S)$ is identical to ρ.

Taking ρ to be the identity relation on $P(S)$ (which is certainly a normal congruence) gives the following result.

Corollary 6.5. *If S is an SP-semigroup satisfying $P(xy) = P(P^2(x)y)$ then the relation*

$$\mu = \{(a,b) \mid P(a) = P(b) \text{ and } P(ea) = P(eb) \text{ for all } e \in P(S)\}$$

is the largest congruence on S separating $P(S)$.

Note that the class of SP-semigroups satisfying $P(xy) = P(P^2(x)y)$ contains SP-semigroups that are not closable, even amongst the class of SP-semilattices. For example the semilattice $S = \{\{1\},\{2\},\varnothing\}$ under $\cap$ with $P(S) = P^2(S) = \{\{1\},\varnothing\}$ is easily verified to be an SP-semilattice satisfying $P(xy) = P(P^2(x)y)$ but $\{2\} \cap P(P(\{2\})) = \varnothing \neq \{2\}$. A second example of a non-closable semigroup to which this theorem applies is any openable SP-semigroup S for which $S \neq P^2(S)$. In this case however, the result becomes somewhat trivial. Indeed Theorem 5.2 implies that if S is an openable SP-semigroup and ρ is any congruence on $P(S)$, then $\rho_{\max}$ can be defined by $x\rho_{\max}y$ if $P(x)\rho P(y)$. This is easily seen to be consistent with the description in Corollary 6.5. Another similarly trivial case is given by the variety of PC-semilattices where we also have $P^2(xy) = P^2(x)P^2(y)$.

Proposition 6.2 also applies to interior semigroups (when U is an interior operation) and while openable SP-semigroups give rise to interior semigroups satisfying $I(I(x)y) = I(xy)$, in the general case, I need not be merely an endomorphism. An example is given as follows. Let S be the semigroup on $\{1,2\} \times \{1,2\} \cup \{0\}$ whose multiplication is given by $\langle i,j\rangle \times \langle \ell,k\rangle = \langle i,k\rangle$ and all other products equal 0 (this makes S a rectangular band with adjoined 0). The subsemilattice $L := \{0, \langle 1,1\rangle\}$ makes S an I-semigroup in which $I(0) = 0$ and $I(\langle i,j\rangle) = \langle 1,1\rangle$ if $i = 1$ and 0 otherwise. This interior semigroup satisfies $I(I(x)y) = I(xy)$ but not $I(xy) = I(x)I(y)$.

Since this interior semigroup congruence has not previously been described we state it formally here.

Proposition 6.6. *If S is an I-semigroup satisfying $I(I(x)y) = I(xy)$ and ρ is a congruence on $I(S)$ for which $I(a)\rho I(b)$ implies $I(ax)\rho I(bx)$ for all $x \in S$, then the relation*

$$\rho_{\max} = \{(a,b) \mid I(a)\rho I(b) \text{ and } I(ea)\rho I(eb)\ \forall e \in I(S)\}$$

is an I-semigroup congruence, and moreover is the maximum congruence on S whose restriction to the subsemilattice $I(S)$ is ρ.

In Jackson and Stokes (2001) the congruence

$$\rho_{\min} = \{(a,b) \mid C(a)\ \rho\ C(b) \text{ and there exists } e \in C(S) \cap C(a)/\rho \text{ such that } ae = be\}$$

is defined for a twisted RC-semigroup and a congruence ρ on $C(S)$ satisfying the implication $C(a)\rho C(b) \Rightarrow C(ax)\rho C(bx)$ for all $x \in S$. This is shown to be the minimum congruence that is identical to ρ on the subsemilattice $C(S)$. As before, by replacing C by P^2, this result can be extended to the case of closable SP-semigroups which satisfy the additional law $P^2(x)y = yP^2(xy)$; it only remains to verify that this equivalence respects the SP-operation. (Of course the current assumption is that ρ is a congruence on the SP-semilattice $P(S)$ and not just the semilattice $C(S)$.) This is true since if $(a,b) \in \rho_{\min}$, then $P^2(a)\rho P^2(b)$ and therefore $P(a) = P^3(a)\rho P^3(b) = P(b)$ (because ρ is a congruence that respects the SP-operation on $P(S)$). The condition $P^2(a)\rho P^2(b) \Rightarrow P^2(ax)\rho P^2(bx)$ is easily seen to be equivalent to the condition of being a normal congruence on $P(S)$ defined above.

As discussed above, the SP-semigroup $\mathscr{P}_X$ satisfies $P^2(x)y = yP^2(xy)$ when X is given the discrete topology. Another example is given by any PC-semilattice, in which $P^2(x)y = yP^2(y)P^2(x) = yP^2(xy)$ (using the PC-semilattice laws $xy = yx$, $xP^2(x) = x$ and $P^2(x)P^2(y) = P^2(xy)$).

7. PSEUDO-COMPLEMENTED RINGS

Closely related to the notions of semilattice pseudo-complemented semigroups, closure semigroups and interior semigroups are corresponding versions in the theory of rings. In this case the relationship between the various different types of unary operations is considerably tighter.

Let us say a ring with identity R is an *SP-ring* if its multiplicative monoid is an SP-semigroup. Note that the replacement rule of Proposition 1.4 extends in the commutative case to derived unaries defined in terms of any of the ring operations and P, because of distributivity of multiplication over addition.

Proposition 7.1. *In any SP-ring R, $P^*(R) = P(R)$, that is, $P(R)$ is closed under multiplication.*

Proof. Let $e \in P^*(R)$. Then, for $f \in P^*(R)$,

$$(P(0) - e)f = 0 \Leftrightarrow f = P(0)f = ef \Leftrightarrow f \leq e\,,$$

so $P(P(0) - e) = e$. Hence $P^*(R) = P(R)$. □

In Gardner and Stokes (1999), Kelarev and Stokes (1999) and Fearnley-Sander and Stokes (2003), closure rings, interior rings and E-rings are defined. Here we restate these definitions. First, recall that the adjoint monoid of a ring R is the monoid with underlying set R and semigroup operation $\circ$ given by $a \circ b := a + b - ab$. In any ring with identity R, the multiplicative and adjoint semigroups are isomorphic via the correspondence $x \leftrightarrow 1 - x$.

- R is a *closure ring* if its adjoint semigroup has a distinguished subsemilattice $C(R)$ such that $C(a) := \min\{e \in C(R) \mid ae = a\}$ exists for all $a \in R$.
- R is an *interior ring* if its multiplicative semigroup has a distinguished subsemilattice $I(R)$ such that $I(a) := \max\{e \in I(R) \mid ae = e\}$ exists for all $a \in R$.
- R is an *E-ring* if its multiplicative semigroup has a distinguished subsemilattice L_R such that $(a =_i b) := \max\{e \in L_R \mid ae = be\}$ exists for all $a, b \in R$.

Each of these enriched ring structures can be defined in the absence of the identity element, and all can be characterised equationally. If a ring R with identity has a multiplicative subsemilattice L, then $L' := \{1 - e \mid e \in L\}$ is a subsemilattice of the adjoint monoid. Note that for $e, f \in L$ (or $e, f \in L'$), $e = ef$ if and only if $f = e \circ f$, so view L as a meet-semilattice and L' as a join-semilattice; then under the above isomorphism, $e \leq f$ in one semilattice if and only if $1 - f \leq 1 - e$ in the other.

Proposition 7.2. *Let R be a ring with identity, with L a distinguished multiplicative subsemilattice and L' the corresponding adjoint subsemilattice. The following are equivalent*:

(i) *R is an SP-ring with $P(R) = L$;*
(ii) *R is a closure ring with $C(R) = L'$;*
(iii) *R is an interior ring with $I(R) = L$;*
(iv) *R is an E-ring with $L_R = L$.*

Proof. Suppose (i) holds. Then for each $a \in R$, $P(a) := \max\{e \in L \mid ae = 0\}$ exists. Now $a(1 - P(a)) = a$ with $1 - P(a) \in L'$, and for any $f \in L'$ for which $af' = a$, $a(1 - f') = a - af' = a - a = 0$, so $1 - f' \leq P(a)$ and so $f' \geq 1 - P(a)$, implying that $C(a) := 1 - P(a) = \min\{f \in L' \mid af = a\}$. Moreover, $aP(1 - a) = aP(1 - a) + (1 - a)P(1 - a) = P(1 - a)$ and if $ae = e$ for $e \in L$ then $(1 - a)e = 0$ and so $e \leq P(1 - a)$, so $I(a) = P(1 - a) = \max\{e \in L \mid ae = e\}$. Finally, for $a, b \in R$, $(a - b)P(a - b) = 0$ so $aP(a - b) = bP(a - b)$, and if $ae = be$ for some $e \in L$, then

$(a-b)e=0$, so $e \leq P(a-b)$, and so $(a =_i b) = P(a-b) = \max\{e \in L \mid ae = be\}$. Thus (ii), (iii) and (iv) hold. The converse directions are all very similar. □

Various term equivalences of rings with additional operations follow. Note that if R is a Boolean ring with identity, each of the above objects is nothing but an $S4$ modal ring, the algebraic models of type $S4$ modal logic; these are also known as closure rings and interior rings in the context of topology (see McKinsey and Tarski, 1944; Rasiowa, 1974).

It follows from a proof in Gardner and Stokes (1999) that the $\circ$-semilattice $C(R)$ in the closure ring R is closed under multiplication also. From Proposition 7.2, it therefore follows that in any SP-ring R with identity and for $e, f \in P(R)$, $e \circ f = 1 - (1-e)(1-f) \in P(R)$, so $P(R)$ is closed under the adjoint operation, and is a distributive lattice under these two operations. In fact this works for SP-rings with or without identity.

Proposition 7.3. *In the SP-ring R, $P(R)$ is closed under the adjoint operation and is in fact a Brouwerian lattice under the ring multiplication* (*meet*) *and adjoint operation* (*join*).

Proof. First note that $P(R)$ generates a distributive lattice under ring multiplication and $\circ$, as is routinely verified; call this $P'(R)$.

Now for $e, f \in P(R)$, as in the proof of Proposition 7.1, $e \circ f = P(P(0) - e \circ f)$, so $e \circ f \in P(R)$ and then $P(R) = P'(R)$.

To show that $P(R)$ is Brouwerian, note that for all $e, f \in P(R)$, $g := P(ef - e)$ is the maximal element of $P(R)$ for which $(ef-e)g = 0$, that is, $efg = eg$, or $eg \leq f$. Hence $P(ef-e)$ is the relative pseudo-complement of e with respect to f. □

Following this theme, some ideals of an SP-ring will (induce congruences which) respect the operation P; call these *SP-ideals*. In closure rings, the analogous notion of a C-ideal can be characterised neatly: it turns out that the ideal I is a C-ideal if and only if $C(i) \in I$ for all $i \in I$. (One direction of the proof is obvious, the other requires a little work.) Now in the ring with identity case, this immediately leads to the characterisation of SP-ideals as ideals I of an SP-ring for which for all $i \in I$, $P(i) - 1 \in I$, via Proposition 7.2. However, the following analog of this result for C-ideals applies to SP-rings with or without identity.

Proposition 7.4. *The ideal I of the SP-ring R is an SP-ideal if and only if $P(i) - P(0) \in I$ for all $i \in I$.*

Proof. Now I is an SP-ideal if and only if $P(a+i) - P(a) \in I$ for all $a \in R$ and $i \in I$. Letting $a = 0$ gives one direction. For the other, assuming that $P(i) - P(0) \in I$ for all $i \in I$, note that for $a \in R$ and $i \in I$,

$$\begin{aligned} P(a+i)P(i) &= P((a+i)P(i))P(i) = P(aP(i) + iP(i))P(i) = P(aP(i)+0)P(i) \\ &= P(aP(i))P(i) = P(a)P(i), \end{aligned}$$

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and so

$$\begin{aligned}P(a) - P(a+i) &= P(a)P(i) - P(a+i) - P(a)P(i) + P(a) \\ &= P(a+i)P(i) - P(a+i)P(0) - P(a)P(i) + P(a)P(0) \\ &= (P(a+i) - P(a))(P(i) - P(0)) \in I,\end{aligned}$$

so I is an SP-ideal as required. □

8. *SP*-SEMILATTICES

We now finish with a brief analysis of the variety of SP-semilattices, which have been shown above to be central to the theory of SP-semigroups and which arise naturally via topological spaces. It is first shown that this variety is not finitely generated and contains infinitely many subvarieties. This is in contrast to the variety of PC-semilattices which contains only one proper nontrivial subvariety (the variety of Boolean algebras, Jones, 1972) and is therefore generated by any given non-Boolean member. We then show that SP-semilattices satisfying the law $P(x) = P^3(x)$ are locally finite, and give a Hasse diagram for the one-generated free SP-semilattice.

Let w_n denote the term $P^2(x_1x_2\cdots x_n)$ and for each $i \leq n$, let $w_{n,i}$ denote the term $P^2(x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n)$.

Lemma 8.1. *Every commutative SP-semigroup satisfies* $w_n \leq w_{n,1}w_{n,2}\cdots w_{n,n}$.

Proof. This follows immediately from repeated applications of the identity $P^2(xy) = P^2(xy)P^2(y)$. □

Lemma 8.2. *If S is an SP-semilattice with fewer than n elements, then S satisfies* $w_n = w_{n,1}w_{n,2}\cdots w_{n,n}$.

Proof. First note that since S has fewer than n elements, for any assignment of elements of S to the variables $x_1, x_2, \ldots, x_n$ there must be two variables x_i and x_j that are assigned the same element of S. Hence w_n takes the same value as $w_{n,i}$ under this assignment and so the value of w_n in S is in fact greater than or equal to the value of $w_{n,1}w_{n,2}\cdots w_{n,n}$ (since this last value is the meet of $w_{n,i}$ with some other elements). This combined with Lemma 8.1 shows that w_n in fact takes the same value as $w_{n,1}w_{n,2}\cdots w_{n,n}$ in S. Since the assignment was arbitrary, the equality $w_n = w_{n,1}w_{n,2}\cdots w_{n,n}$ holds under all assignments. □

Theorem 8.3. *The variety of SP-semilattices is not generated by any finite algebra.*

Proof. By Lemma 8.2 it suffices to prove that for all n, there is an SP-semilattice on which $w_n = w_{n,1}w_{n,2}\cdots w_{n,n}$ fails.

Let n be fixed and let B_n be the 2^n-element Boolean algebra (with top element 1 and bottom element 0) considered as a meet semilattice. Now let $L := \{0, 1\}$ and make B_n L-pseudo-complemented with SP-operation given by $P(x) = 0$ if $x \neq 0$ and $P(0) = 1$. Now B_n has n coatoms, denoted by $a_1, a_2, \ldots, a_n$. These elements are such that $P^2(a_1 a_2 \cdots a_n) = P^2(0) = 0$ while for any $i \leq n$, $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$ is a non-zero element and so

$$P^2(a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n) = 1.$$

Hence under the assignment $x_i \mapsto a_i$, the identity $w_n = w_{n,1} \cdots w_{n,n}$ fails on B_n. □

Corollary 8.4. *The variety of SP-semilattices contains infinitely many subvarieties.*

Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the identity $w_n = w_{n,1} \cdots w_{n,n}$ fails on some finite closable SP-semilattice, while every finite SP-semilattice satisfies $w_n = w_{n,1} \cdots w_{n,n}$ for some n. □

Note that the algebra B_n in the proof of Theorem 8.3 satisfies the law $xP^2(x) = x$ and so the variety of closable SP-semilattices is also not finitely generated and contains infinitely many subvarieties. It will, however, be shown to be locally finite, as is the case for SP-semilattices satisfying $P(x) = P^3(x)$. Some further identities must first be established.

If $t_1, t_2, \ldots, t_n$ are SP-semilattice terms then the term $P(t_1)P(t_2) \cdots P(t_n)$ will be called an *SP-term*, and $t_1, \ldots, t_n$, its *subterms*. Note that every SP-semilattice term can be written in the form wv where w is a semilattice term and v is an SP-term.

Lemma 8.5. *If e is an SP-term then the law $P^2(xP(xe)) = P^2(x)P^3(e)$ holds in any SP-semilattice.*

Proof. Let e be written as $P(t_1) \cdots P(t_n)$. Now

$$\begin{aligned} eP(xP(xe)) &= P(t_1) \cdots P(t_n)P(xP(xP(t_1) \cdots P(t_n))) \\ &= P(t_1) \cdots P(t_n)P(xP(x)) = P(t_1) \cdots P(t_n)P(0) \\ &= P(t_1) \cdots P(t_n) = e. \end{aligned}$$

That is, $e \leq P(xP(xe))$ and therefore $P(e) \geq P^2(xP(xe))$ and $P(e)P^2(x) \geq P^2(xP(xe))P^2(x) = P^2(xP(xe))$. However,

$$\begin{aligned} P^2(xP(xe))P(e)P^2(x) &= P^2(xP(xeP(e)))P(e)P^2(x) \\ &= P^2(xP(0))P(e)P^2(x) = P(e)P^2(x), \end{aligned}$$

showing that $P^2(xP(xe)) \geq P^2(x)P(e)$ also. □

Lemma 8.6. *Let e be an SP-term. Then the identities $P^2(xe) = P^2(xP^2(e)) = P^2(x)P^2(e)$ hold in any SP-semilattice.*

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Proof. First note that $e \le P^2(e)$ and so $xe \le xP^2(e)$ and then $P^2(xe) \le P^2(xP^2(e))$. Also $P^2(xP^2(e)) = P^2(xP^2(e))P^2(e)$ (by Proposition 1.3) which in turn equals $P^2(x)P^2(e)$. To complete the proof, it must be shown that $P^2(xP^2(e)) \le P^2(xe)$. Now, by Lemma 8.5,

$$\begin{aligned} P(xe)P^2(xP^2(e)) &= P(xe)P^2(x)P^2(e) = P^2(xP(xe))P(xe)P^2(e) \\ &= P^2(x)P^3(e)P(xe)P^2(e) = P^2(x)P(xe)P^2(e)P^3(e) = 0, \end{aligned}$$

whence $P^2(xP^2(e)) \le P^2(xe)$ as required. □

Theorem 8.7. *The variety of SP-semilattices satisfying $P(x) = P^3(x)$ is locally finite.*

Proof. Let $X = \{x_1, \ldots, x_n\}$ be a fixed finite alphabet and let $\mathbf{F}_n$ be the relatively free *SP*-semilattice generated by X in the variety of *SP*-semilattices satisfying $P(x) = P^3(x)$. It is now shown that $P(\mathbf{F}_n)$ is a quotient of the free Boolean algebra (as an *SP*-semilattice) on $2^n - 1$ generators (which has $2^{2^{2^n-1}}$ elements). Since every term can be written in the form uv where u is a semilattice term (of which there are $2^n - 1$ choices in the alphabet of X) or empty and v is an *SP*-term or empty (of which there are at most $2^{2^{2^n-1}}$ choices), it follows that there are at most $(2^n) \times (2^{2^{2^n-1}} + 1) - 1$ elements in $\mathbf{F}_n$ (as u and v cannot simultaneously be empty).

To prove that $P(\mathbf{F}_n)$ is a quotient of the free Boolean algebra on $2^n - 1$ generators, it is shown to be a Boolean algebra generated by elements of the form $P(w)$ where w is a semilattice term in X. Since the law $P(x) = P^3(x)$ shows that $P(\mathbf{F}_n)$ is a Boolean algebra, it suffices to show that every *SP*-term u can be reduced using the available axioms to one in which every subterm is either a semilattice term (that is, has no applications of P) or is an *SP*-term.

Let u be an *SP*-term having a subterm of the form vw where v is a semilattice term and w is an *SP*-term. Now the law $P(x) = P^3(x)$ and Lemma 8.6 give $P(vw) = P(P^2(vw)) = P(P^2(v)P^2(w))$. Repeated applications of this procedure reduce u to a term of the desired form. □

It is easy to give slightly improved bounds for the number of elements in $\mathbf{F}_n$. For example if u and v are as in the first part of the proof, then $v \le P(u)$ implies $uv = 0$ which is already considered under the case u is empty and $v = 0$. This reduces the number of choices of v by at least 1 (usually more). This gives at most $(2^n) \times (2^{2^{2^n-1}}) - 1$ elements. It is in fact quite easy to construct $\mathbf{F}_1$ and then to verify that this bound is attained. For $n \ge 2$ this bound is not exact: $P(\mathbf{F}_n)$ is not freely generated since for example $P(xy) \ge P(x)$.

It is not clear if local finiteness continues to hold if the law $P(x) = P^3(x)$ is not present. The law $P^2(xP(xe)) = P^2(x)P^3(e)$ enables terms containing a P^2 to be reduced in the style of the above proof; however terms without double applications of the P operation seem difficult to reduce. However, some further work leads to a complete description of the one-generated free *SP*-semilattice. Already this is considerably more complicated than $\mathbf{F}_1$.

Lemma 8.8. *The following identities hold for all SP-semilattices*:

(i) $P(xP^2(x)) = P^3(x)$;
(ii) $P(x) \leq P(xP^3(x))$ *and* $P^2(x) \leq P(xP^3(x))$;
(iii) $P^2(xP^3(x)) = 0$.

Proof. (i) First, $P(xP^2(x))P^3(x) = P(xP^2(x)P^3(x))P^3(x) = P(0)P^3(x) = P^3(x)$. But then we also have

$$P^3(x)P(xP^2(x)) = P(P^2(x)P(xP^2(x)))P(xP^2(x))$$
$$= P(P^2(x)P(x))P(xP^2(x)) = P(0)P(xP^2(x)) = P(xP^2(x)).$$

(ii) Now $P(xP^3(x))P(x) = P(xP^3(x)P(x))P(x) = P(x)$ and also

$$P(xP^3(x))P^2(x) = P(xP^3(x)P^2(x))P^2(x) = P^2(x).$$

(iii) By Lemma 8.6, $P^2(xP^3(x)) = P^2(x)P^3(x) = 0$. □

Next is an attempt to list the possibly distinct *SP*-semilattice terms on one variable. Starting from the single generator x, there are three terms generated by applying P repeatedly: $P(x), P^2(x)$ and $P^3(x)$ (with $P(x) \leq P^3(x)$). Taking products gives $0, xP^2(x)$ and $xP^3(x)$. Now applying P and (i) and (iii) of Lemma 8.8 gives only two new terms: $P(0)$ and $P(xP^3(x))$ (with $P(0)$ a multiplicative identity for all elements of the form $P(w)$ for some w, and with $P(xP^3(x)) \geq P^2(x), P(x)$). Taking products gives only two new terms: $xP(0)$ and $xP(xP^3(x))$. Finally, consider possible applications of P to these last terms. Now $P(xP(0)) = P(x)$, which is already listed above. Also, $P(xP(xP^3(x)))P^3(x) = P(xP(x))P^3(x) = P^3(x)$, and since $P(xP^3(x)) \geq P^2(x)$, it follows that $xP(xP^3(x)) \geq xP^2(x)$, and so $P(xP(xP^3(x))) \leq P(xP^2(x)) = P^3(x)$. Therefore, $P(xP(xP^3(x))) = P^3(x)$ and so no new terms are obtained at all. This shows that the one-generated free algebra has at most 11 elements. To show that no further collapse takes place, a one-generated *SP*-semilattice with 11 elements is given in Fig. 1 (in which elements of the form $P(a)$ for some a are denoted by hollow vertices).

In fact, it follows that Fig. 1 depicts the Hasse diagram for the one-generated free *SP*-semilattice. The one-generated *SP*-semilattice freely generated within the subvariety defined by $xP(0) = x$ is easily seen to be obtained by collapsing the element x in Fig. 1 onto $xP(0)$.

Note in comparison that the free *PC*-semilattice on one generator has only five elements $(x, x^*, x^{**}, 0, 0^* = 1)$, though this number increases rather rapidly as the number of generators increases: the n-generated free *PC*-semilattice has exactly $1 + \sum_{k=0}^{n} \binom{n}{k}(2^{2^{n-k}} - 1)$ elements (see Balbes, 1973).

Corollary 2.2 combined with Theorem 5.10 of McKinsey and Tarski (1944) shows that the real numbers $\mathbb{R}$ given the usual topology contains a subset x for which each of the ten distinct terms in the one-generated $\{x = xP(0)\}$-free semilattice give distinct sets (where for $A \subseteq \mathbb{R}$, interpret $P(A)$ as A'°). The set A described in

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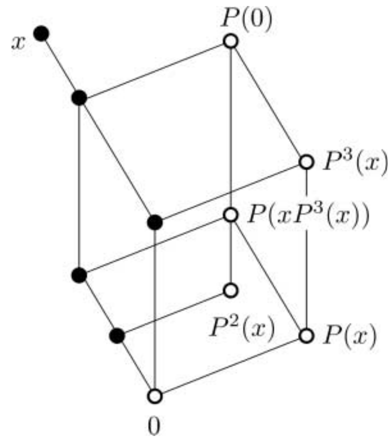

Figure 1. The free *SP*-semilattice on one generator.

Fig. 12 of Steen and Seebach (1970) is one such example. To get a copy of the free *SP*-semilattice in Fig. 1 using subsets of $\mathbb{R}$, let x correspond to $A \cup \{r\}$ where $r < 0$ or $r > 8$ and interpret interior with respect to the subspace $[0, 8]$.

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