Finitely Based, Finite Sets of Words

Marcel Jackson and Olga Sapir

Abstract

For W a finite set of words, we consider the Rees quotient of a free monoid with respect to the ideal consisting of all words that are not subwords of W. This monoid is denoted by $S(W)$. It is shown that for every finite set of words W, there are sets of words $U \supset W$ and $V \supset W$ such that the identities satisfied by $S(V)$ are finitely based and those of $S(U)$ are not finitely based (regardless of the situation for $S(W)$). The first examples of finitely based (not finitely based) aperiodic finite semigroups whose direct product is not finitely based (finitely based) are presented and it is shown that every monoid of the form $S(W)$ with fewer than 9 elements is finitely based and that there is precisely one not finitely based 9 element example.

1 Introduction

This paper is a combination of results obtained independently by the two authors.

A semigroup S (indeed, any algebra) is said to be *finitely based* (FB) if the set, $Id(S)$, of identities it satisfies can be derived from a finite subset of $Id(S)$. Otherwise it is said to be *not finitely based* (NFB). A locally finite semigroup is said to be *inherently not finitely based* (INFB) if it is NFB and any locally finite variety containing it is also NFB. There exist semigroups with every consistent combination of these properties: FB; INFB; and NFB but not INFB. The term "*weakly finitely based*" (WFB) has been introduced in [1] to denote those locally finite algebras that are not INFB. Likewise a semigroup will be called "*weakly not finitely based*" (WNFB) if it is NFB but not INFB (that is, from the intersection of the class of WFB semigroups and the class of NFB semigroups).

The classes of FB and INFB finite algebras were shown to be recursively inseparable by R. McKenzie ([2]), giving a negative solution to one of the most famous problems in the study of varieties, Tarski's Finite Basis Problem. For semigroups there is a very large volume of work devoted to investigating the finite basis problem (see [10] for example) and in contrast with McKenzie's result, a powerful description of the INFB finite semigroups has been obtained by M. V. Sapir ([6] and [7]). Algorithmically classifying the classes of FB and WNFB semigroups still remains a difficult and unsolved problem.

In this paper we investigate an interesting class of finite aperiodic semigroups (that is, semigroups with only trivial subgroups) whose identities are very simple to describe yet exhibit some complicated behavior.

Definition 1.1 *(i) Let X^* be a free monoid on some set of generators X and W be a finite, nonempty set of elements of the free semigroup X^+. Let $S(W)$ be the Rees quotient $X^*/I(W)$, where $I(W)$ is the ideal of X^* consisting of all elements of X^* that are not subwords of W. $S(W)$ is a finite monoid with zero.*
(ii) If $S(W)$ is FB then we will say that W is a FB set of words (or W is FB). Otherwise W is a NFB set of words. If W contains just one word w then we will say that w is a FB (or NFB) word if $W = \{w\}$ is a FB (or NFB, respectively) set of words.

The identities of semigroups of the form $S(W)$ have been of interest since P. Perkins ([3]) showed that $S(\{abcba, acbab, abab, aab\})$ was NFB, one of the first examples of a finite NFB semigroup. It is clear from the results in [6] and [7] however that a semigroup $S(W)$ is never INFB. This means that there does exist a FB, locally finite variety containing $S(\{abcba, acbab, abab, aab\})$ and it is therefore natural to ask whether this FB, locally finite variety can be generated by a semigroup of the form $S(V)$ for some finite set of finite words V. More generally we may ask:

Question 1.2 *(i) If W is a finite set of finite words, are there finite sets of finite words U, V such that $W \cup V$ is FB and $W \cup U$ is NFB?*
(ii) Conversely, do there exist finite sets of words W such that V is FB (or NFB) whenever $V \supseteq W$?

Another natural question (essentially Question 7.1 of [10]) is the following:

Question 1.3 *Which finite sets of words are FB?*

A partial solution to Question 1.3 has been obtained by the second author of this paper:

Theorem 1.4 *(O. Sapir, [9]) If w is an element of $\{a, b\}^*$ then w is a FB word if and only if it is one of the following words: $a^n b^m$, $b^n a^m$, $a^n b a^m$, or $b^n a b^m$ for some n and m.*

This shows that "most" words in a two letter alphabet, are NFB! On the other hand, results in this paper obtained independently by both of the authors show that the general solution to Question 1.3 is likely to be very complicated. In particular we show that for any finite set of finite words W one can find finite sets $V_1, V_2, \ldots$ of finite words with $W \subseteq V_1 \subset V_2 \subset \ldots$ such that V_{2i} is FB and V_{2i-1} is NFB for each $i > 0$. Thus we have a positive solution to Question 1.2 part (i) and consequently a negative solution to part (ii). Also proved are a number of general non-finite basis results for monoids all of which have applications to the finite basis problem for sets of words. Finally we show that there are finite sets of finite words U_1, U_2 and V_1, V_2 such that $S(U_1)$, $S(U_2)$ are FB, $S(V_1)$, $S(V_2)$ are NFB but $S(U_1) \times S(U_2)$ is NFB and $S(V_1) \times S(V_2)$ is (as will be shown, this is equivalent to $U_1 \cup U_2$ being NFB and $V_1 \cup V_2$ being FB). These provide the first examples of FB (or NFB) aperiodic semigroups generating varieties whose join is NFB (or FB respectively).

2 Preliminaries

Elements of a free semigroup will be referred to as *words* and the equality relation on a free semigroup will be denoted "$\equiv$". The *length* of a word w will be the number of (not necessarily distinct) letters appearing in w. Unless otherwise stated, in all arguments to follow we will take "W is a set of words" to mean "W is a finite subset of X^+, the free semigroup on an alphabet X". It is apparent from Definition 1.1 that we may regard $S(W)$ as consisting of 0 and 1 along with all subwords of W, and with the obvious multiplication.

Definition 2.1 *Let w be a word. Then $occ(x, w)$ is the number of occurrences of the letter x in w and $c(w)$ is $\{x : occ(x, w) > 0\}$, the set of all letters occurring in w (the content of w). If $W = \{w_1, \ldots, w_n\}$ is a finite set of words then $c(W) = \bigcup_{i=1}^n c(w_i)$.*

An *identity* is an expression $u \approx v$ where u and v are words and a semigroup S will be said to *satisfy* $u \approx v$ (written $S \models u \approx v$) if for every assignment, θ, of elements of S to the letters in $c(u) \cup c(v)$, $\theta(u)$ takes the same value in S as $\theta(v)$. The set of all identities satisfied by a semigroup S will be denoted by $Id(S)$.

If Σ is a set of identities then we will say that $u \approx v$ can be *derived* from Σ (written $\Sigma \vdash u \approx v$) if there is a sequence of words $u \equiv u_1, u_2, \dots u_{n-1}, u_n \equiv v$ in an alphabet X and homomorphisms $\theta_i : X^+ \to X^+$ so that $u_i \equiv u_i'\theta(p_i)v_i'$ and $u_{i+1} \equiv u_i'\theta(q_i)v_i'$ for some (possibly empty) words u_i' and v_i' and some identity $p_i \approx q_i \in \Sigma$. The homomorphisms θ_i are called *substitutions* and the number $n-1$ is called the *length* of the derivation of $u \approx v$ from Σ.

Definition 2.2 *(i) If $occ(x, w) = n$ then x is said to be* n-occurring *in w. If $m > n$ then we will say x is* less than m-occurring *in w and if $m < n$ then we will say x is* more than m-occurring *in w. The letter x is n-occurring in an identity $p \approx q$ if x is n-occurring in both of the words p and q. The identity is said to be balanced if for every letter x, $occ(x, p) = occ(x, q)$.*
(ii) A word w is n-limited *if $occ(x, w) \leq n$ for all letters x. An identity $p \approx q$ is n-limited if both p and q are n-limited words.*
(iii) The expression ${}_ix$ means the i^{th} occurrence of a letter x in a word.

In the special case when a letter x is 1-occurring in a word w we will also say that x is a *linear letter* in w. Very often it will be necessary to extend some property of words to identities. This will done using the following definition.

Definition 2.3 *If L is a set of words then an identity $u \approx v$ is an* L-identity *if both u and v are words in L.*

So if an *almost linear* word is defined to be one with at most one nonlinear letter, then an almost linear identity $u \approx v$ is one in which both u and v are almost linear words.

Definition 2.4 *If $c(w) = \{x_1, \dots, x_n\}$ then $w(x_{k+1}, x_{k+2}, \dots, x_n)$ is the word obtained from w by assigning 1 to each of the letters in $\{x_1, \dots, x_k\}$. In this case we will say that "w deletes to $w(x_{k+1}, x_{k+2}, \dots, x_n)$". If $p \approx q$ is an identity, τ, with $c(p) = c(q) = \{x_1, \dots, x_n\}$ then we will say $p \approx q$ deletes to $p(x_{k+1}, x_{k+2}, \dots, x_n) \approx q(x_{k+1}, x_{k+2}, \dots, x_n)$ or τ deletes to $\tau(x_{k+1}, x_{k+2}, \dots, x_n)$.*

Since $S(W)$ is always a monoid with zero element, $S(W) \models p \approx q$ implies that $c(p) = c(q)$ and that every identity that $p \approx q$ deletes to is also an identity satisfied by $S(W)$. Because of this, in the arguments to follow we will tacitly assume that all sets of identities are closed under deletion.

Definition 2.5 *A pair of letters (x, y) in an identity $p \approx q$ is called* stable *if $p(x, y) \equiv q(x, y)$. If (x, y) is not stable in $p \approx q$ we will say it is* unstable *in this identity. A pair of letters is stable in a word w with respect to a semigroup S if $S \models w \approx v$ implies (x, y) is stable in $w \approx v$.*

Naturally, if every pair of letters are stable in an identity then that identity is a tautology (trivial identity). We can define a similar notion of stability for pairs of the form $({}_ix, {}_jy)$.

Definition 2.6 *A pair $({}_ix, {}_jy)$ is* stable *in an identity $u \approx v$ if the order of appearance of the i^{th} occurrence of x and the j^{th} occurrence of y is the same in both u and v. If $({}_ix, {}_jy)$ is not stable in $u \approx v$ then we will say it is* unstable *in this identity. An unstable pair $({}_ix, {}_jy)$ is a* critical pair *for $u \approx v$ if it is unstable in $u \approx v$ and $({}_ix)({}_jy)$ is a subword of u. The set of all unstable pairs $({}_ix, {}_jy)$ in an identity $u \approx v$ is denoted* **Chaos**$(u \approx v)$.

Definition 2.7 *A word p is an isoterm relative to a set of identities Σ if $\Sigma \vdash p \approx q \Rightarrow p \equiv q$. When referring to a specific semigroup S, a word will be said to be an isoterm for S if it is an isoterm for $Id(S)$, the set of all identities satisfied by S.*

Note that if w is a subword of a word in W then w is an isoterm for the identities of $S(W)$ though there may be many isoterms for a monoid $S(W)$ that are not subwords of the set W. For example xx, xxy and yxx are all isoterms for $S = S(\{abb, aab\})$ since they are equivalent, up to a change in letters names to the words bb, aab and abb, all of which are words or subwords of words in the set $\{abb, aab\}$. However xyx is also an isoterm for S since if S satisfies an identity $xyx \approx w$ for some word w then because xx is an isoterm for S and xyx is 2-limited, $occ(x, w) = 2$ and $occ(y, w) = 1$. Since xxy and yxx are both isoterms for S, w must be xyx.

3 FB Finite Sets of Words

In this section we find finite bases for a number of sets of words that will be important in Sections 4 and 5. We will be considering words with a large number of linear (1-occurring) letters. To avoid unnecessary confusion with indices, we will always assume t stands for a linear letter and that different occurrences of t in a word have different indices which we shall omit. For example by the word $xtytabct$ we will mean the word $xt_1yt_2abct_3$. We will use $\{t\}$ to denote the set of all linear letters in a word and subwords between successive linear letters in a word will be called *blocks*.

Definition 3.1 *If w is a word $w_1w_2 \dots w_n$ (the w_i are not necessarily distinct letters) then $\lfloor w, t \rfloor$ is the word $w_1tw_2tw_3t \dots w_nt$, where different occurrences of t, as usual, represent distinct linear letters. Likewise $\lfloor t, w \rfloor$ is the word $tw_1tw_2tw_3t \dots w_n$.*

Definition 3.2 *(i) A critical pair $({}_ix, {}_jy)$ of an identity $U \approx V$ is said to be $u \approx v$-*removable *if after applying $u \approx v$ to U we obtain an identity $U' \approx V$ such that* $\mathbf{Chaos}(U' \approx V)$ *does not contain $({}_ix, {}_jy)$ and is a subset of* $\mathbf{Chaos}(U \approx V)$.
(ii) If Σ is a set of identities, we say that a critical pair is Σ-removable if it is σ-removable for some $\sigma \in \Sigma$.

Definition 3.3 *Let S be a semigroup and Σ be a subset of $Id(S)$. We say that a set of identities Γ complements Σ in $Id(S)$ if all identities of S follow from Σ and Γ.*

Definition 3.4 *Let S be a semigroup. If L is a set of words with the property that for any $u \in L$, $S \models u \approx v$ implies $v \in L$ then we say that L is a closed set with respect to S.*

The following lemma from [9] gives us the most economic way of proving that a monoid is FB.

Lemma 3.5 *([9]) Let S be a semigroup satisfying a set of identities Σ. Let L be a closed set for S and suppose that the set of all L-identities of S complements Σ in $Id(S)$. Suppose also that any nontrivial L-identity of S contains a Σ-removable critical pair. Then Σ contains a basis of identities for S.*

Let $\mathcal{A}_n$ denote the system of two identities:

$$\{x^n \approx x^{n+1}, t_1xt_2xt_3x...t_nx \approx x^nt_1t_2...t_n\}$$

and let W_n be the set of all n-limited words in two letters.

Theorem 3.6 *Let S be a monoid satisfying $\mathcal{A}_{n+1}$ for some n and suppose that every word in W_n is an isoterm for S. Then $\mathcal{A}_{n+1}$ is a finite basis of identities for S.*

Proof. Let L be the set of all n-limited words. Since x^n is an isoterm of S, L is a closed set for S. The identities $\mathcal{A}_{n+1}$ can be used to reduce every word w to an $(n+1)$-limited word of the form $w_1 w_2$ where w_1 is n-limited and w_2 is of the form $x_1^{n+1} x_2^{n+1} \dots x_m^{n+1}$ for

$$\{x_1, \dots, x_m\} = \{x \in c(w) : occ(x, w) > n\},$$

the set of more than n-occurring letters in w. Therefore the set of all n-limited identities of S complements $\mathcal{A}_n$ in $Id(S)$. Since all words in W_n are isoterms and S is a monoid, all L-identities are trivial. Thus all conditions of Lemma 3.5 are satisfied.

□

No letter appears in a word from W_n more than n times and therefore $S(W_n) \models \mathcal{A}_{n+1}$. This implies the following

Corollary 3.7 *Let S be a monoid satisfying $\mathcal{A}_n$ for some $n > 0$. Then the identities $\mathcal{A}_n$ are a finite basis for $Id(S \times S(W_{n-1}))$.*

Taking S to be a trivial monoid, gives the following corollary.

Corollary 3.8 *The set of words W_n is FB.*

Another obvious corollary is:

Corollary 3.9 *Let S be a semigroup (or finite semigroup) satisfying the set of identities $\mathcal{A}_n$ for some n. Then S is a subsemigroup of a FB semigroup (or a FB, finite semigroup respectively).*

A semigroup is said to be k-nilpotent if the product of any k elements is 0 and a monoid is said to be k-nilpotent if it is a k-nilpotent semigroup with adjoined identity element. It is clear that if S is a k-nilpotent monoid then S satisfies the conditions of Corollary 3.9, with $n = k$ and so is a subsemigroup of a finitely based finite semigroup. However the direct product of S with $S(W_k)$ is not a nilpotent monoid (it has identity element $(1, 1)$ but $(1, 0)$ is also an idempotent). An alternative construction is as follows. Since S and $S(W_k)$ are nilpotent monoids, $\bar{S} = S \backslash \{1\}$ and $\bar{S}(W_k) = S(W_k) \backslash \{1\}$ are nilpotent semigroups. Now consider the semigroup $\bar{T}$ on the set

$$(\bar{S} \backslash \{0\}) \cup (\bar{S}(W_k) \backslash \{0\}) \cup \{0\}$$

with multiplication within the subsets $\bar{S}$ and $\bar{S}(W_k)$ unchanged and all other products equalling zero (this construction is called the 0-direct join of $\bar{S}$ with $\bar{S}(W_k)$). Finally let T be the semigroup $\bar{T}$ with adjoined identity element. It is clear that T contains both S and $S(W_k)$ as submonoids and that T is a $(2k+1)$-nilpotent monoid (since the longest word in W_k is $2k$ letters long). Finally Theorem 3.6 shows that T is FB. Thus we have shown the following

Corollary 3.10 *The pseudovariety generated by the class of finite, FB, nilpotent monoids (that is, the closure of this class under taking subsemigroups, homomorphic images and finite direct products) contains all nilpotent monoids and nilpotent semigroups.*

The next result uses the fact that the words in W_n are capable of "dominating" smaller collections of words.

Corollary 3.11 *If W is a set of words then there is a finite set of words $W' \supseteq W$ involving no more than $|c(W)|$ letters such that W' is a finitely based set of words.*

Proof. If W is a set of words in one letter, then $S(W)$ is commutative and therefore already finitely based (see [3]). Assume then that $c(W)$ contains two letters a and b. Let n be the maximal number of times a letter appears in words in W and take W' to be the union of W and W_n. Then any word in W_n is an isoterm for $S(W')$ and $S(W')$ satisfies $\mathcal{A}_n$. By Theorem 3.6, $S(W')$ is FB.

$\square$

Theorem 3.12 *The set*

$$\Sigma = \{t_1xt_2xt_3x \approx x^3t_1t_2t_3, x^2 \approx x^3, xxt \approx txx, xt_1xyt_2y \approx xt_1yxt_2y\}$$

is a finite basis for the identities of $S \equiv S(\{abcab, abcba\})$.

Proof. We will use Lemma 3.5. First note that S does indeed satisfy Σ. Now let L be the set of all 2-limited words u with the property that for each 2-occurring letter x in u, u deletes to xtx for some linear letter t. If $u \approx v$ is an identity of S and u is an L-word then since xtx is an isoterm of S, v must be also an L-word. So L is closed with respect to S. We now show that every word w can be transformed by Σ to a word of the form $x_1^2...x_n^2u$ where u is an L-word and does not contain any of the letters $x_1, x_2, ..., x_n$.

Firstly, if x occurs more than 3 times in the word u then we may apply the identity $t_1xt_2xt_3x \approx x^3t_1t_2t_3$ to move all occurrences of it to the left. Applying $x^3 \approx x^2$ then reduces the number of occurrences of x to 2. Thus for any word w, $\Sigma \vdash w \approx w'$ where w' is 2-limited.

Now say that x is 2-occurring in a 2-limited word w and that there is no linear letter t in w for which $w(x,t) \equiv xtx$. So $w \equiv AxBxC$ for some words A, B and C where every letter in B is 2-occurring in w. If B is empty then we may apply $txx \approx xxt$ to move x to the left as required. If B is not empty then we have the following cases.

Case 1. There is a letter y that is 2-occurring in w such that $xBx \equiv xD({}_2y)Ex$ where D contains only first occurrences of letters 2-occurring in w (this includes the situation where E is empty and x is y).
In this case we may move y leftward out of B using $xt_1xyt_2y \approx xt_1yxt_2y$ and $xxt \approx txx$.

Case 2. There is a letter y that is 2-occurring in w such that $xBx \equiv xD({}_1y)Ex$ where E contains only second occurrences of letters 2-occurring in w.
In this case we may move y to the right using $xxt \approx txx$ or $xt_1yxt_2y \approx xt_1xyt_2y$.

In each case the length of B is reduced. Therefore by repeating these steps a word in which xx is a subword is eventually obtained. Both occurrences of the letter x can now be moved to the far left hand end of the word using the identity $xxt \approx txx$. Since this can be done for all 2-occurring letters x in w such that w does not delete to xtx for some t, we have shown (for some n) $\Sigma \vdash w \approx x_1^2...x_n^2u$ where u is in L as required. So if $w \approx v$ is an identity satisfied by S then we may use Σ to derive $w \approx x_1^2...x_n^2u_1$ and $v \approx x_1^2...x_n^2u_2$ where u_1 and u_2 are in L. Since u_1 and u_2 do not contain x_i for $i \leq n$, S must satisfy the L-identity $u_1 \approx u_2$ Therefore L-identities complement Σ in $Id(S)$ and so the first part of the Lemma is satisfied.

In order to complete the proof we are going to show that any critical pair $({}_ix,{}_jy)$ in an L-identity $u \approx v$ is removable by applying Σ to one of the words u or v.

Let $u \approx v$ be an L- identity of S. Critical pairs of the form $({}_2x,{}_1y)$ and $({}_1x,{}_2y)$ are removable by applying the identity $xt_1xyt_2y \approx xt_1yxt_2y$. If $u \approx v$ contains a critical pair of the form $({}_1x,{}_1y)$ then without loss of generality we may assume that $u \equiv AxyBxCyD$ or $AxyByCxD$ for some words A, B, C, D. Since u is in L, B must contain a linear letter, t. But then we can assign a to x, b to y, c to t and 1 to all other letters and u takes the value $abcba$ or $abcab$ which are isoterms for $S(\{abcba, abcab\})$. This contradicts the assumption that $({}_1x,{}_1y)$ was a critical pair and therefore such critical pairs do not exist in $u \approx v$. The case for critical pairs of the form $({}_2x, {}_2y)$ follows by the symmetry of the set $\{abcba, abcab\}$.

Similarly we can show that there are no critical pairs of the form $({}_1x, t)$, $({}_2x, t)$, $(t, {}_1x)$, or $(t, {}_2x)$ (t is a linear letter as usual) since there is a linear letter between every 2-occurring letter in $u \approx v$ and xtx is an isoterm. Thus every L-identity contains a $\{xt_1xyt_2y \approx xt_1yxt_2y\}$-removable critical pair and so by Lemma 3.5, $S(\{abcba, abcab\})$ is FB.

□

To give the shortest proof of the next theorem, we will use a further result from [9].

Definition 3.13 *Let U and V be words from $\{x, y\}^*$. An identity $u \approx v$ is said to be (U, V)-pseudocommutative if*

$$u \equiv \lfloor U, t\rfloor xy\lfloor t, V\rfloor,\ v \equiv \lfloor U, t\rfloor yx\lfloor t, V\rfloor.$$

Definition 3.14 *Let S be a semigroup.*
(i) A pair $({}_ix, {}_jy)$ in a word $u \equiv U({}_ix{}_jy)V$ is said to be commutative in u modulo S if S satisfies the $(U(x, y), V(x, y))$-pseudocommutative identity.
(ii) A critical pair in an identity $u \approx v$ is said to be commutative modulo S if it is commutative in u or in v modulo S.

Recall that a word is called *almost linear* if it contains at most one nonlinear letter.

Lemma 3.15 *([9]) Let W be a finite set of words and let n be the maximum number of times that any letter appears in a word from W. Suppose that all words in W_k are isoterms of $S(W)$. Let L be the set of all n-limited words containing no more than one more than k-occurring letter. Suppose that each L-identity of $S(W)$ contains a commutative critical pair modulo $S(W)$ or a τ-removable critical pair for some almost linear identity $\tau \in Id(S)$. Then W is a finitely based set of words.*

Theorem 3.16 *Let*

$$U = \{abbaa, ababa, aabba\}$$

and

$$V = \{baaab, aabb, abba, abab\}.$$

Then U and V are FB sets of words.

Proof. Let $S = S(U)$ and $T = S(V)$. We will check the conditions of Lemma 3.15. First notice that each letter occurs in U and V no more than 3 times and that all words in W_2 are isoterms for both $S(U)$ and $S(V)$. So for both S and T, the set L described in Lemma 3.15 is the set of all 3-limited words containing no more than one 3-occurring letter. Let x denote the single 3-occurring letter in an L-identity $u \approx v$. The remaining (less than 2-occurring) letters in $u \approx v$ we will denote by $y_i, i = 1, ..., m$. Since every word in W_2 is an isoterm for both S and T, the identity $u(y_1, y_2, ..., y_m) \approx v(y_1, y_2, ..., y_m)$ must be a tautology and $u(y_1, ..., y_m)$ must be an isoterm for both S and T.

Let c_k denote the k^{th} (not necessarily distinct) letter to appear in the 2-limited word $u(y_1, y_2, ..., y_m)$ (so that the initial segment of $u(y_1, y_2, ..., y_m)$ to the left of c_k has length $k-1$). Now for a 3-occurring letter x_i in u, define ${}_jx$ to have coordinate k in u if the j^{th} occurrence of x in u lies between c_k and c_{k+1}. If ${}_jx$ is before c_1 (or after $c_{|u(y_1,...,y_m)|}$) then the coordinate of ${}_jx$ is 0 (or $|u(y_1, ..., y_m)|$ respectively).

To verify the last condition of Lemma 3.15 for S, we will use the following easily verified properties of S:

S1: Let x be 3-occurring in an identity $u(x,y) \approx v(x,y)$ and y be less than 3-occurring in $u(x,y) \approx v(x,y)$. If $u(x,y)$ is *not* one of the words $yxxyx$, $xyxxy$, $xxxy$, $yxxx$, any critical pair is commutative in u.
S2: The critical pairs $({}_1y,{}_1x)$ in $xyxxy \approx yxxyx$ and $({}_2y,{}_3x)$ in $yxxyx \approx xyxxy$ are commutative.
Now let $u \approx v$ be a non trivial L-identity with $occ(x,u) = 3$. S1 and S2 show that any critical pair of the type $({}_1x,{}_iy)$, $({}_iy,{}_2x)$, $({}_2x,{}_iy)$ or $({}_iy,{}_3x)$ is commutative ($i \in \{1,2\}$). Denote this fact by S3. Now since $u \approx v$ is a nontrivial L-identity and W_n are all isoterms for S we can assume without loss of generality that one of the following holds:
Case 1. The coordinate of ${}_1x$ is smaller in u than in v.
Case 2. The coordinate of ${}_3x$ is smaller in u than in v.
Case 3. Both ${}_1x$ and ${}_3x$ have the same coordinates in both u and v but the coordinate of ${}_2x$ is smaller in u than in v.
In Case 1, $u \approx v$ either contains a critical pair of the form $({}_1x,{}_iy)$ or u contains the subword $({}_1x)({}_2x)$. Property S3 shows that the critical pair $({}_1x,{}_iy)$ is commutative. If u contains the subword $({}_1x)({}_2x)$ then either v contains the subword $({}_1x)({}_2x)$ or $v \approx u$ contains a critical pair of the form $({}_iy,{}_2x)$. This critical pair is also commutative by property S3. If both u and v contain the subword $({}_1x)({}_2x)$ then either v contains the subword xxx or $u \approx v$ contains a critical pair of the form $({}_2x,{}_iy)$ which is commutative by property S3. If v contains xxx then either u contains xxx or $u \approx v$ contains a critical pair of the form $({}_iy,{}_3x)$ which is commutative by S3. Finally if both u and v contain xxx then all critical pairs involving x are removable using the identity $xxxt \approx txxx$. Thus in Case 1, the identity $u \approx v$ contains a commutative or a $x^3t \approx tx^3$-removable critical pair. The situation described in Case 2 follows by symmetry. In Case 3 $u \approx v$ must contain a critical pair of the form $({}_2x,{}_iy)$ which is commutative by property S3. Thus any L-identity of S contains either commutative or $x^3t \approx tx^3$-removable critical pair.

To check that every L- identity of T contains a commutative critical pair, consider the following easily verified properties of T:
T1. Let x be 3-occurring in a word $u(x,y)$ and y be less than 3-occurring in $u(x,y)$. If $u(x,y)$ is *not* one of the words $yxxyx, xyxxy$, any critical pair is commutative.
T2: The critical pairs $({}_1x,{}_1y)$ in $xyxxy \approx yxxyx$ and $({}_2y,{}_3x)$ in $yxxyx \approx xyxxy$ are commutative.
Thus if $u \approx v$ is a L-identity for T then any of the critical pairs $({}_iy,{}_1x)$, $({}_iy,{}_2x)$, $({}_2x,{}_iy)$ or $({}_3x,{}_iy)$ are commutative. The rest of the proof is similar to the above proof for S.

□

4 NFB Finite Sets of Words

Lemma 4.1 *Let S be a monoid such that xy is an isoterm of S. Let u be an isoterm of S. Erasing a prefix (suffix) of a block in u gives a new isoterm for S.*

Proof. Let $\{t_1, ..., t_k\}$ be the set of all linear letters in u. Erase a part B' of a block $B \equiv B'B''$ between linear letters t_i and t_{i+1} in u and denote the resulting word by v. Suppose that there exists a nontrivial identity $v \approx w$ satisfied by S. Since xy is an isoterm, w has the same pattern of linear letters as v (otherwise for some j, k, $v(t_j,t_k) \approx w(t_j,t_k)$ would be the identity $t_jt_k \approx t_kt_j$). Since the identity $v \approx w$ is not trivial, w contains a block which does not match the corresponding block in v. Consider the substitution θ which takes t_i to t_iB' and is identical on all other letters. Then $\theta(v) \equiv u$ and $\theta(w)$ contains a block which does not match the corresponding block in u. Therefore, $u \approx \theta(w)$ is a nontrivial identity which contradicts the fact that u is an isoterm. Therefore no such word w exists and v is an isoterm.

□

The following lemma will also be useful.

Lemma 4.2 *Let u be an isoterm for a monoid S that contains a linear letter t_1. If $u \not\equiv t_1$, then the word, v, obtained by adding a linear letter t_2 immediately to the left (or right) of the occurrence of t_1 in u is also an isoterm.*

Proof. Since $occ(t_1, u) = 1$ and $u \not\equiv t_1$, the word xy must be an isoterm. Let v be as in the statement of the lemma. If $v \approx w$ is a nontrivial identity satisfied by S then any unstable pair in $v \approx w$ must include the letter t_2 and not the letter t_1 (note that if (t_1, t_2) was unstable then S would satisfy $t_1t_2 \approx t_2t_1$ which is not the case). Let (x, t_2) be such a pair. Now the word obtained from v by deleting t_1 is equivalent to u up to a change of letter names and therefore is an isoterm. This contradicts the fact that (x, t_2) is an unstable pair in $v \approx w$. Thus no such w exists and v is an isoterm for S.

□

Definition 4.3 *Let $X = \{x_1, x_2, \ldots\}$. Then $[Xn]$ and $[nX]$ denote the words $x_1x_2 \ldots x_n$ and $x_nx_{n-1} \ldots x_1$ respectively. We will use $[\mathcal{X}(2n)]$ to denote the word*

$$x_2x_4 \ldots x_{2n}x_1x_3 \ldots x_{2n-1}.$$

Lemma 4.4 *Let S be a monoid. Suppose that*

(1) for some words A, B and C from the monoid $\{xyt, yxt\}^*$, the word $U \equiv AyxtBxytC$ is an isoterm of S,

(2) for each $n = 1, 2, \ldots$, the semigroup S satisfies the identity τ_n:

$$\phi(A)y[Xn][Yn]xt\phi(B)[nX]xy[nY]t\phi(C) \approx \phi(A)[Yn]yx[Xn]t\phi(B)x[nY][nX]yt\phi(C),$$

where $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$ and ϕ is a substitution (that is, $\phi : \{xyt, yxt\}^* \to X \cup Y \cup \{x, y, t\}$ is a homomorphism) defined by

$$\phi(xyt) \equiv xt[Xn]t[Yn]tyt, \ \phi(yxt) \equiv yt[nY]t[nX]txt.$$

Then S is NFB.

Proof. Let L_n be the left hand side of τ_n and R_n be the right hand side. The following properties of τ_n are easy to check:
P1: The only unstable pairs of letters in τ_n are: (x_i, y_j), (x, x_i), (y, y_j) for $i, j = 1, \ldots, n$.
P2: If (z_1, z_2) is an unstable pair of letters, then $\tau_n(z_1, z_2, \{t\})$ is essentially the following identity modulo renaming letters (that is they differ only by names of letters): $A'xytB'xytC' \approx A'yxtB'yxtC'$, where deleting some linear letters from A', B', C' gives us A, B, C respectively.
P3: If the pair (z_1, z_2) belongs to the set $\{(x_i, y), (x, y), (x, y_i), (x_i, x_j), (y_i, y_j); i, j = 1, \ldots, n, \ i \neq j\}$, then $L_n(z_1, z_2, \{t\})$ is essentially the isoterm U with some extra linear letters added adjacent to existing linear letters in U. So by Lemma 4.2 $L_n(z_1, z_2, \{t\})$ is an isoterm for S and therefore $L_n(z_1, z_2, \{t\}) \equiv R_n(z_1, z_2, \{t\})$.

Fix m and let $n = 2m$. Let us show that there is no derivation of τ_n involving identities of length less than m (where the length of an identity is the maximum of the lengths of its two sides).

Claim 1. If $L_n \approx w$ is a nontrivial identity of S, then $w \equiv R_n$.
Clearly w has the same pattern of linear letters as L_n (otherwise deleting all but an unstable

pair of linear letters from τ_n would give an identity of the form $t_i t_j \approx t_j t_i$, contradicting the fact that U is an isoterm). Now Lemma 4.1 implies $U(x, \{t\})$ is an isoterm. Since $L_n(x, \{t\})$, $L_n(y, \{t\})$, $L_n(x_i, \{t\})$ and $L_n(y_i, \{t\})$ are all equivalent to $U(x, t)$ modulo renaming letters, each block in w is a permutation of the corresponding block in L_n. Condition P3 implies that each block $[Xn]$, $[nX]$, $[Yn]$ or $[nY]$ is identical to the corresponding block in w. Therefore nontrivial permutation can only happen in blocks of w which correspond to blocks $B_1 \equiv y[Xn][Yn]x$ and $B_2 \equiv [nX]xy[nY]$ in L_n. Call these blocks of w B_1' and B_2'. Condition P3 implies that letters in $y[Xn]$, $[Yn]x$, $[nX]y$ and $x[nY]$ do not commute within $L \approx w$. So, the block B_1' must be a shuffle of $y[Xn]$ and $[Yn]x$ and B_2' must be an interleaving of $[nX]y$ and $x[nY]$.

Condition P3 implies that $L_n(x, y, \{t\})$ is an isoterm. Therefore y precedes x in B_1' and x precedes y in B_2'. Now suppose that for some i, j, y_i precedes x_j in B_1'. It follows that x_j cannot precede y_i in B_2' since then $L_n(x_j, y_i, \{t\}) \not\approx w(x_j, y_i, \{t\})$ but $w(x_j, y_i, \{t\})$ is an isoterm for S (it would be of the form described in P3). Therefore y_i precedes x_j in B_2' as well. Since the letters in $y[Xn]$, $[Yn]x$, $[nX]y$ and $x[nY]$ do not commute, we must have that y_n precedes x_1 in B_2' and also that y_n precedes x_1 in B_1'. This implies that for every $i, j \le n$, y_i precedes x_j in B_1' and consequently that y_i precedes x_j in B_2' (else $w(x_j, y_i, \{t\})$ would be an isoterm of the form described in P3). Therefore B_2' and B_1' are the same as the corresponding blocks in R_n and the claim is proved.

Claim 1 shows that without loss of generality any derivation of $L_n \approx R_n$ may be considered as a derivation of length 1. That is there is an identity $u \approx v$ such that $L_n \equiv u_1\theta(u)v_1$ and $R_n \equiv u_1\theta(v)v_1$.

Claim 2. There is no length 1 derivation of τ_n from those identities of S which involve only words with length less than n.
Assume τ can be derived from an identity $u \approx v$ of S where both u and v are words of length less than n. So there are letters z_1 and z_2 occurring $u \approx v$ and a substitution so that $\theta(z_1)$ contains $x_i x_{i+1}$ for some $i = 1, ..., n-1$ and $\theta(z_2)$ contains $y_j y_{j+1}$ for some $j = 1, ..., n-1$. Since (x_i, y_j) is an unstable pair in $L_n \approx R_n$ we may assume that $u(z_1, z_2, \{t\}) \approx v(z_1, z_2, \{t\})$ is nontrivial. However $u(z_1, z_2, \{t\})$ is equivalent, modulo renaming letters, to a word obtained from U by deleting all subwords of the form yx and possibly some subwords of the form xy. This is an isoterm by Lemma 4.1 and so we have obtained a contradiction. Thus S does not satisfy $u \approx v$ and so any basis for S must contain identities of arbitrarily large length.

□

Example 1 *Consider $S(\{abba\})$. Take A, B and C in Lemma 4.4 to be the empty word. Now $xyyx$ and xyx are isoterms for $S(\{abba\})$ so therefore xyt_1yxt_2 is an isoterm. On the other hand, it is easy to verify that*

$$S(\{abba\}) \models y[Xn][Yn]xt_1[nX]xy[nY]t_2 \approx [Yn]yx[Xn]t_1x[nY][nX]yt_2$$

since for any unstable pair (z_1, z_2), $\tau_n(z_1, z_2)$ is simply the identity $z_1z_2z_1z_2 \approx z_2z_1z_2z_1$. By Lemma 4.4, $S(\{abba\})$ is NFB.

The following lemma is similar to one in [9] but uses different identities.

Lemma 4.5 *If $xyxy$ is an isoterm for a monoid S and for every $n > 0$, S satisfies the identity ω_n given by $L_n \equiv [X(2n)]t[\mathcal{X}(2n)] \approx [\mathcal{X}(2n)]t[X(2n)] \equiv R_n$, then S is NFB.*

Proof: Given that $xyxy$ (and consequently $xytxy$) is an isoterm for the monoid S it follows that if (x_i, x_j) is an unstable pair in any identity $L_n \approx w$ satisfied by S then either i is even, j is odd and $j < i$ or j is even, i is odd and $i < j$. Furthermore in this case the identity $L_n(x_i, x_j, t) \approx$

$w(x_i, x_j, t)$ is equivalent up to a change of letter names to the identity $xytyx \approx yxtxy$. We now show that if $L_n \approx w$ is a nontrivial identity of S, then $w \equiv R_n \equiv [\mathcal{X}(2n)]t[X(2n)]$.

Let (x_i, x_j) be an unstable pair in a nontrivial identity $L_n \approx w$ satisfied by S. It is convenient to denote the word to the left of t in L_n by B_1 and the word to the right of t in L_n by B_2. Since xyx is an isoterm for S, (x_i, t) is stable in $L_n \approx w$ for any $i \leq 2n$ and so there are corresponding blocks B_1' and B_2' in w either side of the linear letter t that are permutations of the corresponding blocks B_1 and B_2 in L_n. Without loss of generality, we may assume that x_i precedes x_j in B_1 and x_j precedes x_i in B_1'. As noted above we have that i is odd, j is even and $i < j$ and therefore since $xytxy$ is an isoterm for S we can conclude that $w(x_i, x_j, t) \equiv x_j x_i t x_i x_j$. Now i is odd and so we have that $L_n(x_1, x_i, t) \equiv x_1 x_i t x_1 x_i$, an isoterm for S or $i = 1$. If i is not 1 it follows that x_1 precedes x_i in B_1' and in B_2' and also that x_1 precedes x_j in B_2' (because x_i does). As noted at the start of the proof, the pair (x_j, x_{2n}) is stable in $L_n \approx w$ and so x_{2n} occurs after x_j and therefore after x_1 in B_2'. That is, (x_1, x_{2n}) is an unstable pair in $L_n \approx w$. If x_1 precedes x_{2n} in B_1' (as it does in B_1), then $w(x_1, x_{2n}, \{t\})$ is the word $x_1 x_{2n} t x_1 x_{2n}$, an isoterm for S and so contradicting the fact that (x_1, x_{2n}) was an unstable pair. So we must have x_1 occurring after x_{2n} in B_1'. Since for any odd number j', $(x_1, x_{j'})$ is stable in $L_n \approx w$, we must have $x_{j'}$ occurs after x_{2n} in B_1'. Likewise for any even number i', $x_{i'}$ precedes x_{2n} and therefore x_1 in B_1'. These facts ensure that B_1' is the word $[\mathcal{X}(2n)]$. It now easily follows that in B_2', x_1 precedes x_2, x_2 precedes x_3 and so on, so that B_2' is the word $[X(2n)]$. So $w \equiv R_n \equiv [\mathcal{X}(2n)]t[X(2n)]$.

We now show by contradiction that if Σ is a basis for the identities of S then for every nontrivial identity $L_n \approx R_n$ satisfied by S, Σ contains an identity with at least $2n$ letters. Since S satisfies such an identity for infinitely many n, this implies that Σ is infinite. Any derivation of $L_n \approx R_n$ involves just one step since we showed above that R_n is the only word $w \not\equiv L_n$ for which $S \models L_n \approx w$. Therefore there is an identity $p \approx q \in \Sigma$ such that $L_n \equiv U_1\theta(p)U_2$ and $R_n \equiv U_1\theta(q)U_2$ (indeed it is clear from the form of $L_n \approx R_n$ that U_1 and U_2 can be taken to be empty). Say $p \approx q$ involve fewer than $2n$ distinct letters. The word $[X(2n)]$ involves $2n$ distinct letters and so there must be a letter x in $c(p)$ such that, for some $i \leq 2n - 1$, $x_i x_{i+1}$ is a subword of $\theta(x)$. This subword occurs just once in L_n and w so x must be linear in p and q. Similarly there is a variable y such that $\theta(y)$ contains a subword of $[\mathcal{X}(2n)]$ whose length is at least 2, and y is linear in p and q. However the subword $\theta(x)$ occurs before $\theta(y)$ in L_n and after $\theta(y)$ in R_n. Therefore $p(x, y) \approx q(x, y)$ is the identity $xy \approx yx$, contradicting the fact that $xyxy$ is an isoterm for S. Hence $p \approx q$ must contain at least $2n$ distinct letters as required. Therefore S is NFB. □

Example 2 *Consider $S(\{abab\})$. If we take A, B and C to be the empty word then since xyx and $xyxy$ are isoterms, we have that xyt_1xyt_2 is an isoterm for $S(\{abab\})$. On the other hand we can easily verify that $S(\{abab\}) \models [X(2n)]t_1[\mathcal{X}(2n)]t_2 \approx [\mathcal{X}(2n)]t_1[X(2n)]t_2$ since for any unstable pair (z_1, z_2), $\omega_n(z_1, z_2)$ is the identity $z_1z_2z_2z_1 \approx z_2z_1z_1z_2$. Therefore by Lemma 4.5, $S(\{abab\})$ is NFB.*

The following two lemmas will be useful in Section 5.

Lemma 4.6 *Let A, B be elements of $\{xyt, yxt\}^*$ and ρ be a substitution defined by $\rho(xyt) \equiv \lfloor xy[Xn], t \rfloor$, $\rho(yxt) \equiv \lfloor [nX]yx, t \rfloor$. Let u_1, u_2, v_1, and v_2 be elements of $\{xy, yx\}$ such that u_1u_2 is not $xyxy$ and v_1v_2 is not $xyyx$.*
(a) If for some $m > 1$, Ax^my^mtB, and $AxytxytB$ are isoterms for a monoid S and for every $n > 0$, $S \models \rho(A)xyx_1^mx_2^m \dots x_n^mxyt\rho(B) \approx \rho(A)u_1x_1^mx_2^m \dots x_n^mu_2t\rho(B)$, then S is NFB.
(b) If for some $m > 1$, Ax^my^mtB and $AxytyxtB$ are isoterms for a monoid S and for every n, $S \models \rho(A)xyx_1^mx_2^m \dots x_n^myxt\rho(B) \approx \rho(A)v_1x_1^mx_2^m \dots x_n^mv_2t\rho(B)$ then S is NFB.

Proof. We will only prove part (a) since the proof of (b) is almost identical. Let L_n be the word

$$\rho(A)xyx_1^m x_2^m \dots x_n^m xyt\rho(B)$$

and R_n be the word

$$\rho(A)u_1x_1^m x_2^m \dots x_n^m u_2t\rho(B).$$

Let $L_n \approx w$ be a nontrivial identity satisfied by S. By Lemmas 4.2 and 4.1, for any non-linear letter z, $L_n(z, \{t\})$ is an isoterm. Therefore w differs from L_n only by permutations within blocks. Since there is only one block of length more than 1, the only differences between L_n and w are to be found in this block. We will refer to this block as the central block of L_n and w. Since Ax^my^mtB is an isoterm, Lemma 4.2 implies that $L_n(x_i, x_j, \{t\})$ is an isoterm. Thus it must be the case that $L_n(x_1, \dots, x_n, \{t\})$ is an isoterm. Since Ax^my^mtB is an isoterm, by Lemma 4.1, Ax^myytB and $Axxy^mtB$ are isoterms. Thus for any letter $x_i \in \{x_1, \dots, x_n\}$, the central block of $w(x, x_i, \{t\})$ cannot be of the form xxx_i^m or x_i^mxx. In particular this is true for $i = 1$ and $i = n$. Thus the central block of w is of the form $ux_1x_2^mx_3^m \dots x_{n-1}^m x_nv$, where u is a permutation of xx_1^{m-1} and v is a permutation of $x_n^{m-1}x$.

Now we examine possible derivations of $L_n \approx R_n$ from the identities of S. In any derivation of $L_n \approx R_n$ we have a sequence of identities $I_1 \approx I_2$, $I_2 \approx I_3$, ..., $I_{k-1} \approx I_k$ such that $I_1 \equiv L_n$, $I_k \equiv R_n$ and for each i there is an identity $p_i \approx q_i$ and a substitution θ_i such that $I_i \equiv u\theta_i(p_i)v$ and $I_{i+1} \equiv u\theta_i(q_i)v$ for some words u, v. Let h smallest number such that $I_h(x, y) \not\equiv I_{h+1}(x, y)$ (this exists since by the choice of u_1 and u_2, $L_n(x, y) \not\equiv R_n(x, y)$). Both I_h and I_{h+1} are of the form of w as described above. Consider $p_h \approx q_h$. Since the central block of both words contain $n+2$ distinct letters, if p_h contained less than n letters, there must be a letter z in $c(p_h)$ such that $\theta_i(z)$ contains x_jx_{j+1} for some j. This subword occurs just once in I_h and I_{h+1} so z is linear in p_i. Similarly there a letters x' and y' such that $\theta_i(x')$ contains x and $\theta_i(y')$ contains y. Consider $p_h(x', y', z, \{t\}) \approx q_h(x', y', z, \{t\})$. By the choice of I_h and $I_{(h+1)}$, (x', y') is an unstable pair in this identity. Now if z is a linear letter, $AxyzxytB$ and all subwords of this word are isoterms. Define a new substitution θ' by defining $\theta'(x') \equiv x$, $\theta'(y') \equiv y$, $\theta'(z) \equiv z$ and assigns remaining linear letters in $p_h(x', y', z, \{t\})$ to subwords of $AxyzxytB$ between corresponding occurrences of $\theta'(x')$, $\theta'(y')$ and $\theta'(z')$. That (x', y') is an unstable pair in $p_h(x', y', z, \{t\}) \approx q_h(x', y', z, \{t\})$ now contradicts the fact that $AxyzxytB$ is an isoterm. Thus p_h must contain more than n letters. Since S satisfies

$$\rho(A)xyx_1^m x_2^m \dots x_n^m xyt\rho(B) \approx \rho(A)u_1x_1^m x_2^m \dots x_n^m u_2t\rho(B)$$

for every $n > 0$, any basis for $Id(S)$ must be infinite since for every $n > 0$ it contains an identity with more than n letters.

□

Example 3 *Consider $S(\{abcab, abcba, a^kb^k\})$ for some $k > 2$. Some isoterms for this semigroup are $xytxyt$, $xytyxt$ and x^ky^kt. On the other hand it is easy to verify that $S(\{abcab, abcba, a^kb^k\})$ satisfies $xyx_1^k \dots x_n^kxy \approx xyx_1^k \dots x_n^kyx$. Therefore by Lemma 4.6, $S(\{abcab, abcba, a^kb^k\})$ is NFB.*

Lemma 4.7 *Let A, B be elements of $\{xyt, yxt\}^*$. Say $AxyxytB$ and $AyxxytB$ are isoterms for a monoid S and for every $n > 0$, $\sigma(A)xx[nX][Xn]t\sigma(B)$ is not an isoterm for S, where σ is a substitution defined by $\sigma(xyt) \equiv \lfloor x[Xn], t\rfloor$ and $\sigma(yxt) \equiv \lfloor [nX]x, t\rfloor$. Then S is NFB.*

Proof. The proof will be similar to that of the previous three lemmas. Fix some number n and let L_n be the word $\sigma(A)xx[nX][Xn]t\sigma(B)$. As in the proofs to the previous three lemmas,

Lemmas 4.1 and 4.2 show that for any nonlinear letter y in $c(L_n)$, $L(y, \{t\})$ is an isoterm. Thus if $L_n \approx w$ is a nontrivial identity satisfied by S then w differs from L_n only by a permutation within blocks. The word $xx[nX][Xn]$ forms a block in L_n which we will refer to as the central block B_1. Since B_1 is the only block in L_n with length more than 1, there is a block B_2 in w corresponding to the central block of L_n which is a permutation of $xx[nX][Xn]$. Since $AyxxytB$ is an isoterm, $L_n(x_i, x_j, \{t\})$ is an isoterm for every $i, j \leq n$. Thus the central block is an interleaving of xx and $[nX][Xn]$. Because $AyxxytB$ is an isoterm for S, the two occurrences of x in B_2 cannot lie between the two occurrences of any letter x_i since in that case $w(x, x_i, \{t\})$ would be an isoterm yet (x, x_i) an unstable pair in $L_n \approx w$. Furthermore, for every $i \leq n$, the central block cannot delete to xx_ixx_i since then $w(x, x_i, \{t\})$ is an isoterm and $w(x, x_i, \{t\}) \not\equiv L_n(x, x_i, \{t\})$. Thus w is either the word

$$\sigma(A)[nX]Cxt\sigma(B),$$

where C is a interleaving of x and $[Xn]$, or the word

$$\sigma(A)x[nX][Xn]xt\sigma(B).$$

Now we show that if Σ was a set of identities with fewer than n distinct letters then $\Sigma \vdash L_n \approx w$ only if $S \not\models \Sigma$. Thus any basis for S is infinite.

Let Σ be such a set of identities and let A' and B' be the words $A(x, \{t\})$ and $B(x, \{t\})$ respectively. Since $AyxxytB$ is an isoterm, Lemmas 4.1 and 4.2 imply that $A'xxtB'$ ($\equiv (xt)^{occ(x,A)}xxt(xt)^{occ(x,B)}$) and $A'xtB'$ ($\equiv (xt)^{(occ(x,L)-1)}$) are isoterms. Lemma 4.2 implies that $A'xxztB'$ and $A'xztB'$ are also isoterms if z is a linear letter. Furthermore, if we give z the value xx in the isoterm $A'yxxytB'$, then similar arguments show that $A'xzxtB'$ must be an isoterm as well. Since $\Sigma \vdash L_n \approx w$ there is an identity $p \approx q \in \Sigma$ and a substitution θ such that $L_n \equiv u\theta(p)v$ and $u\theta(q)v$ is of one of the two forms derived above. Given the restricted nature of these two forms, $\theta(p)$ must contain the word $xx[nX][Xn]$. Σ contains only identities involving less than n letters so any substitution θ must assign some letter z in $c(p)$ a value containing as a subword the word x_ix_{i+1}. Since this subword occurs just once in L_n, z is linear in p. Furthermore there must be a letter x' or letters x' and y' such that $\theta(x') \equiv x$ or both $\theta(x') \equiv x$ and $\theta(y') \equiv x$ and (x', z) are an unstable pair in $p \approx q$. In either case we have that $p(x', z, \{t\}) \approx q(x', z, \{t\})$ is not satisfied by S as required because (x', z) is an unstable pair in this identity and we can delete some linear letters so that $p(x', z, \{t\})$ becomes a subword of one of the forms $A'xxztB'$, $A'xztB'$ or $A'xzxtB'$. Thus S is NFB.

□

Example 4 *The semigroup* $S(\{abab, abba\})$ *is NFB.*

Definition 4.8 *i) Let w be a word in a two letter alphabet $\{a, b\}$. The* height *of w (written $h(w)$) is the maximum of the number of distinct subwords ab and the number of distinct subwords ba. For example, the word $aaabbab$ has height 2 because it contains 2 occurrences of ab and 1 occurrence of ba.*
ii) Let w be a word in an arbitrary finite alphabet. Then the height of w is

$$max\{h(w(a, b)); (a, b) \in c(w) \times c(w), a \not\equiv b\},$$

the maximum of the height of $w(a, b)$ for all disjoint pairs (a, b) from $c(w)$.
iii) If W is a finite set of words, then the height of W (written $h(W)$) is the maximum height of the words in W.

Lemma 4.9 *Let A and B be alphabets and w be a word over B of height h. Let (x, y) be a pair of letters in w such that the height of $w(x, y)$ is h. Let θ be a substitution $B \to A^*$ such that none of the following conditions hold:*

1) $\theta(x) \equiv 1$

2) $\theta(y) \equiv 1$

3) $\theta(x) \equiv a^n$ and $\theta(y) \equiv a^m$ for some letter a.

Then the height of $\theta(w)$ is greater or equal to h.

Proof. Case 1. Suppose that $\theta(x)$ or $\theta(y)$ contains the subword ab. Then the number of different subwords ab in $\theta(w)$ is at least the number of occurrences of x or y in w, which is greater than h.

Case 2. Now suppose that $\theta(x) \equiv a^n$ and $\theta(y) \equiv b^m$ (with $a \not\equiv b$). Then $\theta(w)$ contains at least h different subwords of the form $a^n b^m$ and therefore at least h different subwords of the form ab. Thus in this case the height of $\theta(w)$ is greater or equal to h too.

□

The following theorem gives a general sufficient condition for $S(W)$ to be NFB.

Theorem 4.10 *Let W be a (possibly infinite) set of words of (finite) height h in an arbitrary alphabet. Suppose that among the words of height h there exists a word w in two-letter alphabet which starts and ends with a and contains a subword $b^{\alpha_1}a^{\alpha_2}b^{\alpha_3}$, where $\alpha_i \geq 2$ for $i = 1, 2, 3$. Then W is a NFB set of words.*

Proof. $w \equiv w_1 b a^{\alpha_2} b w_2$ for some words w_1 and w_2 such that w_1 starts with a and ends with b and w_2 starts with b and ends with a. Let n be the height of w_1 and $m = h - n$ be the height of w_2. Let us check conditions of Lemma 4.4.

(1) Take $A \equiv yxt_1yxt_2 \dots yxt_n$, $B \equiv 1$, $C \equiv xyt_{n+3}xyt_{n+4} \dots xyt_{n+m+2}$. We will show that $I \equiv Axyt_{n+1}yxt_{n+2}C$ is an isoterm of $S(W)$.

Firstly every pair of linear letters is stable in I with respect to $S(W)$ since otherwise $S(W)$ would satisfy either $xy \approx yx$ or $x \approx x^k$ for some $k > 1$. Now if all maximal subwords of the form b^k in w are replaced by linear letters, we get an isoterm w' otherwise there would be a nontrivial identity $w' \approx v$ satisfied by $S(W)$ which could be applied to w to derive a nontrivial identity involving this word. Now Lemma 4.1 can be applied to the word w' to show that $xt_1xt_2...xt_nx^2t_{n+2}xt_{n+3}...t_{m+n+1}x$ is an isoterm. So (x, t_i) is a stable pair in $I(x, \{t\})$ for all $i \neq n + 1$, $i \leq n + m + 1$. We show that(x, t_{n+1}) and (x, t_{n+m+2}) are also stable pairs for $I(x, \{t\})$ with respect to $S(W)$. If $I(x, \{t\}) \approx J$ is an identity satisfied by $S(W)$ then since $t_1t_2 \dots t_{m+n+2}$ and $xt_1x...t_nx^2t_{n+2}x...xt_{m+n+1}x$ are isoterms, $J(x, t_1, \dots t_{n+m+1})$ must be one of the following words:

$$xt_1x...t_nxt_{n+1}xt_{n+2}x...xt_{m+n+1}x \ (\equiv I(x, t_1, t_2, \dots, t_{n+m+1})),$$

$$xt_1x...t_nt_{n+1}xxt_{n+2}x \dots xt_{m+n+1}, \text{ or}$$

$$xt_1x...t_nxxt_{n+1}t_{n+2}x \dots xt_{m+n+1}x.$$

If we delete t_n in the second of these or t_{n+2} in the third, then we get a word equivalent after a change of letter names to the word $xt_1x...t_nx^2t_{n+2}x...t_{m+n+1}x$ which is an isoterm. Therefore $J(x, t_1, ...t_{n+m+1})$ must be the first of these words, that is

$$J(x, t_1, ...t_{n+m+1}) \equiv I(x, t_1, ..., t_{n+m+1}).$$

Likewise,

$$J(x, t_1, ..., t_n, t_{n+2}, ..., t_{n+m+1}, t_{m+n+2}) \equiv I(x, t_1, ..., t_n, t_{n+2}, ..., t_{n+m+1}, t_{m+n+2}).$$

So therefore $I(x, \{t\})$ is an isoterm. Since $I(y, t)$ is essentially the same word as $I(x, t)$, it is also an isoterm.

To complete the proof that I is an isoterm, it only remains to show that (x, y) is a stable pair in I. To do this, use a substitution θ which takes x to b, y to a, and I to w (that is, assigns to each linear letter a maximal subword between corresponding occurrences of ab or ba). Since $w \in W$ is an isoterm for $S(W)$, condition 1 of Lemma 4.4 is satisfied.

(2) Let τ_n be the identity $L_n \approx R_n$ as defined in Lemma 4.4. By Property P2 in the proof of Lemma 4.4, for any unstable pair of letters (z_1, z_2), $\tau_n(z_1, z_2, \{t\})$ is essentially the identity $AxytxyC = AyxtyxC$. Notice that the height of both parts of this identity is greater than h. Since W has height h, if the identity was to fail on $S(W)$ by Lemma 4.9 we must assign 1 to at least one of the letters in every pair x', y' for which $h(L_n(x', y')) > h$ or $h(R_n(x', y')) > h$. This reduces the identity to a tautology and so τ_n holds on $S(W)$. Thus Lemma 4.4 applies and $S(W)$ is NFB.

□

Theorem 4.11 *Let w be a word of height $h > 0$ and length $l > 2$ in the alphabet $\{a, b\}$ such that w is a NFB word.*
(a) If the height of a set of words W is less than $max(1, h-2)$, then $W \cup \{w\}$ is a NFB set of words.
(b) If the maximum length of the words in W is less than $l - 1$ then $W \cup \{w\}$ is a NFB set of words.

Proof. The proof essentially follows from the proof of Theorem 1.4. In the proof of Theorem 1.4 (see [9]), if $S(\{w\})$ is NFB, it is shown that $S(\{w\})$ satisfies for each $n > 1$ an identity $u_m \approx v_m$ (depending on m) and that within $Id(S(\{w\}))$ these identities have no finite basis (that is, there is no finite subset of $Id(S(\{w\}))$ from which $u_m \approx v_m$ can be derived for every m). Furthermore, for every unstable pair (x, y) (or (y, x)) in the identities $u_m \approx v_m$, the height of both $u_m(x, y)$ and $v_m(x, y)$ is at least $h - 2$. Since the height of W is less than $h - 2$, $S(W) \models u_m \approx v_m$ for every m also and therefore $S(W \cup \{w\}) \models u_m \approx v_m$. Since $Id(S(W \cup \{w\})) \subseteq Id(S(\{w\})$, $S(W \cup \{w\})$ is NFB as required.

The proof for part (b) is very similar since for any unstable pair (x, y) in the identities $u_n \approx v_n$, the words $u_n(x, y)$ and $v_n(x, y)$ have length $l - 1$.

□

Using variant of Lemma 4.4 it is possible to replace the number $max(1, h-2)$ above with $max(1, h-1)$.

Theorems 4.10 and 4.11 easily give us the following:

Corollary 4.12 *If W is a set of words then there is a word w in a two letter alphabet such that $W \cup \{w\}$ is a NFB set of words.*

Corollary 4.13 *If S is a k-nilpotent monoid then S is a subsemigroup of a NFB $max(5, (k+2))$-nilpotent monoid.*

Proof: We will assume that $k \geq 3$ and show that S is a subsemigroup of a $k + 2$ nilpotent monoid. Let k' be the smallest integer such that $2k' \geq k$ for some number k. Consider the monoid $S(\{ba(ab)^{k'-1}\})$. This is certainly $(k+2)$-nilpotent since the length of $ba(ab)^{k'-1}$ is either k or $k+1$ and in both cases $S(\{ba(ab)^{k'-1}\})$ is $(k+2)$-nilpotent. We now use Lemma 4.4. Let A and B be the empty word and C be the possibly empty word $(xyt)^{k'-2}$ (this exists since $k \geq 3$ implies that $k' \geq 2$). It is easily established that $yxtxytC$ is an isoterm for $S(\{ba(ab)^{k'-1}\})$. Thus if $S(\{ba(ab)^{k'-1}\})$ satisfies the identities τ_n of Lemma 4.4 then it is NFB. For any unstable pair (z_1, z_2) in τ_n, the length of $L_n(z_1, z_2)$ and $R_n(z_1, z_2)$ is exactly $2k'$. This shows that S satisfies τ_n since $k \leq 2k'$ and any product of k non identity elements in S is zero. Now assume that τ_n fails on $S(\{ba(ab)^{k'-1}\})$ under some assignment θ. Since $S(\{ba(ab)^{k'-1}\})$ is $(2k'+1)$-nilpotent, all letters in $c(L_n)$ must be assigned the value 1 except for two letters z_1 and z_2 with the property that (z_1, z_2) is unstable in τ_n. Furthermore one of the letters z_1 and z_2 must be assigned a and the other must be assigned b. By property P2 of Lemma 4.4, we may assume that $\tau_n(z_1, z_2)$ is the identity

$$z_1z_2z_1z_2(z_1z_2)^{k'-2} \approx z_2z_1z_2z_1(z_1z_2)^{k'-2}.$$

Evidently $\theta(L_n)$ and $\theta(R_n)$ are not subwords of $ba(ab)^{k'-1}$ and so both take the value 0 on $S(\{ba(ab)^{k'-1}\})$, contradicting the assumption that τ_n failed under the assignment θ. Thus both $S(\{ba(ab)^{k'-1}\})$ and S satisfy τ_n.

Using the same construction as for Corollary 3.10 we arrive at a $(k+2)$-nilpotent monoid T containing both S and $S(\{(ab)^{k'}\})$ as subsemigroups and satisfying τ_n. Thus T is NFB.

□

An immediate corollary of this is

Corollary 4.14 *The pseudovariety generated by the class of finite NFB nilpotent monoids contains all finite nilpotent monoids and all finite nilpotent semigroups.*

Combining Corollaries 3.11 and 4.12 we have the following:

Corollary 4.15 *If W is a set of words then there are sets of words $W = V_0, V_1, V_2, ...$ with $|c(V_i)| = max(2, |c(W)|)$ and $V_i \subseteq V_{i+1}$ for $i \geq 0$ so that V_{2j} is FB and V_{2j+1} is NFB for every $j > 0$.*

This means that if V is a variety generated by a monoid of the form $S(W)$ (where W is a finite set of words) then there is an infinite chain of supervarieties of V, each generated by a finite semigroup $S(V_i)$ but whose identities alternate between being FB and NFB.

We now address the question as to what is the smallest NFB word.

Theorem 4.16 *(i) For any set of words $\{w_1, w_2, ..., w_n\}$ with the length of each w_i strictly less than 4, $\{w_1, ..., w_n\}$ is FB.*
(ii) If $S(W)$ has less than 10 elements it is NFB if and only if $S(W) \cong S(\{abab\})$.

Proof. Let $W = \{w_1, w_2, ..., w_n\}$ be a set of words as in part (i) of the theorem. If there is a word w_i of the form abc (that is, containing three distinct letters) then we can replace this with the word ab and the resulting semigroup will be equationally equivalent to $S(W)$. This is because xy is an isoterm in a monoid if and only if xyz is an isoterm. Furthermore if W contains two words w_1 and w_2 differing only by letter names then we may remove w_1, say, and the resulting semigroup $S(W \backslash \{w_1\})$ is still equationally equivalent to $S(W)$. Up to a change in letter names, this leaves only subsets of the set $\{aaa, aba, aab, baa\}$ to consider (recall that we are assuming this set is closed under taking subwords).

Recall that a word u is almost linear if at most one letter appears in u more than one time. In [9], O. Sapir has shown that if W is a finite set of almost linear words then $S(W)$ is FB.

Since every subset of the set $\{aaa, aba, aab, baa\}$ contains only almost linear words, Theorem 4.16 part (i) is proved.

Now assume that $S(W)$ is a NFB semigroup with less than 10 elements. By (i) we can assume that W contains a four letter word. It is easily verified that a four letter word, w, involving three distinct letters has at least 8 distinct subwords and so $S(\{w\})$ has at least 10 elements. Now the only four letter words w involving two or less distinct letters for which $S(\{w\})$ has less than 10 elements are (up to a change in letter names) $aaaa$ and $abab$. The word $aaaa$ has only 4 distinct subwords. In [5] it is shown that if a semigroup satisfies $xyx \approx xxy$ or $xyx \approx yxx$ then it is FB. If $xxxx$ is an isoterm then in order that $S(W)$ not satisfy one of these identities, either xyx or both xyy and yyx must also be isoterms. However $S(\{aaaa, aba\})$ and $S(\{aaaa, aab, baa\})$ have at least 10 elements (in fact they are FB anyway). The Theorem now follows since by Example 2, $S(\{abab\})$ is NFB.

□

5 Joins of Varieties Generated by Monoids of the Form $S(W)$

Examples found by M. Volkov (see [10]) and M. Sapir ([8]) show that the class of finite FB semigroups and the class of finite NFB semigroups are not closed under taking subsemigroups, homomorphic images, or direct products. The presence of nontrivial subgroups plays a central role in these examples. An aperiodic semigroup is a semigroup with no nontrivial subgroups. Corollaries 3.11 and 4.12 show that the class of finite FB aperiodic semigroups and the classes of finite FB and finite NFB aperiodic semigroups (and in particular the classes of FB or NFB monoids of of the form $S(W)$) are also not closed under taking subsemigroups or homomorphic images. In this section we will address the problem of finding FB finite aperiodic semigroups whose direct product is NFB and NFB finite aperiodic semigroups whose direct product is FB.

The following lemma is useful.

Lemma 5.1 *Let W_1 and W_2 be two sets of words over some alphabet X. Then $S(W_1 \cup W_2)$ has the same identities as the direct product $S(W_1) \times S(W_2)$.*

Proof. Indeed, if an identity $p \approx q$ holds in $S(W_1 \cup W_2)$ then for every substitution θ, if $\theta(p)$ or $\theta(q)$ is a subword of a word in $W_1 \cup W_2$ then $\theta(p) \equiv \theta(q)$. This implies that $p \approx q$ holds on $S(W_1)$ and on $S(W_2)$, so it holds on $S(W_1) \times (W_2)$. On the other hand, if $p \approx q$ does not hold on $S(W_1 \cup W_2)$ then there exists a substitution θ such that $\theta(p)$ or $\theta(q)$ is a subword of a word in W_1 or W_2 but $\theta(L_\tau) \not\equiv \theta(R_\tau)$. But then $p \approx q$ does not hold in one of $S(W_1)$ or $S(W_2)$, and so it does not hold in the direct product of these monoids. The lemma is proved.

□

Definition 5.2 *Let $\mathsf{A_n}$ be the set of all words starting with a in the alphabet $\{ab, ba\}$ whose length n (as words in this alphabet) is greater than 1 and let A be a fixed element of $\mathsf{A_n}$, say $(ab)^{m_1}(ba)^{m_2}...(ab)^{m_k}$, $m_i > 0$ for all $i < k$, $m_k \geq 0$, $\Sigma_{i=1}^k m_i = n$.*

For $n > 2$, at least one of the words $(ab)^{n-1}ba$ and $ab(ba)^{n-1}$ are contained in the set $A_n \backslash \{A\}$. Fix one of them and call it B. For each $m \geq 1$ let ξ_m be a substitution defined by $\xi_m(ab) \equiv [Xm]$, $\xi_m(ba) \equiv [mX]$. We now construct an identity $L_{A,m} \approx R_{A,m}$ or $\tau_{A,m}$ as follows. To make the word $L_{A,m}$, first replace every occurrence of ab by the word abt and every occurrence of the word ba by the word bat. Let the resulting word be denoted by A'. Now replace every occurrence of a in A' by the letter x and every occurrence of b by corresponding occurrences of $\xi_m(ab)$ or $\xi_m(ba)$ from the word $\xi_m(B)$. That is, if the i^{th} letter to appear in B as a word in the alphabet

$\{ab, ba\}$ is ab then the i^{th} occurrence of b in A' is to replaced by $\xi_m(ab)$. Otherwise replace the i^{th} occurrence of b in A' by $\xi_m(ba)$. The same procedure is followed to make the word $R_{A,m}$ except each occurrence of b in A' is replaced with x and each occurrence of a is replaced with the corresponding subwords of $\xi_m(B)$. For example, let $n = 3$, $A \equiv ababba$ and therefore $B \equiv abbaba$. So A' is the word $abtabtbat$ and

$$L_{A,m} \equiv (x(x_1x_2 \dots x_m)t)(x(x_m \dots x_2x_1)t)((x_m \dots x_2x_1)xt)$$

Likewise,

$$R_{A,3} \equiv ((x_1x_2 \dots x_m)xt)((x_m \dots x_2x_1)xt)(x(x_m \dots x_2x_1)t).$$

Lemma 5.3 *If S is a monoid for which $\mathsf{A_n}\backslash\{\mathsf{A}\}$ (for some $n > 2$) are isoterms and for every $m > 0$, $S \models \tau_{A,m}$, then S is NFB.*

Proof. If A is not the word $(ab)^n$ then by assigning a to x and b to respective linear letters t we find that $L_{A,m}(x, \{t\})$ becomes the word $(ab)^n$. Since this is an isoterm, $L_{A,m}(x, \{t\})$ must be too. If $A \equiv (ab)^n$ then both $(ab)^{n-1}ba$ and $(ab)^{n-2}baab$ must be isoterms. By assigning a to x and maximal subwords of the form b^i to corresponding linear letters t we find that $xt_1xt_2...xt_{n-1}x$ and $xt_1xt_2...xt_{n-2}xxt_n$ are isoterms. These two facts combined ensure that $xt_1xt_2...xt_n$ is an isoterm. So for every non-linear letter y in $\tau_{A,m}$, $\tau_{A,m}(y, \{t\})$ is a tautology and the words in this identity are isoterms. Since B is an isoterm for S, $L_{A,m}(x_1, x_2, ..., x_m)$ is an isoterm and for any $i \leq m$, $L_{A,m}(x, x_i)$ is essentially the word A (up to a change in letter names).

Let $L_{A,m} \approx w$ be a nontrivial identity satisfied by S. There is an unstable pair (x, x_i) in $L_{A,m} \approx w$ and since $\mathsf{A_n}\backslash\{\mathsf{A}\}$ and $L_{A,m}(x, \{t\})$ are all isoterms, $w(x, x_i) \equiv R_{A,m}(x, x_i)$. We show that this statement is true for $i = 1$ and $i = m$: this combined with the fact that $xt_1xt_2...xt_n$ and $L_{A,m}(x_1, x_2, ..., x_m)$ are isoterms are enough to show that $w \equiv R_{A,m}$. Since $B \not\equiv A$ there is a subword of $L_{A,m}$ of the form $x_i...x_2x_1x$ (or $xx_1x_2...x_i$) and x_i occurs before x in the corresponding block in w (or x_i occurs after x in w respectively). Since $L_{A,m}(x_1, x_2, ..., x_m)$ is an isoterm, (x, x_1) must also be an unstable pair. Similarly there is a subword $xx_n...x_i$ (or $x_i...x_nx$) in $L_{A,m}$ such that in the corresponding block of w the order of appearance of x and x_i are switched. Thus (x, x_n) is also an unstable pair and consequently $w \equiv R_{A,m}$.

Now we show that there is no derivation of $\tau_{A,m}$ involving identities of S that contain less than n letters. Assume otherwise. So there is an identity $p \approx q$ involving fewer than n letters and a substitution θ such that $L_{A,m} \equiv u\theta(p)v$ and $L_{A,m} \equiv u\theta(q)v$. By the choice of B we can assume that there is only one occurrence of each subword $x_{i+1}x_i$ in $L_{A,m}$. Since we are assuming that $|c(p)| < n$ there must be a linear letter z in $c(p)$ such that $\theta(z)$ contains $x_{i+1}x_i$ as a subword. There is also a letter $x' \in c(p)$ such that $\theta(x')$ contains x and (x', z) is unstable in $p \approx q$. We may further assume that linear letters $\{t\}$ in p are assigned subwords containing the linear letters in $L_{A,m}$. Now for some j, ${}_jx$ occurs before $({}_jx_{i+1})({}_jx_i)$ in $L_{A,m}$ and after $({}_jx_{i+1})({}_jx_i)$ in $R_{A,m}$. Therefore for some j', ${}_{j'}x'$ occurs before z in p and after z in q. So $p(x', z, \{t\}) \approx q(x', z, \{t\})$ is the identity

$$x't_1...({}_{j'}x')z(t_{j'})x't_{j'+1}...x't_{(occ(x',p))} \approx x't_1...z({}_{j'}x')(t_{j'})x't_{j'+1}...x't_{(occ(x',p))}.$$

Since $xt_1...xt_n$ is an isoterm and $occ(x', p) \leq n$, by Lemma 4.2 the left hand side of this is an isoterm, a contradiction. Thus no such identity $p \approx q$ exists. Therefore any basis for S must contain identities involving arbitrarily large numbers of letters and is therefore infinite.

□

Recall that W_n is the set of all words in the alphabet $\{a, b\}$ with at most n occurrences of any letter. For any fixed word A from $\mathsf{A_n}$ with $n > 1$ let $W_{n,A}$ be the result of removing from W_n the word A and the word $\bar{A}$ obtained from A by replacing a by b and b by a.

Corollary 5.4 *For $n > 1$, $W_{n,A}$ is a NFB set of words.*

Proof. For $n > 2$ Lemma 5.3 can be used as follows. Since $\mathsf{A_n} \backslash \{A\}$ is a subset of $W_{n,A}$, every word in $\mathsf{A_n} \backslash \{A\}$ is an isoterm for $S(W_{n,A})$. On the other hand, every word in $W_{n,A}$ has length less than $2n+1$. So if θ was a substitution such that $\theta(L_{m,A})$ was contained in $W_{n,A}$ then $\theta(L_{m,A})$ must have length less than $2n+1$. Therefore θ must assign 1 to at least all but two letters from $\{x\} \cup \{x_i; i \leq m\}$. In this case either $\theta(L_{m,A}) \equiv \theta(R_{m,A})$ or $\theta(L_{m,A})$ is equivalent up to a change of letter names to A and $\theta(R_{m,A})$ is similarly equivalent to $\bar{A}$. Since $S(W_{n,A}) \models A \approx \bar{A}$, $S(W_{n,A}) \models \tau_{m,A}$ for every $m > 1$. Therefore by Lemma 5.3, $S(W_{n,A})$ is NFB.

For $n = 2$, $\mathsf{A_n}$ is the set $\{abab, abba\}$. $S(W_n)$ is equationally equivalent to $S(\{abab,\ abba,\ aabb\})$ since $\{abab, abba, aabb\}$ contains a copy (up to a change of letter names) of every 2-limited word in a two letter alphabet. Thus to prove the result we need to show that $S(\{abab, aabb\})$ and $S(\{abba, aabb\})$ are NFB. For this we can easily apply Lemmas 4.4 and 4.5 respectively. For example xyt_1xyt_2 is an isoterm for $S(\{abab, aabb\})$ since $xyxy$ and xyx are. However for any unstable pair of letters (x, y) in ω_m, $\omega_m(x, y)$ is the identity $xyyx \approx yxxy$ which is a satisfied by $S(\{abab, aabb\})$. Thus $S(\{abab, aabb\}) \models \omega_m$ for every $m > 0$. The Corollary is proved.

□

The following Corollary now follows immediately from Theorem 1.4, Corollary 3.7 and Corollary 5.4.

Corollary 5.5 *For every $n > 1$ and every word $A \in \mathsf{A_n}$, the monoids $S(\{A\})$ and $S(W_{n,A})$ are NFB but $S(\{A\}) \times S(W_{n,A})$ is FB. That is, $\{A\}$ and $W_{n,A}$ are NFB sets of words but $\{A\} \cup W_{n,A}$ is a FB set of words.*

Since $abab$ and $Sabba$ are NFB words, one might wonder if the set $\{abab, abba\}$ is FB, therefore giving a smaller example. Example 4 shows however that this is not true. Nevertheless, we can find two NFB words w_1 and w_2 such that $\{w_1 \cup w_2\}$ is FB. First consider the following lemma.

Lemma 5.6 *If w is an isoterm for a monoid S then $Id(S) \subseteq Id(S(\{w\}))$.*

Proof. Let $p \approx q$ be an identity not satisfied by $S(\{w\})$. This means that there is a substitution θ such that $\theta(p)$ is a subword of w and $\theta(p) \not\equiv \theta(q)$. So $w \equiv u\theta(p)v$ for some words u and v so that $u\theta(p)v \not\equiv u\theta(q)v$. But then $p \approx q \vdash u\theta(p)v \approx u\theta(q)v$ so w is not an isoterm for any semigroup satisfying $p \approx q$. That is, $S \not\models p \approx q$. The lemma is proved.

□

Since $S(\{ababcddee\})$ contains the subsemigroup $S(\{abab\})$ and the subsemigroup $S(\{ddee\})$, $Id(S(\{ababcddee\}))$ is contained in both $Id(S(\{abab\}))$ and $Id(S(\{aabb\}))$ and therefore also in $Id(S(\{abab, aabb\}))$. On the other hand since $xyxy$, $xyyx$, xxy, yxx and xyx are all isoterms for $S(\{abab, aabb\})$, so must be the word $ababcddee$ and so Lemma 5.6 shows that $Id(S(\{ababcddee\})) \supseteq Id(S(\{abab, abba\}))$. Therefore $S(\{ababcddee\})$ is equationally equivalent to $S(\{abab, abba\})$ and in a similar way $S(\{ababcddee, abba\})$ is equationally equivalent to $S(\{abab, aabb, abba\})$. Therefore $ababcddee$ and $abba$ are NFB words but $\{ababcddee, abba\}$ is a FB set of words.

Another simple example of two NFB words w_1 and w_2 such that $\{w_1, w_2\}$ is FB are the words $abcba$ and $abcab$. The arguments used in Examples 1 and 2 apply equally well to the monoids $S(\{abcba\})$ and $S(\{abcab\})$ respectively and so $abcba$ and $abcab$ are also NFB. On the other hand in Theorem 3.12 it was shown that $\{abcab, abcba\}$ (and by Lemma 5.6, $\{abcabdefgfe\}$) is a FB set of words. The relevancy of this example is due to the following theorem:

Theorem 5.7 *For any* $n \geq 2$ *the monoids* $S(\{abcab, abcba\})$ *and* $S(\{a^n b^n\})$ *are FB but the monoid* $S(\{abcab, abcba\}) \times S(\{a^n b^n\})$ *is NFB.*

Thus by Lemma 5.1 we have an example of two finite FB aperiodic semigroups whose direct product is NFB. The problem of finding such an example was raised by M. Sapir about 10 years ago.
Proof. Theorems 1.4 and 3.12 show that $S(\{a^n b^n\})$ and $S(\{abcab, abcba\})$ are FB while Example 3 shows that $S(\{a^n b^n\}) \times S(\{abcab, abcba\})$ is NFB. That is $\{a^n b^n\}$ and $\{abcab, abcba\}$ are FB but $\{a^n b^n, abcab, abcba\}$ is NFB.

□

Finally we present an example of two FB sets of words in the alphabet $\{a, b\}$ whose union (or equivalently, the direct product of the two corresponding monoids) is NFB.

Theorem 5.8 *The monoids*

$$S = S(\{abbaa, ababa, aabba\})$$

and

$$T = S(\{baaab, aabb, abba, abab\})$$

are FB but $S \times T$ *is NFB.*

Proof. Firstly, the monoids S and T are FB by Theorem 3.16. Now let

$$M = S(\{abbaa, ababa, aabba, baaab\}).$$

By Lemma 5.1 it follows that M satisfies the same identities as $S \times T$.

We now show that M is NFB. It is easy to check that for each n, this monoid satisfies

$$[Xn]yt_1yt_2[nX]y \approx y[Xn]t_1yt_2y[nX]$$

Now $[Xn]t[nX]$, xt_1xt_2x, xyt_1xt_2yx and yxt_1xt_2xy are all isoterms for M. Thus if

$$S(\{abbaa, ababa, aabba, baaab\}) \models [Xn]yt_1yt_2[nX]y \approx W$$

for some word W, then $W \equiv y[Xn]t_1yt_2y[nX]$. Any derivation of

$$[Xn]yt_1yt_2[nX]y \approx y[Xn]t_1yt_2y[nX]$$

therefore consists of a single application of an identity to the word $[Xn]yt_1yt_2[nX]y$. If this identity has fewer than n letters then it must assign linear letters to some portion of $[Xn]$ and $[nX]$ (since they contain n distinct letters). Thus

$$S(\{abbaa, ababa, aabba, baaab\}) \models t_3yt_1yt_2t_4y \approx yt_3t_1yt_2yt_4.$$

By assigning a to y, bb to t_3 and 1 to all other letters, we see that the right hand side of this identity takes the value $abbaa$ but the left hand side does not. Since $abbaa$ is an isoterm for M, we have reached a contradiction. Therefore any basis for the identities of M must contain identities involving arbitrarily large numbers of letters, and therefore must be infinite. □

References

[1] O. G. Kharlampovich and M. V. Sapir. Algorithmic problems in varieties, Internat. J. Algebra Comput. **5** (1995), 379-602.

[2] R. McKenzie. Tarski's finite basis problem is undecidable, Internat. J. Algebra and Comput. **6** (1996), 49-104.

[3] P. Perkins. Bases for equational theories of semigroups, J.Algebra **11** (1969), 298-314.

[4] P. Perkins. Basic questions for general algebras, Algebra Universalis **19** (1984), 16-23.

[5] G. Pollak and M. V. Volkov. On almost simple semigroup identities, in Semigroups. Structure and Universal Algebraic Problems, Budapest (1985), 287-323.

[6] M. V. Sapir. Problems of Burnside type and the finite basis property in varieties of semigroups, Math. USSR Izvestiya, **30** (1988) No. 2, 295-314.

[7] M. V. Sapir. Inherently nonfinitely based finite semigroups, Math. USSR Sbornik, **61** (1988), No.1, 155-166.

[8] M. V. Sapir. On cross semigroup varieties and related questions, Semigroup Forum, **42** (1991) 345-361.

[9] O. Sapir. Finitely based words, to appear in Internat. J. Algebra and Comput.

[10] L. N. Shevrin and M. V. Volkov. Identities of semigroups, Russian Math (Iz. VUZ), **29** (1985), No. 11, 1-64.