ALGEBRAS OF PARTIAL MAPS

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Abstract. We examine the abstract algebraic properties of algebras of partial maps on a set. We present several new results, as well as surveying the existing literature.

1. Introduction

In this paper we are going to be interested in algebras related to semigroups that arise as abstract versions of algebras of partial maps and binary relations. The "master objects" of this area of mathematics are the concrete relation algebras of Tarski: the algebras whose elements are binary relations on some set X and that are closed under the operations of union, intersection, composition, converse (= inverse), complementation and including the identity relation id on X, the universal relation $\nabla := X \times X$ and the empty relation $\varnothing$. These algebras have been heavily investigated for more than fifty years and form a fundamental part of universal algebra and algebraic logic. They are unfortunately also rather opaque structures: Tarski showed that it is possible to interpret set theory in the equational theory of the concrete relation algebras; while Hirsch and Hodkinson [16] proved that there is no algorithm to decide if a given finite algebra (of appropriate type) is actually isomorphic to a concrete relation algebra.

Semigroup theorists are familiar with some other algebras of relations, namely: the algebras of injective partial maps under the operation of composition and inverse (inverse semigroups); of permutations under composition and inverse (groups); transformations, binary relations or partial maps under composition (semigroups). Our investigations are going to concentrate on enrichments of these structures, in particular, the algebra of partial maps.

There is already one very extensive and informative survey on this topic: that of Boris Schein [31]. We do not intend to match that survey in spectrum, but rather give an update on just one aspect: algebras isomorphic to systems of (some kind of) partial maps on a set under the operations of composition and other naturally defined operations. We will give almost all of our emphasis to results that have been obtained since Schein's survey, and in particular to how the current authors' investigations into "semilattice-valued" operations fit into the known body of work. We are going to argue that there are several natural "master objects" in the algebra of partial maps within which most of the natural operations that can be conceived exist as term functions. Unfortunately we have no abstract characterisation of these objects. We finish the article with a small contribution toward our understanding

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of these classes: we show that they generate congruence distributive varieties and that the finitely representable members are a pseudovariety.

We are going to begin the survey with a section that paints a more precise picture of the kind of classes we can expect to find. In particular, we recall some basic facts from universal algebra and model theory and present Schein's Fundamental Theorem of Relation Algebras. In fact we are going to present a slightly generalised version of this result, which allows the possibility that some operations are only partially defined. Readers who are unfamiliar with the various notions required in Part 1 should find that the subsequent part reads reasonably comfortably without these technicalities, providing that they are willing to take occasional leaps of faith.

Part 1. **The Fundamental Theorem of Relation Algebras.**

2. Universal Horn classes and elementary logic

In this section we sketch a very brief introduction to the universal Horn logic of partial structures. We stress here that while we include partial operations in our discussion, almost all cases of interest in this article involve only conventional total operations (the main exceptions are Theorems 4.1 and 10.3). Our presentation is reasonably self-contained but we direct the reader to [4, Chapter V] for an easy introduction to some basics of model theory of total structures, and to [3] for further details and technicalities concerning the model theory of partial structures. See also [15] for a comprehensive study of the universal Horn logic and quasivarieties of total structures.

A *similarity type* of partial structures $\mathcal{L}$ (sometimes called a *type* or a *language*) is a set $\langle \mathcal{G}, \mathcal{H}, \mathcal{R} \rangle$ consisting of three families of symbols, each symbol with an associated number from $\omega = \{0, 1, \dots\}$ called the *arity*. The symbols in $\mathcal{G}$ correspond to the fundamental *operations* of the objects of interest, the symbols in $\mathcal{H}$ correspond to fundamental *partial operations*, while the symbols in $\mathcal{R}$ give us the fundamental *relations*. An example suffices: the language of partially ordered monoids has $\mathcal{G} = \{\cdot, 1\}$ and $\mathcal{R} = \{\leq\}$, where $\cdot$ and $\leq$ are of arity 2 (a binary operation and a binary relation respectively), while 1 is an operation of arity 0. The set $\mathcal{H}$ is empty. When $\mathcal{H}$ is empty we say that partial structures of type $\mathcal{L}$ are *total structures*, or simply *structures*, while when $\mathcal{R}$ is empty we talk of *total algebras* or simply *algebras* (if $\mathcal{H}$ is empty) or *partial algebras* (otherwise).

An atomic formula is an expression of one of the following forms:

(1) $s \approx t$, where s and t are terms in the fundamental (partial) operations from $\mathcal{G} \cup \mathcal{H}$;
(2) $(s_1, \dots, s_n) \in r$, where r is an n-ary relation in $\mathcal{R}$ and the s_i are terms in $\mathcal{G} \cup \mathcal{H}$.

A *formula* in a given language $\mathcal{L}$ is any expression built in the usual legal fashion from atomic formulæ using the usual connectives $\ \&\ $, $\mathbb{W}$, $\neg$, $\rightarrow$ and the quantifiers $\forall$ and $\exists$ (we use $\ \&\ $ and $\mathbb{W}$ for "and" and "or" to avoid confusion with the lattice operations $\wedge$ and $\vee$). A formula is a *sentence* if every variable is quantified.

For example, semigroups are a class of structures defined by the satisfaction of a universally quantified atomic formula:

$$\forall x \forall y \forall z \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z.$$

where the fundamental operation $\cdot$ is of arity 2 (binary). It is convention to omit universal quantifiers and to use concatenation in place of $\cdot$. We also frequently omit the use of brackets when associativity is given.

When $s(x_0, \ldots, x_{n-1})$ and $t(x_0, \ldots, x_{n-1})$ are terms involving partial operations we need to be careful about the concept of equality. For a partial structure $\mathbf{M}$ and $m_0, \ldots, m_{n-1}$ in M we say that the equality

$$s^{\mathbf{M}}(m_0, \ldots, m_{n-1}) = t^{\mathbf{M}}(m_0, \ldots, m_{n-1})$$

is true if both sides are defined and equal. Otherwise this interpretation of the terms s and t makes the equality false. Hence if $\langle M; \cdot \rangle$ is a partial structure where $\cdot$ is a partial binary operation symbol, then satisfaction of the sentence

$$\forall x \forall y \forall z \quad x \cdot (y \cdot z) \approx (x \cdot y) \cdot z. \tag{2.1}$$

by $\mathbf{M}$ is sufficient to guarantee that $\cdot$ is totally defined (since if $\cdot$ is undefined at some pair (a, b), then so is, for example, the expression $(a \cdot b) \cdot b$). Hence we can use (2.1) to define the class of "partial algebras with a single binary partial operation which is totally defined and associative", a class that we could reasonably consider to be indistinguishable from the usual variety of semigroups.[1]

We are interested in two special kinds of formula. If $\Phi(x_0, \ldots, x_{n-1})$ is a formula without any quantifiers and whose variables are precisely $x_0, \ldots, x_{n-1}$, then the sentence $\forall x_0 \ldots \forall x_{n-1}\ \Phi(x_0, \ldots, x_{n-1})$ is called a *universal sentence*. Classes defined by the satisfaction of universal formulæ are known as *universal classes* and are closed under taking ultraproducts and substructures. Conversely, every class of similar algebras (that is, of the same similarity type) closed under taking ultraproducts, substructures and isomorphic copies is a universal class.

Now let $\Phi, \Phi_1, \ldots$ be atomic formulæ. A *universal Horn sentence* (abbreviated as *uH-sentence*) is a universally quantified formula of one of the following forms:

(1) $\bigvee\!\!\!\bigvee_{1 \leq i \leq n} \neg \Phi_i$ (for some $n \in \omega \backslash \{0\}$);
(2) $\Phi \mathbin{\bigvee\!\!\!\bigvee} \left(\bigvee\!\!\!\bigvee_{1 \leq i \leq n} \neg \Phi_i \right)$ for some $n \in \omega$.

Formulæ of the second kind are logically equivalent to (universally quantified) formulæ of the form $(\&_{1 \leq i \leq n} \Phi_i) \rightarrow \Phi$, and this is our preferred form; we often omit the brackets. These expressions are called *quasi-identities*. When the number n is equal to 0 we get back to atomic expressions and in the case where $\mathcal{L}$ contains no relations, these are simply identities in the familiar sense.

Classes defined by the satisfaction of uH-sentences are called *universal Horn classes*. Classes defined by the satisfaction of some set Σ of quasi-identities are known as *quasivarieties*. Of course, when $\mathcal{L}$ contains no relations or partial operations and the quasi-identities in Σ are identities, we get *varieties*. UH-formulæ of the first kind are also logically equivalent to ones of the second kind on any structure with more than one element: satisfaction of $\bigvee\!\!\!\bigvee_{1 \leq i \leq n} \neg \Phi_i$ is equivalent to satisfaction of $(\&_{1 \leq i \leq n} \Phi_i) \rightarrow x \approx y$, where x and y are new variables not appearing in the Φ_i. So a uH-class differs from a quasivariety by at most some one-element structures (in fact, by at most one 1-element structure up to isomorphism). To see

[1] A subtle difference is that for structures whose type has $\mathcal{G}$ empty it is conventional and convenient to consider the empty set as defining a structure; see [3]. In conventional algebra it is common to omit this possibility, a point at which both universal algebraists *and* category theorists are in agreement when nullary operations are present! In this article we will not consider any cases where $\mathcal{G}$ is empty and so we will ignore any further reference to the empty structure.

how this distinction manifests itself in the theory of relational representations, consider the class of all isomorphic copies of systems of binary relations on a nonempty set and closed under the operations composition, $\varnothing$ and id. It is not hard to see that this consists of the class of all monoids with 0 satisfying $\neg 0 \approx 1$ and the one element monoid (where $0 = 1$) is not a member. So this class is in fact a proper uH-class (and neither a variety nor a quasivariety).

The notions of substructure, direct products, homomorphic images, ultraproducts and reduced products all have essentially the usual meanings for partial structures (although one must use one of several possible definitions of reduced and ultraproducts; an issue we address when the need arises). So, for example, a substructure of a partial structure $\mathbf{M} := \langle M; G, H, R\rangle$ must be closed under the operations G and the partial operations H. The usual definition of a homomorphism of total structures continues to make sense, but as in the relational case one must be careful: a bijective homomorphism need not be an isomorphism, just as a bijective graph homomorphism need not be a graph isomorphism (we additionally need the inverse map to be a homomorphism).

These constructions enable a second very useful characterisation of uH-classes: a uH-class is a class of similar structures closed under the class operators, $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}^+$ and $\mathbb{P}_{\mathrm{u}}$ corresponding to isomorphic copies, substructures, nonempty direct products and ultraproducts. An even more refined version: $\mathcal{K}$ is a uH-class if and only if $\mathcal{K} = \mathbb{I}\mathbb{S}\mathbb{P}^+\mathbb{P}_{\mathrm{u}}(\mathcal{K})$. A quasivariety is a uH-class containing the one-element structure (with all relations total). Alternatively $\mathcal{K}$ is a quasivariety if $\mathcal{K} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{u}}(\mathcal{K})$ (so that the empty direct product is allowed). The total structure version of these results are usually attributed to Malcev, while the partial structure version can be found in [3] for example.

Example 2.1. *The class of simple graphs is the universal Horn class of binary relational structures (with "edge relation" $\sim$, say) defined by* $x \sim y \rightarrow y \sim x$, $\neg x \sim x$.

(Here we wrote $a \sim b$ in place of the more formal $(a, b) \in \sim$. This convention is used throughout.) The quasivariety generated by the class of simple graphs is the same, except the one-element looped graph is included.

Example 2.2. *The class of partially ordered sets is the quasivariety of binary relational structures (in $\leq$, say) defined by* $x \leq y \ \& \ y \leq x \rightarrow x \approx y$, $x \leq x$ *and* $x \leq y \ \& \ y \leq z \rightarrow x \leq z$.

Example 2.3. *The class of cancellative semigroups is a quasivariety defined (within the variety of semigroups) by* $xy \approx xz \rightarrow y \approx z$ *and* $xy \approx zy \rightarrow x \approx z$.

(Here we have omitted the $\cdot$ symbol in products. We continue this convention whenever associativity is implicit.) Every monoid of injective transformations (acting on the left) is left cancellative. Conversely, the usual left regular representation represents a left cancellative monoid as a monoid of injective transformations.

The following property appears to have been first observed by Gorbunov [14].

Example 2.4. *Every pseudovariety of finite algebras is equal to the finite part of some quasivariety (namely, the quasivariety generated by the pseudovariety).*

All of these examples have no partial operations.

Example 2.5. *The class of partial algebras whose operations are total is given by the identities $f(x_0, \dots, x_{n-1}) \approx f(x_0, \dots, x_{n-1})$, for each fundamental partial operation f.*

In the usual situation for algebras (where operations are always total), the identities in Example 2.5 would be redundant tautologies.

3. The Fundamental Theorem of Relation Algebras

In this section we give an account of Schein's Fundamental Theorem of Relation Algebras, as found in [31, §II] for example. Our presentation follows [31, §II] reasonably closely, although we give a slight extension and describe some special cases where the theorem can be refined further (Theorem 3.2).

In order to state Schein's Fundamental Theorem we need to be clear about what kind of algebras we are trying to classify.

Let $\mathcal{K}$ be a class of structures of the same similarity type; these are to be the underlying sets on which our binary relations are acting. A typical concrete (partial) algebra of relations over a member of $\mathcal{K}$ can be described by

- listing a pair $(\mathbf{X}; F)$ consisting of a member $\mathbf{X} \in \mathcal{K}$ and a set of binary relations F over X,
- listing the operations O, partial operations P and relations R being used to combine the members of F into an algebra.

We could express such an instance as $\langle(\mathbf{X}, F); O, P, R\rangle$. Throughout our following discussion we hold this arbitrary instance of a concrete relation algebra fixed and call it $\mathbf{C}$.

We wish to describe the isomorphic copies of (partial) structures such as $\mathbf{C}$ where $\mathcal{K}$ is some fixed class (for example, sets, partially ordered sets), the relations F satisfy certain properties (for example, arbitrary binary relations, partial maps, order preserving partial maps) and the operations, partial operations and relations are defined in some canonical way (for example, composition). The Fundamental Theorem requires that we have a realistic description of these three facets. The language under which we have to be able to define these properties is the first order language of $\mathcal{K}$ augmented by a countably infinite set of binary predicate variables $\phi_0, \phi_1, \dots$ which we will interpret in F. So a typical atomic expression involving ϕ_i is one of the form $(s, t) \in \phi_i$, where s, t are terms in the language of $\mathcal{K}$. We note here, as in [31], that obvious modifications will give corresponding results for algebras of n-ary relations instead of binary relations.

Properties of $\mathcal{K}$. We ask that $\mathcal{K}$ be closed under ultraproducts. So it is sufficient that $\mathcal{K}$ be axiomatised by some set of first order sentences. The example of primary interest in this survey is the class of sets, which is somewhat degenerate: it is the collection of models of empty similarity type satisfying the empty set of sentences! Other cases of interest are partially ordered sets, algebras from some fixed variety, and so on.

Properties of F. Let $\mathcal{L}$ be the language of $\mathcal{K}$, and $\mathcal{L}^\sharp$ be $\mathcal{L}$ augmented by some countable set $\{\phi_0, \phi_1, \dots\}$ of binary predicate symbols. The symbols ϕ_i are going to be interpreted as elements of F in the system $(\mathbf{X}, F)$ and so we refer to them as the *predicate variables*. We are going to be looking at sentences of the form

$$(Q_0 x_0)(Q_1 x_1) \dots (Q_{m-1} x_{m-1}) \Phi(x_0, \dots, x_{m-1}; \phi_0, \dots, \phi_{n-1}) \tag{3.1}$$

where

- $\Phi(x_0, \ldots, x_{m-1})$ is a quantifier free formula in $\mathcal{L}^\sharp$ whose variables are precisely $x_0, \ldots, x_{m-1}$,
- $\phi_0, \ldots, \phi_{n-1}$ is a complete list of the predicate variables occurring in Φ,
- the Q_i are either $\forall$ or $\exists$.

Let us denote the formula (3.1) by $\Psi(\phi_0, \ldots, \phi_{n-1})$. Every interpretation of the predicate variables ϕ_i in F (say $\phi_i \mapsto f_i \in F$) gives rise to a corresponding truth value of $\Psi(\phi_0, \ldots, \phi_{n-1})$ according to whether or not $\Psi(f_0, \ldots, f_{n-1})$ is true or false in $(\mathbf{X}, F)$; here, the standard variables $x_0, x_1, \ldots$ are quantified over the elements of X. We will say that $\Psi(\phi_0, \ldots, \phi_{n-1})$ is satisfied by $(\mathbf{X}, F)$ if the corresponding statement is true under all interpretations of the ϕ_i amongst F. Satisfaction of $\Psi(\phi_0, \ldots, \phi_{n-1})$ defines a property of the binary relations F and so we will say that a set of formulæ M of the form given in (3.1) defines a *type of relation*. The pair $(\mathbf{X}, F)$ is of this type if all expressions in M are satisfied by $(\mathbf{X}, F)$.

For example, the property that all members of F are partial maps is described by

$$\forall x_0 \forall x_1 \forall x_2 \quad (x_0, x_1) \in \phi_0 \ \& \ (x_0, x_2) \in \phi_0 \to x_1 \approx x_2. \tag{3.2}$$

The property of injectivity can be described by sentence (3.2) and

$$\forall x_0 \forall x_1 \forall x_2 \quad (x_0, x_1) \in \phi_0 \ \& \ (x_2, x_1) \in \phi_0 \to x_0 \approx x_2,$$

while the property of being a transformation can be given by sentence (3.2) and

$$\forall x_0 \exists x_1 \quad (x_0, x_1) \in \phi_0. \tag{3.3}$$

If $\mathbf{X} = \langle X; + \rangle$ is an abelian group, then the property that a binary relation is an endomorphism of $\mathbf{X}$ can be given by sentences (3.2), (3.3) and the following sentence:

$$\begin{aligned} &\forall x_0 \forall x_1 \forall x_2 \forall x_3 \forall x_4 \forall x_5 \\ &\qquad x_0 + x_1 \approx x_2 \ \& \ (x_0, x_3) \in \phi_0 \ \& \ (x_1, x_4) \in \phi_0 \ \& \ (x_2, x_5) \in \phi_0 \\ &\qquad\qquad \to x_3 + x_4 \approx x_5. \end{aligned}$$

Definitions of O, P, R. Lastly, the definition of the operations, partial operations and relations combining the members of F must be canonically defined.

Each fundamental n-ary relation $r \in R$ must be described by a formula of the form of (3.1) which in this context we will call *relational*; as before we abbreviate this as $\Psi(\phi_0, \ldots, \phi_{n-1})$. So for $f_0, \ldots, f_{n-1} \in F$ we have $(f_0, \ldots, f_{n-1}) \in r$ if and only if $\Psi(f_0, \ldots, f_{n-1})$ is true in $(\mathbf{X}, F)$. For example, the binary relation of relational containment between members of F is given by

$$\forall x_0 \forall x_1 \quad (x_0, x_1) \in \phi_0 \to (x_0, x_1) \in \phi_1.$$

A formula similar to that of (3.1) but with *precisely two unquantified standard variables*—say x_0, x_1—is said to be *operational*. An operational formula $\Psi(x_0, x_1; \phi_0, \ldots, \phi_{n-1})$ defines a partial operation o in the following way: for every $f_0, \ldots, f_{n-1} \in F$ the formula defines the binary relation $o(f_0, \ldots, f_{n-1}) =: f$ on X by $(a, b) \in f$ if $\Psi(a, b; f_0, \ldots, f_{n-1})$ is true in $(\mathbf{X}, F)$. In the case where F is closed under applications of o, we say that Ψ is *conformed* with $(\mathbf{X}, F)$.

For example, the following sentence is operational and defines the composition of partial maps (acting on the left of a set):

$$\exists x_2 \quad (x_0, x_2) \in \phi_1 \ \& \ (x_2, x_1) \in \phi_0. \tag{3.4}$$

For convenience, we will consider binary relations as multivalued partial maps and so we use (3.4) to define composition for arbitrary binary relations (which is the reverse of the standard composition of binary relations). The nullary operation constantly equal to $\varnothing$ can be given by any contradiction; for example

$$\neg x_0 \approx x_0 \ \ \& \ \ \neg x_1 \approx x_1,$$

as there are no pairs (a, b) satisfying this expression.

A triple sequence $(O_0, O_1, \ldots; P_0, P_1, \ldots; R_0, R_1, \ldots)$ consisting of two sequences of operational formulæ and one of relational formulæ will be called a *genre*. We will only consider concrete structures in which the first set of operational sentences are conformed with the corresponding binary relations, thus defining total operations. We make no such restriction on the second sequence and so these may define partial operations. In the case where the second operational sequence is empty, we say that G is a *total genre*. We say that a genre is conformed with system $(\mathbf{X}, F)$, if each operational formula in the *first* sequence is conformed with $(\mathbf{X}, F)$. Hence in this case a genre G conformed with $(\mathbf{X}, F)$ defines a corresponding list $(o_0, o_1 \ldots; p_0, p_1, \ldots; r_0, r_1, \ldots)$ of operations, partial operations and relations on $(\mathbf{X}, F)$.[2]

If $\mathcal{K}$ is a class of structures, M is a type of relation over $\mathcal{K}$ and G is a genre, then we let $C(\mathcal{K}, M, G)$ denote the class of concrete relation algebras $\langle(\mathbf{X}, F); O, P, R\rangle$ for which $\mathbf{X} \in \mathcal{K}$, F is a set of binary relations over X of type M and G is a (partial) genre conformed with $(\mathbf{X}, F)$ and defining the operations O, partial operations P and relations R. We let $\mathrm{Rep}(\mathcal{K}, M, G)$ denote the abstract class of (partial) structures isomorphic to a member of $C(\mathcal{K}, M, G)$. Of course, if G is a total genre then $\mathrm{Rep}(\mathcal{K}, M, G)$ consists of conventional structures (that is, no partial operations are present).

We now state a version of Schein's Fundamental Theorem of Relation Algebras. Recall that a set S is *recursive* if there is an algorithm deciding membership in S.

Fundamental Theorem of Relation Algebras 3.1. *Let $\mathcal{K}$ be a class of structures closed under ultraproducts, M be a type of binary relation and G be a genre. Then $\mathrm{Rep}(\mathcal{K}, M, G)$ is a universal class. If G is a total genre, $\mathcal{K}$ is recursively axiomatisable and M and G are recursive sets of formulæ, then there is a recursive axiomatisation for $\mathrm{Rep}(\mathcal{K}, M, G)$ by universal sentences.*

Proof. Proofs of the statements when G is a total genre are given in [31]. We here prove the statement about arbitrary genre, which is very similar to the corresponding statement about total genre.

Certainly $\mathrm{Rep}(\mathcal{K}, M, G)$ is trivially closed under taking substructures. Hence to show it is a universal class it remains to show that it is closed under ultraproducts. It suffices to show that an ultraproduct of members of $C(\mathcal{K}, M, G)$ is contained in $\mathrm{Rep}(\mathcal{K}, M, G)$. Let $\{\mathbf{S}_i : i \in I\}$ be a nonempty family of concrete structures in $C(\mathcal{K}, M, G)$; say each $\mathbf{S}_i$ is $\langle(\mathbf{X}_i, F_i); o_0, \ldots, p_0, \ldots, r_0, \ldots\rangle$. For simplicity, we will assume that there are no operations or relations: these are covered by Schein's proof in [31] (alternatively, the operations are really just partial operations that happen to be total, while we can replace each relation r by the partial operation equal to the first projection on r).

[2]Because both operations and partial operations are defined in exactly the same way by a genre, there is a sense in which our insistence that some operations be predetermined as "total" is artificial: all operations are really partial! Nevertheless, we continue the charade...

Let $\mathcal{U}$ be an ultrafilter on I. When partial operations are present the definition of the ultraproduct $\prod_{\mathcal{U}} \mathbf{S}_i$ (which we abbreviate to $\mathbf{S}$) is slightly different from at least one commonly given total structure version so we briefly recall it here.

For each tuple $a \in \prod_{i\in I} S_i$ we use the notation $[a]$ to denote the set consisting of all $b \in \prod_{i\in I} S_i$ for which $\{i \in I \mid a(i) = b(i)\} \in \mathcal{U}$; these sets obviously form a partition of $\prod_{i\in I} S_i$. In the total structure case the corresponding equivalence $\equiv_{\mathcal{U}}$ is a congruence on the direct product but this is not necessarily the case for partial algebras. Nevertheless, the underlying set S of $\mathbf{S}$ is still the set $(\prod_{i\in I} S_i)/{\equiv_{\mathcal{U}}}$.

For each fundamental operation p (of arity $n \geq 0$ say) the domain of $p^{\mathbf{S}}$ is the subset of S^n consisting of the tuples $([a_0], \ldots, [a_{n-1}])$ for which

$$\{i \in I \mid (a_0(i), \ldots, a_{n-1}(i)) \in \mathrm{dom}(p^{\mathbf{S}_i})\} \in \mathcal{U}.$$

The value of $p^{\mathbf{S}}$ applied to a tuple $([a_0], \ldots, [a_{n-1}])$ in its domain is $[a]$, where a is any tuple for which

$$\{i \in I \mid a(i) = p^{\mathbf{S}_i}(a_0(i), \ldots, a_{n-1}(i))\} \in \mathcal{U}.$$

We need to show that $\mathbf{S}$ is isomorphic to a concrete relation algebra from $C(\mathcal{K}, M, G)$. Let $\mathbf{X}$ denote the ultraproduct $\prod_{\mathcal{U}} \mathbf{X}_i$, which is in $\mathcal{K}$ because $\mathcal{K}$ is closed under taking ultraproducts. The binary relations F on X will correspond to the elements of $\prod_{\mathcal{U}} F_i$ as follows: for $[u], [v] \in X$ and $f \in \prod_{i\in I} F_i$ we write $([u], [v]) \in [f]$ if and only if $\{i \in I \mid (u(i), v(i)) \in f(i)\} \in \mathcal{U}$.

Now $\prod_{\mathcal{U}} F_i$ is the underlying set S of $\mathbf{S}$, so the sets S and F are essentially the same thing. Łos's Theorem shows that $(\mathbf{X}, F)$ is of type M and the genre G defines a partial structure $\mathbf{T}$ from $(\mathbf{X}, F)$ (with underlying set F). We are going to prove that $\mathbf{S} \cong \mathbf{T}$. It remains to show that for all fundamental partial operations p, we have $p^{\mathbf{S}}$ (as defined on the ultraproduct of structures) coincides with $p^{\mathbf{T}}$ (as defined by the genre G).

Let p be the operation symbol defined by some operational formula Φ. Say that $p^{\mathbf{S}}$ is defined at $([f_0], \ldots, [f_{n-1}])$ and equal to $[f]$. We show that $p^{\mathbf{T}}$ is also defined and takes the same value $[f]$. Put

$$A := \{i \mid f(i) = p^{\mathbf{S}_i}(f_0(i), \ldots)\} = \{i \mid f(i) = \{(a, b) \mid \mathbf{S}_i \models \Phi(a, b, f_0(i), \ldots)\}\}.$$

Say that $([u], [v]) \in [f]$ and let $B := \{i \mid (u(i), v(i)) \in f(i)\} \in \mathcal{U}$. So for every i in the set $A \cap B \in \mathcal{U}$ we have $\mathbf{S}_i \models \Phi(u(i), v(i), f_0(i), \ldots)$. Hence $\mathbf{T} \models \Phi([u], [v], [f_0], \ldots)$ by Łos' Theorem. This proves that $p^{\mathbf{S}}([f_0], \ldots, [f_{n-1}])$ is a subset of the solution set to $\Phi(\_, \_, [f_0], \ldots, [f_{n-1}])$ on $\mathbf{T}$. But if $\mathbf{T} \models \Phi([u], [v], [f_0], \ldots, [f_{n-1}])$ then by Łos' Theorem we have

$$D := \{i \mid \mathbf{S}_i \models \Phi(u(i), v(i), f_0(i), \ldots)\} \in \mathcal{U},$$

showing that on $A \cap D$ we have $(u(i), v(i)) \in f(i)$. Hence $([u], [v]) \in [f]$ and so we can deduce that $p^{\mathbf{T}}([f_0], \ldots, [f_{n-1}]) = [f]$.

Now say that $p^{\mathbf{S}}$ is not defined at $([f_0], \ldots, [f_{n-1}])$. So the set

$$A := \{i \mid (f_0(i), \ldots) \notin \mathrm{dom}(p^{\mathbf{S}_i})\}$$

is in $\mathcal{U}$. Let $g(i) \subseteq X_i \times X_i$ denote the solution set to $\Phi(\_, \_, f_0(i), \ldots)$ on $\mathbf{S}_i$. So for each $i \in A$ we have $g(i) \notin F_i$. Let $[f] \in F$ be arbitrary. We show that $[f]$ is not the solution set of $\Phi(\_, \_, [f_0], \ldots)$ on $\mathbf{T}$; hence $p^{\mathbf{T}}$ is also undefined at $([f_0], \ldots, [f_{n-1}])$. For each $i \in A$ we have $f(i) \neq g(i)$ so the sets $B := \{i \in A \mid f(i) \subsetneq g(i)\}$ and $C := \{i \in A \mid f(i) \not\subseteq g(i)\}$ are complementary subsets of the set $A \in \mathcal{U}$. Precisely one of these sets is in $\mathcal{U}$ because $\mathcal{U}$ is an ultrafilter. If it is B, then we may take, for each $i \in B$, a pair $(u(i), v(i)) \in g(i) \backslash f(i)$. If we extend these to I-tuples (u, v) arbitrarily then Łos' Theorem gives $\mathbf{T} \models \Phi([u], [v], [f_0], \ldots)$ but $([u], [v]) \notin [f]$. A dual argument works for case C. So we deduce that $p^{\mathbf{T}}$ is undefined at $([f_0], \ldots, [f_{n-1}])$.

We have shown that $\mathbf{S}$ and $\mathbf{T}$ have precisely the same elements and all partial operations are defined and equal in the same places. Hence $\mathbf{S} \cong \mathbf{T}$ (they differ only in the symbols for the operations), and so $\mathbf{S} \in \mathrm{Rep}(\mathcal{K}, M, G)$ as required. □

In fact all of the classes $\mathrm{Rep}(\mathcal{K}, M, G)$ we are going to encounter are additionally closed under taking nonempty direct products and so are uH-classes. To see why this is the case we are going to make some further definitions. Let $\{X_i : i \in I\}$ be an indexed set of pairwise disjoint sets. If for each $i \in I$, we have a binary relation $f_i \subseteq X_i \times X_i$, then we write $\bigcup_{i \in I} f_i$ for the binary relation on $\bigcup_{i \in I} X_i$ given by the set theoretic union of the f_i. In many key examples of interest—such as for the class of sets, partial orders or quasi-orders (that is, transitive and reflexive relations)—the disjoint union $\bigcup_{i \in I} X_i$ can be made into a member of $\mathcal{K}$ in a canonical way. For example, if each X_i is preordered by some preorder $\preceq_i$, then we can define the preorder $\preceq$ on $\bigcup_{i \in I} X_i$ to be $\bigcup_{i \in I} \preceq_i$. In these situations, we will say that $\mathcal{K}$ is *closed under taking canonical disjoint sums*.

Let

$$\{\langle (\mathbf{X}_i, F_i); o_{i,0}, o_{i,1}, \ldots, p_{i,0}, p_{i,1}, \ldots, r_{i,0}, r_{i,1}, \ldots \rangle \mid i \in I\}$$

be a family of members of $C(\mathcal{K}, M, G)$. We let X abbreviate $\bigcup_{i \in I} X_i$, $\mathbf{C}_i$ abbreviate $\langle (\mathbf{X}_i, F_i); o_{i,0}, \ldots, p_{i,0}, \ldots, r_{i,0} \ldots \rangle$, and F abbreviate $\{\bigcup_{i \in I} f_i \mid (\forall i)\ f_i \in F_i\}$. The map $f \mapsto \cup_{i \in I} f(i)$ is a bijective correspondence between the cartesian product $\prod_{i \in I} C_i$ and the set F. Now define the operations, partial operations and relations on F so that this map is an isomorphism: for an operation or partial operation $q_k \in \{o_k, p_k\}$ we require $q_k(f_0, \ldots, f_{n-1}) = \bigcup_{i \in I} q_{i,k}(f_0(i), \ldots, f_{n-1}(i))$ (here it is implicit that $q_{i,k}(f_0(i), \ldots, f_{n-1}(i))$ is defined for each i). For a relation r_k we have $(f_0, \ldots, f_{n-1}) \in r_k$ if and only if $(f_0(i), \ldots, f_{n-1}(i)) \in r_{i,k}$ for each $i \in I$. We say that $C(\mathcal{K}, M, G)$ is *closed under disjoint sums* if $\mathcal{K}$ is closed under taking canonical disjoint unions and for any such choice of $\{\mathbf{C}_i : i \in I\}$ from $C(\mathcal{K}, M, G)$ where the sets X_i are pairwise disjoint, then $(\mathbf{X}, F)$ is a system of binary relations of type M over $\mathcal{K}$ and the genre G is conformed with $(\mathbf{X}, F)$ and the corresponding operations, partial operations and relations defined by G coincide with those just described.

Theorem 3.2. *If $C(\mathcal{K}, M, G)$ is closed under disjoint sums, then $\mathrm{Rep}(\mathcal{K}, M, G)$ is closed under nonempty direct products and is a uH-class. If $\mathrm{Rep}(\mathcal{K}, M, G)$ contains the one-element structure on the universe $\{0\}$ in which all partial operations are total and all (n-ary) relations are $\{0\}^n$, then $\mathrm{Rep}(\mathcal{K}, M, G)$ is a quasivariety.*

Proof. By definition, if $\mathbf{C}$ is the disjoint union of $\{\mathbf{C}_i \mid i \in I\}$ for $\mathbf{C}_i \in C(\mathcal{K}, M, G)$ then $\prod_{i \in I} \mathbf{C}_i \cong \mathbf{C}$ under the map $f \mapsto \cup_{i \in I} f(i)$. So $\mathrm{Rep}(\mathcal{K}, M, G)$ is closed under nonempty direct products. The second statement ensures that the class is closed under the empty direct product (which is isomorphic to the described one-element structure). □

This theorem only gives one possible way at which the Fundamental Theorem refines to a quasivariety, but it is sufficient to cover all of the examples in this article.

Example 3.3. *Let $\mathcal{K}$ be the class of sets, partially ordered sets, or preordered sets. Let M be any one of the following types of relation: arbitrary binary relations; reflexive binary relations; transitive binary relations; symmetric binary relations; transformations; partial maps; injective partial maps; injective transformations; permutations; order preserving partial (injective) maps on a partial order or a preorder. Let G be a genre defining any of the following operations, partial operations and relations: composition; inversion; intersection; union; the distinguished elements* id, $\varnothing$; *and the binary relation of containment* $\subseteq$.

Whenever these choices of $\mathcal{K}$, M and G make sense, $\mathrm{Rep}(\mathcal{K}, M, G)$ *is a quasivariety.*

Proof. In each case, the class $C(\mathcal{K}, M, G)$ is easily seen to be closed under disjoint sums and the one-element algebra is representable. □

We note that two obvious examples of operations whose definition does not translate across disjoint unions of sets are complementation of binary relations and the nullary operation ∇. Tarski's relation algebras are endowed with operations modelling both of these and the class of representable members fail to form a quasivariety[3].

Part 2. Characterisations of partial map algebras

The reader is directed to [31] for an extensive overview of known characterisations of representable algebras. We concentrate almost entirely on representations as systems of partial maps. There are many examples of these in [31], which we do recall briefly here, however we concentrate most of our efforts on examples found since the appearance of [31]. We will also restrict our attention to genre defining the operation of composition (so all our classes are enrichments of semigroups).

We let $\mathcal{P}(X)$ denote the set of all partial maps on the set X. Likewise $\mathcal{B}(X)$, $\mathcal{I}(X)$, $\mathcal{S}(X)$, and $\mathcal{T}(X)$ denote respectively, binary relations, injective partial maps, permutations and transformations on the set X.

4. Partial maps

The case of partial maps with composition is well known and elementary but also fundamental and so we take the time to list some of the important variants.

Each of $\langle \mathcal{P}(X); \circ\rangle$, $\langle \mathcal{B}(X); \circ\rangle$ and $\langle \mathcal{T}(X); \circ\rangle$ are semigroups, and the usual regular representations show that every semigroup embeds into some $\langle \mathcal{T}(X); \circ\rangle$, whence into $\langle \mathcal{P}(X); \circ\rangle$ and $\langle \mathcal{B}(X); \circ\rangle$. So the class of all semigroups is the class of abstract algebras of binary relations, partial maps or transformations under composition.

The case of semigroups representable as permutations on a set coincides with the class of semigroups embeddable in the semigroup reduct of a group. This class was described by Malcev and is a proper quasivariety with no finite axiomatisation. An infinite system of axioms is known, but too complicated to describe here (see [5]).

The case of semigroups of injective partial maps coincides with the class of semigroups embeddable in the semigroup reduct of an inverse semigroup. This class was described by Schein (see [29] and [37]; an alternative proof can be found

[3]The variety generated by the representable relation algebras (in the sense of Tarski) is often called the *variety of representable relation algebras*, however the notion of representability is weaker than what we are using here: Tarski's variety of representable relation algebras consists of subdirect products (rather than isomorphic copies) of actual algebras of relations.

in [44]) and is a proper quasivariety with no finite axiomatisation. An infinite system of axioms is known, but too complicated to describe here (see [37]). We note that Schein obtains several descriptions of this class, one of which is easily seen to be testable in polynomial time for finite semigroups. This characterisation can be found in [5] (a different algorithm is given in [46]).

We complete this section with a quite elementary but contrasting observation. The Fundamental Theorem of Relation Algebras also allows the possibility that our partial maps are not closed under the operation of composition: indeed, Example 3.3 shows that the corresponding class of partial algebras is a quasivariety. This class consists of some kind of partially defined semigroup: there is a partially defined binary operation $\cdot$ (representing composition) on a set satisfying the property that if $x(yz)$ and $(xy)z$ are both defined then they are both equal. However this property alone is insufficient to describe the class. The class of semigroups is a subvariety of this quasivariety defined by the axiom $xy \approx xy$ (see Example 2.5 for a similar example), but in contrast to the totally defined case we have the following (following standard terminology, an algebra with a single binary operation will be called a *groupoid*, while a partial algebra with a single partial binary operation will be called a *partial groupoid*).

Theorem 4.1. *There is no algorithm to decide if a finite partial groupoid is representable as a system of partial maps under composition.*

Proof. Most of the work is done for us by a well known result of Evans: for any variety of algebras $\mathcal{V}$, the class of partial structures embeddable in a member of $\mathcal{V}$ is recursive if and only if the uniform word problem for $\mathcal{V}$ is decidable. The uniform word problem is undecidable for semigroups, and so there is no algorithm to decide when a finite partial groupoid is embeddable in a semigroup.

Given a partial groupoid $\mathbf{P} = \langle P; \cdot \rangle$, let $\mathcal{P}$ denote the set of partial algebras whose underlying set is P and whose single partial operation $\odot$ extends the operation $\cdot$. The members of $\mathcal{P}$ we will call *partial completions* of $\mathbf{P}$. It is not hard to see that $\mathbf{P}$ is embeddable in a (totally defined) semigroup if and only if only at least one member of $\mathcal{P}$ is representable. Indeed, if $\mathbf{P}$ is embeddable in a semigroup $\mathbf{S}$ (under ι, say), then that embedding followed by the usual regular representation for $\mathbf{S}$ ($\phi : \mathbf{S} \to \langle \mathcal{T}(S^1); \circ \rangle$, say) provides a representation of some partial completion of $\mathbf{P}$ (simply restrict composition to domain $(\phi \circ \iota(P))^2$ and range $\phi \circ \iota(P)$). Conversely, if some partial completion is isomorphically representable as partial maps on some set X, then as $\mathbf{P}$ embeds into each of its partial completions, it follows that $\mathbf{P}$ is embeddable into the semigroup $\langle \mathcal{P}(X); \circ \rangle$.

By the Evans result it follows that there is no algorithm deciding representability for partial groupoids as partial maps (binary operations, transformations) on a set under composition. □

We also note that because the Evans result holds in any variety with unsolvable word problem (groups, rings, inverse semigroups) this result holds in the partial algebra version of many other standard classes of algebras of relations.

5. Other classes covered in Schein's 1970 survey

It is not possible here to give a reasonable account of the many other classes surveyed in Schein's 1970 survey [31]. In the following list we present a definition of each of the characterised algebras of partial maps with composition that are surveyed in [31] (composition is assumed throughout). We stress that there are many more cases in [31] that relate to other types of binary relations, or to partial maps without composition but we do not list these here. We mention also that Schein's definition of composition is dual to that given here so that a representation ψ has $\psi_{x\cdot y} = \psi_y \circ \psi_x$. Because of this, some trivial modifications are required to translate between the results presented in [31] and those given here.

(1) $\subseteq$. The semigroups with a partial order $\leq$ representable in $\langle \mathcal{P}(X); \circ, \subseteq\rangle$ for some X were characterised by Schein (see [31, p. 38]) and are a finitely axiomatisable quasivariety.

(2) $\{\ulcorner\}$ and $\{\llcorner\}$. The binary relation $\ulcorner$ is defined on $\mathcal{P}(X)$ by $f \ulcorner g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$. The relation $\llcorner$ is the same except is defined for ranges of elements. Schein has characterised the corresponding abstract semigroups with any of the combinations $\{\ulcorner\}$, $\{\llcorner\}$, $\{\ulcorner, \llcorner\}$, $\{\ulcorner, \leq\}$; see [31, pp. 39–42]. they are finitely axiomatisable quasivarieties.

(3) $\{\sqcap\}$ and $\{\sqcup\}$. The binary relation $\sqcap$ is the largest equivalence contained in $\ulcorner$; so $f \sqcap g$ if $f \ulcorner g$ and $g \ulcorner f$. The binary relation $\sqcup$ is defined in the same way but using $\llcorner$. Schein has characterised the representable semigroups with any of the combinations $\{\sqcap\}$, $\{\sqcup\}$, $\{\sqcap, \sqcup\}$, $\{\sqcup, \leq\}$; see [31, p. 46]. Each case is described by a necessarily infinite system of quasi-identities (see also [35]). Trokhimenko has described the representable semigroups with $\{\sqcap, \leq\}$.

(4) $\{\downarrow\}$. The binary relation $\downarrow$ is defined by $f \downarrow g$ if $f(a) = g(a)$ whenever both sides of this equality make sense. Schein has characterised the semigroups with either of the two combinations $\{\ulcorner, \downarrow\}$ and $\{\downarrow\}$; see [31, p.48].

(5) $\{\triangleright\}$ and $\{\triangleleft\}$. The binary operations $\triangleright$ and $\triangleleft$ are defined by letting $f \triangleright g$ be the restriction of g in its domain to the domain of f and $f \triangleleft g$ be the restriction of f in its range to the range of g. These operations are known as the first and second restrictive multiplications, and the survey [31] details a number of interesting results concerning these even before composition is introduced. These structures are known as *restrictive bi-semigroups*. In the case when composition is present, both the case of semigroups with $\{\triangleright\}$ and semigroups with $\{\triangleright, \triangleleft\}$ have been characterised by Schein [30] (see [31, p. 52]). The first case (called *1-stacks* by Schein) is a finitely based variety, while the second is a finitely axiomatised quasivariety (*stacks*).

(6) $\{\perp\}$. The binary relation $\perp$ is defined by $f \perp g$ if $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \varnothing$. This relation is not actually discussed in the survey [31], but is characterised by Schein in [32].

This list does not include the class of *function systems* of Schweizer and Sklar [42] (which are considered in [31]) because these also fall into the realm of the semilattice valued operations on which we want to elaborate; see Section 13. We note also that a number of the relations just described have been generalised to "multiplace" functions on a set (functions from $X^n \to X$) and corresponding characterisations have been obtained. These can be found in the survey article by Schein and Trokhimenko [38].

6. Partial maps with intersection

Infimums in the order $\subseteq$ on $\mathcal{P}(X)$ coincide with the operation of intersection. This operation is clearly covered by the Fundamental Theorem and Theorem 3.2.

It was Garvac'kiĭ who first characterised the semigroups with $\wedge$ representable in the concrete algebras of the form $\langle \mathcal{P}(X); \circ, \cap \rangle$. The class is a variety that can be defined within semigroups with a semilattice $\wedge$ by the extra laws:

(6.1) $x(y \wedge z) \wedge xy \approx x(y \wedge z)$ (which simply says that $\cdot$ preserves order from the left);

(6.2) $(x \wedge y)z \approx xz \wedge yz$ (right distributivity, which implies $\cdot$ preserves order from the right);

(6.3) $x(a \wedge b \wedge c) \wedge y(b \wedge c) \approx x(a \wedge c) \wedge y(b \wedge c)$.

Garvac'kiĭ also describe characterised the abstract semigroups of injective partial maps with the semilattice operation of intersection. As well as the left and right distributivity laws for $\cdot$ over $\wedge$, the law $xv \wedge uv \wedge uy \leq xy$ must hold.

7. Partial maps with ∩ and ∪

The partial structure $\langle \mathcal{P}(X); \circ, \cap, \cup \rangle$ has $\circ$ and $\cap$ total but $\cup$ partial, which makes the following problem of some interest if the answer is positive.

Problem 7.1. *Is there a nice axiomatisation of the semigroups with $\wedge$ and partial $\vee$ representing partial maps with composition, intersection (operations) and join (partial operation)?*

Theorem 3.2 is easily used to show that this class is a quasivariety. We note that the domain of definition of $\vee$ on $\mathcal{P}(X)$ is precisely the relation $\downarrow$.

If we ask that $\vee$ be total, then $\langle \mathcal{P}(X); \circ, \cap, \cup \rangle$ is no longer a meaningful example, but the corresponding class of abstract algebras is still a quasivariety. This quasivariety was investigated by Schein and turns out to have a simple axiomatisation.

Theorem 7.2. [33] *A semigroup $\mathbf{S} = \langle S; \cdot, \wedge, \vee \rangle$ with semilattice $\wedge$ and $\vee$ is isomorphic to a semigroup of partial maps under the operations of composition, intersection and union if and only if $\langle S; \wedge, \vee \rangle$ is a distributive lattice, $\cdot$ distributes over $\vee$ and $\wedge$ from the left and right and $x \wedge yz \approx xz \wedge y$ is satisfied.*

The case where $\wedge$ is not present is of similar interest.

Theorem 7.3. [34] *A semigroup $\mathbf{S} = \langle S; \cdot, \vee \rangle$ with semilattice $\vee$ is isomorphic to a semigroup of partial maps under the operations of composition and union if and only if $\cdot$ distributes over $\vee$ from the left and right and the following quasi-identity holds:*

- $s \leq t \vee uv \rightarrow s \leq t \vee us$,

where $x \leq y$ stands for $x \vee y \approx y$.

Schein further shows that the quasivariety described in this theorem is not a variety.

Amongst other results of interest in [34], Schein obtains or states axiomatisations for the corresponding classes representable as injective partial maps.

8. Difference semigroups

The operation of set subtraction is well defined on $\mathcal{P}(X)$ (as a subset of $\mathcal{B}(X)$) and expressive enough to define $\cap$: we have $f \cap g = f \backslash (f \backslash g)$. A characterisation of algebras representable as systems of partial maps with composition and set subtraction is given in [36].

Theorem 8.1. *A semigroup with additional binary operation $\langle S; \cdot, - \rangle$ is representable as a system of partial maps under composition and set subtraction if and only if the following identities hold*:

(8.1) $x - (y - x) \approx x$;
(8.2) $x - (x - y) \approx y - (y - x)$;
(8.3) $(x - y) - z \approx (x - z) - y$;
(8.4) $(x - y)z \approx (xz) - (yz)$;
(8.5) $x \leq y \rightarrow z(y - x) \approx (zy) - (zx)$.

The last quasi-identity is equivalent to the identity $z(x - (x - y)) \approx zx - z(x - y)$ *and so the class is a variety.*

9. Semilattice valued operations

Most of the remaining operations on $\mathcal{P}(X)$ that have been studied have their ranges inside the semilattice of restrictions of the identity map on X; we denote this by $\Delta(X)$. There are two main kinds in the literature: those relating to the domain of maps; and those relating to the range of maps. In this section we set up some general definitions that serve as a useful first approximation for representability.

Let **S** be a semigroup with a subsemilattice **E**. We begin by defining a family of operations that might possibly make sense on **S**, depending on the relationship between **E** and **S**. In the particular case where $\mathbf{S} = \langle \mathcal{P}(X); \circ \rangle$ and $\mathbf{E} = \Delta(X)$ (and more generally where **S** is any subsemigroup of $\langle \mathcal{B}(X); \circ \rangle$ containing $\Delta(X)$), these operations are easily seen to be well defined.

- $R(x) = \min\{e \in E \mid xe = x\}$. On $\mathcal{P}(X)$, we have $R(f)$ equal to the identity on the domain of f.
- $I(x) = \max\{e \in E \mid xe = e\}$. On $\mathcal{P}(X)$, we have $I(f)$ equal to the identity on the fix-set of f.
- (If **S** contains a 0 element and $0 \in E$.) $P(x) = \max\{e \in E \mid xe = 0\}$. On $\mathcal{P}(X)$, we have $P(f)$ equal to the identity on the points where f is undefined.
- (If **S** contains an identity element 1 and $1 \in E$.) $x \bowtie y = \max\{e \in E \mid xe = ye\}$. On $\mathcal{P}(X)$, we have $f \bowtie g$ equal to the identity on the places where f and g do not disagree.
- (If **S** already has R defined on it as above.) $x * y = \max\{e \in E \mid xe = ye \ \& \ eR(x)R(y) = R(x)R(y)\}$. On $\mathcal{P}(X)$, we have $f * g$ equal to the identity on the places where f and g agree.

Each of these operations has a corresponding range version obtained by reversing the products in the definition. We denote these by L, J, Q, $\boxtimes$ and $\star$ respectively.

It is also easy to give a definition of these operations on $\mathcal{P}(X)$ by way of operational formula in the sense of Section 3. For example the operation $\bowtie$ can be given by

$$\forall x_2 \quad x_0 \approx x_1 \ \& \ \left((x_0, x_2) \in \phi_0 \leftrightarrow (x_0, x_2) \in \phi_1\right).$$

Furthermore the operations are closed under disjoint unions in the sense of Theorem 3.2 and the one-element algebra is representable (if we allow representability on the empty set so that id $= \varnothing$). So Theorem 3.2 shows that the class of representable semigroups endowed with any combination of $\circ, R, L, I, J, P, Q, \bowtie, \boxtimes, *, \star, 0, 1$ is a quasivariety.

With the exception of the operation $\vee$ (which is not total on $\mathcal{P}(X)$), all of the previous relations and operations in this article are term operations in some usually quite simple combination of these semilattice valued operations. In fact all of the operations and relations in Section 5 can be expressed using either composition and R, composition and L, or composition R and L. For example

$$f \downarrow g \quad \text{if and only if} \quad fR(g) = gR(f),$$

because the places where both f and g are defined are given by restricting by $R(f)R(g)$; while

$$f \perp g \quad \text{if and only if} \quad R(f)R(g) = \varnothing.$$

The operation $\wedge$ cannot be written in terms of $\circ$ and R, but we do have, for example, $x \wedge y = x(x * y) = x(x \bowtie y)$. There are many other interdependencies between various combinations of the semilattice valued operations. We examine these as they arise, but here observe that if the empty map is present, then on $\mathcal{P}(X)$, each of id, R, I, P, $*$, $\backslash$ and $\wedge$ are terms in the operation $\bowtie$. Indeed the following equalities hold for any $f \in \mathcal{P}(X)$:

- $\mathrm{id} = f \bowtie f$;
- $R(f) = f * f = (f \bowtie \varnothing) \bowtie \varnothing$;
- $I(f) = f * (f * f) = f \bowtie (f \bowtie f)$;
- $P(f) = f \bowtie \varnothing$;
- $f * g = (f \bowtie g)R(f)R(g)$;
- $f \backslash g = f((f \bowtie g) \bowtie \varnothing)$;
- $f \wedge g = f(f * g) = f(f \bowtie g)$.

We can also "dualise" the first five of these equalities, replacing $\bowtie$ by $\boxtimes$, R by L etc and reversing any products (for example, $f * g = (f \bowtie g)R(f)R(g)$ becomes $f \star g = L(g)L(f)(f \boxtimes g)$). These dual equalities happen to be true on $\mathcal{P}(X)$, and show that $\mathrm{id}, L, J, Q, \star$ can be defined using the operation $\boxtimes$. If we dualise the right hand side of the sixth and seventh of the equalities in the list, we obtain term terms modelling a range theoretic version of $\backslash$ and $\wedge$ respectively. The range dual of $\wedge$ will be discussed in Section 14.

A subset E of a semigroup $\mathbf{S}$ is simply a unary relation. In this way, the class of semigroups with distinguished subsemilattice can be thought of as a quasivariety of structures with one binary operation (the multiplication $\cdot$) and one unary relation (the subset E) satisfying the associativity condition for $\cdot$ and the quasi-identities $e \in E \ \& \ f \in E \to (ef \in E \ \& \ ee \approx e \ \& \ ef \approx fe)$ (under this formulation, E may be empty). The semigroups with R, L, I etc., as described above, are all enrichments of semigroups with distinguished idempotents, which motivates the next problem.

Problem 9.1. *Characterise semigroups with a unary relation E isomorphic to a substructure of $\langle \mathcal{P}(X); \cdot, \Delta(X)\rangle$ (for some X).*

The class is a quasivariety by Theorem 3.2, however we announce here that the authors have recently proved that it admits no finite axiomatisation (in preperation).

10. Semigroups with R

Consider a semigroup with subsemilattice $\mathbf{E}$ on which the *right closure* operation $R(x) := \min\{e \in E \mid xe = x\}$ is well defined (as described in Section 9). As

$R(S) = E$, we can omit explicit reference to $\mathbf{E}$ and attempt to axiomatise the corresponding unary semigroups. The class turns out to be a variety which can be defined by the following laws:

(10.R1) $xR(x) \approx x$;
(10.R2) $R(x)R(y) \approx R(y)R(x)$;
(10.R3) $R(R(x)) \approx R(x)$; and
(10.R4) $R(xy)R(y) \approx R(xy)$.

This class has been extensively studied by the authors under the name *RC-semigroups* (right closure semigroups). In fact these structures did not first appear in [18], since they also are introduced under the name *type SL-γ semigroups* in [1] and *right* $\mathbf{E}$-*semiadequate semigroups* in [26]. The main idea in [18] is a general overview of the various properties of RC-semigroups and how they relate to inverse semigroups, however various concrete examples are also considered. Amongst other results, a characterisation of the RC-semigroups representable in $\langle \mathcal{P}(X); \circ, R\rangle$ is given. This is also the $n = 1$ case of the much earlier paper by Trokhimenko [45]. The corresponding class is a variety defined by adjoining the following identity:

(10.R5) $R(x)y \approx yR(xy)$ (the twisted law).

Theorem 10.1. ([45], [18]) *A (finite) algebra* $\mathbf{S} = \langle S; \cdot, R\rangle$ *is representable as a system of partial maps on a (finite) set under the operations of composition and right closure if and only if* $\mathbf{S}$ *is a twisted RC-semigroup. The map* $\psi : S \to \mathcal{P}(S)$ *where for* $x \in S$ *we have* $\psi_x : R(x)S \to S$ *given by* $\psi_x(a) = xa$ *is a faithful representation of* $\mathbf{S}$ *in* $\langle \mathcal{P}(X); \circ, R\rangle$.

The twisted RC-semigroups are closely related to the weakly right ample semigroups (see [13] for example) and the right type A semigroups of Fountain [11]. These two classes are quasivarieties of unary semigroups whose variety coincides with the variety of twisted RC-semigroups. Fountain [11] solved the word problem on the free right type A semigroups, whence the equational theory of the twisted RC-semigroups.

Twisted RC-semigroups were also recently rediscovered by Manes [27] who called them *guarded semigroups*. Manes independently proved some of the results from [18] (including Theorem 10.1), as well as new results, including an extension of Fountain's work in [11]. See also [6], where a category theoretic version of Theorem 10.1 is (independently) established.

We announce here that the subclass of RC-semigroups with R representable as injective partial maps with right closure is a proper quasivariety given by the extra law $xy \approx xz \to R(x)y \approx R(x)z$ [23].

A further identity that appears naturally in any investigation into the general properties of RC-semigroups is

(10.R6) $R(x)y \approx yR(R(x)y)$ (the translucent law).

This condition is known to be equivalent to satisfaction of the following property

$$\forall x \forall y \exists z \quad R(y)x \approx xR(z)$$

and is sufficient to ensure that the partial order $\leq_R$ defined by $x \leq_R y \Leftrightarrow x = yR(x)$ is stable under left and right multiplication. (We note that on $\langle \mathcal{P}(X); \circ, R\rangle$, the order $\leq_R$ coincides with $\subseteq$.)

Another concrete algebra of partial maps is the algebra of all order preserving partial maps of a preordered (= reflexive and transitive) set $\langle X; \preceq\rangle$. We denote this by $\mathcal{P}_{\text{o.p.}}(\langle X;\preceq\rangle)$. Here we may define a variant R' of the unary operation R by the operational formula

$$\exists x_2 \exists x_3 \quad (x_0 \approx x_1) \ \& \ (x_2, x_3) \in \phi_0 \ \& \ x_0 \preceq x_2.$$

In other words, $R(x)$ is the restriction of the identity map to the smallest downset in $\preceq$ containing the domain of x. Because an intersection of any family of downsets is again a downset, the restrictions of the identity to the downsets of X form a complete subsemilattice of $\Delta(X)$, which we denote $\Delta_{\preceq}(\langle X;\preceq\rangle)$. It is easy to see that $R'(x) = \min\{e \in \Delta_{\preceq}(\langle X;\preceq\rangle) \mid xe = x\}$ and hence $\langle \mathcal{P}_{\text{o.p.}}(\langle \mathbf{X};\preceq\rangle); \circ, R'\rangle$ is an RC-semigroup.

Theorem 10.2. [22] *The following are equivalent for an algebra* $\langle S;\cdot,R\rangle$:

(1) $\langle S;\cdot,R\rangle$ *is a (finite) RC-semigroup satisfying (10.R6)*;
(2) $\langle S;\cdot,R\rangle$ *is representable as a system of order preserving partial maps on some (finite) preordered set under composition and the operation* R';
(3) $\langle S;\cdot,R\rangle$ *is representable as a system of continuous partial maps on a (finite) topological space under the operations of composition and with* $R(x)$ *equal to the restriction of the identity map to the topological closure of the domain of* x.

The representations used to prove Theorem 10.2 are variants of the regular twisted RC-semigroup representation ψ mentioned in Theorem 10.1; for example, when the topology in item (3) is discrete and the RC-semigroup is twisted, the two representations coincide.

To complete this section we give our second and last example of the Fundamental Theorem applied to partial algebras. Let us consider the algebras of partial maps on sets endowed with the genre describing the operation of composition $\circ$, right closure R and the partial operation $^{-1}$ of inverse. The Fundamental Theorem allows us to consider the case where R and $\circ$ are totally defined, but inverse may not be. So a concrete example will consist of a family F of partial maps on a set, closed under the operations of composition and R, and whenever a partial map $f \in F$ has an inverse f' in F, then $^{-1}$ is defined at f and gives the value f'. Theorem 3.2 shows that the class of representable partial algebras is a quasivariety. The algebra $\langle \mathcal{P}(X); \circ, R, ^{-1}\rangle$ is obviously one such member.

Theorem 10.3. *A partial algebra* $\langle S;\cdot,R,^{-1}\rangle$ *where* $\cdot$ *and* R *are total but* $^{-1}$ *is partial, is representable as a system of partial maps with composition,* R *and the partially defined operation* $^{-1}$ *if and only if* $\langle S;\cdot,R\rangle$ *is a twisted RC-semigroup and the following quasi-identities hold*:

- $xy \approx R(y) \ \& \ yx \approx R(y) \rightarrow y \approx x^{-1}$;
- $x^{-1} \approx x^{-1} \rightarrow x^{-1}x \approx R(x)$;
- $x^{-1} \approx x^{-1} \rightarrow xx^{-1} \approx R(x^{-1})$.

Proof. Certainly the stated laws are necessary for representability. Now we show they are sufficient.

Consider the usual (faithful) representation ψ of $\mathbf{S}$ as an RC-semigroup (Theorem 10.1). We need to show that an element x gets represented as an injective map with an inverse in $\psi(S)$ if and only if $^{-1}$ is defined at x.

Say that $^{-1}$ is defined at x. The second and third items along with the fact that ψ preserves $\cdot$ and R, show that ψ_x has an inverse and that inverse is $\psi_{x^{-1}}$.

Now say that ψ_x has an inverse, say ψ_y. Then $\psi_{yx} = \psi_y \circ \psi_x = R(\psi_x) = \psi_{R(x)}$ and $\psi_{xy} = \psi_x \circ \psi_y = R(\psi_y) = \psi_{R(y)}$, so because ψ is a faithful representation we can use the first law to deduce that $y = x^{-1}$, hence $^{-1}$ is defined at x. □

It is interesting to compare this result with Theorem 4.1, where the move to partial operations had devastating consequences to any prospect of characterising the representable partial semigroups.

Note that if we insist that $^{-1}$ in Theorem 10.3 is total, then we obtain—up to term equivalence—the class of inverse semigroups; so this class is defined within the quasivariety described in Theorem 10.3 by $x^{-1} \approx x^{-1}$ (or by $R(x) \approx x^{-1}x$).

11. Partial maps with $*, 0, 1$ and their reducts

Let $\mathbf{S}$ be an RC-semigroup. When pairwise infimums exist in the order $\leq_R$ (see Section 10), we can consider the corresponding semilattice $\wedge$. A general investigation into these structures—known as SLORCs (semilattice ordered RC-semigroups)—is given by the authors in [19]. The SLORCs form a variety which contains the concrete algebras $\langle \mathcal{P}(X); \circ, R, \cap \rangle$ (since the order $\leq_R$ coincides with $\subseteq$), however the variety of SLORCs is far broader than the class of SLORCs isomorphic to systems of partial maps. A characterisation of representable (as partial maps) SLORCs is given in [19] and also independently by the $n = 1$ case of a corresponding result by Dudek and Trokhimenko [8] (where a multiplace function version is obtained).

Theorem 11.1. *A semigroup $\langle S; \cdot, R, \wedge \rangle$ is representable as a system of partial maps under composition, right closure and intersection if and only if it is a twisted RC-semigroup with semilattice $\wedge$ satisfying (6.1), (6.2) and the law*

(11.1) $(x \wedge y) = xR(x \wedge y)$.

Let $\mathbf{S} = \langle S; \cdot, R, \wedge \rangle$ be a SLORC and define a new binary operation $*$ by $x * y = R(x \wedge y)$. Then $R(x) = x * x$ and $x \wedge y = x(x * y)$. These relationships define a term equivalence between the variety of SLORCs and a variety of semigroups with a second binary operation $*$ called *agreeable semigroups*. In the case of $\langle \mathcal{P}(X); \circ, R, \wedge \rangle$, the term function $R(x \wedge y)$ agrees with the definition of $*$ as given in Section 9. By a simple translation from $R, \wedge$ to $*$, we get a corresponding equational characterisation of the representable agreeable semigroups. As shown in [19], the required laws (along with associativity of $\cdot$) simplify to

(11.2) $x(x * x) \approx x$;
(11.3) $x * y \approx y * x$;
(11.4) $x(x * y) \approx y(x * y)$;
(11.5) $x(z * w) * y \approx (x * y)(z * w)$; and
(11.6) $(x * y)z \approx z(xz * yz)$ (known as the *twisted law* for agreeable semigroups).

The variety is known as the variety of *twisted agreeable semigroups*. When 0 and 1 are present, and we are required that these represent as $\varnothing$ and id, then the extra axioms are simply the usual multiplicative properties, along with $0*0 \approx 0$ (see [23]).

In [21] the authors have given an algorithmic description of the equational theory of twisted agreeable semigroups (with or without 0 and/or 1). This in turn provides an algorithmic solution to the equational theory of all combinations of domain operations amongst $\wedge, R, I, \triangleright$, as well as for statements involving the relations considered in Section 5, because we have seen in Section 9, that each of these is expressible using $*$.

Now recall the operation I defined in Section 9 on a semigroup with subsemilattice $\mathbf{E}$: $I(x) = \max\{e \in E \mid xe = e\}$. In [24] it is shown that the semigroups admitting such an operation are a variety—*interior semigroups*—given by the following identities:

(11.I1) $I(I(x)) \approx I(x)$;
(11.I2) $xI(x) \approx x$;
(11.I3) $I(x)I(y) \approx I(y)I(x)$;
(11.I4) $I(xy)I(y) \approx I(x)I(y)$.

The authors have recently extended Theorem 11.1 downwards to describe the representable semigroups with R and I: the required identities in addition to the twisted RC-semigroup axioms (10.R1)–(10.R5), are the laws (11.I1)–(11.I4), the laws $R(I(x)) \approx I(x)$ and $I(R(x)) \approx R(x)$ and the extra quasi-identity

(11.I5) $xy \approx y \to I(x)y \approx y$.

The nullaries 0 and 1 corresponding to $\varnothing$ and id can be included by adjoining the usual multiplicative laws for 0 and 1 and the additional law $R(0) \approx 0$ (the law $R(1) \approx 1$ is a corollary of laws (10.R1)–(10.R5)).

No characterisation is currently known for the representable semigroups with I or with I and $\wedge$, although the authors have proved that the I case is a proper quasivariety that cannot be characterised by only finitely many laws (in preparation).

Problem 11.2. *Find a characterisation of the semigroups with I or with I and $\wedge$ that are representable as partial maps with these operations.*

In the case of injective partial maps, we have $I(x) = x \wedge x^2$ and so the semigroups of injective partial maps with $\wedge$ and I are covered by Garvac'kiĭ's results; see Section 6. We report that the case of injective partial maps with I has also recently been characterised by the authors but admits no finite axiomatisation and is rather technical (in preperation).

12. Partial maps with P

A semigroup $\mathbf{S}$ with subsemilattice $\mathbf{E}$ on which the operation P of Section 9 is well defined is called *SP-semigroup* (semilattice pseudocomplemented semigroup). This general class is introduced and studied by the authors in [20]. Unlike in the corresponding case for the operation R or I, the subsemilattice $\mathbf{E}$ is not uniquely determined by P, however the class can still be axiomatised as unary semigroups using the following identities:

(12.P1) $xP(x) \approx 0$;
(12.P2) $P(0)P(x) \approx P(x)$;
(12.P3) $P(x)P(y) \approx P(y)P(x)$;
(12.P4) $P(x)P(y) \approx P(xP(y))P(y)$

(we can choose $\mathbf{E}$ to be the subsemilattice generated by $P(S)$).

No characterisation is known for SP-semigroups representable in $\langle \mathcal{P}(X); \circ, P, 0\rangle$.

Problem 12.1. *Characterise the semigroups with P representable as partial maps with P. Are they a variety?*

The operation R is a term function of $\langle \mathcal{P}(X); \circ, P, 0\rangle$: we have $R(f) = P(P(f))$. Hence the identities (10.R1)–(10.R5) that are necessary and sufficient for representability of RC-semigroups (see Theorem 10.1) translate to necessary laws for representability for SP-semigroups. Some investigation of these laws is made in [20]. We also observe that when $\wedge$ is added, $\langle \mathcal{P}(X); \circ, P, \wedge, 0\rangle$ is term equivalent with $\langle \mathcal{P}(X); \circ, \bowtie, 0\rangle$: for example we have

$$x \bowtie y = P(P(x \wedge y) \wedge P(P(x)P(y))).$$

To see this, note that for $\alpha, \beta \in \Delta(X)$, we have $P(P(\alpha) \wedge P(\beta)) = \alpha \cup \beta$, while $P(P(P(x))) \approx P(x)$ also holds. So the right hand side of the equality is the union of $P(P(x \wedge y))$ with $P(x)P(y)$; that is, the identity map on the places where x and y agree and where neither x nor y are defined.

13. R and L: Function semigroups

We use L to denote the left-sided version of the RC-semigroup operation R. So on a semigroup with subsemilattice $\mathbf{E}$, the operation L, if defined, is given by $L(x) = \min\{e \mid ex = x\}$. Axioms for these unary semigroups are given by reversing all products in laws (10.R1)–(10.R4) and replacing R by L and the resulting structures are called *LC-semigroups* (left closure semigroups). We denote these laws by (13.L1)–(13.L4) respectively.

A semigroup $\mathbf{S}$ with R and L making it an RC- and LC-semigroup is called a *two-sided C-semigroup* if $R(S) = L(S)$ (add the axioms $R(L(x)) \approx L(x)$ and $L(R(x)) \approx R(x)$). The algebra $\langle \mathcal{P}(X); \circ, R, L\rangle$ is a two-sided C-semigroup (with R,L defined relative to $\Delta(X)$), where the operation L is simply the range version of the operation R; that is, $L(f) = \mathrm{id}\restriction_{\mathrm{ran}(f)}$.

The case of representable two sided C-semigroups is one of the most interesting and natural, and has a correspondingly long history. It was Menger [28] who first set down postulates generalising semigroups of functions in a way that would allow capturing of both domain and range information. This programme was taken up by Schweizer and Sklar in a series of four papers, [39, 40, 41, 42].

In [39], the objects considered were partially ordered monoids $\mathbf{S}$ equipped with unary operations R, L such that for all $a, b \in S$,

(1) $e \leq 1 \Rightarrow ae, ea \leq a$,
(2) $b \leq a \Rightarrow b = ae$ for some $e \leq 1$,
(3) $L(a)a = a = aR(a)$,
(4) $L(ab) \leq L(a)$ and $R(ab) \leq R(b)$, and
(5) $e \leq 1 \Rightarrow R(e) \leq e$.

We shall call any such $\mathbf{S}$ a *type 1 function system.*

Proposition 13.1. *Type 1 function systems are exactly two-sided C-semigroups with identity for which the right closure is translucent and $L(S) = R(S)$, with the partial order defined to be $\leq_R$.*

Proof. Let $\mathbf{S}$ by a type 1 function system. From the corollary to Theorem 3 in [39], $R^2(a) = LR(a) = R(a)$ and $RL(a) = L^2(a) = L(a)$ for all $a \in S$. We use these facts repeatedly in what follows.

The identities $xR(x) \approx x$ and $L(x)x \approx x$ are part of the axioms for type 1 function systems, whereas the identities $R^2(x) \approx R(x)$ and $L^2(x) \approx L(x)$ were just given. Now since $R(a), R(b) \leq 1$ by Theorem 2 in [39], it follows from the second corollary to Theorem 6 in [39] that $R(a)R(b) = R(b)R(a)$. Hence also $L(a)L(b) = RL(a)RL(b) = RL(b)RL(a) = L(b)L(a)$. As for the partial order, $a \leq b$ if and only if $a = bR(a)$, so if R is a right closure, the partial order is the usual order associated with R. But for all $a, b \in S$, $L(ab) \leq L(a)$, so $RL(ab) \leq RL(a)$, so $RL(ab) = RL(a)R(RL(ab))$, and so $L(ab) = L(a)L(ab)$, so L is a left closure, and $R(ab) \leq R(b)$ implies that $R(ab) = R(b)R(R(ab)) = R(b)R(ab) = R(ab)R(b)$, and so R is indeed a right closure, and the first direction is complete.

Conversely, if $\mathbf{S}$ is a two-sided C-semigroup with identity, with $R(S) = L(S)$ and R translucent, then most of the type 1 function system axioms are immediate if $\leq$ is taken to be the usual order (induced by the right closure R), the only part which is not being the fact that $L(a) \leq a$ if $a \leq 1$. But this follows since $a \leq 1$ means $a = R(a)$, so $L(a) = LR(a) = R(a) = a$. □

Of course not all two-sided C-semigroups are type 1 function systems: notably, the example $\mathcal{B}(X)$ of binary relations on a set X, considered in [18], is not translucent on either side, although it does satisfy $L(\mathcal{B}(X)) = R(\mathcal{B}(X))$. In any case there are natural examples of one-sided C-semigroups, such as the Cartesian product of monoids example considered in [18].

The follow-up paper [40] narrows the definition further by restricting attention to cases in which the identities

(13.R7) $R(ab) \approx R(R(a)b)$ (the *right congruence condition*) and
(13.L5) $L(ab) \approx L(aL(b))$ (the *left congruence condition*)

hold; call these *type 2 function systems*. It is shown in [40] that any such $\mathbf{S}$ satisfies the twisted law $R(a)b \approx bR(ab)$; that is, $\mathbf{S}$ is twisted as an RC-semigroup with right closure R. It is known [18, Corollary 3.3] that an RC-semigroup is twisted if and only if it is translucent (10.R6) and satisfies the right congruence condition (13.R7), whence we have the following.

Proposition 13.2. *A type 2 function system is exactly a two-sided C-semigroup with identity which is twisted as an RC-semigroup and satisfies the left congruence condition as an LC-semigroup.*

A third paper [41] introduces definitions and results that appear only to make sense for two-sided cases. The results all have immediate translation via Proposition 13.2 into the language of two sided C-semigroups.

A fourth paper [42] takes an approach much closer to that of the present authors in [18] in that the partial order is made secondary and all axioms are equational in terms of L and R. Both type 1 and type 2 function systems get equational definitions, though the defining identities used are slightly different (but equivalent) to ours.

In [42], a "proof" that any type 2 function system can be embedded in the two-sided C-semigroup of all partial maps on a set is given. The representation used is the same as the one we used in [18] (see ψ of Theorem 10.1), as we now show.

Thus as in [42], for an F-system $\mathbf{S}$ and any $a \in S$, define $f_a : S'_a \to S$ by setting $f_a(x) = ax$ for all $x \in S'_a := \{x \in S \mid L(x) \leq R(x)\}$. This then is the same mapping as $\psi_a : S_a \to S$ given by $\psi_a(x) = ax$ for all $x \in S_a = \{x \in S \mid R(a)x = x\}$, used to represent any twisted RC-semigroup in [18] (see Theorem 10.1), providing only that $S_a = S'_a$. We now show this. If $x = R(a)x$, then $L(x) = L(R(a)x) = L(R(a)L(x)) = R(a)L(x)$, using the left congruence condition. Conversely, if $L(x)R(a) = L(x)$, then $x = L(x)x = R(a)L(x)x = R(a)x$.

This shows that at least the twisted RC-semigroup structure is correctly represented in [42]. However, Schein has shown (see [30] and [31]) that type 2 function systems are not representable in terms of partial maps in general: the left closure L is not representable unless an additional implication holds, namely

(13.L6) $ac \approx bc \rightarrow aL(c) \approx bL(c)$.

So the class of representable C-semigroups is a quasivariety which Schein has shown not to be a variety.

Theorem 13.3. (*Schein* [31]) *a semigroup with R and L is representable as a system of partial maps with composition, R and L if and only if it is right twisted* (*that is,* (*10.R5*)), *satisfies the left congruence condition* (*13.L5*), *the laws $R(L(x)) \approx L(x)$, $L(R(x)) \approx R(x)$, and quasi-identity* (*13.L6*).

Extending Schweizer and Sklar's notation, we will say that such a system is a *type 3 function system*.

Schein's representation is too complicated to describe here, but we note that it represents every type 3 function system as partial maps on an infinite set. The answer to the following question is unknown.

Question 13.4. *Is every finite type 3 function system finitely representable* (*that is, representable as partial maps on a finite set*)*?*

14. Partial maps with $*, \star, 0, 1$ and their reducts

The operation $\cap$ on $\mathcal{P}(X)$ also has a range version which we denote by $\curlywedge$. So $f \curlywedge g$ is the restriction of f (or g) in its range to the places where the multivalued maps f^{-1} and g^{-1} agree. Alternatively, $\curlywedge$ is simply the infimum operation for the order $\leq_L$ defined on the LC-semigroup $\langle \mathcal{P}(X); \circ, L\rangle$. Hence $\langle \mathcal{P}(X); \circ, \curlywedge, L\rangle$ is a SLOLC: the left hand version of a SLORC (in the sense of [19]; see Section 11)! The corresponding left agreeable operation $\star$ is easily seen to be the semilattice valued operation $\star$ as defined in Section 9. That is, $f \star g$ is the restriction of the identity to the points at which f^{-1} and g^{-1} are identical.

The operation J is likewise the range dual to the interior operation I. So $J(f)$ is equal to the identity on the points that are fixed by f^{-1} (that is, the points a for which $f^{-1}(a) = \{a\}$). The algebra $\langle \mathcal{P}(X); \circ, J\rangle$ is an example of a "left interior semigroup". The general class of left interior semigroups can be axiomatised by identities dual to (11.I1)–(11.I4). We denote these laws by (14.J1)–(14.J4) respectively.

The authors have recently shown that Schein's representation for type 3 function systems extends to include any combination of $I, J, \wedge, \curlywedge$; whence also to the operations $*$ and $\star$. We list only two of the most natural cases.

Theorem 14.1. [23] *A semigroup with R, L, I, J is representable in some concrete algebra $\langle \mathcal{P}(X); \circ, R, L, I, J\rangle$ if and only if it is representable as a semigroup with R and L* (*that is, is a type 3 function system*), *the laws* (*11.I1*)–(*11.I5*) *and* (*14.J1*)–(*14.J4*) *hold and each of R, L, I, J fix each others values* (*so $R(J(x)) \approx J(x)$ etc.*). *This class is a proper quasivariety.*

A semigroup with $\wedge, \curlywedge, R, L$ is representable in a concrete algebra of the form $\langle \mathcal{P}(X); \circ, \wedge, \curlywedge, R, L\rangle$ if and only if it satisfies

- *laws* (*10.R1*)–(*10.R5*),
- *laws* (*13.L1*)–(*13.L5*),

- *the laws stating that $\cdot$ preserves the order of both $\wedge$ and $\curlywedge$,*
- *right distributivity of $\cdot$ over $\wedge$,*
- *law* (6.2) *and its $\curlywedge$ dual $L(x \curlywedge y)y \approx x \curlywedge y$, and*
- *the law $acL(ac \curlywedge bc) \approx aL(c)(aL(c) \curlywedge bL(c))$.*

Note in particular that the representable semigroups with $R, L, \wedge$ and $\curlywedge$ (equivalently, $*$ and $\star$) form a variety; quasi-identity (13.L6) becomes a consequence of the given identities.

Question 14.2. *Is the equational theory of the representable semigroups with $R, L, \wedge$ and $\curlywedge$ decidable? If so, is there a good algorithm for it?*

15. Equality monoids

Finally we turn to semigroups with $\bowtie$. As established in Section 9, all of the domain functions considered in this article are term functions in $\bowtie$ and $\varnothing$, while the same is true for range operations in relation to $\boxtimes$ and $\varnothing$. Unfortunately, no characterisation of the representable algebras (even without 0) are known.

Problem 15.1. *Characterise the semigroups with $\bowtie$ that are representable as systems of partial maps with the operations of composition and $\bowtie$. Do the same for $\bowtie, 0$ and $\bowtie, \boxtimes, 0$. Is there an algorithm describing the equational theory of the corresponding algebras?*

We conjecture here that the class is a variety; indeed we show that the algebras representable over finite sets form a pseudovariety (closed under homomorphic images, subalgebras and finitary direct products). In [22], Theorem 10.2 is extended by the authors to include natural versions of both $\bowtie$ and $\boxtimes$ if both R and L are present. Also, the second author has recently adapted a result of Schein [36] to characterise the semigroups with $\bowtie$ and $^{-1}$ that are representable as injective partial maps [43] (the class is a finitely axiomatised variety).

A monoid $\mathbf{M}$ with an additional binary operation $\bowtie$ is an *Eq-monoid* if there is a subsemilattice $\mathbf{E} \leq \mathbf{M}$ with $1 \in E$ and for every $x, y \in S$ we have

(15.0) $\quad x \bowtie y = \max\{e \in E \mid xe = ye\}$.

Clearly, $\langle \mathcal{P}(X); \circ, \bowtie, \mathrm{id}\rangle$ is an Eq-monoid. The class of Eq-monoids is introduced by Fearnley-Sander and the second author [10] as part of a wider class of objects. As shown in [10], the Eq-monoids can be characterised equationally (amongst monoids with binary $\bowtie$) by the following set of identities:

(15.1) $x \bowtie x \approx 1$;
(15.2) $x(x \bowtie y) \approx y(x \bowtie y)$;
(15.3) $(u \bowtie v)(x \bowtie y) \approx (x \bowtie y)(u \bowtie v)$;
(15.4) $(u \bowtie v)(x \bowtie y) \approx (u(x \bowtie y) \bowtie v(x \bowtie y))(x \bowtie y)$.

The law $(x \bowtie y) \approx (y \bowtie x)$ is a corollary of these axioms.

Note that the semilattice $\mathbf{E}$ is uniquely determined by $\bowtie$, as it is the solution set to the equation $x = (x \bowtie 1)$.

The second author has given a far more detailed investigation into Eq-monoids in [43], including some characterisations of congruences that are useful below. For our purposes, it suffices to know that each congruence θ on an Eq-monoid $\mathbf{M}$ is determined by the congruence class containing the identity element 1 restricted to E; that is $1/\theta \cap E$. This can be proved ([43]; see also [17]) from the fact that $x \bowtie y = 1$ if and only if $x = y$, which in turn follows from (15.0).

The set $1/\theta \cap E$ is a filter of $\mathbf{E}$ (considered as a meet semilattice), but it is also closed under the implication $a \bowtie b \in 1/\theta \to ac \bowtie bc \in 1/\theta$ because the first condition says that $a/\theta = b/\theta$. We call a filter F of $\mathbf{E}$ satisfying the implication $a \bowtie b \in F \Rightarrow ac \bowtie bc \in F$ a *normal* filter. So we have seen that every congruence defines a normal filter F however it is also known that every normal filter uniquely defines a congruence $x \; \theta_F \; y \Leftrightarrow x \bowtie y \in F$. Hence the lattice of congruences of $\langle M; \cdot, \bowtie, 1\rangle$ is isomorphic with the lattice of normal filters of $\mathbf{E}$ (relative to $\mathbf{M}$).

We begin by showing that the lattice of normal filters on $\langle M; \cdot, \bowtie, 1\rangle$ is isomorphic to a sublattice of the lattice of normal filters on the sub-Eq-monoid on $\mathbf{E}$. This will show that the variety of Eq-monoids is congruence distributive, because it is known that this latter Eq-monoid is (term equivalent to) a Brouwerian semilattice, and these are already known to be congruence distributive; see [2, Section 1] (actually, when 0 is present, $\mathbf{E}$ is easily seen to be term equivalent to a Boolean algebra).

Let $\mathbf{M}$ be an EQ-monoid with subsemilattice of equality elements $\mathbf{E}$. For each $c \in M$, we have the operator $\Box_c : E \to E$ given by

$$\Box_c(e) = (ec \bowtie c).$$

It is easy to see that $\Box_c$ is an endomorphism of $\mathbf{E}$. Indeed we have

$$\begin{aligned}(c \bowtie ec)(c \bowtie fc) &= (c \bowtie efc)(c \bowtie fc)\\ &= (c \bowtie efc)(c \bowtie efc)\\ &= (c \bowtie efc),\end{aligned}$$

as follows from repeated use of (15.4) and the "replacement rule" (15.2).

Lemma 15.2. *A filter $F \subseteq E$ is normal if and only if for every $e \in F$ we have $\Box_c(e) \in F$ for every $c \in M$.*

Proof. Suppose F is normal, with $e \in F$. So $(1 \bowtie e) \in F$ and then $\Box_c(e) = (c \bowtie ec) \in F$ by normality.

Conversely, suppose F is a filter such that whenever $e \in F$, $(c \bowtie ec) \in F$. Let $e = (a \bowtie b) \in F$ and $c \in M$. Then $ae = be$, and $(c \bowtie ec) \in F$ by assumption. Hence

$$ac(c \bowtie ec) = aec(c \bowtie ec) = bec(c \bowtie ec) = bc(c \bowtie ec).$$

Hence $(ac \bowtie bc) \geq (c \bowtie ec) \in F$, so $(ac \bowtie bc) \in F$ as required. □

Lemma 15.3. *The lattice of normal filters in $\mathbf{M}$ is a sublattice of the lattice of filters of the Eq-submonoid $\mathbf{E} = \langle E; \cdot, \bowtie\rangle$.*

Proof. The meet of two normal filters is just their usual intersection.

Now for joins. For filters I, J of $\mathbf{E}$, their filter join $F := I \vee J$ is the upset of $\{ef \mid e \in I, f \in J\}$. We show that F is normal in $\mathbf{M}$ (whence $\mathbf{E}$). Now if $g \in F$, then $g \geq ef$ for some $e \in I$ and $f \in J$, so $ef = efg$. Hence if I, J are normal, $\Box_c(e) \in I$ and $\Box_c(f) \in J$, and so

$$\Box_c(g) \geq \Box_c(g)\Box_c(ef) = \Box_c(efg) = \Box_c(ef) = \Box_c(e)\Box_c(f) \in F.$$

So $\Box_c(g) \in F$ also, and so F is normal. □

Corollary 15.4. *The variety of eq-monoids is congruence distributive.*

Proof. Let $\mathbf{M}$ be an Eq-monoid. The congruence lattice of $\mathbf{M}$ is isomorphic to the normal filter lattice of $\mathbf{M}$, which by Lemma 15.3 is a sublattice of the distributive lattice of normal filters on the subalgebra of equality elements $\mathbf{E}$. □

Theorem 15.5. *The class $\mathcal{E}_{\text{fin}}$ of a finitely representable Eq-monoids is a pseudovariety.*

Proof. $\mathcal{E}_{\text{fin}}$ is certainly closed under taking subalgebras, while the concrete Eq-monoids of partial maps are closed under taking finite disjoint sums. Hence $\mathcal{E}_{\text{fin}}$ is closed under taking finitary direct products. It remains to show that homomorphic images of members of $\mathcal{E}_{\text{fin}}$ are in $\mathcal{E}_{\text{fin}}$.

Let $\mathbf{M}$ be a concrete Eq-monoid of partial maps; that is, $\mathbf{M}$ is a subalgebra of $\langle \mathcal{P}(X); \circ, \bowtie, \mathrm{id}\rangle$ for some set X. We let $\mathbf{E}$ denote the semilattice elements $M \cap \Delta(X)$. Let F be a normal filter of $\mathbf{M}$ that is principal; that is, there is $f \in E$ such that $F = \{f \in E \mid f \geq e\})$. If $\mathbf{M} \in \mathcal{E}_{\text{fin}}$ then every normal filter is principal, so it will be enough to prove that $\mathbf{M}/\theta_F$ is representable.

Let $\psi : M \to \mathcal{P}(\mathrm{dom}(f))$ be defined by $x \mapsto x\restriction_{\mathrm{dom}(f)}$.

We show that this is a Eq-monoid homomorphism with kernel $\psi^{-1}(\mathrm{id}) = 1/\theta_F$. Hence $\mathbf{M}/\theta_F$ is isomorphic to $\psi(\mathbf{M})$.

Claim. For every $x \in M$ we have $x(\mathrm{dom}(f)) \subseteq \mathrm{dom}(f)$.

Proof. Say that there is $a \in \mathrm{dom}(f)$ such that $x(a) \notin \mathrm{dom}(f)$. Then $\Box_x(f) \notin F$ as $f \circ x$ does not agree with f at the point $a \in \mathrm{dom}(f)$. By Lemma 15.2, F is not a normal filter, a contradiction. □

This claim shows that ψ preserves $\circ$. Let $\bowtie_f$ denote the $\bowtie$ operation on $\mathcal{P}(\mathrm{dom}(f))$. Then $\psi(x \bowtie y) = (x \bowtie y)\restriction_{\mathrm{dom}(f)} = x\restriction_{\mathrm{dom}(f)} \bowtie_f y\restriction_{\mathrm{dom}(f)} = \psi_x \bowtie_f \psi_y$. Finally, $\psi(\varnothing) = \varnothing$, so ψ is an Eq-monoid homomorphism.

To complete the proof we need to show that ψ has θ as its kernel. Now $e\,\theta_F\,1$ is equivalent to $e \geq f$ because F is principal. This in turn is equivalent to $\psi_e = e\restriction_{\mathrm{dom}(f)} = f\restriction_{\mathrm{dom}(f)} = \psi_1$ as required. □

The corresponding result also holds when both of $\bowtie$ and $\boxtimes$ are present; we omit the details but it is a minor modification of the proof of Theorem 15.5.

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