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INTERPRETING GRAPH COLORABILITY IN FINITE SEMIGROUPS

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We show that a number of natural membership problems for classes associated with finite semigroups are computationally difficult. In particular, we construct a 55-element semigroup **S** such that the finite membership problem for the variety of semigroups generated by **S** interprets the graph 3-colorability problem.

Keywords: Computational complexity; semigroups; variety membership problem; quasi-varieties; finite equational bases for finite algebras.

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1. Introduction

During his lectures at the conference on Structural Theory of Automata, Semigroups and Universal Algebra (a NATO Advanced Study Institute) held at the Université de Montréal from 7 to 18 July, 2003, Mikhail Volkov introduced the problem, "does there exist a finite monoid $\mathbf{M}$ such that the problem, to determine of any finite monoid $\mathbf{M}'$ whether $\mathbf{M}' \in \mathsf{HSP}(\mathbf{M})$ (the finite membership problem for $\mathsf{HSP}(\mathbf{M})$) is NP-complete?" Volkov recalled that the corresponding problem for finite general algebras was solved by Zoltan Szekely who produced (see [15]) a seven-element algebra $\mathbf{A}$ such that the finite membership problem for $\mathsf{HSP}(\mathbf{A})$ is NP-complete. In this paper, we modify Szekely's example to obtain a 55-element semigroup $\mathbf{S}$ and a 56-element monoid $\mathbf{S}^1$ such that the finite membership problems, both for the variety of semigroups generated by $\langle S, \cdot\rangle$, and for the variety of monoids generated by $\langle S^1, \cdot, 1\rangle$, are at least as difficult as determining if a finite graph is 3-colorable.

Table 1. Complexity results for semigroups and monoids: lower bounds.

K	$\mathbf{A} \in \mathsf{K}(\star)$	$\star \in \mathsf{K}(\mathbf{A})$	$\star \in \mathsf{K}(\star)$	$\mathsf{K}(\star) = \mathsf{K}(\mathbf{A})$	$\mathsf{K}(\star) = \mathsf{K}(\star)$
HSP	?	NP-hard	NP-hard	NP-hard	NP-hard
SP	?	NP↑	NP↑	NP↑	NP↑
HS	?	—	NP↑	—	GI
H	NP↑	—	NP↑	—	GI
S	?	—	NP↑	—	GI

We also look at a number of other membership problems on finitely generated classes related to a class operator K amongst S, H, HS, SP and HSP (here S, H and P denote respectively isomorphic copies of subalgebras, homomorphic images and products).

Most of these problems do not appear to have been investigated for semigroups or monoids, so we present our results in Table 1. The rows of the table correspond to choices of K. An instance of one of these problems is a finite algebra $\mathbf{B}$ (or algebras) in place of the star (or stars). When the symbol $\mathbf{A}$ appears in the column title, we mean that we can find $\mathbf{A}$ (depending on the column and on the choice of K) for which the corresponding problem has the stated complexity. NP↑ abbreviates NP-complete, P abbreviates polynomial-time, while GI abbreviates the graph isomorphism problem (the exact complexity of which is a long standing open problem; see [2]). All the results presented in Table 1 are the same for semigroups as for monoids and, except for the final three rows of Column 5, are new. We note that most of the solutions in the third and fifth column can be found for general algebras and unary algebras in [1].

For example, in the first row, column two asserts that there is a finite semigroup (monoid) $\mathbf{A}$ such that the finite membership problem for $\mathsf{HSP}(\mathbf{A})$ polynomially interprets the NP-complete graph 3-colorability problem. In the fifth row, the entry GI in the fifth column indicates that the problem of deciding if two finite semigroups (monoids) share the same subalgebras is polynomially equivalent to the problem of deciding when two finite graphs are isomorphic (this is a well-known result of Booth [2]). The entries containing question marks appear to be open, even for general algebras (note that the problem $\mathbf{A} \in \mathsf{S}(\star)$ is known to lie in P — see below — but we do not know precisely where in this class the problem lies). The blank entries of Table 1 indicate that no interesting lower bounds are possible. This is explained by Table 2, which lists the known upper bounds for the complexity of the problems in Table 1. In Table 2, when the symbol $\mathbf{A}$ appears in the column title we mean that *for every* finite algebra $\mathbf{A}$ the corresponding problem lies in the given complexity class. Thus the blank entries in Table 1 correspond to problems that can be solved in constant time for any fixed finite algebra $\mathbf{A}$. The notation 2-ET abbreviates the complexity class 2-EXPTIME corresponding to those problem solvable in doubly exponential time ($O(2^{2^{p(n)}})$ for some polynomial p). The entries in rows 2–5 of Table 2 are all quite easy and are discussed in relevant sections below. The bounds

Table 2. Complexity results for finite algebras: upper bounds.

K	$\mathbf{A} \in \mathsf{K}(\star)$	$\star \in \mathsf{K}(\mathbf{A})$	$\star \in \mathsf{K}(\star)$	$\mathsf{K}(\star) = \mathsf{K}(\mathbf{A})$	$\mathsf{K}(\star) = \mathsf{K}(\star)$
HSP	2-ET	2-ET	2-ET	2-ET	2-ET
SP	NP	NP	NP	NP	NP
HS	NP	constant	NP	constant	GI
H	NP	constant	NP	constant	GI
S	P	constant	NP	constant	GI

given in the first row (and many of the others) can be found in [1]. We note that Table 2 shows that all bounds given in rows 2–5 of Table 1 are sharp.

For further definitions and details on complexity, see [5] for example.

The most involved of our proofs are associated with the first row of Table 1; these arguments are given in Sec. 3. Section 5 contains proofs of the results in the second row, while the last three rows of both Tables 1 and 2 are given in Sec. 6. Our results from Sec. 3 also have some interesting applications to the finite basis problem for semigroups that are investigated in Sec. 4.

2. Graphs and Relational Structures

A *universal Horn class* is a class of relational structures of one signature axiomatized by a set of universal Horn sentences, that is, first-order sentences of the following kinds: $(\forall\bar{x})\,(\&_{i\in I}\Phi_i \to \Phi)$ and $(\forall\bar{x})\left(\bigvee_{i\in I}\neg\Phi_i\right)$, where I is a finite set and the Φ_i and Φ are atomic formulas. If the set of axioms consists entirely of sentences of the first kind, the axiomatized class is called a *quasi-variety*.

Let $G = \langle V_G, E_G\rangle$ be a relational structure where $E_G \subseteq V_G \times V_G$ is a binary relation on V_G. In the case where E_G is irreflexive and symmetric, this is of course a *simple graph* (that is, without loops or multiple edges). We will also use the symbol $a \sim b$ to denote $(a, b) \in E_G$. If $H = \langle V_H, E_H\rangle$ is another relational structure with E_H a binary relation on V_H, then by a *homomorphism* from G to H is meant any mapping $\varphi : V_G \to V_H$ with the property that $(\varphi(a), \varphi(b)) \in E_H$ whenever $(a, b) \in E_G$.

An *atomic formula* in the first-order language of binary relations is an expression $u \approx v$ or $u \sim v$ where u and v are variables. It is a well-known result of Mal'cev that the universal Horn class generated by G (that is, the class of all binary relational structures that satisfy all the universal Horn sentences that are valid in G) is the class of all isomorphic copies of substructures of non-empty direct products of ultrapowers of G, that is $\mathsf{SP}^+\mathsf{P}_\mathsf{u}(G)$. Similarly, the quasi-variety generated by G is the class $\mathsf{SPP}_\mathsf{u}(G)$. Note that $\mathsf{SPP}_\mathsf{u}(G)$ contains the one-element relational structure $I = \langle\{0\}, E\rangle$ with $(0, 0) \in E$ (isomorphic to the product of an empty family of structures), and we have $\mathsf{SPP}_\mathsf{u}(G) = \mathsf{SP}^+\mathsf{P}_\mathsf{u}(G)$ if and only if $I \in \mathsf{SP}^+\mathsf{P}_\mathsf{u}(G)$ if and only if $(a, a) \in E_G$ for some $a \in V_G$. If G is finite, the universal Horn class and the quasi-variety generated by G reduce to $\mathsf{SP}^+(G)$ and $\mathsf{SP}(G)$, respectively.

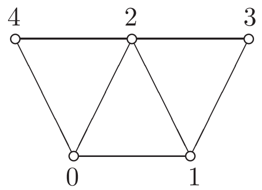
Thus, if G is finite, then a binary relational structure $H = \langle V_H, E_H\rangle$ lies in the quasi-variety generated by G if and only if for every pair $a, b \in V_H$ with $a \neq b$, there is a homomorphism $\varphi : H \to G$ with $\varphi(a) \neq \varphi(b)$ and for every pair $c, d \in V_H$ with $(c, d) \notin E_H$, there is a homomorphism $\psi : H \to G$ with $(\psi(c), \psi(d)) \notin E_G$. Also, H lies in the universal Horn class generated by G precisely if, in addition, there is at least one homomorphism from H to G (this follows from the SP, SP^+ descriptions).

The analogous notions and results are meaningful and valid for algebras, except here atomic formulas are of the form $s \approx t$ (for terms s and t), and homomorphisms are the usual thing, that is, mappings that preserve the truth of atomic formulas.

Both irreflexivity $(\forall x)(\neg x{\sim}x)$ and symmetry $(\forall x, y)(x{\sim}y \to y{\sim}x)$ of binary relations are universal Horn sentences and hence the class of all simple graphs is a universal Horn class. We note that disjunctions of negated atomic formulas are in fact equivalent to implications in structures with more than one-element: $(\forall \bar{x}) \left(\bigvee_{i\in I} \neg\Phi_i\right)$ becomes $(\forall \bar{x}, y, z)\, (\&_{i\in I} \Phi_i \to y \approx z)$, where y and z are two different variables distinct from those contained in $\bar{x}$. However, the one-element looped graph (which is not in $\mathsf{SP}^+(G)$ for any simple graph G) satisfies all implications, but fails the sentence $(\forall x)(\neg x{\sim}x)$.

Let K_n denote the complete simple graph on vertices $\{0, 1, \ldots, n-1\}$. An *n-coloring* of a graph G is a homomorphism $c : G \to K_n$. The class of all n-colorable simple graphs will be denoted by $\mathcal{C}_n$. The key to our approach (and also that of Szekely in [15]) lies in the fact that $\mathcal{C}_n$ is a universal Horn class. Finite graphs generating $\mathcal{C}_n$ were found by Nešetřil and Pultr [10] and also by Wheeler [16]. The more efficient of these constructions is the first.

For $n \geq 2$ let C_n denote the graph on vertices $\{0, 1, \ldots, n, n+1\}$ with edges making $\{0, 1, \ldots, n-1\}$ a complete graph and with additional edges (i, n), (n, i) for each $i \in \{1, \ldots, n-1\}$ and $(j, n+1)$, $(n+1, j)$ for $j \in \{0, 2, 3, \ldots, n-1\}$. This construction produces a graph isomorphic to those given in [10]. The graph C_3 has a special role in this paper.



The graph C_3

Lemma 2.1 [10]. *For $n \in \mathbb{N}$, the class $\mathsf{SP}^+(C_n)$ is equal to $\mathcal{C}_n$.*

Proof. The map $i \mapsto i \bmod(n)$ is an n-coloring of C_n. Now let $G \in \mathsf{SP}^+(C_n)$. So there is at least one graph homomorphism $\varphi : G \to C_n$. Thus G is n-colorable via the map $c : G \to K_n$ given by $v \mapsto \varphi(v) \bmod(n)$.

Now suppose that G is n-colorable, and let $c : G \to K_n$ be a coloring. If u and v are distinct elements of V_G and $u \sim v$, then c is a homomorphism into C_n separating u and v. Now assume that $\{u, v\} \subseteq V_G$ and $(u, v) \notin E_G$. If $u = v$, then $(c(u), c(v)) \notin E_{C_n}$. So we can assume that $u \neq v$. We can also assume that $c(u) = 0$. Define $\varphi : V_G \to V_{C_n}$ so that $\varphi(u) = n$ and $\varphi(x) = c(x)$ for all $x \in V_G \backslash \{u\}$. Now φ is a homomorphism of G into C_n, for if $x \in V_G$ and $(u, x) \in E_G$ then $c(x) \in \{1, \ldots, n-1\}$ and so $(\varphi(u), \varphi(x)) \in E_{C_n}$. Moreover, $\varphi(u) \neq \varphi(v)$.

We have yet to find a homomorphism $c' : G \to C_n$ with $(c'(u), c'(v)) \notin E_{C_n}$. If $c(v) = 0$ $(= c(u))$, then we can take $c' = c$. If $c(v) \neq 0$ then we can assume that $(c(u), c(v)) = (0, 1)$. Define $c'(u) = n$, $c'(v) = n + 1$, and put $c'(x) = c(x)$ for all $x \in V_G \backslash \{u, v\}$. Since $(u, v) \notin E_G$, then c' is a homomorphism $G \to C_n$, and $(c'(u), c'(v)) \notin E_{C_n}$, as required. □

For $n > 1$ it is easily seen that C_n has a minimal number of elements with respect to the property of generating $\mathcal{C}_n$. Indeed if the simple graph G generates $\mathcal{C}_n$, then as $C_n \in \mathcal{C}_n$, there exists a graph homomorphism $\varphi : C_n \to G$ with $(\varphi(n + 1), \varphi(n)) \notin E_G$. If $|G| < n + 2$ then φ must identify at least two elements of $\{0, 1, \ldots, n + 1\}$. Now φ cannot identify n and $n + 1$, because identifying these produces a complete graph on $n + 1$ vertices, which cannot be homomorphically mapped into the n-colorable graph G. As G has no loops, the only remaining identifications possible are $\varphi(n) = \varphi(0)$ or $\varphi(n + 1) = \varphi(1)$. However under either of these identifications, we obtain $\varphi(n) \sim \varphi(n + 1)$.

Szekely [15] constructs a seven-element groupoid generating a variety with NP-complete finite membership problem from a six-element graph which generates $\mathcal{C}_3$. We wish to observe that a six-element groupoid with this property can be produced in the same way, from the five-element graph C_3.

For a given simple graph $G = \langle V_G, E_G \rangle$, let G_Δ denote the graph obtained from G by adding a new vertex $w_{\{u,v\}}$ for each unordered pair of vertices u, v with $(u, v), (v, u) \in E_G$ and by adding the new edges $(u, w_{\{u,v\}})$, $(w_{\{u,v\}}, u)$ and $(v, w_{\{u,v\}})$, $(w_{\{u,v\}}, v)$. Let G_{C_n} be a graph obtained by taking the disjoint union of C_n with G and then connecting one vertex of C_n to a vertex of G (any pair will suffice). It is easy to see that if $n \geq 3$, then G is n-colorable if and only if G_Δ is n-colorable. Also, G is n-colorable if and only if $\mathsf{SP}^+(G_{C_n}) = \mathsf{SP}^+(C_n)$. Summarizing, we have the following.

Lemma 2.2. *These statements are pairwise equivalent, for a simple graph G:*

(i) *G is n-colorable*;
(ii) $G \in \mathsf{SP}^+(C_n)$;
(iii) $|\hom(G, C_n)| \geq 1$;
(iv) $|\hom(G, K_n)| \geq 1$;
(v) *G_Δ is n-colorable* (*so long as $n \geq 3$*);
(vi) $\mathsf{SP}^+(G_{C_n}) = \mathsf{SP}^+(C_n)$.

3. Variety Membership

3.1. *The construction*

Recall that B_2 denotes the five-element Brandt semigroup with zero generated by A, B subject to the relations $ABA = A$, $BAB = B$ and $A^2 = B^2 = 0$. For a given binary relational structure $G = \langle V_G, E_G\rangle$ we are going to construct a semigroup $S(G)$ embedding B_2 as follows. We assume that $V_G \cap \{A, B\} = \varnothing$. Then $S(G)$ is the semigroup with zero generated by $\{A, B\} \cup V_G$ subject to the relations:

$$A^2 = B^2 = Bx = xB = AxA = 0 \quad (\text{when } x \in V_G)$$

$$ABA = A, \quad BAB = B$$

$$xy = B \quad (\text{when } (x, y) \in E_G)$$

$$xy = 0 \ \ (\text{when } \{x, y\} \subseteq V_G \text{ and } (x, y) \notin E_G).$$

To make the construction more transparent, we recall the definition of a Rees matrix semigroup. Because our construction turns out to have only trivial subgroups, the following definition will suffice. Let I and J be sets and P be a $J \times I$ matrix over $\{0, 1\}$. The Rees matrix semigroup $\mathcal{M}[P]$ over P is the set $(I \times J) \cup \{0\}$ endowed with the multiplication $(i, j)(k, \ell) = (i, \ell)$ if $P_{j,k} = 1$ and all other products equal 0.

Let $I = V_G \cup \{A, B\}$ and let P $(= P_G)$ denote the $I \times I$ matrix with entries from the submatrix $P_{V_G \times V_G}$ corresponding to the adjacency matrix of G and all remaining entries 0, except for $P_{A,B} = P_{B,A} = 1$ (so P is the direct sum of the adjacency matrix of G with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). Then $S(G)$ is isomorphic to the semigroup defined on the set $\mathcal{M}[P] \cup V_G$ with multiplication extending that of $\mathcal{M}[P]$ by setting, for $\{x, y\} \subseteq V_G$:

$$xy = \begin{cases} (B, B) & \text{if } (x, y) \in E_G, \\ 0 & \text{otherwise,} \end{cases}$$

$$x(i, j) = \begin{cases} (B, j) & \text{if } i \in V_G \quad \text{and} \quad (x, i) \in E_G, \\ (x, j) & \text{if } i = A, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(i, j)x = \begin{cases} (i, B) & \text{if } j \in V_G \quad \text{and} \quad (j, x) \in E_G, \\ (i, x) & \text{if } j = A, \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked that the following is a one-to-one list of all the elements of $S(G)$ under the first definition:

$$A, B\ (= xy \text{ when } (x, y) \in E_G), AB, BA, 0\ (= AA)$$

$$\text{and for } \{x, y\} \subseteq V_G: \ x, Ax, BAx, xA, xAB, xAy.$$

Each element is represented above in shortest form with respect to the set of generators $\{A, B\} \cup V_G$; and for all elements except 0, the given representations are the unique shortest representations. The isomorphism between the first and second definitions is determined by its action on the generators as follows: $x \mapsto x$ for $x \in V_G$, $C \mapsto (C, C)$ for $C \in \{A, B\}$. For example, $BAx \mapsto (B, B)(A, A)x = (B, A)x = (B, x)$.

A final approach to our construction is to let $\Sigma = \{A\} \cup V_G$ and write Σ^+ for the free semigroup consisting of all finite non-void sequences from Σ, under the operation of concatenation. For $u, v \in \Sigma^+$, write $u \leq v$ to denote that we have $v = rus$ for some pair of possibly empty words r, s. Define SEQ to be the subsemigroup of Σ^+ generated by all words Axy where $(x, y) \in E_G$. Define J to be the set of all $u \in \Sigma^+$ such that $u \leq v$ holds for no $v \in$ SEQ. Now J is a two-sided ideal in Σ^+ and so we have the ideal congruence $\theta_1 = (J \times J) \cup \mathrm{id}_{\Sigma^+}$. Let θ_2 be the congruence on Σ^+ generated by all pairs (xy, uv) with $(x, y), (u, v) \in E_G$, together with all pairs $(AxyA, A)$, $(xyAxy, xy)$ with $(x, y) \in E_G$.

Then J is a union of equivalence classes for θ_2 and so the equivalence relation join of θ_1 and θ_2 is the congruence

$$\theta = (J \times J) \cup (\theta_2 \cap [(\Sigma^+ \times \Sigma^+) \backslash J]) .$$

It can be checked that $S(G) \cong \Sigma^+/\theta$. Furthermore, if we assume that no two distinct elements of V_G have identical adjacencies with respect to $\sim$, then θ can be seen to be the largest congruence on Σ^+ for which SEQ is a union of congruence classes. Hence $S(G)$ is in this case the syntactic semigroup of the language SEQ.

Note that $S(G)$ has precisely $|V_G|^2 + 5|V_G| + 5$ elements, and the construction can be created from an efficient encoding of G (say, its adjacency matrix) in polynomial time.

We write $S^1(G)$ for the monoid obtained by adjoining a unit element 1 to $S(G)$. The following lemma collects together some useful information about the subsemigroup $S^1(G)\backslash V_G$.

Lemma 3.1. *Let $x, a, a_1, \ldots, a_n \in S(G)\backslash V_G$ where $n \geq 1$, and let $b, c, d \in S^1(G)\backslash V_G$.*

(i) *$a_1 a_2 \cdots a_n \neq 0$ if and only if $a_i a_{i+1} \neq 0$ for each $i \leq n - 1$.*
(ii) *If $a_1 \cdots a_n \neq 0$ and $a_1 x a_n \neq 0$ then $a_1 \cdots a_n = a_1 x a_n$.*
(iii) *If $abca$, $abda$ and $acda$ are non-zero, then $abca = abda = acda = abcda = a$.*

Proof. The first two claims follow immediately from the definition of a Rees matrix semigroup with trivial subgroups and the second definition of $S(G)$. Now consider (iii). By (ii) (with $n = 1$ and $a_1 = a$), if these products are all non-zero, then they all equal a. If one or more of b, c, d are 1, then $abcda$ is in fact equal to one of the three given non-zero products. Otherwise $abcda$ is a product within the Rees matrix semigroup $\mathcal{M}[P]$. □

Here is the chief result of this section.

Theorem 3.2. *Let G be any finite graph and n be a positive integer. The following are equivalent*:

(i) G *is* n*-colorable*;
(ii) $S(G) \in \mathsf{HSP}(S(C_n))$;
(iii) $S^1(G_\Delta) \in \mathsf{HSP}(S^1(C_n))$ (*for* $n \geq 3$);
(iv) $\mathsf{HSP}(S(G_{C_n})) = \mathsf{HSP}(S(C_n))$;
(v) $\mathsf{HSP}(S^1((G_{C_n})_\Delta)) = \mathsf{HSP}(S^1(C_n))$.

If G is the 5-element graph C_3, then $S(C_3)$ is a 55-element semigroup. Since the problem to determine if G is 3-colorable is NP-complete, we obtain the following corollary.

Corollary 3.3. *The following problems are NP-hard*:

(i) $\star \in \mathsf{HSP}(S(C_3))$;
(ii) $\star \in \mathsf{HSP}(S^1(C_3))$ (*for semigroups or monoids*);
(iii) $\mathsf{HSP}(\star) = \mathsf{HSP}(S(C_3))$;
(iv) $\mathsf{HSP}(\star) = \mathsf{HSP}(S^1(C_3))$ (*for semigroups or monoids*).

In order that we can derive some other interesting corollaries, we prove Theorem 3.2 by way of the following two lemmas.

Lemma 3.4. *Let G and H be binary relational structures. If $H \in \mathsf{SP}^+(G)$ then $S(H) \in \mathsf{HSP}(S(G))$ and $S^1(H) \in \mathsf{HSP}(S^1(G))$.*

Lemma 3.5. *Let $H = \langle V_H, E_H \rangle$ be a finite non-directed graph*, *and $G = \langle V_G, E_G \rangle$ be a finite binary relational structure. If $S(H) \in \mathsf{HSP}(S(G))$*, *then $|\mathrm{hom}(H,G)| \geq 1$. If $S^1(H) \in \mathsf{HSP}(S^1(G))$ and every pair of adjacent elements in V_H forms part of a triangle*, *then $|\mathrm{hom}(H,G)| \geq 1$.*

In view of the equivalences of Lemma 2.2, Theorem 3.2 will follow from Lemmas 3.4 and 3.5.

3.2. *Proof of Lemma 3.4*

Let $G = \langle V_G, E_G \rangle$ and $H = \langle V_H, E_H \rangle$ be binary relational structures with $H \in \mathsf{SP}^+(G)$. We are going to find $S(H)$ as a quotient of a subsemigroup of $S(G)^{\mathrm{hom}(H,G)}$ which is the direct power of $S(G)$ consisting of all maps from $\mathrm{hom}(H,G)$ into $S(G)$. We denote this semigroup by $S_{H \to G}$.

If $\alpha : H \to G$ is a homomorphism, there is an associated map $\bar{\alpha}$ from $S(H)$ into $S(G)$ defined as follows. We let $\bar{\alpha}$ agree with α on V_H, and be the identity on $\{A, B, 0\}$. On elements of the form uAv, where u and v are each either empty, elements of V_H or equal to B, we let $\bar{\alpha}(uAv) = \alpha(u)A\alpha(v)$ (where of course, this product is calculated in $S(G)$). Note that α is well defined but may not be a homomorphism, because if $x, y \in V_H$ have $(x,y) \notin E_H$ and $(\alpha(x), \alpha(y)) \in E_G$, then $\bar{\alpha}(xy) = \bar{\alpha}(0) = 0$ while $\bar{\alpha}(x)\bar{\alpha}(y) = B$. Now define a map $\bar{\varphi} : S(H) \to S_{H \to G}$ by $\bar{\varphi}(w)(\alpha) = \bar{\alpha}(w)$, for $\alpha \in \mathrm{hom}(H,G)$. This again will rarely be a homomorphism, however if $w_1, w_2 \in S(H)\backslash\{0\}$ and $w_1w_2 \neq 0$, then we do have $\bar{\varphi}(w_1w_2) = \bar{\varphi}(w_1)\bar{\varphi}(w_2)$. This is easily proved by considering the various possible forms for w_1 and w_2. For example, if $w_1 = Ax$ and $w_2 = y \in V_H$, then $w_1w_2 \neq 0$ implies $(x, y) \in E_H$ and $w_1w_2 = AB$ so that for every $\alpha \in \hom(H, G)$ we have $\bar{\varphi}(w_1w_2)(\alpha) = AB$, while $[\bar{\varphi}(w_1)\bar{\varphi}(w_2)](\alpha) = A\alpha(x) \cdot \alpha(y) = AB$ (because $(\alpha(x), \alpha(y)) \in E_G$). The other cases are all similar and we leave them to the reader.

Now consider the situation when $w_1, w_2 \in S(H)\backslash\{0\}$ are such that $w_1w_2 = 0$. We claim that there is $\alpha : H \to G$ such that $[\bar{\varphi}(w_1)\bar{\varphi}(w_2)](\alpha) = 0$. To prove this, one must again consider the possible ways in which w_1w_2 may equal 0. Let us assume that w_1 and w_2 are written in their shortest forms. If w_1 finishes with $x \in V_H$ and w_2 begins with $y \in V_H$, but $(x, y) \notin E_H$, then we can find a homomorphism $\alpha : H \to G$ with $(\alpha(x), \alpha(y)) \notin E_G$ and then $\bar{\alpha}(w_1)\bar{\alpha}(w_2) = 0$. For all other cases, any homomorphism α in $\hom(H, G)$ will suffice.

Now let T denote the subsemigroup of $S_{H\to G}$ generated by the image of $\bar{\varphi}$, and J denote the subset of T consisting of all elements which take the value 0 somewhere. By the above observations, we have for $w_1, w_2 \in S(H)\backslash\{0\}$ that $w_1w_2 \neq 0$ implies $\bar{\varphi}(w_1w_2) = \bar{\varphi}(w_1)\bar{\varphi}(w_2)$ and $w_1w_2 = 0$ implies that $\bar{\varphi}(w_1)\bar{\varphi}(w_2) \in J$. Hence the map $\varphi : S(H) \to T/J$ defined by $w \mapsto \bar{\varphi}(w)/J$ is a surjective homomorphism. Because for distinct $x, y \in V_H$, there is $\alpha : H \to G$ with $\alpha(x) \neq \alpha(y)$, the map φ can also be seen to be injective. This completes the proof of Lemma 3.4 in the non-monoid case. For the monoid case, note that $S(H) \in \mathsf{HSP}(S(G))$ implies $S^1(H) \in \mathsf{HSP}(S^1(G))$.

3.3. *Proof of Lemma 3.5 — without unit*

Assume that $S(H) \in \mathsf{HSP}(S(G))$. We wish to prove that $|\hom(H, G)| \geq 1$. We first prove this under the assumption that H is connected and $E_H \neq \varnothing$.

Since $S(H) \in \mathsf{HSP}(S(G))$ and $S(G)$ and $S(H)$ are finite, there is a finite set L, a semigroup $D \leq [S(G)]^L$ and a surjective homomorphism $\varphi : D \to S(H)$.

We shall write Λ for the set of $f \in D$ such that $\varphi(f) \neq 0$, and we put $\Omega = \varphi^{-1}(AB) \subseteq \Lambda$. Now note that for every $w \in S(H)\backslash\{0\}$, there are w_1 and w_2 (possibly empty) such that $w_1ww_2 = AB$ (that is, w *divides* AB). (This requires only that every element of V_H is edge-related to some element of V_H.) Hence for every $f \in \Lambda$, we have that $DfD \cap \Omega \neq \varnothing$. Writing the finite set Ω as $\{f_0, \ldots, f_k\}$ and putting $\varepsilon = (f_0f_1 \cdots f_k)^2$, it follows that for every $f \in \Lambda$, there are $g, h \in D$ with $gfh = \varepsilon$. Note that $S(G)$ satisfies the equation $x^2 \approx x^3$, and from this it follows that $\varepsilon = \varepsilon^2$.

For $f \in D$, write $\operatorname{supp}(f)$ for the set $\{\ell \in L : f(\ell) \neq 0\}$. From the above considerations, it follows that for all $f \in \Lambda$,

$$\operatorname{supp}(\varepsilon) \subseteq \operatorname{supp}(f);$$

that is, if $f \in \Lambda$ and $\ell \in \operatorname{supp}(\varepsilon)$, then $f(\ell) \neq 0$.

Let us now choose, for each $x \in V_H$, an element $f_x \in D$ with $\varphi(f_x) = x$. Since $\varepsilon^2 = \varepsilon$ and $ABA = A$ in $S(H)$, we can also choose $\alpha \in D$ with $\varphi(\alpha) = A$ and

$\varepsilon\alpha = \alpha$. The elements ε, α and f_x $(x \in V_H)$ are held fixed for the remainder of this argument. Note that $\text{supp}(\varepsilon) = \text{supp}(\alpha)$.

Our objective now is to prove that there exists $\ell \in \text{supp}(\varepsilon)$ such that $f_x(\ell) \in V_G$, for all $x \in V_H$. For such an ℓ, the map $x \mapsto f_x(\ell)$ will clearly be the desired homomorphism from H to G — for if $(x,y) \in E_H$ then $f_x f_y \in \Lambda$, implying that $[f_x f_y](\ell) \neq 0$, that is, $(f_x(\ell), f_y(\ell)) \in E_G$. Let M denote $S(G)\backslash(V_G \cup \{0\})$.

For an element $w \in M$, we let the *left character*, $L(w)$ of w be defined as follows

$$L(w) = \begin{cases} 0 & \text{if } w \in AS \\ 1 & \text{if } w \in xAS \\ 2 & \text{if } w \in BS \end{cases}$$

(where $S = S(G)$ and $x \in V_G$). We define the right character $R(w)$ dually. Two elements will have the same *character* if they have identical left and right characters. The following observation is trivial.

Observation 1. If $a_1 \cdots a_n$ is a product of elements of M that does not equal 0, then for each $i \leq n-1$, $R(a_i) + L(a_{i+1}) = 2$. Similarly, if $a, c \in M$ and $b \in V_G$, then $abc \neq 0$ implies that $R(a) + 1 + L(c) = 2$.

Claim 1. *For* $\ell \in \text{supp}\,(\varepsilon), \alpha(\ell) \in M$.

To see this, note that $\alpha(\ell) \neq 0$ by definition of $\text{supp}(\varepsilon)$, while $\varepsilon\alpha = \alpha$, which shows that $\alpha(\ell) \notin V_G$.

Claim 2. *Let* $\ell \in \text{supp}\,(\varepsilon)$. *If* $x \in V_H$ *has* $f_x(\ell) \in V_G$, *then* $f_y(\ell) \in V_G$, *for all* $y \in V_H$.

Let y be adjacent to x, where $f_x(\ell) \in V_G$. We prove the claim for y and then the claim will follow for all elements of V_H because H is connected. Now in $S(H)$ we have $AxyA = A = AyxA$, so $\alpha(\ell)f_x(\ell)f_y(\ell)\alpha(\ell) \neq 0$ and $\alpha(\ell)f_y(\ell)f_x(\ell)\alpha(\ell) \neq 0$. If $f_y(\ell) \notin V_G$, then the first equation, Observation 1 and Claim 1 give $R(\alpha(\ell)) + 1 + L(f_y(\ell)) = 2$ while the second equation gives $R(\alpha(\ell)) + L(f_y(\ell)) = 2$, a contradiction. Claim 2 is proved.

Claim 3. *Let* $\ell \in \text{supp}(\varepsilon)$. *If* $x \in V_H$ *has* $f_x(\ell) \in M$, *then for every* $y \in V_H$, *the elements* $f_x(\ell)$, $f_y(\ell)$ *and* $\alpha(\ell)$ *have the same character. Furthermore,* $L(f_y(\ell)) + R(f_y(\ell)) = L(\alpha(\ell)) + R(\alpha(\ell)) = 2$.

Fix some $x \in V_H$ with $f_x(\ell) \in M$. By connectivity it will suffice to prove the claim for an arbitrary $y \in V_H$ adjacent to x. We again have that both $\alpha(\ell)f_x(\ell)f_y(\ell)\alpha(\ell)$ and $\alpha(\ell)f_y(\ell)f_x(\ell)\alpha(\ell)$ are non-zero, while Claim 2 implies $f_y(\ell) \in M$. By Observation 1 and Claim 1, we have the following equations:

$$\begin{aligned} R(\alpha(\ell)) + L(f_x(\ell)) &= 2, \quad & R(\alpha(\ell)) + L(f_y(\ell)) &= 2, \\ R(f_x(\ell)) + L(f_y(\ell)) &= 2, \quad & R(f_y(\ell)) + L(f_x(\ell)) &= 2, \\ R(f_y(\ell)) + L(\alpha(\ell)) &= 2, \quad & R(f_x(\ell)) + L(\alpha(\ell)) &= 2. \end{aligned}$$

Solving these (over integers) easily gives the claim.

We are now ready to prove that there is $\ell \in \text{supp}(\varepsilon)$ with $f_x(\ell) \in V_G$ for every $x \in V_H$. Assume otherwise. By Claim 2, for every $\ell \in \text{supp}(\varepsilon)$ and every $x \in V_H$ we have $f_x(\ell) \in M$. Let $(x, y) \in E_H$. Then both $f_x f_y$ and $f_y f_x$ are in Λ so that for every $\ell \in \text{supp}(\varepsilon)$ we have $f_x(\ell)f_y(\ell) \neq 0$ and $f_y(\ell)f_x(\ell) \neq 0$. By Lemma 3.1(i) and (ii) we have $f_x(\ell)f_y(\ell)f_x(\ell)f_y(\ell) = f_x(\ell)f_y(\ell)$. Hence $\alpha f_x f_y \alpha = \alpha f_x f_y f_x f_y \alpha$, because these elements agree on $\text{supp}(\varepsilon)$ but are zero elsewhere. This contradicts the fact that $\alpha f_x f_y \alpha \in \varphi^{-1}(A)$, while $\alpha f_x f_y f_x f_y \alpha \in \varphi^{-1}(0)$. Thus the desired $\ell \in \text{supp}(\varepsilon)$ exists, and we have a homomorphism from H into G. This completes the proof under the assumption that H is connected and not a one-element simple graph.

If H is a one-element simple graph, then certainly $|\text{hom}(H, G)| \geq 1$.

Now say that H is not connected and let $\{H_i : i \in I\}$ be the set of connected components of H. For each $i \in I$, the semigroup $S(H_i)$ is a subsemigroup of $S(H)$. Hence if $S(H) \in \mathsf{HSP}(S(G))$, then $S(H_i) \in \mathsf{HSP}(S(G))$. But then, there is a homomorphism $\phi_i : H_i \to G$. As H is a disjoint union of the subgraphs $\{H_i : i \in I\}$, the family of maps $\{\phi_i : i \in I\}$ are easily seen to give a homomorphism from H into G.

3.4. *The monoid case*

We now consider the case where $S^1(H) \in \mathsf{HSP}(S^1(G))$ and every edge of H forms part of a triangle.

We wish to extend our proof of the previous section to the present case. Again, it suffices to prove the result under the assumption that H is connected and $E_H \neq \varnothing$. Everything up to the definition of $L(w)$ and $R(w)$ holds with only trivial modification. Note that if $\varepsilon(\ell) = 1$ for some $\ell \in \text{supp}(\varepsilon)$, then $f(\ell) = 1$ for every $f \in \Lambda$. Now there must be $\ell \in \text{supp}(\varepsilon)$ such that $\varepsilon(\ell) \neq 1$, because $\alpha^2 \neq \alpha$, yet $\varepsilon\alpha = \alpha$. Let K denote $\{\ell \in L : \varepsilon(\ell) \notin \{0, 1\}\}$; thus, $K = \{\ell \in L : \alpha(\ell) \notin \{0, 1\}\} \neq \varnothing$.

Claim 4. *For $\ell \in K, \alpha(\ell) \in M$.*

To see this, note that $\{\varepsilon(\ell), \alpha(\ell)\} \subseteq S(G) \backslash \{0, 1\}$ and $\varepsilon(\ell)\alpha(\ell) = \alpha(\ell)$, implying that $\alpha(\ell) \notin V_G$.

Claim 5: *Let $(x, y) \in E_H$, $\ell \in K$, and $f_x(\ell) \in V_G$. Then $f_y(\ell) \in V_G$.*

To prove this claim, observe that we have $\alpha(\ell) \in M$ and $\alpha(\ell)f_x(\ell)f_y(\ell)\alpha(\ell) \neq 0 \neq \alpha(\ell)f_y(\ell)f_x(\ell)\alpha(\ell)$. Observation 1 easily yields that $f_y(\ell) \notin M$. It remains to show that $f_y(\ell) \neq 1$. So suppose that $f_y(\ell) = 1$.

Choose $z \in V_H$ so that $\{x, y, z\}$ is a triangle. Since $f_y(\ell) = 1$, the products $\alpha(\ell)f_x(\ell)\alpha(\ell)$, $\alpha(\ell)f_x(\ell)f_z(\ell)\alpha(\ell)$ and $\alpha(\ell)f_z(\ell)\alpha(\ell)$ are non-zero. The first expression gives $R(\alpha(\ell)) + L(\alpha(\ell)) = 1$. Then $f_z(\ell) \neq 1$, else Observation 1 implies $R(\alpha(\ell)) + L(\alpha(\ell)) = 2$. Also, if $f_z(\ell) \in M$, then $\alpha(\ell)f_x(\ell)f_z(\ell) \neq 0$ implies $R(\alpha(\ell)) + L(f_z(\ell)) = 1$; but also, $\alpha(\ell)f_z(\ell) \neq 0$ implies $R(\alpha(\ell)) + L(f_z(\ell)) = 2$. This contradiction shows that $f_z(\ell) \notin M$. We conclude that $f_z(\ell) \in V_G$. Finally, since $R(\alpha(\ell)) + L(\alpha(\ell)) = 1$ then $\alpha(\ell) \in SAw$ or $\alpha(\ell) \in wAS$ for some $w \in V_G$ — either way, $\alpha(\ell)f_x(\ell)f_z(\ell)\alpha(\ell) = 0$ since $\{f_x(\ell), f_z(\ell)\} \subseteq V_G$. This contradiction finishes our proof of Claim 5.

Now to complete the proof of the monoid case, we need to show that there is an $\ell \in K$ such that $f_x(\ell) \in V_G$ for every $x \in V_H$. Assume that this is not the case. By Claim 5 and connectivity, we have $f_x(\ell) \in M \cup \{1\}$ for every $x \in V_H$ and every $\ell \in K$. Let $\{x, y, z\}$ be a triangle in H. For every two-element subset $\{a, b\} \subset \{x, y, z\}$, the product $\alpha(\ell)f_a(\ell)f_b(\ell)\alpha(\ell)$ is non-zero. By Lemma 3.1(iii), we have $\alpha(\ell)f_x(\ell)f_y(\ell)f_z(\ell)\alpha(\ell) = \alpha(\ell)$ and then (as $[\alpha f_x f_y f_z \alpha](\ell)$ is 0 if $\ell \notin \text{supp}(\varepsilon)$ and 1 if $\ell \in \text{supp}(\varepsilon)\backslash K$) we have $\alpha f_x f_y f_z \alpha = \alpha$, contradicting the fact that $\varphi(\alpha f_x f_y f_z \alpha) = 0$, while $\varphi(\alpha) = A$.

Our proof of Lemmas 3.4 and 3.5, and of Theorem 3.2, is now complete. The theorem has this corollary.

Corollary 3.6. *For each of the algebras* $\mathbf{S} = \langle S(C_3), \cdot\rangle$, $\langle S^1(C_3), \cdot\rangle$, $\langle S^1(C_3), \cdot, 1\rangle$, *the finite algebra membership problem and variety equivalence problem for* $\mathsf{HSP}(\mathbf{S})$ *interpret the graph* 3*-colorability problem.*

3.5. *The syntactic approach*

If H is a non-3-colorable graph, then we have shown that $S(H) \notin \mathsf{HSP}(S(C_3))$ and so it follows that there must be an equation satisfied by $S(C_3)$ that fails on $S(H)$. We are going to find such an equation.

We use an idea from [11]. Let $H = \langle V_H, E_H\rangle$ be a finite connected graph. We construct an equation $p_H \approx q_H$ that fails in $S(H)$, and for any binary relational structure G, holds (as a law) in $S(G)$ if and only if $\text{hom}(H, G)$ is empty. Let $|H| = n$, say $V_H = \{a_1, \ldots, a_n\}$, and let $v_1, \ldots, v_n$ be distinct variables. It is trivial that $\text{hom}(H, G)$ is empty if and only if G satisfies the universal Horn sentence

$$d(H): \quad (\forall v_1, \ldots, v_n)\left(\bigvee\{\neg v_i{\sim}v_j : (a_i, a_j) \in E_H\}\right).$$

Essentially, we convert the sentence $d(H)$ into the desired semigroup equation.

Because H is symmetric, we may consider it as a directed graph in which every vertex has equal indegree and outdegree. Under this directed graph interpretation we may find, in polynomial time, an Eulerian circuit. Considered in the non-directed sense, this is a *bi-Eulerian circuit* — a path through H that passes through each edge exactly once in each direction. See Fig. 1.

Let $b_1, b_2, \ldots, b_m = b_1$ be a bi-Eulerian circuit for H, so that $E_H = \{(b_i, b_{i+1}) : 1 \leq i < m\}$. Let $H = \{a_1, \ldots, a_n\}$ as above, let $v_1, \ldots, v_n$ be distinct variables, and choose a variable x distinct from all v_i. For $1 \leq i \leq m$, say $b_i = a_{\pi_i}$.

We define p_H to be the semigroup word

$$v_{\pi_1}xv_{\pi_1}v_{\pi_2}xv_{\pi_2}\cdots v_{\pi_m}xv_{\pi_m}$$

and q_H to be the word $p_H v_{\pi_{m-1}}v_{\pi_m}$. For example, if H denotes the graph in Fig. 1 with the given bi-Eulerian circuit, then p_H is the word

$$v_0xv_0\,v_1xv_1\,v_2xv_2\,v_3xv_3\,v_1xv_1\,v_0xv_0\,v_3xv_3\,v_2xv_2\,v_1xv_1\,v_3xv_3\,v_0xv_0.$$

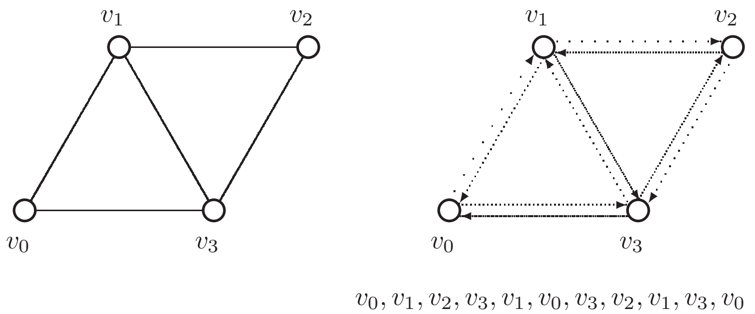


Fig. 1. A graph and a bi-Eulerian circuit.

Lemma 3.7. *Let G be a binary relational structure and H be a finite connected graph. Then $|\mathrm{hom}(H, G)| = 0$ if and only if $S(G) \models p_H \approx q_H$.*

Proof. ($\Rightarrow$) Suppose that $\mathrm{hom}(H, G) = \varnothing$ and let $\theta : \{v_1, \dots, v_n, x\} \to S(G)$ be an assignment. Note that p_H and q_H start and finish with the same letter and contain the same sets of two letter subwords, so Lemma 3.1(ii) shows that $\theta(p_H) = \theta(q_H)$ if θ maps into $M = S(G)\backslash V_G$. So we may assume that some letter appearing in p_H is mapped by θ into V_G.

If $\theta(p_H) = 0$ then obviously $\theta(q_H) = 0$, so we assume that $\theta(p_H) \neq 0$. Now consider the case where $\theta(x) \in V_G$. If $a_i \in V_H$ has $\theta(v_i) \in V_G$, then $\theta(v_i x v_i) \neq 0$ contradicts the fact that products of length 3 from V_G equal 0. Thus we have $\theta(v_i) \in M$ for every i. Now let $a_i = b_{m-1}$ and $a_j = b_m$ be the final two vertices visited in the bi-Eulerian circuit used for p_H (so p_H finishes $\cdots v_i v_j x v_j$). As $v_i v_j$ and $v_j v_i$ appear in p_H and both $\theta(v_i)$ and $\theta(v_j)$ lie in $S(G)\backslash V_G$, it follows from Lemma 3.1(ii) that $\theta(v_j) = \theta(v_j v_i v_j)$. This gives $\theta(p_H) = \theta(q_H)$ again.

Finally, we consider the case where $\theta(x) \in M$. We have that for every edge $(a_r, a_s) \in E_H$, the value of $\theta(x v_r v_s x)$ and $\theta(x v_s v_r x)$ are non-zero. This situation was encountered in the proof of Lemma 3.5, and we proved that if $\theta(v_i) \in V_G$ for some $a_i \in V_H$, then $\theta(v_j) \in V_G$ for every $a_j \in V_H$. As we are assuming that θ maps some letter of p_H into V_G, it follows that $\theta(\{v_1, \dots, v_n\}) \subseteq V_G$. However, whenever $(a_r, a_s) \in E_H$, the product $\theta(v_r)\theta(v_s)$ is non-zero, and this means that we have a graph homomorphism from H into G, a contradiction.

($\Leftarrow$) We prove the contrapositive. Say $\varphi : H \to G$ is a graph homomorphism. Then the assignment $\bar{\varphi}$ given by $\bar{\varphi}(v_i) = \varphi(a_i)$ for $1 \leq i \leq n$ and $\bar{\varphi}(x) = A$ gives q_H the value 0 and p_H a non-zero value. □

For example, there is no graph homomorphism from the one-element looped graph $\mathbf{1}$ into a simple graph. Thus for any simple graph G we have $S(G) \models yxyyxy \approx yxyyxyyy$ while $S(\mathbf{1})$ fails this equation. (In fact, we can use the equation $yy \approx yyy$ for this H.)

Remark 3.8. With some slight modifications, one can also give a monoid version of Lemma 3.7 under the additional assumption that H is triangulated, however we omit this here.

Recall that the term-equivalence problem for an algebra **A** is the problem of deciding for two terms s, t in the signature of **A**, if $\mathbf{A} \models s \approx t$. This problem is known to be in co-NP and there are now a number of known semigroups for which this problem is co-NP-complete (such as B_2^1; see [14] and [8]).

Corollary 3.9. *The term equivalence problem for $S(C_3)$ is co-NP-complete.*

Proof. Given a connected graph G, we find that G is 3-colorable if and only if $S(C_3) \not\models p_G \approx q_G$. This reduction is polynomial because the construction of $p_G \approx q_G$ is of polynomial complexity (as discussed during the definition of p_G). □

4. The Finite Basis Problem

If $\mathcal{V}$ is a variety with a finite basis of equations, then the finite membership problem for $\mathcal{V}$ can be solved in polynomial time (simply test for satisfaction of the finite set of equations). Assuming that $\mathrm{P} \neq \mathrm{NP}$, it follows that the semigroup variety generated by $S(C_3)$ is not finitely based.

The same holds in the monoid case and in this case the absence of a finite equational basis is easily established using the fact that for any binary relational structure G, the monoid $S^1(G)$ contains a submonoid isomorphic to the inherently non-finitely based semigroup B_2^1 (see [13]) and hence has no finite equational basis itself. Results of [13] also show that the semigroup $S(G)$ is never inherently non-finitely based. However, we will show that in most cases $S(G)$ is non-finitely based.

Theorem 4.1. *Let G be a graph with finite chromatic number that is not a disjoint union of complete bipartite graphs. Then $S(G)$ is not finitely based.*

Proof. We use an idea of Nešetřil and Pultr [10] and then Caicedo [3] who proved that the graph G generates a non-finitely axiomatizable graph quasi-variety. Caicedo also proved that a graph that is not a disjoint union of complete bipartite graphs generates a quasi-variety containing $\mathcal{C}_2$ (the quasi-variety of all 2-colorable graphs).

Let k be the chromatic number of the simple graph G and let n be an arbitrary positive integer. Erdös proved in [4] that there is a graph $G_{n,k}$ that is not k-colorable and that has no cycle of length at most $2n$. By Lemma 3.5, $S(G_{n,k}) \notin \mathsf{HSP}(S(G))$. Let T be an n-generated subsemigroup of $S(G_{n,k})$. It is clear that T is also a subsemigroup of $S(H)$ for some $2n$ vertex substructure H of $G_{n,k}$. By the choice of $G_{n,k}$, H has no cycles so is a forest and hence 2-colorable. By the result of Caicedo, $H \in \mathsf{SP}(G)$. Then by Lemma 3.4, we have $T \in \mathsf{HSP}(S(H)) \subseteq \mathsf{HSP}(S(G))$. Hence $S(G_{n,k})$ satisfies all n-variable equations of $S(G)$ but is not in $\mathsf{HSP}(S(G))$, and since n was arbitrary, it follows that there is no finite basis of equations for $S(G)$. □

Erdös used probabilistic methods to establish the existence of the graphs $G_{n,k}$, but a constructive approach was later found by Lovász [9]. These graphs are large and complicated, however for the particular graphs C_k, one can substitute some easily constructed graphs for the $G_{n,k}$ — one simply needs for each n, a non-k-colorable graph whose n-vertex subgraphs are k-colorable. A non-finitely based system of equations corresponding to these graphs may be constructed using Lemma 3.7.

Consider the graph $\mathbf{2} = \langle\{0,1\}, E\rangle$ with $E = \{0,1\}^2 \backslash \{(1,1)\}$.

Lemma 4.2. *If G is a simple graph, then $G \in \mathsf{SP}^+(\mathbf{2})$.*

Proof. This proof is quite easy, and we leave it to the reader. □

By Lemma 3.4, if G is any simple graph, then $S(G) \in \mathsf{HSP}(S(\mathbf{2}))$.

Proposition 4.3. *The semigroup variety generated by $S(\mathbf{2})$ has $2^{\aleph_0}$ subvarieties.*

Proof. This again will follow an idea of Caicedo in [3]. We say that two graphs G, H are *homomorphism independent* if $|\mathrm{hom}(G,H)| = |\mathrm{hom}(H,G)| = 0$. In [3], Caicedo finds an infinite homomorphism independent family $\mathcal{F} := \{G_i : i \in \mathbb{N}\}$ of finite, connected simple graphs with the properties that if $i > j$ then the chromatic number of G_i is greater than that of G_j, and the smallest odd cycle in G_i has strictly larger length than the smallest odd cycle in G_j. Let P be any subset of $\mathbb{N}$, and let G_P denote the (possibly infinite) graph obtained by taking the disjoint union of G_i for each $i \in P$. Now by Lemma 3.4, for each $i \in P$ we have $S(G_i) \in \mathsf{HSP}(S(G_P))$. However, for $j \notin P$, there is no homomorphism $\varphi : G_j \to G_P$, and so by Lemma 3.5, $S(G_j) \notin \mathsf{HSP}(S(G_P))$. It follows that distinct subsets $P, Q \subseteq \mathbb{N}$ give distinct varieties $\mathsf{HSP}(S(G_P))$ and $\mathsf{HSP}(S(G_Q))$. As these graphs are loopless, Lemmas 4.2 and 3.4 show that the $\mathsf{HSP}(S(G_P))$ are subvarieties of $\mathsf{HSP}(S(\mathbf{2}))$. □

Results of [7] show that the semigroup variety generated by $S^1(\mathbf{2})$ also has continuum many (semigroup) subvarieties, but this is simply by virtue of the fact that B_2^1 embeds into $S^1(\mathbf{2})$. However the existence of finite monoids whose *monoid* variety has uncountably many subvarieties has been an open question. We can modify the above proof to show that the monoid $S^1(\mathbf{2})$ has this property.

A technical lemma is needed first. Let Δ denote the triangle on $\{0,1,2\}$. For a given graph G, let $G \bowtie \Delta$ denote the graph on the disjoint union $V_G \dot\cup \{0,1,2\}$ with edge set $E_G \cup E_\Delta \cup V_G \times \{0,1,2\} \cup \{0,1,2\} \times V_G$.

Lemma 4.4. *Let G and H be simple connected graphs containing no triangles and with chromatic number at least 5. Then $|\mathrm{hom}(H,G)| \geq 1$ if and only if $|\mathrm{hom}(H \bowtie \Delta, G \bowtie \Delta)| \geq 1$.*

Proof. Clearly any graph homomorphism from H to G extends to a graph homomorphism from $H \bowtie \Delta$ to $G \bowtie \Delta$. Now assume that $\varphi : H \bowtie \Delta \to G \bowtie \Delta$ is a graph homomorphism. We are going to show that the restriction of φ to H is a graph homomorphism into G.

Now, as G contains no triangles, $\varphi(\{0,1,2\}) \nsubseteq V_G$. Say $\varphi(\{0,1\}) \subseteq V_G$ and $\varphi(2) \in \{0,1,2\}$. Let v be any vertex in V_H. Then as $\{0,1,v\}$ is a triangle, we must have $\varphi(v) \in \{0,1,2\}$. But then φ maps H into Δ, contradicting non-3-colorability. Now say that $\varphi(2) \in V_G$ and $\varphi(\{0,1\}) \subseteq \{0,1,2\}$. Define a map $c : V_H \to \{0,1,2,3\}$ by $c(u) = \varphi(u)$ if $\varphi(u) \in \{0,1,2\}$ and $c(u) = 3$ otherwise. We claim this is a valid coloring of H (contradicting the fact that H is not 4-colorable). Let $(u,v) \in E_H$. Now $\{u,v,2\}$ is a triangle in $H \bowtie \Delta$, and $\varphi(2) \in V_G$, so at least one of u or v must map into $\{0,1,2\}$. As $\varphi(u) \neq \varphi(v)$ (because $G \bowtie \Delta$ contains no loops), it follows that $c(u) \neq c(v)$ as required.

By symmetry, we have proved that $\varphi(\{0,1,2\}) \subseteq \{0,1,2\}$. In fact φ is a bijection on $\{0,1,2\}$ because there are no loops. Now let $u \in V_H$. As u is adjacent to all the vertices 0, 1, 2, and there are no loops, it follows that $\varphi(u) \in V_G$. That is, φ maps V_H into V_G, giving us the desired element of $\hom(H,G)$. □

Proposition 4.5. *The monoid variety* $\mathsf{HSP}(S^1(\mathbf{2}))$ *has* $2^{\aleph_0}$ *subvarieties.*

Proof. Consider the family $\mathcal{F}$ of homomorphism independent graphs found by Caicedo (see proof of Proposition 4.3). By dropping off the first few members (and relabelling) if necessary, we may assume that these graphs contain no triangles and are not 4-colorable. Now let $\mathcal{F}_\Delta$ denote $\{G_i \bowtie \Delta : i \in \mathbb{N}\}$. By Lemma 4.4, this family is also homomorphism independent, and furthermore, in every member, every edge is contained in a triangle. Hence the monoid version of Lemma 3.5 is available, and we can repeat the proof of Proposition 4.3. □

Note that if $\mathbf{2}^-$ is the connected, non-directed, simple graph on $\{0,1\}$, then $S^1(\mathbf{2}^-) \in \mathsf{HSP}(S^1(G_i \bowtie \Delta))$, so there is a continuum of monoid varieties between $\mathsf{HSP}(S^1(\mathbf{2}))$ and $\mathsf{HSP}(S^1(\mathbf{2}^-))$.

5. Quasi-Variety Membership

5.1. *Semigroups*

The maximal degree of complexity of the variety membership problem is currently unknown; however testing membership in finitely-generated quasi-varieties is known to be in NP. Indeed, to test membership of an algebra $\mathbf{A}$ in the quasi-variety generated by some finite algebra $\mathbf{B}$, it suffices to guess separating homomorphisms for each pair of distinct elements in $\mathbf{A}$ (see [1]). This gives the upper bounds listed in the second row of Table 2. In this section we give a 12-element semigroup with NP-complete finite membership problem for its quasi-variety.

We again use graph 3-colorability and the graph C_3. Let $G = \langle V_G, E_G\rangle$ be a finite graph without loops. We define a semigroup $T(G)$ as follows. Let E_G^c denote $(V_G \times V_G)\backslash(E_G \cup \Delta_G)$ where $\Delta_G = \{(x,x) : x \in G\}$. The universe of $T(G)$ is $V_G \cup \{0,a,f,e\} \cup \{f_{\{i,j\}} : (i,j) \in E_G^c\}$ (these unions are assumed to be disjoint — we may need to relabel the vertices of G). For $u, v \in V_G$, let

$$uv := \begin{cases} e & \text{if } (u,v) \in E_G, \\ f & \text{if } u = v, \\ f_{\{u,v\}} & \text{if } (u,v) \in E_G^c, \end{cases}$$

let $av = va = e$ and all other products equal 0. It is easy to verify that $T(G)$ is a semigroup because every product of more than 2 elements (under any bracketing) gives the value 0 (such a semigroup is called 3-nilpotent). It is obvious that $T(G)$ can be constructed in polynomial time from G. Note that $T(C_3)$ has 12 elements.

Lemma 5.1. *Let G be a simple graph. The following are equivalent*:

(i) G *is* 3-*colorable*;
(ii) $T(G) \in \mathsf{SP}(T(C_3))$;
(iii) $T^1(G) \in \mathsf{SP}(T^1(C_3))$.

Proof. First assume that $T(G) \in \mathsf{SP}(T(C_3))$. So there is a homomorphism $\varphi : T(G) \to T(C_3)$ with $\varphi(e) \neq \varphi(0)$. Now in $T(G)$, we have for each $v \in V_G$, $va = e$ and also $a^2 = 0$, so $\varphi(a)$ has the property $(\exists b)\, b\varphi(a) \neq 0$ & $\varphi(a)\varphi(a) = 0$. Thus $\varphi(a) = a$. But then, as $\varphi(v)a = \varphi(e) \neq \varphi(0) = 0$, we have $\varphi(v)a = e$ showing that $\varphi(e) = e$ and for each $v \in V_G$, $\varphi(v) \in V_{C_3}$. Now let $(u, v) \in E_G$. Then in $T(G)$, we have $uv = e$. Hence $\varphi(u)\varphi(v) = e$ in $T(C_3)$, showing that $(\varphi(u), \varphi(v)) \in E_{C_3}$. So the restriction of φ to V_G is a graph homomorphism from G into C_3, showing that G is 3-colorable.

A very similar argument holds in the monoid case. Indeed if $\varphi : T^1(G) \to T^1(C_3)$ has $\varphi(e) \neq \varphi(0)$, then we must have $\varphi(T(G)) \subseteq T(C_3)$ and $\varphi(1) = 1$. The above argument now shows that G is 3-colorable.

Now say that G is 3-colorable. As (ii) implies (iii), it will suffice to show that $T(G) \in \mathsf{SP}(T(C_3))$. So, for every pair $x \neq y$ in $T(G)$ we need to find a homomorphism $\varphi : T(G) \to T(C_3)$ with $\varphi(x) \neq \varphi(y)$. If one of x or y (say, x) is in $V_G \cup \{a\}$, then this is easy: map $x \mapsto e$ and send all other elements to 0. Thus we may assume that $x, y \in T(G)\backslash(V_G \cup \{a\})$. We are going to construct our homomorphisms from graph homomorphisms between G and C_3. Given a graph homomorphism $\psi : G \to C_3$, let $\bar{\psi} : T(G) \to T(C_3)$ be given by

$$\bar{\psi}(w) = \begin{cases} \psi(w) & \text{if } w \in V_G \\ w & \text{if } w \in \{0, a, e, f\} \\ e & \text{if } w = f_{\{i,j\}} \text{ and } (\psi(i), \psi(j)) \in E_{C_n} \\ f & \text{if } w = f_{\{i,j\}} \text{ and } \psi(i) = \psi(j) \\ f_{\{\psi(i),\psi(j)\}} & \text{if } w = f_{\{i,j\}} \text{ and } (\psi(i), \psi(j)) \in E_{C_n}^c. \end{cases}$$

It is routine to check that $\bar{\psi}$ is well defined and a homomorphism.

If one of x or y is 0, then any homomorphism $\psi : G \to C_3$ has $\bar{\psi}(x) \neq \bar{\psi}(y)$, so we assume that $x, y \neq 0$. Say that x is e. If $y = f_{\{i,j\}}$, then any graph homomorphism $\psi : G \to C_n$ for which $(\psi(i), \psi(j)) \notin E_{C_n}$ gives $\bar{\psi}(x) \neq \bar{\psi}(y)$. Likewise if $x = e$ and

$y = f$, then any graph homomorphism gives rise to a semigroup homomorphism separating x and y. By symmetry, we may assume that x and y are either f or of the form $f_{\{i,j\}}$.

Now say that $x = f$ and $y = f_{\{i,j\}}$. So $i \neq j$ and there is a graph homomorphism $\psi : G \to C_3$ with $\psi(i) \neq \psi(j)$. Then $\bar{\psi}$ separates x and y. So it remains to consider the case when $x = f_{\{u_1,v_1\}}$ and $y = f_{\{u_2,v_2\}}$, where $\{u_1, v_1\} \neq \{u_2, v_2\}$ and $u_i \neq v_i$ for $i = 1, 2$. Clearly it is sufficient to find a graph homomorphism $\psi : G \to C_3$ with $\{\psi(u_1), \psi(v_1)\}$ and $\{\psi(u_2), \psi(v_2)\}$ non-equal, not both edges and not both singletons.

Recall that the elements of C_3 are $\{0, 1, 2, 3, 4\}$ with edges obtained from the complete graph by removing $(3, 4)$, $(0, 3)$ and $(1, 4)$ and their reverse. As before, we let Δ be the triangle on $\{0, 1, 2\}$. By relabeling if necessary, we may assume that either $u_1 = u_2$ or all four vertices are distinct. Let $c : G \to \Delta$ be a 3-coloring. If $c(u_1) \neq c(v_1)$ then we can arrange that $c(u_1) = 0$ and $c(v_1) = 1$. The map ψ with $\psi(u_1) = 3$, $\psi(v_1) = 4$ and $\psi(w) = c(w)$ for all $w \in V_G \backslash \{u_1, v_1\}$ is a homomorphism $G \to C_3$, and $\{\psi(u_1), \psi(v_1)\}$ is a non-singleton, non-edge while $\{\psi(u_2), \psi(v_2)\} \neq \{\psi(u_1), \psi(v_1)\}$ — so we are done in this case. If $c(u_1) = c(v_1)$ then we can arrange that $c(u_1) = c(v_1) = 0$. Then the map ψ with $\psi(v_1) = 3$ and $\psi(w) = c(w)$ otherwise is a homomorphism, and $\bar{\psi}(f_{\{u_1,v_1\}}) = f_{\{0,3\}}$ while $\bar{\psi}(f_{\{u_2,v_2\}}) \in \{e, f\}$. □

Recall the graph G_{C_3} from Sec. 2. It is easy to see that $T(C_3)$ embeds into $T(G_{C_3})$ giving $\mathsf{SP}(T(G_{C_3})) \supseteq \mathsf{SP}(T(C_3))$. Therefore $\mathsf{SP}(T(G_{C_3})) = \mathsf{SP}(T(C_3))$ if and only if $T(G_{C_3}) \in \mathsf{SP}(T(C_3))$, if and only G is 3-colorable.

Corollary 5.2. *The following problems are NP-complete*:

(i) $\star \in \mathsf{SP}(T(C_3))$;
(ii) $\star \in \mathsf{SP}(T^1(C_3))$;
(iii) $\mathsf{SP}(\star) = \mathsf{SP}(T(C_3))$;
(iv) $\mathsf{SP}(\star) = \mathsf{SP}(T^1(C_3))$.

This gives row 3 of Table 1.

As in the variety case, (assuming that $\mathrm{P} \neq \mathrm{NP}$) we must have $\mathsf{SP}(T(C_3))$ not finitely axiomatizable (this can be proved directly). We also note that Sapir [12] has shown that the three element semigroup with presentation $\langle a : a^3 = a^4 \rangle$ generates a not finitely axiomatizable quasi-variety. However in contrast with $\mathsf{SP}(T(C_3))$, an algorithm for testing membership is given in [12] and this is routinely seen to be polynomial time.

5.2. *Unary algebras*

Problem 5.6 of [1] asks for the complexity of the problem $\mathsf{SP}(\star) = \mathsf{SP}(\star)$ for unary algebras (algebras whose operations are all unary operations). It is not hard to use methods given in [1] to show that there is in fact a fixed unary algebra $\mathbf{U}$ with two unary operations (*bi-unary*) for which $\mathsf{SP}(\star) = \mathsf{SP}(\mathbf{U})$ is NP-complete.

Given a graph G, a construction due to Hedrlín and Pultr [6] produces (in polynomial time) a bi-unary algebra $U(G)$ on the set $V_G \,\dot{\cup}\, E_G \,\dot{\cup}\, \{u, v\}$ such that every graph homomorphism $\psi : G \to H$ extends uniquely to a homomorphism $\psi^* : U(G) \to U(H)$, and every homomorphism $\varphi : U(G) \to U(H)$ arises in this fashion. In [1] it is shown that $G \in \mathsf{SP}(H)$ implies $U(G) \in \mathsf{SP}(U(H))$ ([1, Proposition 5.4]; actually this is done for two particular graphs, but the arguments are general). On the other hand, if $U(G) \in \mathsf{SP}(U(H))$ there is at least one homomorphism $\psi^* : U(G) \to U(H)$, showing that there is at least one homomorphism $\psi : G \to H$. Choosing H to be C_3 and using Lemma 2.2, we find that G is 3-colorable if and only if $U(G) \in \mathsf{SP}(U(C_3))$ if and only if $\mathsf{SP}(U(G_{C_3})) = \mathsf{SP}(U(C_3))$ (note that $\mathsf{SP}(U(G_{C_3})) \supseteq \mathsf{SP}(U(C_3))$ follows because $U(C_3)$ embeds into $U(G_{C_3})$).

Corollary 5.3. *The following problems are NP-complete for bi-unary algebras*: $\star \in \mathsf{SP}(U(C_3))$; *and* $\mathsf{SP}(\star) = \mathsf{SP}(U(C_3))$.

6. Other Membership Problems

The last three rows of Tables 1 and 2 are more easily established.

We first consider the upper bounds listed in rows 3–5 of Table 2. We omit the obvious proof that all of these problems are in NP. Let K be amongst $\{\mathsf{HS}, \mathsf{H}, \mathsf{S}\}$. For a fixed finite semigroup $\mathbf{A}$, membership in $\mathsf{K}(\mathbf{A})$ is only possible for algebras up to the size of $|A|$, and this gives (large!) constant time complexity. This gives the second column of Table 2. Similar arguments apply for column 4. For column 5 and $\mathsf{K} \in \{\mathsf{H}, \mathsf{S}, \mathsf{HS}\}$, note that $\mathsf{K}(\mathbf{B}) = \mathsf{K}(\mathbf{A})$ if and only if $\mathbf{B} \cong \mathbf{A}$. Booth [2] showed that this is polynomially equivalent to the graph isomorphism problem (thought to be easier than NP-complete). This gives the entries in column 5, rows 3–5 in both Tables 1 and 2.

Now we consider the last remaining entry of Table 2; row 5, column 1. To determine if $\mathbf{A} \in \mathsf{S}(\mathbf{B})$ for finite semigroups or monoids $\mathbf{A}$ and $\mathbf{B}$, one can simply check all possible embeddings of $\mathbf{A}$ into $\mathbf{B}$. If $\mathbf{A}$ is fixed, then this is a polynomial time algorithm because there are fewer than $|B|^{|A|}$ possible embeddings, and each can be checked in at most $O(|B|^2)$ time. So for fixed $\mathbf{A}$, the problem $\mathbf{A} \in \mathsf{S}(\star)$ is in P (row 5, column 1 of Table 2). In contrast we now find a semigroup $\mathbf{A}$ for which $\mathbf{A} \in \mathsf{H}(\star)$ is NP-complete (row 4, column 1 of Table 1). We use the construction of Sec. 5. Recall that Δ denotes the triangle graph. For notational purposes, we now rename its vertices as $\{v_0, v_1, v_2\}$. Recall that G_Δ is the graph obtained from a graph G by triangulating each edge.

Lemma 6.1. *Let G be a simple graph with at least one edge. The following are equivalent*:

(i) G *is* 3-*colorable*;
(ii) $T(\Delta) \in \mathsf{H}(T(G_\Delta))$;
(iii) $T^1(\Delta) \in \mathsf{H}(T^1(G_\Delta))$.

Proof. By Lemma 2.2, G is 3-colorable if and only if $\hom(G_\Delta, \Delta)$ is non-empty. As G_Δ contains a triangle, any homomorphism into Δ must be surjective. If φ is such a homomorphism, then the map $\bar{\varphi}$ from the proof of Lemma 5.1 is a surjective homomorphism from $T(G_\Delta)$ onto $T(\Delta)$.

Now say that G is a graph and $\varphi : T(G_\Delta) \to T(\Delta)$ is a surjective homomorphism. If $a' \in \varphi^{-1}(a)$, then ${a'}^2 \in \varphi^{-1}(0)$, while there is b' with $a'b' \in \varphi^{-1}(e)$. This means that $a' \in V_{G_\Delta} \cup \{a\}$. However, if $a' \in V_{G_\Delta}$, then $0 = \varphi({a'}^2) = \varphi(f)$, implying that for every $v \in V_{G_\Delta}$, $\varphi(v)^2 = \varphi(v^2) = \varphi(f) = 0$. This means that $\varphi^{-1}(\{v_0, v_1, v_2\})$ is empty, a contradiction. Hence $\varphi(a) = a$. Likewise, for $v \in V_{G_\Delta}$, we must have $\varphi(v) \in \{v_0, v_1, v_2\}$. As in the proof of Lemma 5.1, this means that when restricted to V_{G_Δ}, φ is a graph homomorphism. Therefore G is 3-colorable. The monoid case is obtained by trivial modifications once it is noted that any surjective homomorphism from $T^1(G_\Delta)$ onto $T^1(\Delta)$ must have $\varphi(1) = 1$. □

Corollary 6.2. *The problems $T(\Delta) \in \mathsf{H}(\star)$ for semigroups and $T^1(\Delta) \in \mathsf{H}(\star)$ for monoids are NP-complete.*

This gives row 3, columns 1 and 3 of Table 1.

Remark 6.3. It is clear from the discussion in Sec.5.2 that the corresponding problem for bi-unary algebras is also NP-complete (using $U(\Delta)$).

The last remaining claims from the tables are the NP-completeness of $\star \in \mathsf{HS}(\star)$ and $\star \in \mathsf{S}(\star)$ (column 3, rows 3 and 5 of Table 1). We again encode a graph theoretic problem. If $G = \langle V_G, E_G \rangle$ is a finite simple graph and $\ell \leq |V_G|$ then (G, ℓ) is an instance of the problem *clique*. The pair (G, ℓ) is a yes instance if G contains a complete subgraph with at least ℓ vertices.

For a given graph $G = \langle V_G, E_G \rangle$, construct a 3-nilpotent semigroup $R(G)$ on the set $V_G \cup \{0, e\}$ (these sets are assumed to be disjoint) by setting

$$xy = \begin{cases} e & \text{if } \{x, y\} \subseteq V_G \quad \text{and} \quad (x, y) \in E_G \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.4. *Let G be a simple graph on n vertices, $n \geq \ell \geq 3$. Then the following are equivalent:*

(i) *G has an ℓ vertex complete subgraph;*
(ii) $R(K_\ell) \in \mathsf{S}(R(G))$;
(iii) $R(K_\ell) \in \mathsf{HS}(R(G))$;
(iv) $R^1(K_\ell) \in \mathsf{S}(R^1(G))$;
(v) $R^1(K_\ell) \in \mathsf{HS}(R^1(G))$.

Proof. The proof that (i) implies the other conditions is trivial. Clearly also, any of (ii), (iii) or (iv) imply (v).

Now say that $R^1(K_\ell) \in \mathsf{HS}(R^1(G))$ holds (our argument will work in both the language of semigroups and of monoids). Now, any subsemigroup of $R^1(G)$ is of one of the following forms: $R^1(H)$ or $R(H)$ for some subgraph H of G; a semigroup with null multiplication or such a semigroup with adjoined identity element. It is clear that $R^1(K_\ell)$ is not a homomorphic image of these latter semigroups and also that $R^1(K_\ell)$ is not a quotient of $R(H)$ for any H (which lack an identity element). So we may assume that we have a subgraph H of G and a surjective homomorphism $\varphi : R^1(H) \to R^1(K_\ell)$. Note that $\varphi(1) = 1$, regardless of whether or not this element is distinguished. For u a vertex in K_ℓ, choose some element $u' \in \varphi^{-1}(u)$. Then $u' \in V_H$ and for every pair of distinct vertices u, v in K_ℓ we have $uv = e$ so that $u'v' \neq 0$ and then $u'v' = e$. Thus the vertices $\{u' : u' \in \varphi^{-1}(V_{K_\ell})\}$ form a complete subgraph of H, and therefore also of G. That is, (i) holds. □

Clique is NP-complete, and the reduction of an instance (G, ℓ) of clique to the pair $(R(K_\ell), R(G))$ is clearly polynomial.

Corollary 6.5. *The problem* $\star \in \mathsf{K}(\star)$ *is NP-complete for finite semigroups or monoids when* $\mathsf{K} \in \{\mathsf{S}, \mathsf{HS}\}$.

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