

HOW FINITE IS A THREE-ELEMENT UNARY ALGEBRA?

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We show that, within the class of three-element unary algebras, there is a tight connection between a finitely based quasi-equational theory, finite rank, enough algebraic operations (from natural duality theory) and a special injectivity condition.

Keywords: Natural duality; strong duality; injectivity; quasi-equations; unary algebras.

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0. Introduction

A full duality gives us a natural dual equivalence between a quasi-variety generated by a finite algebra and a class of structured topological spaces. There are many well-known examples of full dualities: Stone's duality for Boolean algebras [12], Priestley's duality for distributive lattices [11], and the Hofmann, Mislove, Stralka duality for semilattices [6]. A finite algebra \mathbf{M} is called fully dualizable if it is possible to set up a full duality for the quasi-variety $\mathbb{ISP}(\mathbf{M})$. Given the naturalness of the definition of a full duality, it is perhaps surprising that the property of full dualizability is so difficult to work with. The subtlety of full dualizability is illuminated by the work done by Hyndman and Willard [8] to prove that a particular three-element unary algebra is not fully dualizable.

There is a property that is stronger than full dualizability but easier to work with. Strong dualizability is more well behaved than full dualizability, and it is often easier to establish. Indeed, most full dualities are established by setting up a strong duality. It is not presently known whether every full duality is strong.

Recently, Willard [13] provided a new method for showing that an algebra which is known to be dualizable is also strongly dualizable. He proved that a finite algebra is strongly dualizable if it is dualizable and has finite rank. The definition of finite rank is fairly technical and the concept is still not properly understood. Lampe, McNulty and Willard [9] introduced a simpler, sufficient condition for finite rank, which they call having enough algebraic operations.

In this paper, we attempt to elucidate the difficult concepts of rank and enough algebraic operations by investigating them within a simple class of algebras: three-element unary algebras. We find that within the class of three-element unary algebras, these concepts are not only equivalent to each other, they are also equivalent to some other nice finiteness conditions. We should not expect such a tight connection between these properties outside the realm of three-element unary algebras. Nevertheless, the results, and the methods used to establish them, do give us some intuition into these properties. Our results reinforce the idea that finite rank and enough algebraic operations are related to finiteness and injectivity.

We consider a three-element unary algebra to be “bad” if there is a natural finiteness condition that it does not satisfy. For example, if an algebra’s quasi-equational theory is not finitely based, or if an algebra does not have finite rank, then it is “bad”. There are three particular three-element unary algebras that between them seem to capture everything that can make a three-element unary algebra “bad”. These are the algebras **V**, **L** and **D**, which are shown in Fig. 1. (We denote a unary operation $u: \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ by the string $u(0)u(1)u(2)$.)

The main aim of this paper is to prove the following theorem. The definitions of the concepts involved are reviewed in Sec. 1. An algebra **A** is called an **isoreduct** of another algebra **B** if **A** is isomorphic to a term reduct of **B**.

Main Theorem. *Let **M** be a three-element unary algebra. Then the following are equivalent:*

- (1) **M** has neither **V**, **L** nor **D** as an isoreduct;
- (2) **M** has finite rank;
- (3) **M** has enough algebraic operations;
- (4) **M** is quasi-injective.

*Moreover, if **M** has finite type, then these conditions are equivalent to:*

- (5) *the quasi-equational theory of **M** is finitely based.*

Note that, although the properties of finite rank and enough algebraic operations come from duality theory, the equivalences in our main theorem are independent of dualizability. We shall find several three-element unary algebras that have enough algebraic operations, even though they are not dualizable. For those algebras that are dualizable, we use known results from duality theory to establish the equivalences.

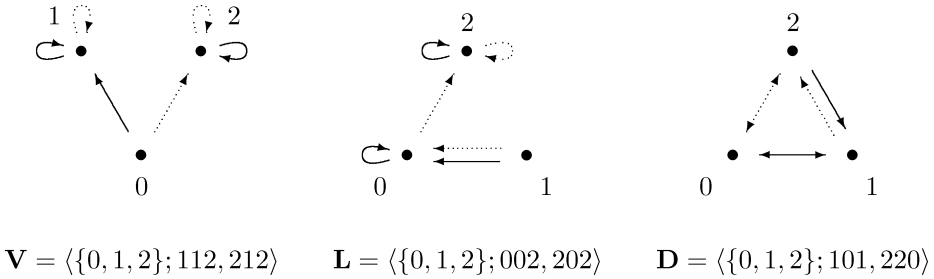


Fig. 1.

The algebras \mathbf{V} , \mathbf{L} and \mathbf{D} are not global obstacles to finite rank or enough algebraic operations. In the last section of this paper, we exhibit a unary algebra that has \mathbf{V} as a subalgebra but still has enough algebraic operations. Indeed, there can be no global obstacles to finite rank or enough algebraic operations amongst unary algebras. Within every locally finite variety of unary algebras, each finite unary algebra can be embedded into a finite injective algebra [1]. So any finite unary algebra can be embedded into a finite algebra that is injective in the quasi-variety it generates. An algebra which is injective in the quasi-variety it generates has finite rank and enough algebraic operations.

Bestsenyi [2] has already proven that conditions (1) and (5) are equivalent for algebras of finite type, and Lampe, McNulty and Willard [9] have shown that (3) implies (2) in general. We establish that the rest of the equivalences hold. We do this by proving that condition (1) is equivalent to each of conditions (2), (3) and (4). In Sec. 1, we provide some preliminary definitions and results. In Sec. 2, we show that every three-element unary algebra that satisfies condition (1) also satisfies conditions (3) and (4) (and therefore also satisfies condition (2)). In Sec. 3, we show that a three-element unary algebra that does not satisfy (1) also does not satisfy (2) or (4) (and so does not satisfy (3)). In the final section of the paper, we give examples to show that at least some of the implications in our main theorem do not hold for unary algebras in general.

1. Dualizability and Three-Element Unary Algebras

In this section, we provide the definitions we need for this paper. We also review the necessary results from [5] and [10] on three-element unary algebras. Let \mathbf{M} be a finite algebra. A **quasi-equation** is an expression of the form

$$(s_1 \approx t_1 \ \& \ s_2 \approx t_2 \ \& \ \dots \ \& \ s_n \approx t_n) \Rightarrow s \approx t.$$

For any finite algebra \mathbf{M} , the **quasi-variety** $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ is determined by the **quasi-equational theory** of \mathbf{M} . We begin by showing briefly how we set up a full duality for the quasi-variety \mathcal{A} . The text [4], by Clark and Davey, explains the theory of natural dualities in detail. The process of creating a full duality starts by choosing a topological structure $\underline{\mathbf{M}} = \langle M; G, H, R, T \rangle$ on the same underlying set

as \mathbf{M} . We allow the structure on $\widetilde{\mathbf{M}}$ to include a set G of finitary operations on M , a set H of finitary partial operations on M and a set R of finitary relations on M . We insist that the structure in the type of $\widetilde{\mathbf{M}}$ is compatible with \mathbf{M} in the following way:

- (1) each operation $g \in G$ is a homomorphism $g: \mathbf{M}^n \rightarrow \mathbf{M}$, for some $n \in \omega$;
- (2) each partial operation $h \in H$ is a homomorphism $h: \mathbf{A} \rightarrow \mathbf{M}$, for some $n \in \omega \setminus \{0\}$ and $\mathbf{A} \leq \mathbf{M}^n$;
- (3) each relation $r \in R$ is a subuniverse of \mathbf{M}^n , for some $n \in \omega \setminus \{0\}$;
- (4) \mathcal{T} is the discrete topology on M .

We want to set up a natural dual equivalence between \mathcal{A} and the topological quasi-variety $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\widetilde{\mathbf{M}})$ of all isomorphic copies of closed substructures of non-zero powers of $\widetilde{\mathbf{M}}$.

We can define a pair of contravariant functors $D: \mathcal{A} \rightarrow \mathcal{X}$ and $E: \mathcal{X} \rightarrow \mathcal{A}$ such that

- (1) for all $\mathbf{A} \in \mathcal{A}$, the structure $D(\mathbf{A})$ is the closed substructure of $\widetilde{\mathbf{M}}^{\mathbf{A}}$ on the set $\mathcal{A}(\mathbf{A}, \mathbf{M})$ of all homomorphisms from \mathbf{A} to \mathbf{M} ,
- (2) for all $\mathbf{X} \in \mathcal{X}$, the algebra $E(\mathbf{X})$ is the subalgebra of $\mathbf{M}^{\mathbf{X}}$ on the set $\mathcal{X}(\mathbf{X}, \widetilde{\mathbf{M}})$ of all morphisms from \mathbf{X} to $\widetilde{\mathbf{M}}$,
- (3) for each homomorphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ in \mathcal{A} , the morphism $D(\varphi): D(\mathbf{B}) \rightarrow D(\mathbf{A})$ is given by $D(\varphi)(x) := x \circ \varphi$,
- (4) for each morphism $\psi: \mathbf{X} \rightarrow \mathbf{Y}$ in \mathcal{X} , the homomorphism $E(\psi): E(\mathbf{Y}) \rightarrow E(\mathbf{X})$ is given by $E(\psi)(\alpha) := \alpha \circ \psi$.

For each $\mathbf{A} \in \mathcal{A}$ there is a natural embedding $e_{\mathbf{A}}: \mathbf{A} \hookrightarrow E(D(\mathbf{A}))$, given by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in \mathbf{A}$ and $x \in D(\mathbf{A})$. Similarly, for each $\mathbf{X} \in \mathcal{X}$, we define an embedding $\varepsilon_{\mathbf{X}}: \mathbf{X} \hookrightarrow D(E(\mathbf{X}))$ by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$. We say that $\widetilde{\mathbf{M}}$ **yields a duality on \mathcal{A}** if $e_{\mathbf{A}}$ is an isomorphism, for all $\mathbf{A} \in \mathcal{A}$. Finally $\widetilde{\mathbf{M}}$ **yields a full duality on \mathcal{A}** if $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are isomorphisms, for all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. In the latter case, there is a dual equivalence between the categories \mathcal{A} and \mathcal{X} . The algebra \mathbf{M} is called **(fully) dualizable** if there is a structure $\widetilde{\mathbf{M}}$ that yields a (full) duality on \mathcal{A} .

As mentioned in the introduction, full dualizability is rather complicated. It is usually easier to work with strong dualizability instead. In order to give the definition of a strong duality, we need to introduce the notion of term closure. Let S be a non-empty set and let $F_{\mathbf{M}}(S)$ denote the set of all S -ary term functions of \mathbf{M} . For all $s, t \in F_{\mathbf{M}}(S)$, define $\text{eq}(s, t) = \{x \in M^S : s(x) = t(x)\}$. A set $X \subseteq M^S$ is said to be **term closed** if

$$X = \cap \{\text{eq}(s, t) \mid s, t \in F_{\mathbf{M}}(S) \text{ and } s \upharpoonright_X = t \upharpoonright_X\}.$$

In [4], it is shown that $\widetilde{\mathbf{M}}$ yields a full duality on \mathcal{A} if and only if $\widetilde{\mathbf{M}}$ yields a duality on \mathcal{A} and every closed substructure of a non-zero power of \mathbf{M} is isomorphic to a term-closed substructure of a power of $\widetilde{\mathbf{M}}$. This result leads us to say that $\widetilde{\mathbf{M}}$ **yields**

a **strong duality on \mathcal{A}** if $\underline{\mathbf{M}}$ yields a duality on \mathcal{A} and every closed substructure of a non-zero power of \mathbf{M} is term closed. The algebra \mathbf{M} is called **strongly dualizable** if there is a structure $\underline{\mathbf{M}}$ that yields a strong duality on \mathcal{A} . Clearly every strong duality is also a full duality. At present, there are no examples of full dualities that are not strong.

In general, it is also reasonably difficult to show that a finite algebra is strongly dualizable. Willard [13] recently invented a method for lifting dualizability up to strong dualizability using the concept of rank. We provide some preliminary definitions, and then give Willard’s definition of finite rank. (Actually, we give a definition of rank that is easily seen to be equivalent to that of Willard.)

A **coordinate embedding** is a map $\sigma: \mathbf{M}^k \rightarrow \mathbf{M}^{k+l}$, for some $k \in \omega \setminus \{0\}$ and $l \in \omega$, that embeds \mathbf{M}^k into \mathbf{M}^{k+l} by repetition of coordinates. More precisely, the map σ is a coordinate embedding if there is a surjective map $\tau: \{1, \dots, k+l\} \rightarrow \{1, \dots, k\}$ such that $\sigma(a)(i) = a(\tau(i))$, for all $a \in \mathbf{M}^k$ and $i \in \{1, \dots, k+l\}$. For each $Y \subseteq \mathcal{A}(\mathbf{A}, \mathbf{M})$, where $\mathbf{A} \in \mathcal{A}$, define the natural product homomorphism $\sqcap Y: \mathbf{A} \rightarrow \mathbf{M}^Y$ by $(\sqcap Y(a))(y) = y(a)$, for all $a \in \mathbf{A}$ and $y \in Y$. Finally, given algebras $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and homomorphisms $h: \mathbf{A} \rightarrow \mathbf{M}$ and $\varphi: \mathbf{A} \rightarrow \mathbf{B}$, we say that h **lifts to \mathbf{B} through φ** if there is a homomorphism $h': \mathbf{B} \rightarrow \mathbf{M}$ such that $h = h' \circ \varphi$.

We now give the definition of finite rank. Let $\mathbf{A} \leq \mathbf{M}^k$, for some $k \in \omega \setminus \{0\}$ and let $h: \mathbf{A} \rightarrow \mathbf{M}$ be a homomorphism. Then $\text{rank}(h) \leq 0$ if h is the restriction of a projection. For each $n \in \omega$, we have $\text{rank}(h) \leq n + 1$ if there is some $m \in \omega \setminus \{0\}$ for which the following condition holds:

for each $l \in \omega$ and for each coordinate embedding $\sigma: \mathbf{M}^k \rightarrow \mathbf{M}^{k+l}$, whenever we have $\sigma(\mathbf{A}) \leq \mathbf{B} \leq \mathbf{C} \leq \mathbf{M}^{k+l}$ such that $h: \mathbf{A} \rightarrow \mathbf{M}$ lifts to \mathbf{C} through $\sigma \upharpoonright_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{C}$, there exists $Y \subseteq \mathcal{A}(\mathbf{C}, \mathbf{M})$ with

- (1) $|Y| \leq m$,
- (2) $h: \mathbf{A} \rightarrow \mathbf{M}$ lifts to $\sqcap Y(\mathbf{B})$ through $\sqcap Y \circ \sigma \upharpoonright_{\mathbf{A}}: \mathbf{A} \rightarrow \sqcap Y(\mathbf{B})$, and
- (3) $\text{rank}(g \upharpoonright_{\mathbf{B}}) \leq n$, for all $g \in Y$.

This definition is illustrated in Fig. 2. The homomorphism h **has finite rank** if $\text{rank}(h) \leq n$, for some $n \in \omega$. If we need to emphasize the algebra \mathbf{M} , then we write $\text{rank}_{\mathbf{M}}(h)$ instead of $\text{rank}(h)$. For $n \in \omega$, we write $\text{rank}(\mathbf{M}) \leq n$ if $\text{rank}(h) \leq n$, for each homomorphism $h: \mathbf{A} \rightarrow \mathbf{M}$ such that $\mathbf{A} \leq \mathbf{M}^k$ for some $k \in \omega \setminus \{0\}$. We say that \mathbf{M} **has finite rank** if $\text{rank}(\mathbf{M}) \leq n$, for some $n \in \omega$. The following theorem is due to Willard [13].

Theorem 1.1. *Let \mathbf{M} be a finite algebra. If \mathbf{M} is dualizable and has finite rank, then \mathbf{M} is strongly dualizable.*

Every finite algebra that is known to have finite rank also satisfies a simpler condition developed by Lampe, McNulty and Willard [9]. We say that the finite algebra \mathbf{M} **has enough algebraic operations** if there is a map $f: \omega \rightarrow \omega$ for which the following condition holds:

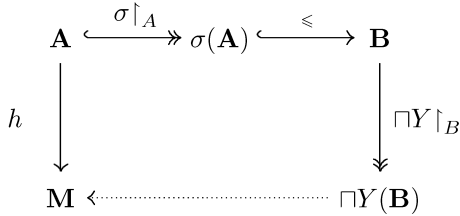
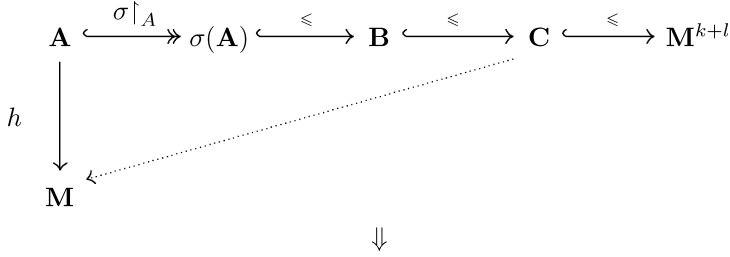


Fig. 2. Rank.

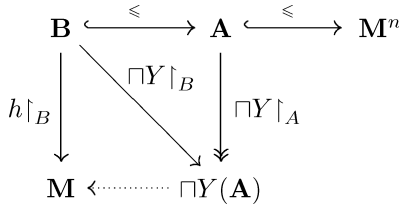


Fig. 3. Enough algebraic operations.

for all $n \in \omega \setminus \{0\}$, all algebras $\mathbf{B} \leq \mathbf{A} \leq \mathbf{M}^n$ and all homomorphisms $h: \mathbf{A} \rightarrow \mathbf{M}$, there exists a set $Y \subseteq \mathcal{A}(\mathbf{M}^n, \mathbf{M})$, with $|Y| \leq f(|B|)$, such that $h \upharpoonright_B$ lifts to $\sqcap Y(\mathbf{A})$ through $\sqcap Y \upharpoonright_B: \mathbf{B} \rightarrow \sqcap Y(\mathbf{A})$.

This definition is illustrated in Fig. 3. The following theorem comes from [9].

Theorem 1.2. *Let \mathbf{M} be a finite algebra. If \mathbf{M} has enough algebraic operations, then $\text{rank}(\mathbf{M}) \leq 2$. So, if \mathbf{M} is dualizable and has enough algebraic operations, then \mathbf{M} is strongly dualizable.*

We now turn our attention to three-element unary algebras. The dualizable three-element unary algebras have been described by Clark, Davey and Pitkethly [5]. The strongly dualizable three-element unary algebras are classified by Pitkethly in [10]. In order to give the characterizations of dualizability and strong dualizability, we divide the class of three-element unary algebras into four types. Consider a unary algebra \mathbf{M} . We define a **kernel of \mathbf{M}** to be an equivalence relation on M that is the kernel of a unary term function of \mathbf{M} that is not a constant map or a permutation. We say that \mathbf{M} is an **n -kernel unary algebra** if n is the number of

different kernels of \mathbf{M} . There are three non-trivial proper equivalence relations on a three-element set. So the class of three-element unary algebras splits into zero-, one-, two- and three-kernel algebras.

The classification of the dualizable and strongly dualizable three-element unary algebras is most complicated within the class of two-kernel algebras. To give our description, we use the fact that every two-kernel three-element unary algebra is isomorphic to a unary algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. (The proof of this fact is very easy, see Lemma 4.1 [5].)

The following theorems, from [5] and [10], give the classifications of dualizable and strongly dualizable three-element unary algebras.

Theorem 1.3. *Let \mathbf{M} be a three-element unary algebra.*

- (1) *If \mathbf{M} is a zero-kernel or one-kernel algebra, then \mathbf{M} is dualizable.*
- (2) *Assume that \mathbf{M} is a two-kernel algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. Then \mathbf{M} is dualizable if and only if none of the following conditions hold:*
 - (a) *ppq and pqq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$, but 010 or 002 is not a term function of \mathbf{M} ;*
 - (b) *010 , 001 and 110 are term functions of \mathbf{M} , but 222 is not;*
 - (c) *002 , 020 and 202 are term functions of \mathbf{M} , but 111 is not.*
- (3) *If \mathbf{M} is a three-kernel algebra, then \mathbf{M} is not dualizable.*

Theorem 1.4. *Let \mathbf{M} be a three-element unary algebra.*

- (1) *If \mathbf{M} is a zero-kernel or one-kernel algebra, then \mathbf{M} is strongly dualizable.*
- (2) *Assume that \mathbf{M} is a two-kernel algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. Then \mathbf{M} is strongly dualizable if and only if 010 and 002 are both term functions of \mathbf{M} and neither of the following conditions holds:*
 - (a) *ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$;*
 - (b) *101 and 220 are term functions of \mathbf{M} .*
- (3) *If \mathbf{M} is a three-kernel algebra, then \mathbf{M} is not strongly dualizable.*

The only method used in [10] to establish strong dualizability was to show that the algebra concerned has enough algebraic operations. So the next theorem also follows from [10] and Theorem 1.2.

Theorem 1.5. *Let \mathbf{M} be a dualizable three-element unary algebra. Then \mathbf{M} is strongly dualizable if and only if \mathbf{M} has enough algebraic operations.*

Within the class of dualizable three-element unary algebras, there is another simpler injectivity condition that is equivalent to strong dualizability. To state this condition, which was introduced in [10], we first need to give some definitions. Let \mathbf{B} be a unary algebra. We define the directed graph $G(\mathbf{B}) = \langle B; E_{\mathbf{B}} \rangle$ by

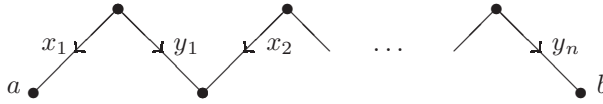
$$E_{\mathbf{B}} := \{(a, b) \mid a \in B \text{ and } b \in \text{sg}_{\mathbf{B}}(a)\},$$

where $\text{sg}_{\mathbf{B}}(a)$ denotes the subuniverse of \mathbf{B} generated by a , for each $a \in B$. We say that the algebra \mathbf{B} is **connected** if $G(\mathbf{B})$ is connected as a simple graph. A **connected component of \mathbf{B}** is a maximal connected subalgebra of \mathbf{B} .

Let \mathbf{M} be a finite unary algebra and let \mathbf{A} belong to the quasi-variety $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$. The **center of \mathbf{A}** is defined to be the subuniverse of \mathbf{A} given by

$$C_{\mathbf{A}} := \{m^{\mathbf{A}} \mid m \in M \text{ is the value of a constant term function of } \mathbf{M}\}.$$

Let $G^*(\mathbf{A})$ denote the induced subgraph of $G(\mathbf{A})$ with vertex set $A \setminus C_{\mathbf{A}}$. For all $a, b \in A \setminus C_{\mathbf{A}}$ and $n \in \omega$, we say that there is a **fence from a to b in \mathbf{A} of length n** if there are edges $x_1, y_1, \dots, x_n, y_n$ of $G^*(\mathbf{A})$ such that



in $G^*(\mathbf{A})$. A subalgebra \mathbf{P} of \mathbf{A} is called a **petal of \mathbf{A}** if $P \setminus C_{\mathbf{A}}$ is the vertex set of a connected component of the graph $G^*(\mathbf{A})$. So $a, b \in A \setminus C_{\mathbf{A}}$ belong to the same petal of \mathbf{A} if and only if there is a fence from a to b in \mathbf{A} . Every non-trivial algebra in \mathcal{A} is the coproduct of its petals, see Lemma 2.3 [5]. The notion of a petal was useful in both [5] and [10] when finding the dualizable and strongly dualizable three-element unary algebras.

Every element of $A \setminus C_{\mathbf{A}}$ belongs to some petal of \mathbf{A} . Now let \mathbf{P} be a petal of \mathbf{A} and let $n \in \omega \setminus \{0\}$. Define the distance function $d_{\mathbf{P}}$ on $P \setminus C_{\mathbf{A}}$ such that $d_{\mathbf{P}}(a, b)$ is the length of the shortest fence from a to b in \mathbf{A} . For each $a \in P \setminus C_{\mathbf{A}}$ the set

$$n_{\mathbf{A}}(a) := \{ b \in P \setminus C_{\mathbf{A}} : d_{\mathbf{P}}(a, b) \leq n \} \cup C_{\mathbf{A}}$$

determines a subalgebra of \mathbf{A} . For $a \in C_{\mathbf{A}}$, we assign $n_{\mathbf{A}}(a) := C_{\mathbf{A}}$. For all subsets $B \subseteq A$ define $n_{\mathbf{A}}(B) := \cup \{ n_{\mathbf{A}}(b) \mid b \in B \}$. The algebra \mathbf{M} is said to be **n -quasi-injective** if, for all finite algebras $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ such that $\mathbf{B} \leq \mathbf{A}$, every homomorphism $x : \mathbf{B} \rightarrow \mathbf{M}$ that extends to $n_{\mathbf{A}}(B)$ also extends to \mathbf{A} . We call the algebra \mathbf{M} **quasi-injective** if there is some $n \in \omega \setminus \{0\}$ such that \mathbf{M} is n -quasi-injective. The theorem below comes from [10].

Theorem 1.6. *Let \mathbf{M} be a dualizable three-element unary algebra. Then \mathbf{M} is strongly dualizable if and only if \mathbf{M} is quasi-injective.*

2. Three-Element Unary Algebras That Have Neither \mathbf{V}, \mathbf{L} Nor \mathbf{D} as an Iso-reduct

In this section, we show that each three-element unary algebra that has neither \mathbf{V}, \mathbf{L} nor \mathbf{D} as an iso-reduct has enough algebraic operations and is quasi-injective.

Consider a unary algebra $\mathbf{M} = \langle \{0, 1, 2\}; F \rangle$ and let $u : \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ be a permutation. We define the new unary algebra $\mathbf{M}_u = \langle \{0, 1, 2\}; F_u \rangle$ such that $F_u := \{ u \circ v \circ u^{-1} \mid v \in F \}$. Since $u \circ v = (u \circ v \circ u^{-1}) \circ u$, for $v \in F$, the

map $u: \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ is an isomorphism between \mathbf{M} and \mathbf{M}_u . Moreover, every isomorphic copy of \mathbf{M} on the set $\{0, 1, 2\}$ can be obtained by conjugation, or relabeling, in this way.

The next two lemmas provide conditions under which \mathbf{V}, \mathbf{L} or \mathbf{D} (see Fig. 1) is an isoreduct of \mathbf{M} .

Lemma 2.1. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$. Assume that ppq and qpq are both term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$. Then \mathbf{M} has \mathbf{V} or \mathbf{L} as an isoreduct.*

Proof. Since $\mathbf{V} = \langle \{0, 1, 2\}; 112, 212 \rangle$ and $\mathbf{L} = \langle \{0, 1, 2\}; 002, 202 \rangle$, if $q = 2$ then either \mathbf{V} or \mathbf{L} is an isoreduct of \mathbf{M} . So assume that $q \neq 2$. First consider the case that $p = 2$. We have

$$ppq \circ ppq = 22q \circ 22q = qq2 \quad \text{and} \quad ppq \circ qpq = 22q \circ q2q = 2q2.$$

So $qq2$ and $2q2$ are term functions of \mathbf{M} . Therefore \mathbf{M} has \mathbf{V} or \mathbf{L} as an isoreduct.

Now assume that $p \neq 2$. The algebra $\mathbf{L}_{021} = \langle \{0, 1, 2\}; 010, 110 \rangle$ is isomorphic to \mathbf{L} . As $101 \circ 101 = 010$ and $101 \circ 001 = 110$, it follows that \mathbf{L} is an isoreduct of \mathbf{M} . □

Lemma 2.2. *Let \mathbf{M} be a two-kernel unary algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. Then*

- (1) \mathbf{M} has \mathbf{V} or \mathbf{L} as an isoreduct if and only if ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$, and
- (2) \mathbf{M} has \mathbf{D} as an isoreduct if and only if 101 and 220 are term functions of \mathbf{M} .

Proof. We prove (1) and (2) simultaneously. Let F be the set of unary term functions of \mathbf{M} . First assume that \mathbf{M} has \mathbf{V} as an isoreduct. If $u: \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ is a permutation such that \mathbf{V}_u has kernels $\{01|2\}$ and $\{02|1\}$, then u is either 012 or 021 . So \mathbf{M} has either $\mathbf{V} = \langle \{0, 1, 2\}; 112, 212 \rangle$ or $\mathbf{V}_{021} = \langle \{0, 1, 2\}; 212, 112 \rangle$ as a term reduct. Therefore $\{112, 212\} \subseteq F$. Similarly, if \mathbf{M} has \mathbf{L} as an isoreduct, then $\{002, 202\} \subseteq F$ or $\{010, 110\} \subseteq F$, and, if \mathbf{M} has \mathbf{D} as an isoreduct, then $\{101, 220\} \subseteq F$.

To finish (1) assume that there is some $p, q \in M$, with $p \neq q$, such that $\{ppq, qpq\} \subseteq F$. By Lemma 2.1, it follows that \mathbf{M} has \mathbf{V} or \mathbf{L} as an isoreduct. Clearly, if $\{101, 220\} \subseteq F$, then \mathbf{M} has \mathbf{D} as an isoreduct so (2) holds. □

We also use the following lemma from [10].

Lemma 2.3. *Let \mathbf{M} be a two-kernel unary algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. Let F denote the set of all unary term functions of \mathbf{M} . Then at least one of the following is true:*

- (1) $\{010, 002\} \not\subseteq F$, and $\{ppq, qpq\} \subseteq F$, for some $p, q \in M$ with $p \neq q$;
- (2) $\{ppq, qpq\} \subseteq F$, for some $p, q \in M$ with $p \neq q$;

- (3) $\{101, 220\} \subseteq F$;
- (4) $\{010, 002\} \subseteq F$, and conditions (2) and (3) both fail.

The next lemma summarizes the parts of our main theorem that already follow from results in the literature.

Lemma 2.4. *Let \mathbf{M} be a three-element unary algebra and consider the following conditions:*

- (1) \mathbf{M} has neither \mathbf{V}, \mathbf{L} nor \mathbf{D} as an isoreduct;
- (2) \mathbf{M} has finite rank;
- (3) \mathbf{M} has enough algebraic operations;
- (4) \mathbf{M} is quasi-injective;
- (5) the quasi-equational theory of \mathbf{M} is finitely based.

Condition (3) implies condition (2). If \mathbf{M} has finite type, then (1) and (5) are equivalent. If the algebra \mathbf{M} is dualizable, then conditions (1) through (4) are equivalent. If \mathbf{M} is dualizable and has finite type, then all five conditions are equivalent.

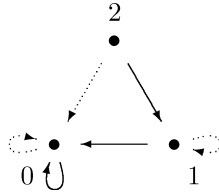
Proof. The property “enough algebraic operations” was developed by Lampe, McNulty and Willard [9] to be a simpler sufficient condition for “finite rank”. (See Theorem 1.2.) So (3) implies (2). The equivalence of (1) and (5) was established for algebras of finite type by Bestsennyi [2].

Now assume that the algebra \mathbf{M} is dualizable. Here we show the equivalence of conditions (2) to (4). The dualizable algebra \mathbf{M} is strongly dualizable if and only if \mathbf{M} has enough algebraic operations (by Theorem 1.5) if and only if \mathbf{M} is quasi-injective (by Theorem 1.6). Thus conditions (3) and (4) are equivalent. However, a dualizable algebra with finite rank is strongly dualizable by Theorem 1.1 so condition (2) implies (3) and (4). Since it is always true (by Theorem 1.2) that condition (3) implies condition (2) we have conditions (2), (3), and (4) are equivalent.

It remains to check that (1) and (3) are equivalent. We do this by considering the four cases provided by kernels. Since \mathbf{M} is dualizable, we know that \mathbf{M} is not a three-kernel algebra (Theorem 1.3). If \mathbf{M} is a zero- or one-kernel algebra, then \mathbf{M} is strongly dualizable by Theorem 1.4. Hence, by Theorem 1.5, \mathbf{M} has enough algebraic operations. As \mathbf{V}, \mathbf{L} , and \mathbf{D} are two-kernel, they cannot be an isoreduct of \mathbf{M} . This shows that both (1) and (3) hold for zero- or one-kernel algebras.

Finally, assume that \mathbf{M} is a two-kernel algebra on the set $\{0, 1, 2\}$ with kernels $\{01|2\}$ and $\{02|1\}$. Our algebra \mathbf{M} must satisfy one of the four conditions in Lemma 2.3. Condition (1) of Lemma 2.3 is exactly condition (2)(a) of Theorem 1.3 which does not hold because \mathbf{M} is dualizable.

Using Theorem 1.4, the dualizable algebra \mathbf{M} is strongly dualizable if and only if \mathbf{M} satisfies Lemma 2.3(4), that is, if and only if Lemma 2.3(2) and (3) fail to hold. Since Lemma 2.2 tells us that \mathbf{M} has \mathbf{V}, \mathbf{L} or \mathbf{D} as an isoreduct if and only if \mathbf{M} satisfies Lemma 2.3(2) or (3), we have that \mathbf{M} is strongly dualizable exactly



$$\mathbf{N} = \langle \{0, 1, 2\}; 001, 010 \rangle$$

Fig. 4.

when \mathbf{M} does not have \mathbf{V}, \mathbf{L} or \mathbf{D} as an isoreduct. Finally, by Theorem 1.5, \mathbf{M} having enough algebraic operations is equivalent to \mathbf{M} not having \mathbf{V}, \mathbf{L} or \mathbf{D} as an isoreduct. Therefore (1) and (3) are equivalent. \square

Now define the unary algebra $\mathbf{N} = \langle \{0, 1, 2\}; 001, 010 \rangle$ as in Fig. 4. The term functions of \mathbf{N} are 001, 010, 012 and 000 so, by Theorem 1.3 part (2)(a), \mathbf{N} is non-dualizable. We shall show that, up to polynomial isomorphism, the algebra \mathbf{N} is the only non-dualizable three-element unary algebra that has neither \mathbf{V}, \mathbf{L} nor \mathbf{D} as an isoreduct. (An algebra \mathbf{M} is **polynomially isomorphic** to \mathbf{N} if \mathbf{M} is isomorphic to an algebra that has the same set of polynomial functions as \mathbf{N} .)

Lemma 2.5. *Every three-kernel three-element unary algebra has \mathbf{V} or \mathbf{L} as an isoreduct.*

Proof. Let \mathbf{M} be a three-kernel unary algebra on the set $\{0, 1, 2\}$ and let F be the set of unary term functions of \mathbf{M} . Suppose that \mathbf{M} has neither \mathbf{V} nor \mathbf{L} as an isoreduct. We begin by proving the following two claims.

- (1) For all $p, q \in M$ with $p \neq q$, we have $\{ppq, qpq\} \not\subseteq F$ and $\{qpp, qqp\} \not\subseteq F$.
- (2) If $p, q \in M$ such that $p \neq q$ and $ppq \in F$, then $p = 2$.

To prove (1), let $p, q \in M$ with $p \neq q$. By Lemma 2.1, we have $\{ppq, qpq\} \not\subseteq F$, as neither \mathbf{V} nor \mathbf{L} is an isoreduct of \mathbf{M} . Now define $\tilde{p} := 201(p)$ and $\tilde{q} := 201(q)$. Then $\{qpp, qqp\}_{201} = \{\tilde{p}\tilde{p}\tilde{q}, \tilde{q}\tilde{p}\tilde{q}\}$. It follows, again by Lemma 2.1, that $\{qpp, qqp\} \not\subseteq F$. So (1) holds.

To show that (2) holds, let $p, q \in M$ with $p \neq q$ and $ppq \in F$. Since \mathbf{M} is a three-kernel algebra, there exist $r, s, t, u \in M$, with $r \neq s$ and $t \neq u$, such that $rsr, tuu \in F$. We have $\{ssr, rsr\} \not\subseteq F$, by (1). So $rsr \circ ppq \neq ssr$ and therefore $p \neq 1$. Similarly, $\{tuu, ttu\} \not\subseteq F$, so $tuu \circ ppq \neq ttu$ and therefore $p \neq 0$. We have shown that $p = 2$. Thus (2) is satisfied.

We can now derive a contradiction. As \mathbf{M} is a three-kernel algebra, there is some $p, q \in M$, with $p \neq q$, such that $ppq \in F$. Using claim (2), we must have $p = 2$. So $220 \in F$ or $221 \in F$. This implies that $002 = 220 \circ 220 \in F$ or $112 = 221 \circ 221 \in F$, which contradicts (2). \square

Lemma 2.6. *Any non-dualizable three-element unary algebra that has neither \mathbf{V} , \mathbf{L} nor \mathbf{D} as an isoreduct is polynomially isomorphic to \mathbf{N} .*

Proof. Let \mathbf{M} be such an algebra. Theorem 1.3 shows that if \mathbf{M} is zero-kernel or one-kernel then it is dualizable while Lemma 2.5 shows that if \mathbf{M} is three-kernel it would have \mathbf{V} or \mathbf{L} as an isoreduct. Thus \mathbf{M} is a two-kernel algebra. Assume that \mathbf{M} has underlying set $\{0, 1, 2\}$ and kernels $\{01|2\}$ and $\{02|1\}$. Let F be the set of all unary term functions of \mathbf{M} . By Lemma 2.2, for all $p, q \in M$ such that $p \neq q$ we know that $\{ppq, qpq\} \not\subseteq F$. Since $\{010, 110\} \not\subseteq F$ and $\{002, 202\} \not\subseteq F$ imply that \mathbf{M} does not satisfy (2)(b) or (2)(c) from Theorem 1.3, condition (2)(a) must be satisfied by \mathbf{M} . That is, there exist $p, q \in M$, with $p \neq q$, such that $ppq, pqp \in F$ but $\{010, 002\} \not\subseteq F$.

We now determine which operations are allowed to belong to F . Note that, since \mathbf{M} has kernels $\{01|2\}$ and $\{02|1\}$, we have

$$F \subseteq \{012, 021\} \cup \{rrs, rsr \mid r, s \in M\},$$

(see Lemma 4.2 [5]). Let $r, s, a, b \in M$ with $r \neq s$ and $a \neq b$. As $pqp \in F$ and $\{qqp, pqp\} \not\subseteq F$, we have $qqp \notin F$. If rrs is in F then $ppq \circ rrs \neq qqp$ and $pqp \circ rrs \neq qqp$. Therefore $r \neq 2$ and $r \neq 1$ so $r = 0$. Similarly $ppq \in F$ and $\{ppq, qpq\} \not\subseteq F$, so if $aba \in F$ then $ppq \circ aba \neq qpq$ and $pqp \circ aba \neq qpq$. This implies that $a \neq 2$ and $a \neq 1$. Thus $a = 0$ and we have

$$F \subseteq \{012, 021, 001, 002, 010, 020, 000, 111, 222\}.$$

Since $ppq, pqp \in F$, we have either $001, 010 \in F$ or $002, 020 \in F$. Thus, one of the two isomorphic algebras \mathbf{M} and its conjugate \mathbf{M}_{021} has 001 and 010 as terms functions, so assume that $001, 010 \in F$. Since \mathbf{M} satisfies (2)(a) of Theorem 1.3, this implies that $002 \notin F$. As $021 \circ 001 = 020 \circ 001 = 002 \notin F$, we have $021, 020 \notin F$. Thus

$$\{001, 010\} \subseteq F \subseteq \{012, 001, 010, 000, 111, 222\},$$

whence \mathbf{M} is polynomially isomorphic to \mathbf{N} . □

To finish the proof that (1) implies (3) and (4) of our main theorem, it remains to show that every three-element unary algebra that is polynomially isomorphic to \mathbf{N} has enough algebraic operations and is quasi-injective. As the algebra $\mathbf{N} = \langle \{0, 1, 2\}; 001, 010 \rangle$ already has $000 = 001 \circ 001$ as a term function, we only need consider adding subsets of $\{111, 222\}$ to the operations of \mathbf{N} . Since $001 \circ 222 = 111$ there are really only two new algebras polynomially isomorphic to \mathbf{N} , those created by adding either 111 or 222 to the operations of \mathbf{N} . Thus, the next two lemmas refer to algebras that are polynomially isomorphic to \mathbf{N} .

Lemma 2.7. *Let $\mathbf{M} = \langle \{0, 1, 2\}; \{001, 010\} \cup F \rangle$ be a unary algebra such that $F \subseteq \{111, 222\}$. Let $\mathbf{B} \leq \mathbf{A}$ in $\mathbb{ISP}(\mathbf{M})$ and let $x: \mathbf{B} \rightarrow \mathbf{M}$. Then the following are*

equivalent:

- (1) x extends to \mathbf{A} ;
- (2) x extends to $\mathbf{1}_{\mathbf{A}}(B)$;
- (3) $x(001(a)) = 0$ or $x(010(a)) = 0$, for each $a \in A$ such that $001(a) \in B$ and $010(a) \in B$.

In particular, the algebra \mathbf{M} is 1-quasi-injective.

Proof. Clearly (1) implies (2). To see that (2) implies (3), assume that x extends to a homomorphism $\bar{x}: \mathbf{1}_{\mathbf{A}}(B) \rightarrow \mathbf{M}$. Let $a \in A$, with $001(a) \in B$ and $010(a) \in B$, such that $x(001(a)) \neq 0$. If $001(a) \in C_{\mathbf{A}}$ then

$$x(001(a)) = x(010 \circ 001(a)) = 010(x(001(a))) \in \{0, 1\}$$

and we have $111 \in F$ and $001(a) = 1^{\mathbf{A}}$. The algebra \mathbf{M} satisfies the quasi-equation $001(z) \approx 1 \Rightarrow 010(z) \approx 0$, and therefore $x(010(a)) = x(0^{\mathbf{A}}) = 0$. Now assume that $001(a) \notin C_{\mathbf{A}}$. We have $a \in \mathbf{1}_{\mathbf{A}}(B)$ and $001(\bar{x}(a)) = x(001(a)) \neq 0$. So $\bar{x}(a) = 2$ and $x(010(a)) = 010(\bar{x}(a)) = 0$.

Now assume that (3) holds. For $a \in A$, if $001(a) \in x^{-1}(1)$ and $010(a) \in B$ then by (3) we have $010(a) \in x^{-1}(0)$. So the map $\bar{x}: A \rightarrow M$ defined by

$$\bar{x}(a) = \begin{cases} 2 & \text{if } 001(a) \in x^{-1}(1), \\ 1 & \text{if } 010(a) \in x^{-1}(1), \\ 0 & \text{otherwise} \end{cases}$$

is well defined. For all $b \in B$, we have

$$x(b) = 2 \quad \text{iff} \quad 001(x(b)) = 1 \quad \text{iff} \quad x(001(b)) = 1 \quad \text{iff} \quad \bar{x}(b) = 2$$

and, similarly, $x(b) = 1$ if and only if $\bar{x}(b) = 1$. So \bar{x} is an extension of x . As x preserves every constant operation in F , so does \bar{x} .

To see that \bar{x} is a homomorphism, let $a \in A$. Then

$$\begin{aligned} 001(\bar{x}(a)) = 1 & \quad \text{iff} \quad \bar{x}(a) = 2 \quad \text{iff} \quad 001(a) \in x^{-1}(1) \\ & \quad \text{iff} \quad 010 \circ 001(a) \in x^{-1}(1) \quad \text{iff} \quad \bar{x}(001(a)) = 1. \end{aligned}$$

Since $001 \circ 001 = 000$ and $x(0^{\mathbf{A}}) = 0$, we have $\bar{x}(001(a)) \neq 2$. It now follows that $001(\bar{x}(a)) = \bar{x}(001(a))$ and, similarly, that $010(\bar{x}(a)) = \bar{x}(010(a))$. So $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}$ is a homomorphism extending x . □

For any set S and $r \in \{0, 1, 2\}$, we use \hat{r} to denote the constant map in $\{0, 1, 2\}^S$ with value r .

Lemma 2.8. *Let $\mathbf{M} = \langle \{0, 1, 2\}; \{001, 010\} \cup F \rangle$ be a unary algebra such that $F \subseteq \{111, 222\}$. Then \mathbf{M} has enough algebraic operations.*

Proof. Define $f: \omega \rightarrow \omega$ by $f(k) := k$. Let $n \in \omega \setminus \{0\}$, let $\mathbf{B} \leq \mathbf{A} \leq \mathbf{M}^n$ and let $h: \mathbf{A} \rightarrow \mathbf{M}$. The set $C := \{0, 1\}^n \cup \{\hat{2}\}$ determines a subalgebra of \mathbf{M}^n . For each $a \in \{0, 1\}^n \setminus \{\hat{0}\}$, the map $g_a: \mathbf{C} \rightarrow \mathbf{M}$ given by

$$g_a(c) = \begin{cases} 2 & \text{if } c = \hat{2}, \\ 1 & \text{if } c = a \text{ or } c = \hat{1}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $c \in C$, is a homomorphism. To see this, first note that g_a respects the constant valued functions 222 and 111. For all $c \in C$, since $010(c) \neq \hat{2}$ and $010(g_a(c)) \in \{0, 1\}$, we have

$$\begin{aligned} 010(g_a(c)) = 1 & \text{ iff } g_a(c) = 1 & \text{ iff } c = a \text{ or } c = \hat{1} \\ & \text{ iff } 010(c) = a & \text{ or } 010(c) = \hat{1} & \text{ iff } g_a(010(c)) = 1 \end{aligned}$$

which implies that $010(g_a(c)) = g_a(010(c))$. For $c \in C$, we have $001(c) = \hat{0}$ or $\hat{1}$ so $g_a(001(c)) = 1$ if and only if $001(c) = \hat{1}$, that is, $c = \hat{2}$. This occurs exactly when $g_a(c) = 2$ or $001(g_a(c)) = 1$. Thus g_a is a homomorphism. We use (3) of the previous lemma to show that for each $a \in \{0, 1\}^n \setminus \{\hat{0}\}$, the homomorphism $g_a: \mathbf{C} \rightarrow \mathbf{M}$ extends to a homomorphism $\bar{g}_a: \mathbf{M}^n \rightarrow \mathbf{M}$. For $d \in M^n$ the elements $001(d)$ and $010(d)$ are in C and are both fixed by the term function 010 so $g_a(001(d))$ and $g_a(010(d))$ are in $\{0, 1\}$. If $g_a(001(d)) = 1$ then either $001(d) = a$ or $001(d) = \hat{1}$. In the latter case $d = \hat{2}$ so $g_a(010(d)) = 0$. In the former case, as \mathbf{M} satisfies the quasi-equation $001(z) \approx 010(z) \Rightarrow 001(z) \approx 0$, we have $010(d) \neq a$. If $010(d) = \hat{1}$ then $d = \hat{1}$ so $g_a(001(d)) = 0$ which is false. We must again have $g_a(010(d)) = 0$. By Lemma 2.7 the map g_a extends to a homomorphism $\bar{g}_a: \mathbf{M}^n \rightarrow \mathbf{M}$. Now define the subset

$$Y := \{\bar{g}_b \mid b \in B \cap (\{0, 1\}^n \setminus \{\hat{0}\})\}$$

of $\mathcal{A}(\mathbf{M}^n, \mathbf{M})$. Then $|Y| \leq |B| - 1 < f(|B|)$. Define the homomorphism $\mu: \mathbf{M}^n \rightarrow \mathbf{M}^Y$ by $\mu := \cap Y$. To show that \mathbf{M} has enough algebraic operations it is sufficient to show that $h \upharpoonright_B$ lifts to $\mu(\mathbf{A})$ through $\mu \upharpoonright_B$. We first show that $\mu \upharpoonright_B$ is an embedding.

The homomorphisms in Y separate the elements of $B \cap \{0, 1\}^n$. To see this, let $a, b \in B \cap \{0, 1\}^n$ with $a \neq b$. If $a = \hat{0}$ or $b = \hat{1}$ then $\bar{g}_b \in Y$ and $\bar{g}_b(b) = 1 \neq \bar{g}_b(a)$ so we may assume that $a \neq \hat{0}$ and $b \neq \hat{1}$. Thus $\bar{g}_a \in Y$ and $\bar{g}_a(a) = 1 \neq 0 = \bar{g}_a(b)$. It follows that for $a, b \in B \cap \{0, 1\}^n$ with $a \neq b$ we have $\mu(a) \neq \mu(b)$ so μ is one-to-one on $B \cap \{0, 1\}^n$. Now let $c, d \in B$ with $c \neq d$. We have either $001(c) \neq 001(d)$ or $010(c) \neq 010(d)$. This implies that $\mu(001(c)) \neq \mu(001(d))$ or $\mu(010(c)) \neq \mu(010(d))$. Thus $\mu(c) \neq \mu(d)$, whence $\mu \upharpoonright_B$ is an embedding.

We prove that $x := h \circ (\mu \upharpoonright_B)^{-1}: \mu(\mathbf{B}) \rightarrow \mathbf{M}$ extends to $\mu(\mathbf{A})$. Choose some $a \in A$ for which $001(\mu(a)) \in \mu(B)$ and $010(\mu(a)) \in \mu(B)$. By Lemma 2.7, it now suffices to show that $x(001(\mu(a))) = 0$ or $x(010(\mu(a))) = 0$. There is some $b_0 \in B$

with $001(\mu(a)) = \mu(b_0)$. Define the element $b := 010(b_0)$ which is in $B \cap \{0, 1\}^n$. Then

$$\mu(b) = \mu(010(b_0)) = 010(\mu(b_0)) = 010 \circ 001(\mu(a)) = 001(\mu(a)).$$

Similarly, there is some $c \in B \cap \{0, 1\}^n$ such that $\mu(c) = 010(\mu(a))$. Recall that we are trying to prove that $x(001(\mu(a))) = 0$ or $x(010(\mu(a))) = 0$. We have

$$x(001(\mu(a))) = x(\mu(b)) = h \circ (\mu \upharpoonright_B)^{-1}(\mu(b)) = h(b)$$

and, similarly, $x(010(\mu(a))) = h(c)$. So we really only need to show that $h(b) = 0$ or $h(c) = 0$. As 000 is a term operation of \mathbf{M} , we have $h(\hat{0}) = 0$ so if $b = \hat{0}$ or $c = \hat{0}$ then we are done. Assume that $b, c \neq \hat{0}$. This gives us $\mu(b), \mu(c) \neq \mu(\hat{0})$ as $\mu \upharpoonright_B$ is an embedding. Since $\mu(001(a)) = \mu(b) \neq \mu(\hat{0})$ and $\mu(010(a)) = \mu(c) \neq \mu(\hat{0})$, we know that $a \notin \{\hat{1}, \hat{2}\}$. As $\mu = \sqcap Y$, we now have

$$\bar{g}_b(001(a)) = \bar{g}_b(b) = 1 \quad \text{and} \quad \bar{g}_c(010(a)) = \bar{g}_c(c) = 1,$$

so $001(a) = b$ and $010(a) = c$. By Lemma 2.7, it follows that $h(b) = 0$ or $h(c) = 0$, as $h \upharpoonright_B$ extends to \mathbf{A} . Thus $x(001(\mu(a))) = 0$ or $x(010(\mu(a))) = 0$. Again by Lemma 2.7, x extends to $\bar{x}: \mu(\mathbf{A}) \rightarrow \mathbf{M}$. Hence $\bar{x} \circ \mu \upharpoonright_B = h \upharpoonright_B$ and \mathbf{M} has enough algebraic operations. □

Lemma 2.6 says that if \mathbf{M} is a non-dualizable three-element unary algebra without \mathbf{V} , \mathbf{L} or \mathbf{D} as an isoreduct then \mathbf{M} is polynomially isomorphic to \mathbf{N} . Lemma 2.7 shows that such an algebra is 1-quasi-injective while Lemma 2.8 shows it has enough algebraic operations. For any finite algebra enough algebraic operations implies finite rank so condition (1) of the main theorem implies condition (2) through (4) for non-dualizable algebras. These implications for dualizable algebras are covered in Lemma 2.4.

3. Three-Element Unary Algebras That Have \mathbf{V} , \mathbf{L} or \mathbf{D} as an Isoreduct

In this section, we show that each three-element unary algebra that has \mathbf{V} , \mathbf{L} or \mathbf{D} as an isoreduct does not have finite rank and is not quasi-injective. We begin by considering the three-element unary algebras that have \mathbf{V} or \mathbf{L} as an isoreduct.

The following construction, for building algebras out of (partially) ordered sets, comes from [10]. For the purposes of the construction, we give the three-element set $\{0, 1, 2\}$ the non-standard order $2 \preceq 0 \preceq 1$.

Definition 3.1. Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ and let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set. Let \triangleleft be a subset of \leq that contains the diagonal Δ_P and define

$\mathbf{P}' = \langle P; \triangleleft \rangle$. Define the set $P^+ := P \cup \{\top, \perp\}$. For all $a, b \in P$ with $a \triangleleft b$, define $\widehat{ab} \in M^{P^+}$ by $\widehat{ab}(\perp) = 2, \widehat{ab}(\top) = 1$ and

$$\widehat{ab}(c) = \begin{cases} 2 & \text{if } c \leq a, \\ 0 & \text{if } c \leq b \text{ and } c \not\leq a, \\ 1 & \text{otherwise,} \end{cases}$$

for all $c \in P$. Now define the algebra

$$\widehat{\mathbf{P}}_{\triangleleft} := \mathbf{sg}_{M^{P^+}}(\{\widehat{ab} \mid a, b \in P \text{ and } a \triangleleft b\}).$$

Where we need to be precise, given $a, b \in P$ with $a \triangleleft b$, we write $\widehat{ab}_{\mathbf{P}}$ instead of \widehat{ab} .

We shall be using the previous construction in a slightly more general setting than it was used in [10]. The following few lemmas tell us that the construction still works the way we want it to, even in this more general setting.

Lemma 3.2. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ and let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set. Let \triangleleft be a subset of \leq that contains Δ_P and define $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft}$.*

- (1) *Then $\mathbf{sg}_{\mathbf{A}}(\widehat{aa}) \cap \mathbf{sg}_{\mathbf{A}}(\widehat{bb}) \subseteq \{\hat{0}, \hat{1}, \hat{2}\}$, for all $a, b \in P$ with $a \neq b$.*
- (2) *Assume that ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$. Then $\mathbf{sg}_{\mathbf{A}}(\widehat{aa}) \cap \mathbf{sg}_{\mathbf{A}}(\widehat{bb}) \subseteq C_{\mathbf{A}}$, for all $a, b \in P$ with $a \neq b$.*

Proof. We prove (2), the proof of (1) is easier. Assume that ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$. Let $a, b \in P$ with $a \neq b$. Now let u and v be unary term functions of \mathbf{M} such that $u(\widehat{aa}) = v(\widehat{bb})$. We wish to show that $u(\widehat{aa}) \in C_{\mathbf{A}}$. We have $u(1) = u(\widehat{aa}(\top)) = v(\widehat{bb}(\top)) = v(1)$ and, similarly, $u(2) = v(2)$. Since $\widehat{aa}, \widehat{bb} \in \{1, 2\}^{P^+}$, this gives us $u(\widehat{aa}) = u(\widehat{bb})$. Therefore $u(1) = u(2)$, as $a \neq b$. There exist $r, s \in M$ such that $u = rss$. Table 1 illustrates that $w := sss$ is a constant term function of \mathbf{M} . Thus $u(\widehat{aa}) = w(\widehat{aa}) \in C_{\mathbf{A}}$. □

Lemma 3.3. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ and let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set. Let \triangleleft be a subset of \leq that contains Δ_P . Define $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft}$ and $A_* := \cup\{\mathbf{sg}_{\mathbf{A}}(\widehat{aa}) \mid a \in P\}$. Then*

Table 1. The term function sss as a composition of rss, ppq and qpq .

p	q	sss
1	2	$rss \circ 112$
2	1	$rss \circ 221$
0	1	$rss \circ 101 \circ 001 \circ 001$
1	0	$rss \circ 110 \circ 010$
0	2	$rss \circ 202 \circ 202$
2	0	$rss \circ 220 \circ 020 \circ 020$

- (1) $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap A_* \subseteq \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})$, for all $(a, b) \in \triangleleft$, and
- (2) $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cd}) \subseteq \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}, \widehat{cc}, \widehat{dd}\})$, for all $(a, b), (c, d) \in \triangleleft$ with $(a, b) \neq (c, d)$.

Proof. For each $w \in M^{P^+}$, define the partition

$$\mathcal{P}(w) := \{w^{-1}(m) \mid m \in M\} \setminus \{\emptyset\}$$

of P^+ . For all $r, s \in M$ and all $a, b \in P$ with $a \triangleleft b$, we have $rrs(\widehat{ab}) = rrs(\widehat{aa})$ and $rsr(\widehat{ab}) = rsr(\widehat{bb})$. For $r \neq s$ in M , we have $\mathcal{P}(rss(\widehat{ab})) = \mathcal{P}(011(\widehat{ab}))$. It follows that, for all $a, b \in P$ with $a \triangleleft b$, the subuniverse $\text{sg}_{\mathbf{A}}(\widehat{ab})$ of \mathbf{A} is contained in the set

$$T_{ab} = \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\}) \cup \{w \in M^{P^+} : \mathcal{P}(w) = \mathcal{P}(\widehat{ab}) \text{ or } \mathcal{P}(w) = \mathcal{P}(011(\widehat{ab}))\}.$$

To prove (1), let $a, b, c \in P$ with $a \triangleleft b$. We want to prove that $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cc}) \subseteq \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})$. If $a = b$ we are done. Otherwise, $|\mathcal{P}(\widehat{cc})| = 2$ and $|\mathcal{P}(\widehat{ab})| = 3$ so $\mathcal{P}(\widehat{cc}) \neq \mathcal{P}(\widehat{ab})$. Let $w \in \text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cc})$. Since $\text{sg}_{\mathbf{A}}(\widehat{ab})$ is a subset of T_{ab} , we need to show that $\mathcal{P}(w) \neq \mathcal{P}(\widehat{ab})$ and $\mathcal{P}(w) \neq \mathcal{P}(011(\widehat{ab}))$. As $w \in \text{sg}_{\mathbf{A}}(\widehat{cc})$, we have $\mathcal{P}(w) = \mathcal{P}(\widehat{cc})$ or $\mathcal{P}(w) = \{P^+\}$. Thus $\mathcal{P}(w) \neq \mathcal{P}(\widehat{ab})$.

It remains to show $\mathcal{P}(w) \neq \mathcal{P}(011(\widehat{ab}))$ which we do by showing $\mathcal{P}(011(\widehat{ab})) \neq \mathcal{P}(\widehat{cc})$ and $\mathcal{P}(011(\widehat{ab})) \neq \{P^+\}$. Since $011(\widehat{ab})(\top) = 1 = 011(\widehat{ab})(\perp)$ and $\widehat{cc}(\top) = 1 \neq 2 = \widehat{cc}(\perp)$, we know that $\mathcal{P}(011(\widehat{ab})) \neq \mathcal{P}(\widehat{cc})$. Moreover, $011(\widehat{ab})(b) = 0$ so $\mathcal{P}(011(\widehat{ab})) \neq \{P^+\}$. Hence $\mathcal{P}(w) \neq \mathcal{P}(011(\widehat{ab}))$ and it follows that $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cc}) \subseteq \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})$. Thus (1) holds.

To prove (2), choose $(a, b), (c, d) \in \triangleleft$ such that $(a, b) \neq (c, d)$. We want to show that $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cd}) \subseteq \text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}, \widehat{cc}, \widehat{dd}\})$. If $a = b$ or $c = d$ we are done. Otherwise, $|\mathcal{P}(\widehat{ab})| = |\mathcal{P}(\widehat{cd})| = 3$ and $|\mathcal{P}(011(\widehat{ab}))| = |\mathcal{P}(011(\widehat{cd}))| = 2$, so $\mathcal{P}(\widehat{ab}) \neq \mathcal{P}(011(\widehat{cd}))$ and $\mathcal{P}(\widehat{cd}) \neq \mathcal{P}(011(\widehat{ab}))$. Since $\text{sg}_{\mathbf{A}}(\widehat{ab}) \cap \text{sg}_{\mathbf{A}}(\widehat{cd}) \subseteq T_{ab} \cap T_{cd}$, we want to show the intersection

$\{w : \mathcal{P}(w) = \mathcal{P}(\widehat{ab}) \text{ or } \mathcal{P}(w) = \mathcal{P}(011(\widehat{ab}))\} \cap \{w : \mathcal{P}(w) = \mathcal{P}(\widehat{cd}) \text{ or } \mathcal{P}(w) = \mathcal{P}(011(\widehat{cd}))\}$ is empty. To do this it is now enough to show that $\mathcal{P}(\widehat{ab}) \neq \mathcal{P}(\widehat{cd})$ and $\mathcal{P}(011(\widehat{ab})) \neq \mathcal{P}(011(\widehat{cd}))$.

First suppose that $\mathcal{P}(\widehat{ab}) = \mathcal{P}(\widehat{cd})$. Then

$$\begin{aligned} & \widehat{ab}(\perp) = 2 = \widehat{cd}(\perp) \quad \text{implies} \quad \widehat{ab}^{-1}(2) = \widehat{cd}^{-1}(2) \\ \text{so} & \quad \{p \in P : p \leq a\} = \{p \in P : p \leq c\} \\ \text{thus} & \quad a = c, \\ \text{and} & \quad \widehat{ab}(\top) = 1 = \widehat{cd}(\top) \quad \text{implies} \quad \widehat{ab}^{-1}(1) = \widehat{cd}^{-1}(1) \\ \text{so} & \quad \{p \in P : p \not\leq b\} = \{p \in P : p \not\leq d\} \\ \text{thus} & \quad b = d, \end{aligned}$$

which is a contradiction. Now suppose that $\mathcal{P}(011(\widehat{ab})) = \mathcal{P}(011(\widehat{cd}))$. Then

$$\begin{aligned} & 011(\widehat{ab})(\top) = 1 = 011(\widehat{cd})(\top) \quad \text{implies} \quad 011(\widehat{ab})^{-1}(1) = 011(\widehat{cd})^{-1}(1), \\ \text{so} & \quad \{p \in P : p \leq a \text{ or } p \not\leq b\} = \{p \in P : p \leq c \text{ or } p \not\leq d\}, \end{aligned}$$

and hence $a = c$ and $b = d$, which is again a contradiction. So (2) holds. □

Let \mathcal{G} denote the category of directed graphs. Then \mathcal{G} includes the two-element ordered set $\mathbf{2} = \langle \{1, 2\}; \leq \rangle$ such that $1 \leq 2$. The reader should note the difference between this order and the non-standard order $2 \preccurlyeq 0 \preccurlyeq 1$ used in the construction of $\widehat{\mathbf{P}}_{\triangleleft}$.

The proofs of Lemmas 3.4, 3.7 and 3.10 are essentially the same as proofs in [10] (cf. [10, Lemmas 4.6, 4.10 and 4.8] respectively). We provide them here for the completeness of this section.

Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set and let \triangleleft be a subset of \leq that contains Δ_P . The injective map $\iota_{\mathbf{P}}: P \rightarrow \widehat{P}_{\triangleleft}$ is given by $\iota_{\mathbf{P}}(a) := \widehat{aa}$ for all $a \in P$. Define $\eta: \mathcal{A}(\widehat{\mathbf{P}}_{\triangleleft}, \mathbf{M}) \rightarrow \{0, 1, 2\}^P$ by $\eta(x) = x \circ \iota_{\mathbf{P}}$. For $x \in \mathcal{A}(\widehat{\mathbf{P}}_{\triangleleft}, \mathbf{M})$ and $a \in P$ we have $\eta(x)(a) = (x \circ \iota_{\mathbf{P}})(a) = x(\widehat{aa}) \in \{0, 1, 2\}$.

Lemma 3.4. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ such that ppq and qqq are term functions of \mathbf{M} for some $p, q \in M$ with $p \neq q$. Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set, let \triangleleft be a subset of \leq that contains Δ_P , and define $\mathbf{P}' := \langle P; \triangleleft \rangle$. Then $\eta(x) = x \circ \iota_{\mathbf{P}}$ is in $\mathcal{G}(\mathbf{P}', \mathbf{2})$ for each homomorphism $x: \widehat{\mathbf{P}}_{\triangleleft} \rightarrow \mathbf{M}$. Moreover the map $\eta: \mathcal{A}(\widehat{\mathbf{P}}_{\triangleleft}, \mathbf{M}) \rightarrow \mathcal{G}(\mathbf{P}', \mathbf{2})$ is a bijection.*

Proof. Define $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft}$. Let $x \in \mathcal{A}(\mathbf{A}, \mathbf{M})$. We first show that $\eta(x) \in \{1, 2\}^P$. For all $a \in P$, as $\widehat{aa} \in \{1, 2\}^{P^+}$, we have

$$\begin{aligned} ppq(\eta(x)(a)) &= ppq(x(\widehat{aa})) = x(ppq(\widehat{aa})) = x(qpq(\widehat{aa})) \\ &= qpq(x(\widehat{aa})) = qpq(\eta(x)(a)) \end{aligned}$$

and therefore $\eta(x)(a) \in \{1, 2\}$. To see that $\eta(x)$ preserves order, let $a, b \in P$ such that $a \triangleleft b$. Since $ppq(\widehat{ab}) = ppq(\widehat{aa})$ and $qpq(\widehat{ab}) = ppq(\widehat{bb})$ we have

$$\begin{aligned} \eta(x)(a) = 2 &\Rightarrow x(\widehat{aa}) = 2 \Rightarrow ppq(x(\widehat{aa})) = q \Rightarrow x(ppq(\widehat{aa})) = q \\ &\Rightarrow x(ppq(\widehat{ab})) = q \Rightarrow ppq(x(\widehat{ab})) = q \Rightarrow qpq(x(\widehat{ab})) = q \\ &\Rightarrow x(qpq(\widehat{ab})) = q \Rightarrow x(ppq(\widehat{bb})) = q \Rightarrow ppq(x(\widehat{bb})) = q \\ &\Rightarrow x(\widehat{bb}) = 2 \Rightarrow \eta(x)(b) = 2. \end{aligned}$$

So $\eta(x) \in \mathcal{G}(\mathbf{P}', \mathbf{2})$ as required.

To see that η is one-to-one, let $x, y \in \mathcal{A}(\mathbf{A}, \mathbf{M})$ such that $\eta(x) = \eta(y)$. Choose any $a, b \in P$ with $a \triangleleft b$. Then

$$\begin{aligned} x(\widehat{ab}) = 2 &\text{ iff } ppq(x(\widehat{ab})) = q \text{ iff } x(ppq(\widehat{ab})) = q \text{ iff } x(ppq(\widehat{aa})) = q \\ &\text{ iff } ppq(x(\widehat{aa})) = q \text{ iff } x(\widehat{aa}) = 2. \end{aligned}$$

By symmetry, we have $y(\widehat{ab}) = 2$ if and only if $y(\widehat{aa}) = 2$. Since $x(\widehat{aa}) = \eta(x)(a) = \eta(y)(a) = y(\widehat{aa})$, this tells us that $x(\widehat{ab}) = 2$ if and only if $y(\widehat{ab}) = 2$. In a similar way, we can show that $x(\widehat{ab}) = 1$ if and only if $y(\widehat{ab}) = 1$. Therefore $x = y$, whence η is one-to-one.

Finally, to prove that η is onto, let $z \in \mathcal{G}(\mathbf{P}', \mathbf{2})$. Consider the subuniverse

$$A_* := \cup\{\text{sg}_{\mathbf{A}}(\widehat{aa}) \mid a \in P\}$$

of \mathbf{A} . For all $a, b \in P$ with $a \neq b$, we have $\text{sg}_{\mathbf{A}}(\widehat{aa}) \cap \text{sg}_{\mathbf{A}}(\widehat{bb}) \subseteq \{\widehat{0}, \widehat{1}, \widehat{2}\}$, by Lemma 3.2(1). So we can define the homomorphism $z_* : \mathbf{A}_* \rightarrow \mathbf{M}$ by $z_* := \pi_{\top} \upharpoonright_{A_1} \cup \pi_{\perp} \upharpoonright_{A_2}$, where

$$A_1 = \cup\{\text{sg}_{\mathbf{A}}(\widehat{aa}) \mid a \in z^{-1}(1)\} \quad \text{and} \quad A_2 = \cup\{\text{sg}_{\mathbf{A}}(\widehat{aa}) \mid a \in z^{-1}(2)\}.$$

Let $a \in P$. If $z(a) = 1$, then we have $z_*(\widehat{aa}) = \pi_{\top}(\widehat{aa}) = \widehat{aa}(\top) = 1 = z(a)$. Otherwise, $z(a) = 2$ and $z_*(\widehat{aa}) = \pi_{\perp}(\widehat{aa}) = \widehat{aa}(\perp) = 2 = z(a)$. So, if z_* extends to a homomorphism $\bar{z} : \mathbf{A} \rightarrow \mathbf{M}$, we will have $\eta(\bar{z})(a) = \bar{z}(\widehat{aa}) = z_*(\widehat{aa}) = z(a)$, and therefore $\eta(\bar{z}) = z$.

To see that z_* extends to a homomorphism $\bar{z} : \mathbf{A} \rightarrow \mathbf{M}$, let $a, b \in P$ with $a \triangleleft b$ and $a \neq b$. We want to find some $c \in P^+$ such that $z_* \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})} = \pi_c \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})}$. By Lemma 3.3, we then are able to define $\bar{z} \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{ab})} = \pi_c \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{ab})}$.

If $z(a) = z(b) = 1$, then we have $z_* \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})} = \pi_{\top} \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})}$. If $z(a) = z(b) = 2$, then $z_* \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})} = \pi_{\perp} \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})}$. Otherwise, $z(a) = 1$ and $z(b) = 2$, as $z(a) \leq z(b)$. We now have $z_*(\widehat{aa}) = \pi_{\top}(\widehat{aa}) = 1 = \widehat{aa}(b)$ and $z_*(\widehat{bb}) = \pi_{\perp}(\widehat{bb}) = 2 = \widehat{bb}(b)$, which implies that $z_* \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})} = \pi_b \upharpoonright_{\text{sg}_{\mathbf{A}}(\{\widehat{aa}, \widehat{bb}\})}$. Hence η is a bijection. \square

When \triangleleft is a subset of \leq containing Δ_P we have $\mathcal{G}(\mathbf{P}, \mathbf{2})$ is a subset of $\mathcal{G}(\mathbf{P}', \mathbf{2})$. In the next lemma we are interested in those homomorphisms $x \in \mathcal{A}(\mathbf{A}, \mathbf{M})$ for which $x \circ \iota_{\mathbf{P}}$, though in $\mathcal{G}(\mathbf{P}', \mathbf{2})$, is not in $\mathcal{G}(\mathbf{P}, \mathbf{2})$.

Lemma 3.5. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ such that ppq and qpq are term functions of \mathbf{M} for some $p, q \in M$ with $p \neq q$. Let $\mathbf{P} = \langle P; \leq \rangle$ be a finite ordered set and let \triangleleft be a subset of \leq containing Δ_P . Assume that $x : \widehat{\mathbf{P}}_{\triangleleft} \rightarrow \mathbf{M}$ is a homomorphism with $x \circ \iota_{\mathbf{P}} \notin \mathcal{G}(\mathbf{P}, \mathbf{2})$. Then x does not have finite rank.*

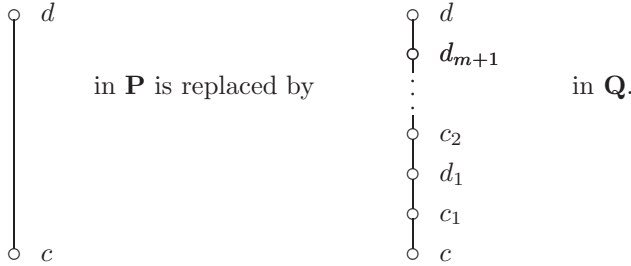
Proof. We prove, by induction, that the following claim holds for all $n \in \omega$.

(*)_n For each finite ordered set $\mathbf{P} = \langle P; \leq \rangle$, each subset \triangleleft of \leq containing Δ_P , and each homomorphism $x : \widehat{\mathbf{P}}_{\triangleleft} \rightarrow \mathbf{M}$ such that $x \circ \iota_{\mathbf{P}} \notin \mathcal{G}(\mathbf{P}, \mathbf{2})$, we have $\text{rank}(x) \not\leq n$.

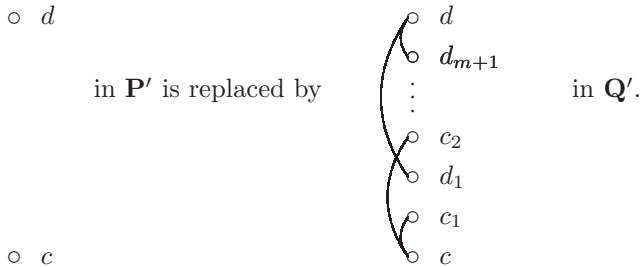
Recall that on $\mathbf{2}$, we have 1 is less than 2, but in $\widehat{\mathbf{P}}_{\triangleleft}$ the order is $2 \preceq 0 \preceq 1$. To see that (*)₀ holds, let $\mathbf{P} = \langle P; \leq \rangle$ be a finite ordered set, let \triangleleft be a subset of \leq containing Δ_P , and let $c \in P^+$. As the homomorphisms with rank 0 are restrictions of projections, it is enough to prove that $\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}} \in \mathcal{G}(\mathbf{P}, \mathbf{2})$. By Lemma 3.4, we know that $\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}} \in \{1, 2\}^P$. We just need to show that $\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}}$ is order preserving. So let $a, b \in P$ with $a \leq b$ and assume that $(\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}})(a) = 2$. Then $\widehat{aa}(c) = \pi_c(\widehat{aa}) = 2$. This implies that $c = \perp$ or that $c \in P$ with $c \leq a \leq b$. Therefore $(\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}})(b) = \pi_c(\widehat{bb}) = 2$. We have shown that $\pi_c \upharpoonright_{\widehat{P}_{\triangleleft}} \circ \iota_{\mathbf{P}} \in \mathcal{G}(\mathbf{P}, \mathbf{2})$. So (*)₀ holds.

Now assume that $(*)_n$ holds, for some $n \in \omega$. Let $\mathbf{P} = \langle P; \leq \rangle$ be a finite ordered set, let \triangleleft be a subset of \leq containing Δ_P , and let $x: \widehat{\mathbf{P}}_{\triangleleft} \rightarrow \mathbf{M}$ be a homomorphism such that $x \circ \iota_{\mathbf{P}} \notin \mathcal{G}(\mathbf{P}, \mathbf{2})$. We want to show that $\text{rank}(x) \not\leq n + 1$. In order to do this, fix $m \in \omega$. We shall define an extension of $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft}$ that is inconsistent with $|Y| \leq m$ in the definition of $\text{rank}(x) \leq n + 1$.

By Lemma 3.4, we have $x \circ \iota_{\mathbf{P}} \in \mathcal{G}(\mathbf{P}', \mathbf{2})$, where $\mathbf{P}' := \langle P; \triangleleft \rangle$. Since $x \circ \iota_{\mathbf{P}} \notin \mathcal{G}(\mathbf{P}, \mathbf{2})$, there is some $c, d \in P$, with $c \leq d$ and $c \not\triangleleft d$, such that $x(\widehat{c\hat{c}}_{\mathbf{P}}) = 2$ and $x(\widehat{d\hat{d}}_{\mathbf{P}}) = 1$. As \mathbf{P} is finite, we can assume that c is covered by d in \mathbf{P} . Let $c_1, d_1, \dots, c_{m+1}, d_{m+1}$ be distinct elements not in P^+ and define the set $Q := P \cup \{c_1, d_1, \dots, c_{m+1}, d_{m+1}\}$. We define two new directed graphs $\mathbf{Q} = \langle Q; \leq \rangle$ and $\mathbf{Q}' = \langle Q; \triangleleft \rangle$ that are extensions of \mathbf{P} and \mathbf{P}' respectively. The ordered set \mathbf{Q} is the extension of \mathbf{P} such that



(More precisely, define the relation $\leq^{\mathbf{Q}}$ to be the reflexive transitive closure of the relation $\leq^{\mathbf{P}} \cup \{(c, c_1), (d_{m+1}, d)\} \cup \{(c_i, d_i) \mid i \in \{1, \dots, m + 1\}\}$ on Q .) Define the reflexive directed graph \mathbf{Q}' to be the extension of \mathbf{P}' such that



(More precisely, define the relation $\triangleleft^{\mathbf{Q}'}$ to be the reflexive closure of the relation $\triangleleft^{\mathbf{P}'} \cup \{(c, c_i), (d_i, d) \mid i \in \{1, \dots, m + 1\}\}$ on Q .)

Now define the algebras $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft} \leq \mathbf{M}^{P^+}$ and $\mathbf{B} := \widehat{\mathbf{Q}}_{\triangleleft} \leq \mathbf{M}^{Q^+}$, as in Definition 3.1. Note that $P^+ \subseteq Q^+$. Define the subalgebra

$$\mathbf{A}' := \text{sg}_{\mathbf{B}}(\{\widehat{ab}_{\mathbf{Q}} \mid a, b \in P \text{ and } a \triangleleft b\})$$

of \mathbf{B} . We can define the coordinate embedding $\sigma: \mathbf{M}^{P^+} \rightarrow \mathbf{M}^{Q^+}$ by

$$\sigma(w)(j) = \begin{cases} w(j) & \text{if } j \in P^+, \\ w(d) & \text{if } j \in \{c_1, d_1, \dots, c_{m+1}, d_{m+1}\}, \end{cases}$$

for all $w \in M^{P^+}$ and $j \in Q^+$. Let $a, b \in P$ with $a \triangleleft b$. We show that $\sigma(\widehat{ab}_{\mathbf{P}}) = \widehat{ab}_{\mathbf{Q}}$. It will then follow that $\sigma(\mathbf{A}) = \mathbf{A}'$. We have $\sigma(\widehat{ab}_{\mathbf{P}})(\top) = \widehat{ab}_{\mathbf{P}}(\top) = \widehat{ab}_{\mathbf{Q}}(\top)$ and, similarly, $\sigma(\widehat{ab}_{\mathbf{P}})(\perp) = \widehat{ab}_{\mathbf{Q}}(\perp)$. As the orders $\leq^{\mathbf{P}}$ and $\leq^{\mathbf{Q}}$ agree on P , we have, $\sigma(\widehat{ab}_{\mathbf{P}})(j) = \widehat{ab}_{\mathbf{P}}(j) = \widehat{ab}_{\mathbf{Q}}(j)$, for all $j \in P$. Finally, for each $j \in \{c_1, d_1, \dots, c_{m+1}, d_{m+1}\}$, we have

$$\begin{aligned} \sigma(\widehat{ab}_{\mathbf{P}})(j) = \widehat{ab}_{\mathbf{P}}(d) &= \begin{cases} 2 & \text{if } d \leq^{\mathbf{P}} a, \\ 0 & \text{if } d \leq^{\mathbf{P}} b \text{ and } d \not\leq^{\mathbf{P}} a, \\ 1 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 2 & \text{if } j \leq^{\mathbf{Q}} a, \\ 0 & \text{if } j \leq^{\mathbf{Q}} b \text{ and } j \not\leq^{\mathbf{Q}} a, \\ 1 & \text{otherwise,} \end{cases} \\ &= \widehat{ab}_{\mathbf{Q}}(j). \end{aligned}$$

So $\sigma(\widehat{ab}_{\mathbf{P}}) = \widehat{ab}_{\mathbf{Q}}$, whence $\sigma(\mathbf{A}) = \mathbf{A}'$.

We now have $\sigma(\mathbf{A}) \leq \mathbf{B} \leq \mathbf{B} \leq \mathbf{M}^{Q^+}$ to use in the definition of rank. We want to show that $x: \mathbf{A} \rightarrow \mathbf{M}$ lifts to \mathbf{B} through $\sigma \upharpoonright_A: \mathbf{A} \rightarrow \mathbf{B}$. We know that $x \circ \iota_{\mathbf{P}} \in \mathcal{G}(\mathbf{P}', \mathbf{2})$ with $x \circ \iota_{\mathbf{P}}(c) = x(\widehat{cc}_{\mathbf{P}}) = 2$ and $x \circ \iota_{\mathbf{P}}(d) = x(\widehat{dd}_{\mathbf{P}}) = 1$. So there is a graph homomorphism $z \in \mathcal{G}(\mathbf{Q}', \mathbf{2})$, given by

$$z(a) = \begin{cases} x \circ \iota_{\mathbf{P}}(a) & \text{if } a \in P, \\ 2 & \text{if } a \in \{c_1, \dots, c_{m+1}\}, \\ 1 & \text{if } a \in \{d_1, \dots, d_{m+1}\}, \end{cases}$$

that extends $x \circ \iota_{\mathbf{P}}$. By the “onto” part of Lemma 3.4, there is some $\bar{x}: \mathbf{B} \rightarrow \mathbf{M}$ with $\bar{x} \circ \iota_{\mathbf{Q}} = z$. The homomorphism $\bar{x} \circ \sigma \upharpoonright_A: \mathbf{A} \rightarrow \mathbf{M}$ satisfies $\bar{x} \circ \sigma \upharpoonright_A \circ \iota_{\mathbf{P}} = \bar{x} \circ \iota_{\mathbf{Q}} \upharpoonright_P = z \upharpoonright_P = x \circ \iota_{\mathbf{P}}$. So $\bar{x} \circ \sigma \upharpoonright_A = x$, by the “one-to-one” part of Lemma 3.4. Thus $x: \mathbf{A} \rightarrow \mathbf{M}$ lifts to \mathbf{B} through $\sigma \upharpoonright_A: \mathbf{A} \rightarrow \mathbf{B}$.

Let $Y \subseteq \mathcal{A}(\mathbf{B}, \mathbf{M})$ with $|Y| \leq m$. Define the homomorphism $\mu: \mathbf{B} \rightarrow \mathbf{M}^Y$ by $\mu := \sqcap Y$. Assume that the homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}$ lifts to $\mu(\mathbf{B})$ through $\mu \circ \sigma \upharpoonright_A: \mathbf{A} \rightarrow \mu(\mathbf{B})$. Thus there is a homomorphism $x^+: \mu(\mathbf{B}) \rightarrow \mathbf{M}$ such that $x = x^+ \circ \mu \circ \sigma \upharpoonright_A$. To see that $\text{rank}(x) \not\leq n + 1$, it now suffices to show that $\text{rank}(g) \not\leq n$, for some $g \in Y$. So suppose, by way of contradiction, that $\text{rank}(g) \leq n$, for all $g \in Y$.

By $(*)_n$, we have $g \circ \iota_{\mathbf{Q}} \in \mathcal{G}(\mathbf{Q}, \mathbf{2})$, for all $g \in Y$. If $g \in Y$ with $g \circ \iota_{\mathbf{Q}}(c_i) \neq g \circ \iota_{\mathbf{Q}}(d_i)$, for some $i \in \{1, \dots, m + 1\}$, then $g \circ \iota_{\mathbf{Q}}(c_i) = 1$ and $g \circ \iota_{\mathbf{Q}}(d_i) = 2$. It follows that $g \circ \iota_{\mathbf{Q}}(c_j) = g \circ \iota_{\mathbf{Q}}(d_j)$, for all $j \in \{1, \dots, m + 1\} \setminus \{i\}$. Since $|Y| \leq m$,

there is some $k \in \{1, \dots, m + 1\}$ such that, for all $g \in Y$, we have

$$g(\widehat{c_k c_k \mathbf{Q}}) = g \circ \iota_{\mathbf{Q}}(c_k) = g \circ \iota_{\mathbf{Q}}(d_k) = g(\widehat{d_k d_k \mathbf{Q}}).$$

Therefore $\mu(\widehat{c_k c_k \mathbf{Q}}) = \mu(\widehat{d_k d_k \mathbf{Q}})$.

We know that $h := x^+ \circ \mu: \mathbf{B} \rightarrow \mathbf{M}$ is a homomorphism. We have

$$\begin{aligned} h(\widehat{cc \mathbf{Q}}) &= x^+ \circ \mu \circ \sigma(\widehat{cc \mathbf{P}}) = x(\widehat{cc \mathbf{P}}) = 2, \text{ and} \\ h(\widehat{dd \mathbf{Q}}) &= x^+ \circ \mu \circ \sigma(\widehat{dd \mathbf{P}}) = x(\widehat{dd \mathbf{P}}) = 1. \end{aligned}$$

Using Lemma 3.4, we have $h \circ \iota_{\mathbf{Q}} \in \mathcal{G}(\mathbf{Q}', \mathbf{2})$. Since $c \leq^{\mathbf{Q}} c_k$ and $h(\widehat{cc \mathbf{Q}}) = 2$, this implies that $h(\widehat{c_k c_k \mathbf{Q}}) = 2$. Similarly, since $d_k \leq^{\mathbf{Q}} d$ and $h(\widehat{dd \mathbf{Q}}) = 1$, we have $h(\widehat{d_k d_k \mathbf{Q}}) = 1$. Thus

$$2 = h(\widehat{c_k c_k \mathbf{Q}}) = x^+ \circ \mu(\widehat{c_k c_k \mathbf{Q}}) = x^+ \circ \mu(\widehat{d_k d_k \mathbf{Q}}) = h(\widehat{d_k d_k \mathbf{Q}}) = 1,$$

which is a contradiction. We have shown that $(*)_{n+1}$ holds. □

Lemma 3.6. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ such that ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$. Then \mathbf{M} does not have finite rank.*

Proof. Define the two ordered sets

$$\begin{array}{ccc} \begin{array}{c} \circ & d \\ | \\ \circ & c \end{array} & \text{and} & c \circ \quad \circ d \\ \\ \mathbf{P} = \langle \{c, d\}; \leq \rangle & & \mathbf{P}' = \langle \{c, d\}; \triangleleft \rangle. \end{array}$$

Define $\mathbf{A} := \widehat{\mathbf{P}}_{\triangleleft} \leq \mathbf{M}^{P^+}$, as in Definition 3.1. We have $\mathbf{A} = \text{sg}_{\mathbf{M}^{P^+}}(\{\widehat{cc}, \widehat{dd}\})$. Since $\text{sg}_{\mathbf{A}}(\widehat{cc}) \cap \text{sg}_{\mathbf{A}}(\widehat{dd}) \subseteq \{\widehat{0}, \widehat{1}, \widehat{2}\}$, by Lemma 3.2(1), we can define the homomorphism $h: \mathbf{A} \rightarrow \mathbf{M}$ by

$$h := \pi_{\perp} \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{cc})} \cup \pi_{\top} \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{dd})}.$$

We have $h \circ \iota_{\mathbf{P}}(c) = h(\widehat{cc}) = \widehat{cc}(\perp) = 2$ and $h \circ \iota_{\mathbf{P}}(d) = h(\widehat{dd}) = \widehat{dd}(\top) = 1$. So $h \circ \iota_{\mathbf{P}} \notin \mathcal{G}(\mathbf{P}, \mathbf{2})$. By Lemma 3.5, the homomorphism h does not have finite rank. So \mathbf{M} does not have finite rank. □

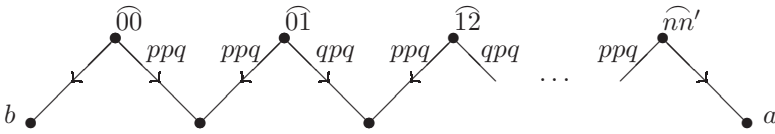
Lemma 3.7. *Let \mathbf{M} be a unary algebra on the set $\{0, 1, 2\}$ such that ppq and qpq are term functions of \mathbf{M} , for some $p, q \in M$ with $p \neq q$. Then \mathbf{M} is not quasi-injective.*

Proof. Let $n \in \omega \setminus \{0\}$ and define $k := 2n + 1$. We shall prove that \mathbf{M} is not n -quasi-injective. Let $\mathbf{P} = \langle \{0, \dots, k\}; \leq \rangle$ be a $(k + 1)$ -element chain with $0 \leq \dots \leq k$. Define the relations

$$\blacktriangleleft := \{(i, i) \mid i \in P\} \cup \{(i, i + 1) \mid i \in P \setminus \{k\}\}$$

and $\triangleleft := \blacktriangleleft \setminus \{(n, n + 1)\}$ on P . Using Definition 3.1, we can define $\mathbf{A} := \widehat{\mathbf{P}}_{\blacktriangleleft}$ and $\mathbf{D} := \widehat{\mathbf{P}}_{\triangleleft}$, whence \mathbf{D} is a subalgebra of \mathbf{A} . Let \mathbf{B} be the subalgebra of \mathbf{A} with the underlying set $B := \text{sg}_{\mathbf{A}}(\{00, kk\})$. Using Lemma 3.2(1), we know that $\text{sg}_{\mathbf{A}}(\widehat{00}) \cap \text{sg}_{\mathbf{A}}(\widehat{kk}) \subseteq \{\widehat{0}, \widehat{1}, \widehat{2}\}$. So we can define the homomorphism $x: \mathbf{B} \rightarrow \mathbf{M}$ by $x := \pi_{\perp} \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{00})} \cup \pi_{\top} \upharpoonright_{\text{sg}_{\mathbf{A}}(\widehat{kk})}$. We want to show that x extends to $n_{\mathbf{A}}(B)$ but not to \mathbf{A} .

For $1 \leq j \leq 2n$ define $j_l := j - 1$ and $j' := j + 1$. Pick $b \in B \setminus C_{\mathbf{A}}$. Without loss of generality we may assume that $b \in \text{sg}_{\mathbf{A}}(\widehat{00})$. For $a \in A \setminus D$ we have $a \in \text{sg}_{\mathbf{A}}(\widehat{nn'}) \setminus \text{sg}_{\mathbf{A}}(\{\widehat{nn}, \widehat{n'n'}\})$. As $ppq(\widehat{00}) = ppq(\widehat{01})$ and $qpq(\widehat{j_l j}) = qpq(\widehat{j' j'})$, the fence



is of length $n + 2$. If $b \in \text{sg}_{\mathbf{A}}(\widehat{00}) \cap \text{sg}_{\mathbf{A}}(\widehat{01})$ then there is a fence from b to a of length $n + 1$. There is no shorter fence between a and b because $\text{sg}_{\mathbf{A}}(\widehat{j_l j}) \cap \text{sg}_{\mathbf{A}}(\widehat{j' j'}) \subseteq \text{sg}_{\mathbf{A}}(\widehat{j' j})$ by Lemma 3.3 and $\text{sg}_{\mathbf{A}}(\widehat{j' j'}) \cap \text{sg}_{\mathbf{A}}(\widehat{j' j'}) \setminus C_{\mathbf{A}} = \emptyset$ by Lemma 3.2(2). Thus $n_{\mathbf{A}}(B) \subseteq D$.

We have $x \circ \iota_{\mathbf{P}}(0) = x(\widehat{00}) = 2$ and $x \circ \iota_{\mathbf{P}}(k) = x(\widehat{kk}) = 1$. It follows that the map $x \circ \iota_{\mathbf{P}} \upharpoonright_{\{0, k\}}: \{0, k\} \rightarrow \{1, 2\}$ does not extend to a graph homomorphism from $\langle P; \blacktriangleleft \rangle$ to $\mathbf{2}$. So, by Lemma 3.4, the homomorphism x does not extend to \mathbf{A} . Since $n \not\triangleleft n'$, the map $x \circ \iota_{\mathbf{P}} \upharpoonright_{\{0, k\}}$ extends to a graph homomorphism $z: \langle P; \triangleleft \rangle \rightarrow \mathbf{2}$. By Lemma 3.4, there is a homomorphism $\bar{x}: \mathbf{D} \rightarrow \mathbf{M}$ such that $\bar{x} \circ \iota_{\mathbf{P}} = z$. Now $\bar{x}(\widehat{00}) = z(0) = x \circ \iota_{\mathbf{P}}(0) = x(\widehat{00})$ and, similarly, $\bar{x}(\widehat{kk}) = x(\widehat{kk})$. This implies that $\bar{x}: \mathbf{D} \rightarrow \mathbf{M}$ is an extension of x . Thus x extends to $n_{\mathbf{A}}(B)$, whence \mathbf{M} is not n -quasi-injective. □

Thus, by Lemmas 3.6 and 3.7, if \mathbf{M} is a unary algebra with \mathbf{V} or \mathbf{L} as an isoreduct, then \mathbf{M} does not have finite rank and \mathbf{M} is not quasi-injective. It remains to consider the three-element unary algebras that have \mathbf{D} as an isoreduct. We use the following two general lemmas to lift up the results we have already obtained to cover this last case.

Whenever we have a fixed algebra \mathbf{M} , an algebra \mathbf{M}^{\sharp} that has \mathbf{M} as a reduct, and an algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}^{\sharp})$, we will use \mathbf{A}^{\flat} to denote the reduct of \mathbf{A} in $\mathbb{ISP}(\mathbf{M})$.

Lemma 3.8. *Let \mathbf{M} be a finite algebra such that \mathbf{M}^{\sharp} has \mathbf{M} as a reduct. Assume that, for every $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}^{\sharp})$, each homomorphism $x: \mathbf{A}^{\flat} \rightarrow \mathbf{M}$ is a homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}^{\sharp}$. For all $n \in \omega$, if $\text{rank}(\mathbf{M}) \leq n$, then $\text{rank}(\mathbf{M}^{\sharp}) \leq n$.*

Proof. Recall $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ and define $\mathcal{A}^\sharp := \mathbb{ISP}(\mathbf{M}^\sharp)$. Note that every homomorphism in \mathcal{A}^\sharp is also a homomorphism in \mathcal{A} . Using induction, we prove that the following statement is true for all $n \in \omega$.

$(*)_n$ For each $k \in \omega \setminus \{0\}$, each $\mathbf{A} \leq (\mathbf{M}^\sharp)^k$ and each $h: \mathbf{A} \rightarrow \mathbf{M}^\sharp$ such that $\text{rank}_{\mathbf{M}}(h) \leq n$, we have $\text{rank}_{\mathbf{M}^\sharp}(h) \leq n$.

The claim follows as the rank of \mathbf{M}^\sharp is defined in terms of the ranks of homomorphisms in \mathcal{A}^\sharp .

It is clear that $(*)_0$ holds as the only maps with rank 0 are projections. Assume that $(*)_n$ holds, for some $n \in \omega$. Now let $k \in \omega \setminus \{0\}$, let $\mathbf{A} \leq (\mathbf{M}^\sharp)^k$ and let $h: \mathbf{A} \rightarrow \mathbf{M}^\sharp$ such that $\text{rank}_{\mathbf{M}}(h) \leq n + 1$. We want to show that $\text{rank}_{\mathbf{M}^\sharp}(h) \leq n + 1$.

There is a maximum number $m \in \omega$ of homomorphisms of rank at most n needed to show that $h: \mathbf{A}^b \rightarrow \mathbf{M}$ has rank at most $n + 1$. Let $\sigma: (\mathbf{M}^\sharp)^k \rightarrow (\mathbf{M}^\sharp)^{k+l}$ be a coordinate embedding, for some $l \in \omega$, and assume that $\sigma(\mathbf{A}) \leq \mathbf{B} \leq \mathbf{C} \leq (\mathbf{M}^\sharp)^{k+l}$ such that $h: \mathbf{A} \rightarrow \mathbf{M}^\sharp$ lifts to \mathbf{C} through $\sigma|_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{C}$. Then $\sigma(\mathbf{A}^b) = \sigma(\mathbf{A})^b \leq \mathbf{B}^b \leq \mathbf{C}^b \leq \mathbf{M}^{k+l}$ such that $h: \mathbf{A}^b \rightarrow \mathbf{M}$ lifts to \mathbf{C}^b through $\sigma|_{\mathbf{A}}: \mathbf{A}^b \rightarrow \mathbf{C}^b$. As $\text{rank}_{\mathbf{M}}(h) \leq n + 1$, there exists $Y \subseteq \mathcal{A}(\mathbf{C}^b, \mathbf{M})$, with $|Y| \leq m$, such that

- (1) $h: \mathbf{A}^b \rightarrow \mathbf{M}$ lifts to $\Pi Y(\mathbf{B}^b)$ through $\Pi Y \circ \sigma|_{\mathbf{A}}: \mathbf{A}^b \rightarrow \Pi Y(\mathbf{B}^b)$, and
- (2) $\text{rank}_{\mathbf{M}}(g|_B) \leq n$, for all $g \in Y$.

By assumption, each homomorphism $g: \mathbf{C}^b \rightarrow \mathbf{M}$ in Y is also a homomorphism $g: \mathbf{C} \rightarrow \mathbf{M}^\sharp$. So $Y \subseteq \mathcal{A}^\sharp(\mathbf{C}, \mathbf{M}^\sharp)$. Using $(*)_n$, we know that $\text{rank}_{\mathbf{M}^\sharp}(g|_B) \leq n$, for all $g \in Y$. There is a homomorphism $h': \Pi Y(\mathbf{B}^b) \rightarrow \mathbf{M}$ with $h = h' \circ \Pi Y \circ \sigma|_{\mathbf{A}}$. Clearly $\Pi Y(\mathbf{B}^b) = \Pi Y(\mathbf{B})^b$. So $h': \Pi Y(\mathbf{B}) \rightarrow \mathbf{M}^\sharp$ is a homomorphism. It now follows that $\text{rank}_{\mathbf{M}^\sharp}(h) \leq n + 1$. Thus $(*)_{n+1}$ holds. □

Lemma 3.9. *Let \mathbf{M} be a finite unary algebra such that \mathbf{M}^\sharp has \mathbf{M} as a reduct. Assume that \mathbf{M} and \mathbf{M}^\sharp have the same constant term functions. Assume further that, for every $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}^\sharp)$, each homomorphism $x: \mathbf{A}^b \rightarrow \mathbf{M}$ is a homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}^\sharp$. For all $n \in \omega$, if \mathbf{M} is n -quasi-injective, then \mathbf{M}^\sharp is n -quasi-injective.*

Proof. Let $n \in \omega \setminus \{0\}$ and assume that \mathbf{M} is n -quasi-injective. Let $\mathbf{B} \leq \mathbf{A}$ in $\mathbb{ISP}(\mathbf{M}^\sharp)$ and let $x: \mathbf{B} \rightarrow \mathbf{M}^\sharp$ be a homomorphism that extends to $n_{\mathbf{A}}(B)$. We need to show that x extends to \mathbf{A} . As \mathbf{M} and \mathbf{M}^\sharp have the same constant term functions, we have $C_{\mathbf{A}} = C_{\mathbf{A}^b}$. Since \mathbf{A} is a term extension of \mathbf{A}^b , it follows that $n_{\mathbf{A}^b}(B) \subseteq n_{\mathbf{A}}(B)$. So $x: \mathbf{B}^b \rightarrow \mathbf{M}$ is a homomorphism that extends to $n_{\mathbf{A}^b}(B)$. As \mathbf{M} is n -quasi-injective, there is a homomorphism $\bar{x}: \mathbf{A}^b \rightarrow \mathbf{M}$ extending x . By assumption, the map \bar{x} is also a homomorphism $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}^\sharp$. So \mathbf{M}^\sharp is n -quasi-injective. □

An algebra $\mathbf{A} \leq \mathbf{M}^S$ is **weakly balanced** if each homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}$ is the restriction of a projection.

Lemma 3.10. *Let \mathbf{M} be a three-element unary algebra that has \mathbf{D} as an isoreduct. There is a three-element unary algebra \mathbf{M}^\sharp such that*

- (1) \mathbf{V} is an isoreduct of \mathbf{M}^\sharp ,
- (2) \mathbf{M} is a reduct of \mathbf{M}^\sharp ,
- (3) \mathbf{M}^\sharp and \mathbf{M} have the same constant term functions,
- (4) for all $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}^\sharp)$, each homomorphism $x: \mathbf{A}^b \rightarrow \mathbf{M}$ is a homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}^\sharp$.

Proof. Let F be the set of unary term functions of \mathbf{M} . Without loss of generality, we assume that \mathbf{M} is a unary algebra on the set $\{0, 1, 2\}$ and that $101, 220 \in F$. Define

$$F^\sharp := F \cup \{ppq, qpq \mid p, q \in M\}$$

and $\mathbf{M}^\sharp = \langle \{0, 1, 2\}; F^\sharp \rangle$. As $000, 111$ and 222 are all constant term functions of \mathbf{D} , they are also constant term functions of \mathbf{M} . So \mathbf{M} and \mathbf{M}^\sharp have the same constant term functions. Clearly \mathbf{M}^\sharp has \mathbf{V} as an isoreduct. It remains to show that condition (4) holds.

We begin by proving that \mathbf{M}^2 is weakly balanced. To do this, let $x: \mathbf{M}^2 \rightarrow \mathbf{M}$ be a homomorphism. Since x preserves $010 = 101 \circ 101$, we have $x(0, 1) \in \{0, 1\}$. First assume that $x(0, 1) = 1$. Then, in \mathbf{M}^2 , we have

$$(0, 1) \xrightarrow{101} (1, 0) \xleftarrow{101} (2, 1) \xrightarrow{220} (0, 2) \xleftarrow{220} (2, 0) \xleftarrow{220} (1, 2).$$

Applying the homomorphism x gives us

$$1 \xleftrightarrow{101} 0 \xleftarrow{101} 1 \xrightarrow{220} 2 \xleftarrow{220} 0 \xleftarrow{220} 2$$

in \mathbf{M} . Since all the constant maps on M are term functions of \mathbf{M} , this implies that $x = \pi_2$. Now assume that $x(0, 1) = 0$. Then $x(1, 0) = 101(x(0, 1)) = 1$ and, by symmetry, we have $x = \pi_1$. Thus \mathbf{M}^2 is weakly balanced.

Let $\mathbf{A} \leq (\mathbf{M}^\sharp)^S$, for some non-empty set S . For all $a \in A$, define the partition $\mathcal{P}(a) := \{a^{-1}(m) \mid m \in M\} \setminus \{\emptyset\}$ of S . Now let $a \in A$ with $|\mathcal{P}(a)| = 2$. We have

$$\text{sg}_{\mathbf{A}}(a) = \{b \in M^S \mid \mathcal{P}(b) = \mathcal{P}(a) \text{ or } \mathcal{P}(b) = \{S\}\},$$

since every map in $\{ppq, qpq \mid p, q \in M\}$ is an operation of \mathbf{M}^\sharp . Thus \mathbf{M}^2 is isomorphic to $\text{sg}_{\mathbf{A}}(a)^b$ via a coordinate embedding, and it follows that $\text{sg}_{\mathbf{A}}(a)^b$ is weakly balanced.

Now let $y: \mathbf{A}^b \rightarrow \mathbf{M}$ be a homomorphism. We know that y preserves the constant operations $000, 111$ and 222 . To prove that the map y is a homomorphism $y: \mathbf{A} \rightarrow \mathbf{M}^\sharp$, it is enough to show that $y \upharpoonright_{\text{sg}_{\mathbf{A}}(a)}$ is the restriction of a projection for all $a \in A \setminus \{\hat{0}, \hat{1}, \hat{2}\}$.

Let $a \in A \setminus \{\hat{0}, \hat{1}, \hat{2}\}$. First assume that $|\mathcal{P}(a)| = 2$. Then $y \upharpoonright_{\text{sg}_{\mathbf{A}}(a)}$ is the restriction of a projection, since $\text{sg}_{\mathbf{A}}(a)^b$ is weakly balanced. Now assume that $|\mathcal{P}(a)| = 3$. We can choose some $s \in S$ with $a(s) = y(a)$. We want to prove that $y \upharpoonright_{\text{sg}_{\mathbf{A}}(a)} = \pi_s \upharpoonright_{\text{sg}_{\mathbf{A}}(a)}$.

First let u be a unary term function of \mathbf{M}^\sharp such that $\ker(u)$ has two blocks. We show that there must be some $v \in F$ with $\ker(u) = \ker(v)$. As $\ker(220) = \{01|2\}$ and $\ker(101) = \{02|1\}$ where $220, 101$ are in F , we only need consider the case when $\ker(u) = \{12|0\}$. Let $w_1, w_2 \in F^\sharp$ then $\ker(w_2 \circ w_1)$ is at least as coarse as $\ker(w_1)$ so if F has no function with kernel $\{12|0\}$ then F^\sharp , whence \mathbf{M}^\sharp , has no term function with kernel $\{12|0\}$. Thus we may pick $v \in F$ with $\ker(u) = \ker(v)$. So $\mathcal{P}(u(a)) = \mathcal{P}(v(a))$ and therefore $\text{sg}_{\mathbf{A}}(u(a)) = \text{sg}_{\mathbf{A}}(v(a))$. Since $\text{sg}_{\mathbf{A}}(u(a))^\flat$ is weakly balanced and $u(a), v(a) \in \text{sg}_{\mathbf{A}}(u(a))$, there is some $t \in S$ such that $y \upharpoonright_{\{u(a), v(a)\}} = \pi_t \upharpoonright_{\{u(a), v(a)\}}$. As $a(s) = y(a)$ and $v \in F$ we have

$$v(a)(s) = v(y(a)) = y(v(a)) = v(a)(t).$$

But $v(a)$ and $u(a)$ determine the same partition of S , and so $u(a)(s) = u(a)(t)$. This implies that $y(u(a)) = u(a)(s)$. Now let w be a unary term function of \mathbf{M}^\sharp that is a permutation. There are no permutations in $F^\sharp \setminus F$. So it follows that $w \in F$ and therefore $y(w(a)) = w(y(a)) = w(a(s))$. Thus $y \upharpoonright_{\text{sg}_{\mathbf{A}}(a)} = \pi_s \upharpoonright_{\text{sg}_{\mathbf{A}}(a)}$. \square

We summarize the results of the last several lemmas to finish the proof of the main theorem. If \mathbf{M} is a three-element unary algebra with \mathbf{D} as a isoreduct then by Lemma 3.10, there exists an algebra \mathbf{M}^\sharp with \mathbf{M} as a reduct and \mathbf{V} as an isoreduct such that \mathbf{M} and \mathbf{M}^\sharp satisfy the hypotheses of Lemmas 3.8 and 3.9. Since \mathbf{V} has term functions 112 and 212, the algebra \mathbf{M}^\sharp will have term functions ppq and qqq for some $p \neq q$. By Lemma 3.6, \mathbf{M}^\sharp does not have finite rank and, by Lemma 3.7, it is not quasi-injective. Invoking Lemmas 3.8 and 3.9 shows that \mathbf{M} does not have finite rank nor is it quasi-injective.

Every three-element unary algebra with \mathbf{V} or \mathbf{L} as an isoreduct does not have finite rank and is not quasi-injective, by Lemmas 3.6 and 3.7. Thus, if an algebra fails to satisfy (1) of the main theorem it must also fail to satisfy (2) through (4).

4. Bigger Unary Algebras

In the final section of this paper, we show that some of the implications of our main theorem do not hold for unary algebras in general. In particular, we give examples to show that

- enough algebraic operations $\not\Rightarrow$ quasi-injective,
- quasi-injective $\not\Rightarrow$ enough algebraic operations,
- finitely based quasi-equations $\not\Rightarrow$ quasi-injective,
- quasi-injective $\not\Rightarrow$ finitely based quasi-equations.

These counterexamples come from two unary algebras:

$$\mathbf{M}_1 = \langle 0, 1, 2, 3, 4 \rangle; u, v, w \quad \text{and} \quad \mathbf{M}_2 = \langle 0, 1, 2, 3, 4, 5, 6 \rangle; u, v,$$

shown in Figs. 5 and 6. We show that \mathbf{M}_1 is quasi-injective, but that \mathbf{M}_1 does not have enough algebraic operations and that the quasi-equational theory of \mathbf{M}_1 is

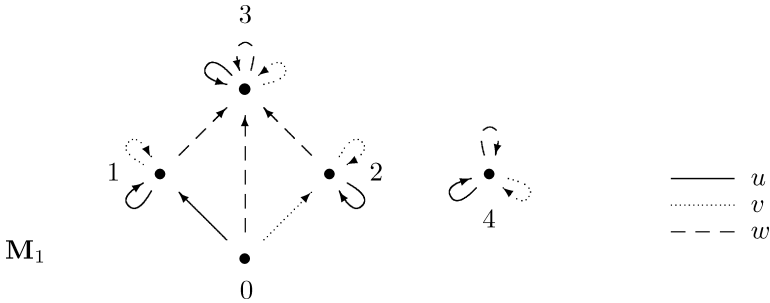


Fig. 5.

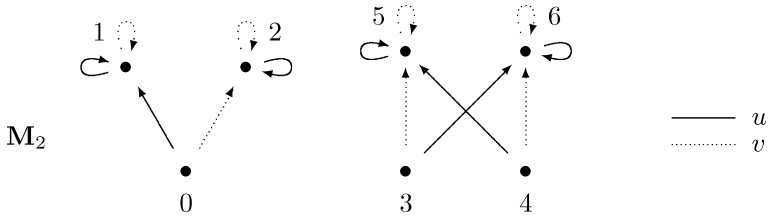


Fig. 6.

not finitely based. Then we prove that M_2 has enough algebraic operations and that the quasi-equational theory of M_2 is finitely based, but that M_2 is not quasi-injective. The algebra M_2 is well behaved, even though it has the bad algebra V as a subalgebra. Indeed, the algebras M_1 and M_2 are both closely related to the algebra V . We use what we have already learnt about V to tell us something about M_1 and M_2 .

Given a unary algebra $M = \langle M; F \rangle$ with $\star \notin M$, we define the **one-point extension** $M_\star = \langle M \cup \{\star\}; F \rangle$ of M such that $u(\star) = \star$, for all $u \in F$. We also consider the **pointed one-point extension** $M_\otimes = \langle M \cup \{\star\}; F \cup \{\underline{\star}\} \rangle$ of M , where the extra operation $\underline{\star} : M \cup \{\star\} \rightarrow M \cup \{\star\}$ is the constant map with value \star .

The algebra M_1 is constructed from the algebra V by taking a pointed one-point extension followed by an ordinary one-point extension. From the main theorem, V does not have enough algebraic operations. We will use this fact to prove that M_1 does not have enough algebraic operations.

Any finite unary algebra with a constant term function satisfies the hypothesis of the next lemma. In particular, a pointed one-point extension of a finite unary algebra will satisfy the hypothesis. Lemma 4.2 shows that V also satisfies the hypothesis.

Lemma 4.1. *Let M be a finite unary algebra such that M^n is connected, for all $n \in \omega$.*

- (1) If the one-point extension of \mathbf{M} has enough algebraic operations, then \mathbf{M} has enough algebraic operations.
- (2) If the pointed one-point extension of \mathbf{M} has enough algebraic operations, then \mathbf{M} has enough algebraic operations.

Proof. We prove claim (2). The proof of (1) is similar. Assume that $\star \notin M$ and let \mathbf{M}_\otimes be the pointed one-point extension of \mathbf{M} . Define the quasi-variety $\mathcal{A}_\otimes := \text{ISP}(\mathbf{M}_\otimes)$. Assume that \mathbf{M}_\otimes has enough algebraic operations. We want to show that \mathbf{M} has enough algebraic operations.

There is some $f_\otimes: \omega \rightarrow \omega$ such that

for all $n \in \omega \setminus \{0\}$, all $\mathbf{B} \leq \mathbf{A} \leq (\mathbf{M}_\otimes)^n$ and all $h: \mathbf{A} \rightarrow \mathbf{M}_\otimes$, there exists a set $Y \subseteq \mathcal{A}_\otimes((\mathbf{M}_\otimes)^n, \mathbf{M}_\otimes)$, with $|Y| \leq f_\otimes(|B|)$, such that $h \upharpoonright_B$ lifts to $\sqcap Y(\mathbf{A})$ through $\sqcap Y \upharpoonright_B: \mathbf{B} \rightarrow \sqcap Y(\mathbf{A})$.

Define $f: \omega \rightarrow \omega$ by $f(n) := f_\otimes(n + 1)$. Now let $\mathbf{B} \leq \mathbf{A} \leq \mathbf{M}^n$, for some $n \in \omega \setminus \{0\}$, and let $h: \mathbf{A} \rightarrow \mathbf{M}$ be a homomorphism. For each subalgebra \mathbf{C} of \mathbf{M}^n , we define \mathbf{C}_\otimes to be the subalgebra of $(\mathbf{M}_\otimes)^n$ with the underlying set $C \cup \{\hat{\star}\}$. We now have $\mathbf{B}_\otimes \leq \mathbf{A}_\otimes \leq (\mathbf{M}^n)_\otimes \leq (\mathbf{M}_\otimes)^n$, and we can define the homomorphism $h_\otimes: \mathbf{A}_\otimes \rightarrow \mathbf{M}_\otimes$ by $h_\otimes := h \cup \{(\hat{\star}, \star)\}$. (Hint for (1): Do not add the extra elements. We have $\mathbf{B} \leq \mathbf{A} \leq (\mathbf{M}_\star)^n$ and $h: \mathbf{A} \rightarrow \mathbf{M}_\star$ is a homomorphism. Define $f: \omega \rightarrow \omega$ by $f := f_\star$.)

As \mathbf{M}_\otimes has enough algebraic operations, there is some $Y \subseteq \mathcal{A}_\otimes((\mathbf{M}_\otimes)^n, \mathbf{M}_\otimes)$, with $|Y| \leq f_\otimes(|B| + 1)$, such that $h_\otimes \upharpoonright_{B \cup \{\hat{\star}\}}$ lifts to $\sqcap Y(\mathbf{A}_\otimes)$ through the homomorphism $\sqcap Y \upharpoonright_{B \cup \{\hat{\star}\}}: \mathbf{B}_\otimes \rightarrow \sqcap Y(\mathbf{A}_\otimes)$. This means that there is a homomorphism $\varphi: \sqcap Y(\mathbf{A}_\otimes) \rightarrow \mathbf{M}_\otimes$ with $h_\otimes \upharpoonright_{B \cup \{\hat{\star}\}} = \varphi \circ \sqcap Y \upharpoonright_{B \cup \{\hat{\star}\}}$.

Now $M^n \subseteq (M \cup \{\star\})^n$. Since the algebra \mathbf{M}^n is connected, for all $y \in Y$ we have $y(M^n) \subseteq M$ or $y(M^n) = \{\star\}$. Define the subset

$$Z := \{y \upharpoonright_{M^n} \mid y \in Y \text{ with } y(M^n) \subseteq M\}$$

of $\mathcal{A}(\mathbf{M}^n, \mathbf{M})$. Then $|Z| \leq |Y| \leq f_\otimes(|B| + 1) = f(|B|)$. To finish the proof, we need to find a homomorphism $\psi: \sqcap Z(\mathbf{A}) \rightarrow \mathbf{M}$ such that $h \upharpoonright_B = \psi \circ \sqcap Z \upharpoonright_B$.

For all $y \in Y \setminus Z$, the set $y(A \cup \{\hat{\star}\}) = \{\star\}$ determines a one-element subalgebra of \mathbf{M}_\otimes . It follows that $\rho: \sqcap Z(\mathbf{A}_\otimes) \rightarrow \sqcap Y(\mathbf{A}_\otimes)$, given by $\rho(\sqcap Z(a)) = \sqcap Y(a)$ is a well-defined isomorphism. We use the homomorphism $\varphi \circ \rho: \sqcap Z(\mathbf{A}_\otimes) \rightarrow \mathbf{M}_\otimes$ to define a homomorphism $\psi: \sqcap Z(\mathbf{A}) \rightarrow \mathbf{M}$ such that $h \upharpoonright_B = \psi \circ \sqcap Z \upharpoonright_B$.

Define the subset S of $\sqcap Z(A)$ by $S := \{x \in \sqcap Z(A): \varphi \circ \rho(x) = \star\}$. Then S is a union of connected components of $\sqcap Z(\mathbf{A})$. Furthermore, for all $b \in B$, we have

$$\varphi \circ \rho \circ \sqcap Z(b) = \varphi \circ \sqcap Y(b) = h_\otimes(b) = h(b) \neq \star$$

and therefore $\sqcap Z(b) \notin S$. So, when lifting the homomorphism h up to $\sqcap Z(\mathbf{A})$, it does not matter where we map the subalgebra \mathbf{S} of $\sqcap Z(\mathbf{A})$. We have shown that, for each $b \in B$, we have $\varphi \circ \sqcap Y(b) \neq \star$, which implies that $\sqcap Y(b) \neq \hat{\star}$. It follows that the set Z is not empty. (Hint for (1): The set Z may be empty. In this case, for

any $b \in B$, the element $h(b) = \varphi \circ \sqcap Y(b) = \varphi(\hat{\star})$ forms a one-element subalgebra of \mathbf{M} .)

Choose some $z \in Z$ and define the homomorphism $\psi : \sqcap Z(\mathbf{A}) \rightarrow \mathbf{M}$ by

$$\psi := \varphi \circ \rho \upharpoonright_{\sqcap Z(A) \setminus S} \cup \pi_z \upharpoonright_S.$$

(Hint for (1): If Z is empty, then map S onto a one-element subalgebra of \mathbf{M} .) For all $b \in B$, we have $\sqcap Z(b) \notin S$, which implies that

$$h(b) = h_{\otimes}(b) = \varphi \circ \sqcap Y(b) = \varphi \circ \rho \circ \sqcap Z(b) = \psi \circ \sqcap Z(b).$$

Hence \mathbf{M} has enough algebraic operations. □

Lemma 4.2. *For each $n \in \omega$, the algebra \mathbf{V}^n is connected.*

Proof. Recall that $\mathbf{V} = \langle \{0, 1, 2\}; 112, 212 \rangle$. Let $n \in \omega \setminus \{0\}$ and let $a, b \in \{0, 1, 2\}^n$. Define $a', b' \in \{0, 1, 2\}^n$ by

$$a'(i) = \begin{cases} 1 & \text{if } a(i) \neq 2, \\ 2 & \text{if } a(i) = 2 \text{ and } b(i) = 2, \\ 0 & \text{if } a(i) = 2 \text{ and } b(i) \neq 2, \end{cases} \quad \text{and} \quad b'(i) = \begin{cases} 1 & \text{if } b(i) \neq 2, \\ 2 & \text{if } b(i) = 2 \text{ and } a(i) = 2, \\ 0 & \text{if } b(i) = 2 \text{ and } a(i) \neq 2. \end{cases}$$

Then there exist unique a_1, a_2 and a_3 such that

$$a \xrightarrow{112} a_1 \xleftarrow{212} a' \xrightarrow{112} a_2 \xleftarrow{112} b' \xrightarrow{212} a_3 \xleftarrow{112} b$$

holds in \mathbf{V}^n . Hence \mathbf{V}^n is connected. □

Since \mathbf{V} does not have enough algebraic operations and \mathbf{M}_1 is formed from \mathbf{V} by taking a pointed one-point extension followed by an ordinary one-point extension, by Lemmas 4.1 and 4.2, it follows that \mathbf{M}_1 does not have enough algebraic operations.

We use the next lemma to show that the quasi-equational theory of \mathbf{M}_1 is not finitely based. Recall that, for any finite algebra \mathbf{M} , the quasi-variety $\mathbb{ISP}(\mathbf{M})$ is determined by the quasi-equational theory $\text{Th}_{\text{qe}}(\mathbf{M})$ of \mathbf{M} .

Lemma 4.3. *Let \mathbf{M} be a finite unary algebra of finite type that either has a constant operation or has two one-element subalgebras.*

- (1) *If the quasi-equational theory of the one-point extension of \mathbf{M} is finitely based, then the quasi-equational theory of \mathbf{M} is finitely based.*
- (2) *If the quasi-equational theory of the pointed one-point extension of \mathbf{M} is finitely based, then the quasi-equational theory of \mathbf{M} is finitely based.*

Proof. We prove (2). Assume that $\star \notin M$ and let \mathbf{M}_{\otimes} be the pointed one-point extension of \mathbf{M} . Assume that $\text{Th}_{\text{qe}}(\mathbf{M}_{\otimes})$ is finitely based. There is some $n \in \omega$, with $n \geq 2$, such that the set $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M}_{\otimes})$ of all n -variable quasi-equations satisfied by \mathbf{M}_{\otimes} is a basis for $\text{Th}_{\text{qe}}(\mathbf{M}_{\otimes})$. As \mathbf{M} is a unary algebra of finite type, the set of

n -variable quasi-equations, $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M})$, is finite. We show that $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M})$ is a basis for $\text{Th}_{\text{qe}}(\mathbf{M})$, whence it is a finite basis.

Let \mathbf{A} be a model of $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M})$. Then every n -generated subalgebra of \mathbf{A} belongs to $\mathbb{ISP}(\mathbf{M})$. It follows that every n -generated subalgebra of \mathbf{A}_{\otimes} belongs to $\mathbb{ISP}(\mathbf{M}_{\otimes})$, where \mathbf{A}_{\otimes} is the pointed one-point extension of \mathbf{A} . So \mathbf{A}_{\otimes} is a model of $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M}_{\otimes})$ and therefore $\mathbf{A}_{\otimes} \in \mathbb{ISP}(\mathbf{M}_{\otimes})$.

Let $a, b \in A$ with $a \neq b$. We find a homomorphism from \mathbf{A} to \mathbf{M} that separates a and b . It will then follow that $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, and therefore that $\text{Th}_{\text{qe}}^{(n)}(\mathbf{M})$ is a basis for $\text{Th}_{\text{qe}}(\mathbf{M})$. As $\mathbf{A}_{\otimes} \in \mathbb{ISP}(\mathbf{M}_{\otimes})$, there is a homomorphism $x: \mathbf{A}_{\otimes} \rightarrow \mathbf{M}_{\otimes}$ such that $x(a) \neq x(b)$.

Case (a). \mathbf{M} has a constant operation. Since every 2-generated subalgebra of \mathbf{A} belongs to $\mathbb{ISP}(\mathbf{M})$, the algebra \mathbf{A} has a constant operation. So \mathbf{A} is connected. This implies that $x(A) \subseteq M$ or $x(A) = \{\star\}$. Since $x(a) \neq x(b)$, we know that $x(A) \subseteq M$. Thus $x|_A: \mathbf{A} \rightarrow \mathbf{M}$ is a homomorphism separating a and b .

Case (b). \mathbf{M} has two one-element subalgebras. Without loss of generality assume that $x(a) \neq \star$. There is some $m \in M$ such that $\{m\}$ determines a one-element subalgebra of \mathbf{M} and $m \neq x(a)$. Define $x': \mathbf{A} \rightarrow \mathbf{M}$ by

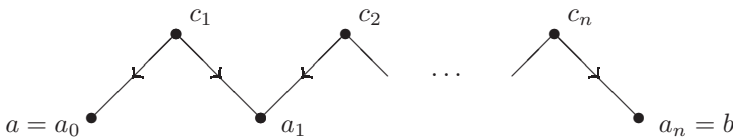
$$x'(c) = \begin{cases} m & \text{if } x(c) = \star, \\ x(c) & \text{otherwise,} \end{cases}$$

for all $c \in A$. Then $x'(a) \neq x'(b)$. □

Example 4.4. The unary algebra \mathbf{M}_1 , defined in Fig. 5, is quasi-injective. But \mathbf{M}_1 does not have enough algebraic operations, does not have finite rank, and the quasi-equational theory of \mathbf{M}_1 is not finitely based.

Proof. From our main theorem \mathbf{V} does not have enough algebraic operations, so \mathbf{M}_1 does not have enough algebraic operations, by Lemmas 4.1 and 4.2. We also know that \mathbf{V} does not have a finitely based quasi-equational theory, and consequently \mathbf{M}_1 does not have a finitely based quasi-equational theory, by Lemma 4.3. We now show that \mathbf{M}_1 is quasi-injective.

Let \mathbf{P} be a petal of $\mathbb{ISP}(\mathbf{M}_1)$. To prove that \mathbf{M}_1 is 2-quasi-injective, we show that $d_{\mathbf{P}}(a, b) \leq 2$, for all $a, b \in P \setminus C_{\mathbf{P}}$. So let $a, b \in P$. (Since \mathbf{M}_1 has no constant term functions, we have $C_{\mathbf{P}} = \emptyset$.) There is a fence



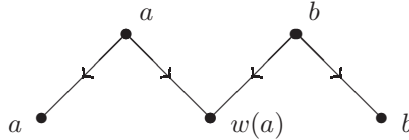
in $G(\mathbf{P})$, for some $n \in \omega$. Let $i \in \{0, \dots, n - 1\}$. There are unary term functions s and t of \mathbf{M}_1 such that $a_i = s(c_{i+1})$ and $a_{i+1} = t(c_{i+1})$. Since \mathbf{M}_1 satisfies the equation $w \circ s(x) \approx w \circ t(x)$, we have

$$w(a_i) = w(s(c_{i+1})) = w(t(c_{i+1})) = w(a_{i+1}).$$

So

$$w(a) = w(a_0) = w(a_1) = \dots = w(a_n) = w(b).$$

This implies that there is a fence



in $G(\mathbf{P})$ with length 2, and therefore $d_{\mathbf{P}}(a, b) \leq 2$. Since every algebra in $\mathbb{ISP}(\mathbf{M}_1)$ is the coproduct of its petals, it follows that \mathbf{M}_1 is 2-quasi-injective.

That \mathbf{M}_1 does not have finite rank follows from the stronger result that \mathbf{M}_1 has rank infinity which is shown in [3]. □

We now consider the algebra \mathbf{M}_2 , defined in Fig. 6. The subalgebra of \mathbf{M}_2 on the set $\{0, 1, 2\}$ is equal to \mathbf{V} and is therefore not quasi-injective, by our main theorem. It follows by the next lemma that \mathbf{M}_2 is also not quasi-injective. We will also prove that \mathbf{M}_2 has enough algebraic operations and that its quasi-equational theory is finitely based.

Lemma 4.5. *Let \mathbf{M} be a finite unary algebra. Assume there is a connected component \mathbf{C} of \mathbf{M} such that no term function is constant on \mathbf{C} . For all $n \in \omega$, if \mathbf{M} is n -quasi-injective, then \mathbf{C} is n -quasi-injective.*

Proof. Let $n \in \omega \setminus \{0\}$ and assume that \mathbf{M} is n -quasi-injective. Let $\mathbf{B} \leq \mathbf{A}$ in $\mathbb{ISP}(\mathbf{C}) \subseteq \mathbb{ISP}(\mathbf{M})$ and let $x: \mathbf{B} \rightarrow \mathbf{C}$ be a homomorphism. Distance in \mathbf{A} depends upon whether we view \mathbf{A} as a member of $\mathbb{ISP}(\mathbf{C})$ or $\mathbb{ISP}(\mathbf{M})$, since a fence cannot intersect the center. But here the algebras \mathbf{C} and \mathbf{M} have no constant term functions. So \mathbf{A} has no center elements, either as a member of $\mathbb{ISP}(\mathbf{C})$ or as a member of $\mathbb{ISP}(\mathbf{M})$. This means that we can use $n_{\mathbf{A}}(B)$ unambiguously to denote the ball in \mathbf{A} around B with radius n , both in $\mathbb{ISP}(\mathbf{C})$ and in $\mathbb{ISP}(\mathbf{M})$. Now assume that the homomorphism $x: \mathbf{B} \rightarrow \mathbf{C}$ extends to a homomorphism from $n_{\mathbf{A}}(B)$ to \mathbf{C} . Since \mathbf{M} is n -quasi-injective, there is an extension $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}$ of x . We want to prove that x extends to a homomorphism from \mathbf{A} to \mathbf{C} . If $|A| = 1$, then $\mathbf{A} = \mathbf{B}$ and we are done, so assume $|A| > 1$. Since $\mathbf{A} \in \mathbb{ISP}(\mathbf{C})$, this implies that there is at least one homomorphism $y: \mathbf{A} \rightarrow \mathbf{C}$. Now let $\bar{\mathbf{B}}$ denote the subalgebra of \mathbf{A} consisting of all the connected components of \mathbf{A} that intersect with \mathbf{B} . As connectivity is preserved by homomorphisms and $\bar{x}(B) \subseteq C$, we have $\bar{x}(\bar{\mathbf{B}}) \subseteq C$. So we can define the extension $\bar{x}|_{\bar{\mathbf{B}}} \cup y|_{\mathbf{A} \setminus \bar{\mathbf{B}}}: \mathbf{A} \rightarrow \mathbf{C}$ of x . Hence \mathbf{C} is n -quasi-injective. □

To see that the variety $\text{Var}(\mathbf{M}_2)$ is determined by the equations

$$u^2(x) \approx u(x), \quad v^2(x) \approx v(x), \quad u \circ v(x) \approx v(x), \quad v \circ u(x) \approx u(x)$$

first note that \mathbf{M}_2 satisfies them, so any term $\tau(x)$ satisfies $\tau(x) \approx u(x)$, $\tau(x) \approx v(x)$ or $\tau(x) \approx x$. As $u(0) \neq 0$, $v(0) \neq 0$ and $u(0) \neq v(0)$, the equations $u(x) \approx x$, $v(x) \approx x$ and $u(x) \approx v(x)$ do not hold in \mathbf{M}_2 . Thus the displayed equations form a basis of the equations of \mathbf{M}_2 .

There is a directed graph naturally associated with every algebra in $\text{Var}(\mathbf{M}_2)$. Consider some $\mathbf{A} \in \text{Var}(\mathbf{M}_2)$. The operations u and v on \mathbf{A} are idempotent. Furthermore, the operation u fixes each element of $v(A)$, and v fixes each element of $u(A)$. So the subset $A_{\text{fix}} := u(A) = v(A)$ of A contains all the fixed points of \mathbf{A} (that is, all the elements of A that are fixed by every operation of \mathbf{A}). Define the relation $\xrightarrow{\mathbf{A}_{\text{fix}}}$ on A_{fix} by

$$a \xrightarrow{\mathbf{A}_{\text{fix}}} b \text{ means } (\exists c \in A) u(c) = a \ \& \ v(c) = b.$$

For example, we have $\xrightarrow{(\mathbf{M}_2)_{\text{fix}}} = \{5, 6\}^2 \cup \{(1, 1), (1, 2), (2, 2)\}$.

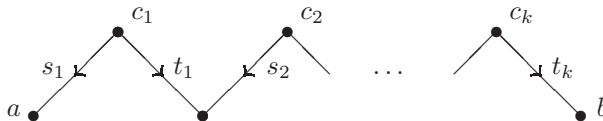
Now let $\xrightarrow{\mathbf{A}_{\text{fix}}^t}$ denote the transitive closure of $\xrightarrow{\mathbf{A}_{\text{fix}}}$. Then $\xrightarrow{\mathbf{A}_{\text{fix}}^t}$ is the quasi-order associated with $\xrightarrow{\mathbf{A}_{\text{fix}}}$. Let $\widetilde{\xrightarrow{\mathbf{A}_{\text{fix}}}}$ denote the smallest equivalence relation on A_{fix} containing the relation $\xrightarrow{\mathbf{A}_{\text{fix}}}$. The blocks of $\widetilde{\xrightarrow{\mathbf{A}_{\text{fix}}}}$ are the connected components of the quasi-ordered set $\langle A_{\text{fix}}; \xrightarrow{\mathbf{A}_{\text{fix}}^t} \rangle$. For example, the equivalence relation $\widetilde{\xrightarrow{(\mathbf{M}_2)_{\text{fix}}}}$ has the two blocks $\{1, 2\}$ and $\{5, 6\}$ whereas the connected components of \mathbf{M}_2 are $\{0, 1, 2\}$ and $\{3, 4, 5, 6\}$.

If the algebra \mathbf{A} belongs to the quasi-variety $\text{ISP}(\mathbf{M}_2)$, then the directed graph $\langle A_{\text{fix}}; \xrightarrow{\mathbf{A}_{\text{fix}}} \rangle$ encodes all of the structure of \mathbf{A} . Since $\xrightarrow{\mathbf{A}_{\text{fix}}}$ is reflexive, we can recover \mathbf{A} as follows. Let $\mathbf{B} = \langle \xrightarrow{\mathbf{A}_{\text{fix}}}; u^{\mathbf{B}}, v^{\mathbf{B}} \rangle$, where $u^{\mathbf{B}}((a, b)) := (a, a)$ and $v^{\mathbf{B}}((a, b)) := (b, b)$ for all $a, b \in A_{\text{fix}}$ with (a, b) in $\xrightarrow{\mathbf{A}_{\text{fix}}}$. The algebra \mathbf{M}_2 satisfies $u(x) \approx u(y) \ \& \ v(x) \approx v(y) \Rightarrow x \approx y$ so, given $a \xrightarrow{\mathbf{A}_{\text{fix}}} b$ there is a unique witness c with $u(c) = a$ and $v(c) = b$. The map $h: \mathbf{A} \rightarrow \mathbf{B}$, given by $h(c) = (u(c), v(c))$ for $c \in \mathbf{A}$ is an isomorphism.

Lemma 4.6. *Let $\mathbf{A} \in \text{Var}(\mathbf{M}_2)$ and $a, b \in A_{\text{fix}}$. Then a and b are in the same connected component of \mathbf{A} if and only if $a \widetilde{\xrightarrow{\mathbf{A}_{\text{fix}}}} b$.*

Proof. Assume $a, b \in A_{\text{fix}}$. As $a \xrightarrow{\mathbf{A}_{\text{fix}}} b$ implies a and b are in the same connected component of \mathbf{A} , by induction $a \xrightarrow{\mathbf{A}_{\text{fix}}^t} b$ implies a and b are in the same connected component of \mathbf{A} . Thus $a \widetilde{\xrightarrow{\mathbf{A}_{\text{fix}}}} b$ implies a and b are in the same connected component of \mathbf{A} .

Now assume $a \neq b$ and $a, b \in A_{\text{fix}}$ are in the same connected component of \mathbf{A} . This means that for some term operations s_i and t_i and elements c_i with $1 \leq i \leq k$ we have a minimal path



from a to b . By the defining equations of $\text{Var}(\mathbf{M}_2)$ we may assume that $s_i, t_i \in \{u, v, \text{id}\}$ for all i . We wish to show that $\{s_i, t_i\} = \{u, v\}$ as this implies $a \widetilde{\mathbf{A}_{\text{fix}}} b$. If $s_i = t_i$ for some i then a shorter path from a to b may be formed by removing c_i . This contradicts the minimality of the path so for all i we have $s_i \neq t_i$.

If $t_i = s_{i+1} = \text{id}$ for some i then $c_i = c_{i+1}$ and c_i may be removed again contradicting the minimality of the path. Thus if $t_i = \text{id}$ then $s_{i+1} \in \{u, v\}$. This means $c_i = t_i(c_i) = s_{i+1}(c_{i+1}) \in A_{\text{fix}}$ so $c_i = u(c_i) = v(c_i)$. We may replace t_i by $t'_i \in \{u, v\} \setminus \{s_i\}$. Since $t_i = \text{id}$ implies $s_i \neq \text{id}$ we have $s_i \in \{u, v\}$ so $\{t'_i, s_i\} = \{u, v\}$. Similarly, if $s_i = \text{id}$ we may replace s_i by s'_i so that $\{s'_i, t_i\} = \{u, v\}$. Thus $a \widetilde{\mathbf{A}_{\text{fix}}} b$. \square

Lemma 4.7. *Let $\mathbf{B} \leq \mathbf{A}$ in $\text{Var}(\mathbf{M}_2)$ and let $x: \mathbf{B} \rightarrow \mathbf{M}_2$ be a homomorphism. Then x extends to \mathbf{A} if and only if*

- (1) *for each connected component \mathbf{C} of \mathbf{A} , we either have $x(B \cap C) \subseteq \{0, 1, 2\}$ or $x(B \cap C) \subseteq \{3, 4, 5, 6\}$, and*
- (2) *for all $b, c \in B_{\text{fix}}$ such that $x(b) = 2$ and $b \xrightarrow{\mathbf{A}_{\text{fix}}} c$, we have $x(c) = 2$.*

Proof. First assume that x extends to a homomorphism $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}_2$. Then condition (1) must hold, since \mathbf{M}_2 divides into two connected components, on the sets $\{0, 1, 2\}$ and $\{3, 4, 5, 6\}$. We show that the subset $\bar{x}^{-1}(2) \cap A_{\text{fix}}$ of A_{fix} is $\xrightarrow{\mathbf{A}_{\text{fix}}}$ -closed (that is, for all $a, b \in A_{\text{fix}}$ with $\bar{x}(a) = 2$ and $a \xrightarrow{\mathbf{A}_{\text{fix}}} b$, we have $\bar{x}(b) = 2$). Let $a, b \in A_{\text{fix}}$ such that $\bar{x}(a) = 2$ and $a \xrightarrow{\mathbf{A}_{\text{fix}}} b$. There is some $d \in A$ such that $u(d) = a$ and $v(d) = b$. We have $u(\bar{x}(d)) = \bar{x}(u(d)) = \bar{x}(a) = 2$ and therefore $\bar{x}(d) = 2$. This implies that $\bar{x}(b) = \bar{x}(v(d)) = v(\bar{x}(d)) = v(2) = 2$. So $\bar{x}^{-1}(2) \cap A_{\text{fix}}$ is $\xrightarrow{\mathbf{A}_{\text{fix}}}$ -closed. Since \bar{x} is an extension of x , it follows by induction that (2) holds.

Now assume that x satisfies (1) and (2). We want to show that x extends to \mathbf{A} . So we can assume that \mathbf{A} is connected. (Otherwise, we can extend x on each of the components of \mathbf{A} independently.) By (1), this implies that $x(B) \subseteq \{0, 1, 2\}$ or $x(B) \subseteq \{3, 4, 5, 6\}$. For each $m \in M_2$, define $B_m := x^{-1}(m)$.

Case (a). $x(B) \subseteq \{0, 1, 2\}$. The left-hand picture in Fig. 7 illustrates the partition $\{B_0, B_1, B_2\}$ of B within the algebra \mathbf{A} . The set A_{fix} of fixed points of \mathbf{A} is shaded. Note that x maps every fixed point of \mathbf{B} to a fixed point of \mathbf{M}_2 . So $B_0 \cap A_{\text{fix}} = \emptyset$. First we want to find all the elements of A_{fix} that must be mapped to 2 by any extension of x to \mathbf{A} . If $y: \mathbf{A} \rightarrow \mathbf{M}_2$ is a homomorphism, then the subset $y^{-1}(2) \cap A_{\text{fix}}$ of A_{fix} is $\xrightarrow{\mathbf{A}_{\text{fix}}}$ -closed by the argument given above. So we define

$$A'_2 := \{a \in A_{\text{fix}} : (\exists b \in B_2 \cap A_{\text{fix}}) b \xrightarrow{\mathbf{A}_{\text{fix}}} a\} \quad \text{and} \quad A'_1 := A_{\text{fix}} \setminus A'_2.$$

(The set A'_2 is the smallest $\xrightarrow{\mathbf{A}_{\text{fix}}}$ -closed subset of A_{fix} that contains $B_2 \cap A_{\text{fix}}$.) We have $A'_2 \cap B = B_2 \cap A_{\text{fix}}$, by (2). As $B_0 \cap A_{\text{fix}} = \emptyset$ this implies $A'_1 \cap B = B_1 \cap A_{\text{fix}}$. This is illustrated in the middle picture in Fig. 7.

Now define the three sets

$$A_2 := u^{-1}(A'_2), \quad A_1 := v^{-1}(A'_1) \quad \text{and} \quad A_0 := A \setminus (A_1 \cup A_2).$$

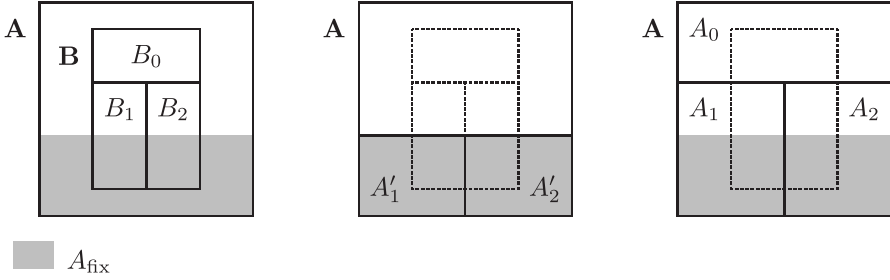


Fig. 7. Partitions in proof of Lemma 4.7.

Since u and v fix A'_2 and A'_1 , we have $A'_2 \subseteq A_2$ and $A'_1 \subseteq A_1$. Suppose that there is some $a \in A_1 \cap A_2$. Then $u(a) \in A'_2$ and $v(a) \in A'_1$. But $u(a) \xrightarrow{\bar{x}} v(a)$ and A'_2 is \bar{x} -closed, which is a contradiction. So $\{A_0, A_1, A_2\}$ is a partition of A .

Define the map $\bar{x}: A \rightarrow M_2$ such that $\bar{x}(A_m) = \{m\}$, for all $m \in \{0, 1, 2\}$. We prove that \bar{x} is a homomorphism from \mathbf{A} to \mathbf{M}_2 that extends x . For all $b \in B$, we have $x(b) = 2$ if and only if $x(u(b)) = 2$. So $B \cap u^{-1}(B_2 \cap A_{\text{fix}}) = B_2$. Therefore

$$A_2 \cap B = u^{-1}(A'_2) \cap B = u^{-1}(A'_2 \cap B) \cap B = u^{-1}(B_2 \cap A_{\text{fix}}) \cap B = B_2.$$

Similarly, we have $A_1 \cap B = B_1$. Thus $A_0 \cap B = B_0$, whence \bar{x} extends x . The right-hand picture in Fig. 7 illustrates this.

It remains to prove that \bar{x} is a homomorphism. Clearly, $u(A_2) \subseteq A'_2 \subseteq A_2$. To see that $v(A_2) \subseteq A_2$, let $a \in A_2$. Then $u(a) \xrightarrow{\bar{x}} v(a)$. Since $u(a) \in A'_2$ and A'_2 is \bar{x} -closed, this implies that $v(a) \in A'_2 \subseteq A_2$. Thus A_2 is closed under u and v . Similarly, the set A_1 is closed under u and v . For all $a \in A_0$, we have $u(a) \in A_{\text{fix}} \setminus A'_2$ and $v(a) \in A_{\text{fix}} \setminus A'_1$. So $u(A_0) \subseteq A'_1 \subseteq A_1$ and $v(A_0) \subseteq A'_2 \subseteq A_2$. Hence $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}_2$ is a homomorphism.

Case (b). $x(B) \subseteq \{3, 4, 5, 6\}$. Define $A'_5 := B_5 \cap A_{\text{fix}}$ and $A'_6 := A_{\text{fix}} \setminus B_5$. Now define

$$\begin{aligned} A_3 &:= u^{-1}(A'_6) \cap v^{-1}(A'_5), & A_5 &:= u^{-1}(A'_5) \cap v^{-1}(A'_6), \\ A_4 &:= u^{-1}(A'_5) \cap v^{-1}(A'_6), & A_6 &:= u^{-1}(A'_6) \cap v^{-1}(A'_5). \end{aligned}$$

It is straightforward to check that we can define an extension $\bar{x}: \mathbf{A} \rightarrow \mathbf{M}_2$ of x such that $\bar{x}(A_m) = \{m\}$, for all $m \in \{3, 4, 5, 6\}$. □

Lemma 4.8. *The quasi-equational theory of \mathbf{M}_2 has the equations*

$$u^2(x) \approx u(x), \quad v^2(x) \approx v(x), \quad u \circ v(x) \approx v(x), \quad v \circ u(x) \approx u(x)$$

and the quasi-equation

$$u(x) \approx u(y) \ \& \ v(x) \approx v(y) \Rightarrow x \approx y$$

as a finite basis.

Proof. As discussed after Lemma 4.5, the four equations form a basis of $\text{Var}(\mathbf{M}_2)$. Let $\mathbf{A} \in \text{Var}(\mathbf{M}_2)$ satisfy $u(x) \approx u(y) \ \& \ v(x) \approx v(y) \Rightarrow x \approx y$. We show that the elements of \mathbf{A} are separated by homomorphisms from \mathbf{A} into \mathbf{M}_2 . Hence \mathbf{A} is in $\mathbb{ISP}(\mathbf{M}_2)$. Choose $a, b \in \mathbf{A}$ with $a \neq b$ and assume, by symmetry, that $u(a) \neq u(b)$. Then $B := \{u(a), u(b)\}$ determines a subalgebra \mathbf{B} of \mathbf{A} and we can define $x: \mathbf{B} \rightarrow \mathbf{M}_2$ by $x(u(a)) = 5$ and $x(u(b)) = 6$. By Lemma 4.7, this homomorphism extends to a homomorphism from \mathbf{A} to \mathbf{M}_2 that separates $u(a)$ and $u(b)$, and therefore also separates a and b . \square

Lemma 4.9. *The unary algebra \mathbf{M}_2 has enough algebraic operations.*

Proof. Define $f: \omega \rightarrow \omega$ by $f(k) := 3k$. Let $n \in \omega \setminus \{0\}$, let $\mathbf{B} \leq \mathbf{A} \leq (\mathbf{M}_2)^n$ and let $h: \mathbf{A} \rightarrow \mathbf{M}_2$. As 5 and 6 are fixed points of \mathbf{M}_2 , any map from $(M_2^n)_{\text{fix}}$ to $\{5, 6\}$ is a homomorphism. Thus by Lemma 4.7, for each $b \in B_{\text{fix}}$ there is a homomorphism $g_b: (\mathbf{M}_2)^n \rightarrow \mathbf{M}_2$ such that, for all $a \in (M_2^n)_{\text{fix}}$, we have

$$g_b(a) = \begin{cases} 5 & \text{if } a = b, \\ 6 & \text{otherwise.} \end{cases}$$

For each $b \in B_{\text{fix}}$ there is a homomorphism $g'_b: (\mathbf{M}_2)^n \rightarrow \mathbf{M}_2$ such that, for all $a \in (M_2^n)_{\text{fix}}$, we have

$$g'_b(a) = \begin{cases} 5 & \text{if } a \in A_{\text{fix}} \text{ and } b \xrightarrow{A_{\text{fix}}} a, \\ 6 & \text{otherwise.} \end{cases}$$

Finally, for each $b \in B_{\text{fix}}$ define $g''_b: (\mathbf{M}_2)^n \rightarrow \mathbf{M}_2$ such that, for all $a \in (M_2^n)_{\text{fix}}$, we have

$$g''_b(a) = \begin{cases} 5 & \text{if } a \in A_{\text{fix}} \text{ and } a \widetilde{A_{\text{fix}}} b, \\ 6 & \text{otherwise.} \end{cases}$$

Now define

$$Y := \{g_b, g'_b, g''_b \mid b \in B_{\text{fix}}\}.$$

Then $|Y| \leq 3|B| = f(|B|)$.

Define the homomorphism $\mu: \mathbf{M}^n \rightarrow \mathbf{M}^Y$ by $\mu := \sqcap Y$. The operations u and v separate the elements of B , by Lemma 4.8, and $B_{\text{fix}} = u(B) = v(B)$. So the homomorphisms in $\{g_b \mid b \in B_{\text{fix}}\}$ separate the elements of B . Thus $\mu \upharpoonright_B$ is an embedding. We use Lemma 4.7 to show that $h \circ (\mu \upharpoonright_B)^{-1}: \mu(\mathbf{B}) \rightarrow \mathbf{M}_2$ extends to $\mu(\mathbf{A})$. Note that $\mu(B_{\text{fix}}) = (\mu(B))_{\text{fix}}$. First we prove a series of claims.

(1) Let $b \in B_{\text{fix}}$ and $a, c \in A_{\text{fix}}$ with $b \xrightarrow{A_{\text{fix}}} a$ and $\mu(a) \xrightarrow{\mu(A)_{\text{fix}}} \mu(c)$. Then $b \xrightarrow{A_{\text{fix}}} c$.

There is some $d \in A$ such that $\mu(u(d)) = \mu(a)$ and $\mu(v(d)) = \mu(c)$. Since $g'_b \in Y$ and $b \xrightarrow{A_{\text{fix}}} a$, we have $g'_b(u(d)) = g'_b(a) = 5$. As $u(d) \in A_{\text{fix}}$, this implies that $b \xrightarrow{A_{\text{fix}}} u(d)$. However d witnesses $u(d) \xrightarrow{A_{\text{fix}}} v(d)$, so by transitivity $b \xrightarrow{A_{\text{fix}}} v(d)$. As $g'_b(c) = g'_b(v(d)) = 5$, it follows that $b \xrightarrow{A_{\text{fix}}} c$.

(2) Let $b, c \in B_{\text{fix}}$ with $\mu(b) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}^t} \mu(c)$. Then $b \xrightarrow{\mathbf{A}_{\text{fix}}^t} c$.

This claim follows by induction from (1), since $b \xrightarrow{\mathbf{A}_{\text{fix}}^t} b$, for all $b \in B_{\text{fix}}$.

(3) Let $b \in B_{\text{fix}}$ and $a, c \in A_{\text{fix}}$ such that $b \xrightarrow{\mathbf{A}_{\text{fix}}} a$ and either $\mu(a) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(c)$ or $\mu(c) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(a)$. Then $b \xrightarrow{\mathbf{A}_{\text{fix}}} c$.

Assume that $\mu(a) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(c)$. There is some $d \in A$ such that $\mu(u(d)) = \mu(a)$ and $\mu(v(d)) = \mu(c)$. We have $g_b''(u(d)) = g_b''(a) = 5$ and therefore $b \xrightarrow{\mathbf{A}_{\text{fix}}} u(d)$. Since $u(d) \xrightarrow{\mathbf{A}_{\text{fix}}} v(d)$, we have $b \xrightarrow{\mathbf{A}_{\text{fix}}} v(d)$. Therefore $b \xrightarrow{\mathbf{A}_{\text{fix}}} c$, as $g_b''(c) = g_b''(v(d)) = 5$. Similarly, if we assume that $\mu(c) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(a)$, then we can show that $b \xrightarrow{\mathbf{A}_{\text{fix}}} c$.

(4) Let $b, c \in B_{\text{fix}}$ with $\mu(b) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(c)$. Then $b \xrightarrow{\mathbf{A}_{\text{fix}}} c$.

This claim follows by induction from (3).

We can now show that conditions (1) and (2) of Lemma 4.7 are satisfied for $h \circ (\mu \upharpoonright_B)^{-1}$ by using Lemma 4.7 on $h \upharpoonright_B$. Let $b, c \in B$ such that $\mu(b)$ and $\mu(c)$ belong to the same connected component of $\mu(\mathbf{A})$. Then the fixed points $u(\mu(b))$ and $u(\mu(c))$ belong to the same connected component so, by Lemma 4.6, we have $\mu(u(b)) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}} \mu(u(c))$. By claim (4), this implies that $u(b) \xrightarrow{\mathbf{A}_{\text{fix}}} u(c)$. Again by Lemma 4.6, the elements b and c belong to the same connected component of \mathbf{A} . Since $h \upharpoonright_B$ extends to \mathbf{A} , we know that $h(b) = h \circ (\mu \upharpoonright_B)^{-1}(\mu(b))$ and $h(c) = h \circ (\mu \upharpoonright_B)^{-1}(\mu(c))$ both belong to $\{0, 1, 2\}$ or both belong to $\{3, 4, 5, 6\}$, by Lemma 4.7. So (1) holds for $h \circ (\mu \upharpoonright_B)^{-1}$.

Now assume that $b, c \in B_{\text{fix}}$ such that $h \circ (\mu \upharpoonright_B)^{-1}(\mu(b)) = 2$ and $\mu(b) \xrightarrow{\mu(\mathbf{A})_{\text{fix}}^t} \mu(c)$. By claim (2), we have $b \xrightarrow{\mathbf{A}_{\text{fix}}^t} c$. Since $h(b) = h \circ (\mu \upharpoonright_B)^{-1}(\mu(b)) = 2$ and $h \upharpoonright_B$ extends to \mathbf{A} , this implies that $h(c) = 2$, by Lemma 4.7. Therefore $h \circ (\mu \upharpoonright_B)^{-1}(\mu(c)) = h(c) = 2$ and 4.7(2) holds. Thus $h \circ (\mu \upharpoonright_B)^{-1}$ extends to $\mu(\mathbf{A})$, whence \mathbf{M}_2 has enough algebraic operations. □

The result below summarizes what we have discovered about \mathbf{M}_2 using our main theorem and Lemmas 4.5, 4.8 and 4.9.

Example 4.10. The unary algebra \mathbf{M}_2 , defined in Fig. 6, has enough algebraic operations and has a finitely based quasi-equational theory, but is not quasi-injective.

For the properties from our main theorem,

- (2) \mathbf{M} has finite rank,
- (3) \mathbf{M} has enough algebraic operations,
- (4) \mathbf{M} is quasi-injective, and
- (5) the quasi-equational theory of \mathbf{M} is finitely based,

Table 2. The status of the general questions.

Implication	Status
(2) \rightarrow (3)	Open problem.
(2) \rightarrow (4)	\mathbf{M}_2 is a counterexample. See Example 4.10.
(2) \rightarrow (5)	False. Lattice counterexample.
(3) \rightarrow (4)	\mathbf{M}_2 is a counterexample. See Example 4.10.
(3) \rightarrow (5)	False. Lattice counterexample.
(3) \rightarrow (2)	Always true. See [9].
(4) \rightarrow (2)	\mathbf{M}_1 is a counterexample. See Example 4.4.
(4) \rightarrow (3)	\mathbf{M}_1 is a counterexample. See Example 4.4.
(4) \rightarrow (5)	\mathbf{M}_1 is a counterexample. See Example 4.4.
(5) \rightarrow (2)	Open problem.
(5) \rightarrow (3)	Open problem.
(5) \rightarrow (4)	\mathbf{M}_2 is a counterexample. See Example 4.10.

we summarize what is known for arbitrary finite algebras in Table 2. Partial results on the question of the equivalence of (3) and (5) can be found in [7]. Neither (2) nor (3) implies (5): there is a finite lattice whose quasi-equational theory is not finitely based [14], but every finite lattice has enough algebraic operations [15].

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