FINITE DEGREE: ALGEBRAS IN GENERAL AND SEMIGROUPS IN PARTICULAR

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Abstract. An algebra **A** has *finite degree* if its term functions are determined by some finite set of finitary relations on A. We study this concept for finite algebras in general and for finite semigroups in particular. For example, we show that every finite nilpotent semigroup has finite degree (more generally, every finite algebra with bounded p_n-sequence), and every finite commutative semigroup has finite degree. We give an example of a five-element unary semigroup that has infinite degree. We also give examples to show that finite degree is not preserved in general under taking subalgebras, homomorphic images, direct products or subdirect factors.

1. Introduction

Given a semigroup $\mathbf{S}$, each word $t(x_1,\dots,x_n)$ in n variables induces an n-ary *term function* $t^{\mathbf{S}}\colon S^n \to S$ via evaluation. For a fixed n, the set of all such term functions forms a subsemigroup $\mathbf{F_S}(n)$ of $\mathbf{S}^{S^n}$. Indeed, $\mathbf{F_S}(n)$ is the free semigroup on n generators in the variety generated by $\mathbf{S}$. The set of all term functions of $\mathbf{S}$,

$$\mathrm{Clo}(\mathbf{S}) := \bigcup \{\, F_{\mathbf{S}}(n) \bigm| n \in \mathbb{N} \,\},$$

is called the *clone* of $\mathbf{S}$. For each $k \in \mathbb{N}$, a k-ary relation r on S is *compatible* with $\mathbf{S}$ if it forms a subsemigroup of $\mathbf{S}^k$. A standard result from clone theory tells us that, if $\mathbf{S}$ is finite, then $\mathrm{Clo}(\mathbf{S})$ is determined by the set of all finitary compatible relations on $\mathbf{S}$ (that is, for all $n \in \mathbb{N}$, an operation $f\colon S^n \to S$ is a term function of $\mathbf{S}$ if and only if f preserves every finitary compatible relation on $\mathbf{S}$).

The set of all compatible relations on a semigroup is always infinite. So we are interested in the question:

> Is the clone $\mathrm{Clo}(\mathbf{S})$ of all term functions of a finite semigroup $\mathbf{S}$ necessarily determined by a *finite* set of relations?

We shall give two classes of finite semigroups within which the answer is yes: nilpotent semigroups (more generally, semigroups with a bounded p_n-sequence) and commutative semigroups. We also give an example of a finite *unary semigroup* for which the answer is no. The general question remains unanswered.

Since many of our results for semigroups will be obtained via more general universal-algebraic considerations, we present the following definition for algebras in general. Note that the definitions and claims above for a semigroup $\mathbf{S}$ carry over directly to an arbitrary algebra $\mathbf{A} = \langle A; F\rangle$; we need only replace the semigroup *word* $t(x_1,\dots,x_n)$ by a *term* in the signature F. In particular, a k-ary relation r

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FIGURE 1. The implication algebra $\mathbf{I} = \langle \{0,1\}; \to \rangle$

on A is compatible with $\mathbf{A}$ if it is a non-empty subuniverse of $\mathbf{A}^k$. An operation $f\colon A^k \to A$ is *compatible* with $\mathbf{A}$ if its graph is a compatible relation on $\mathbf{A}$ (or, equivalently, if $f\colon \mathbf{A}^k \to \mathbf{A}$ is a homomorphism).

Definition 1.1. Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ has *finite degree* if $\mathrm{Clo}(\mathbf{A})$ is determined by a finite set of finitary relations on A; otherwise, we say that $\mathbf{A}$ has *infinite degree.* (Algebras of finite degree are also said to be *finitely related.*) If every finitary operation on A is a term function of $\mathbf{A}$ (that is, $\mathbf{A}$ is *primal*), then $\mathbf{A}$ has *relational degree* 0. Otherwise, if $\mathbf{A}$ has finite degree, then the *relational degree* of $\mathbf{A}$ is the smallest $d \in \mathbb{N}$ such that $\mathrm{Clo}(\mathbf{A})$ is determined by the d-ary compatible relations on $\mathbf{A}$.

We begin with an overview of what is already known.

Groups and semigroups. Every finite group has finite degree [1]. While there is no bound on the relational degree of finite groups in general [24], small bounds have been obtained within the following classes:

- a finite group with abelian Sylow subgroups has relational degree at most 3 [17];
- every abelian group has relational degree at most 3 [11];
- abelian groups of relational degree 2 have been completely described [17, 15];
- every finite 2-step nilpotent group has relational degree at most 4 [17, 16].

Note that a finite group is term-equivalent to its associated semigroup, and so has the same degree. Some further examples of finite-degree semigroups come out of the theory of natural dualities [5]:

- every finite semilattice has relational degree at most 3 [7];
- every finite rectangular band has relational degree at most 3 [5, 9].

We shall give more examples of finite-degree semigroups in this paper. It is not known whether a finite semigroup must have finite degree.

Other algebras. Two major results have been announced recently concerning finite degree. One yields a wealth of finite-degree algebras: if a finite algebra has few subpowers, then it has finite degree [1]. As a finite algebra with a Mal'cev term has few subpowers, this result covers all finite groups. A finite algebra with an $(m+1)$-ary near-unanimity term, for some $m \geqslant 2$, also has few subpowers; but in fact it is easy to prove directly that such algebras have relational degree at most m [2].

The other announced result yields examples of infinite-degree algebras: if a finite algebra $\mathbf{A}$ does not have a near-unanimity term but the variety $\mathrm{V}(\mathbf{A})$ generated by $\mathbf{A}$ is congruence distributive, then $\mathbf{A}$ has infinite degree [3]. For example, the two-element implication algebra has infinite degree [21]; see Figure 1.

There are further natural examples of finite-degree algebras outside the realm of 'few subpowers'. For example:

- for any set F of compatible operations on a finite semilattice $\langle A; \wedge \rangle$, the enriched semilattice $\mathbf{A} = \langle A; \wedge, F \rangle$ has relational degree at most $|A| + 1$ [8];

- a finite unary algebra **A** has relational degree at most $\max(|A|, 3)$ [20];
- a commutative monoid has relational degree at most 3 [18].

Overview of the paper. Wherever possible, we aim to derive our results for semigroups as special cases of more general universal-algebraic results.

In Section 2, we present and reinterpret known clone-theoretic characterisations of finite degree. As a by-product, we can easily show that finite degree is a 'variety property' (that is, if two finite algebras generate the same variety, then both have finite degree or neither does). So we can view finite degree as a very weak finiteness condition on finitely-generated varieties.

In Section 3, we show that every finite nilpotent semigroup and every finite commutative semigroup has finite degree. We shall also see that there is no fixed upper bound on the relational degree of these semigroups.

The fact that every finite nilpotent semigroup has finite degree is generalised in Section 4, where we prove that every finite algebra with a bounded p_n-sequence has finite degree.

As finite semigroups of infinite degree remain elusive, we present in Section 5 an example of a finite unary semigroup that has infinite degree: the Reese matrix semigroup $\mathbf{A}_2$ enriched with the natural involution. We also show that the semigroup $\mathbf{A}_2$ itself has finite degree.

In Section 6, we show that finite degree is not closed under the familiar constructs of forming subalgebras, homomorphic images, products and subdirect factors. In particular, we show that the two-element implication algebra **I** is not inherently of infinite degree: it embeds into a three-element groupoid with finite degree.

2. Background theory

In this section we give a characterisation of finite degree (Theorem 2.9) from which it immediately follows that finite degree is a 'variety property' (that is, if two finite algebras generate the same variety, then both have finite degree or neither does). This characterisation will also help us give examples of finite-degree and infinite-degree algebras in later sections.

The following basic lemma provides a direct method for showing that an algebra has finite degree. We shall mostly use a more indirect method.

Lemma 2.1. *Let $d \in \mathbb{N}$ and let* **A** *be a finite algebra. The following are equivalent:*

1. **A** *has relational degree at most d;*
2. *for all $n \in \mathbb{N}$, an operation $f\colon A^n \to A$ is a term function of* **A** *provided it agrees with a term function on every set $S \subseteq A^n$ with $|S| \leqslant d$.*

Notation 2.2. Let $n \in \mathbb{N}$ and define $\underline{n} := \{1, 2, \dots, n\}$. For each equivalence relation α on $\underline{n}$, we define the map $\widehat{\alpha}\colon \underline{n} \to \underline{n}$ so that $\widehat{\alpha}(i)$ is the largest number in the α-block of i. For $k < n$, we use $\mathrm{Eq}_k(n)$ to denote the set of all k-block equivalence relations on $\underline{n}$.

Now fix a variety $\mathcal{V}$. Let $n, k \in \mathbb{N}$ such that $k < n$, and consider an n-ary term $t(x_1, \dots, x_n)$. Each equivalence relation $\alpha \in \mathrm{Eq}_k(n)$ produces a new n-ary term t_α from t by identifying variables in the same α-block:

$$t_\alpha(x_1, \dots, x_n) := t(x_{\widehat{\alpha}(1)}, \dots, x_{\widehat{\alpha}(n)});$$

note that at most k different variables occur in the term t_α.

We shall call the family of n-ary terms $\mathcal{F}_t := \{\, t_\alpha(x_1, \dots, x_n) \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ the *(n,k)-scheme coming from t*. This family of terms clearly satisfies the following two conditions, which we call 'dependency' and 'consistency':

(D) for all $\alpha \in \mathrm{Eq}_k(n)$,
$$\mathcal{V} \models t_\alpha(x_1, \dots, x_n) \approx t_\alpha(x_{\widehat{\alpha}(1)}, \dots, x_{\widehat{\alpha}(n)});$$

(C) for all $\alpha, \beta \in \mathrm{Eq}_k(n)$,
$$\mathcal{V} \models t_\alpha(x_{\widehat{\gamma}(1)}, \dots, x_{\widehat{\gamma}(n)}) \approx t_\beta(x_{\widehat{\gamma}(1)}, \dots, x_{\widehat{\gamma}(n)}),$$
where $\gamma := \alpha \vee \beta$.

We now give a more general definition.

Definition 2.3. Let $n, k \in \mathbb{N}$ such that $k < n$. We define an *(n,k)-scheme* for a variety $\mathcal{V}$ to be any indexed family of n-ary terms $\{\, t_\alpha(x_1, \dots, x_n) \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ such that the dependency condition (D) and the consistency condition (C) above hold. We say that two (n,k)-schemes $\{\, s_\alpha \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ and $\{\, t_\alpha \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ for $\mathcal{V}$ are *equivalent* if $\mathcal{V} \models s_\alpha \approx t_\alpha$, for all $\alpha \in \mathrm{Eq}_k(n)$. We say that an (n,k)-scheme for $\mathcal{V}$ *comes from a term* if it is equivalent to $\mathcal{F}_t$, for some n-ary term t.

We will see in Example 2.8 that not every term scheme comes from a term.

The following easy lemma says that, if the variety $\mathcal{V}$ is finitely generated, then there exists $k \in \mathbb{N}$ such that n-ary terms are determined by their (n,k)-schemes.

Lemma 2.4. *The following are equivalent, for each locally finite variety $\mathcal{V}$:*

1. *$\mathcal{V}$ is finitely generated;*
2. *there exists $k \in \mathbb{N}$ such that, for all $n > k$ and all n-ary terms s and t, the equation $s \approx t$ holds in $\mathcal{V}$ if and only if the two (n,k)-schemes coming from s and t are equivalent.*

Proof. (1) ⇒ (2): Assume $\mathcal{V} = \mathrm{V}(\mathbf{A})$, for a finite algebra $\mathbf{A}$, and define $k := |A|$. Let s, t be n-ary terms, for some $n > k$, such that the (n,k)-schemes $\mathcal{F}_s$ and $\mathcal{F}_t$ are equivalent. Now let $a_1, \dots, a_n \in A$. Since $|A| = k$, there is a k-block $\alpha \in \mathrm{Eq}_k(n)$ that satisfies $(i,j) \in \alpha \implies a_i = a_j$. We have
$$s^{\mathbf{A}}(a_1, \dots, a_n) = s^{\mathbf{A}}_\alpha(a_1, \dots, a_n) = t^{\mathbf{A}}_\alpha(a_1, \dots, a_n) = t^{\mathbf{A}}(a_1, \dots, a_n).$$
It follows that $\mathbf{A} \models s \approx t$.

(2) ⇒ (1): Assume that (2) holds. Let $\mathbf{A}$ be the k-generated free algebra in $\mathcal{V}$, that is, define $\mathbf{A} := \mathbf{F}_{\mathcal{V}}(k)$. We want to show that $\mathcal{V} = \mathrm{V}(\mathbf{A})$. Let $n > k$ and let s, t be n-ary terms such that $\mathbf{A} \models s \approx t$. It suffices to show that $\mathcal{V} \models s \approx t$. So, by (2), it suffices to show that the (n,k)-schemes $\mathcal{F}_s$ and $\mathcal{F}_t$ are equivalent.

Let $\alpha \in \mathrm{Eq}_k(n)$. We are assuming that $\mathbf{A} \models s \approx t$ and so $\mathbf{F}_{\mathcal{V}}(k) \models s_\alpha \approx t_\alpha$. Since the terms s_α and t_α involve at most k variables, it follows that $\mathcal{V} \models s_\alpha \approx t_\alpha$. So $\mathcal{F}_s$ and $\mathcal{F}_t$ are equivalent, as required. □

We shall see how the existence of term schemes that do not come from a term relates to infinite degree, using the following lemma.

Notation 2.5. Let $n, k \in \mathbb{N}$ with $k < n$, and consider an operation $f \colon A^n \to A$. For $\alpha \in \mathrm{Eq}_k(n)$, define the operation $f_\alpha \colon A^n \to A$ by
$$f_\alpha(a_1, \dots, a_n) := f(a_{\widehat{\alpha}(1)}, \dots, a_{\widehat{\alpha}(n)});$$
note that f_α depends on at most k coordinates.

Lemma 2.6. *Let* $\mathbf{A}$ *be a finite algebra and let* $\mathcal{F} = \{\, t_\alpha \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ *be an* (n,k)*-scheme for* $\mathrm{V}(\mathbf{A})$*, where* $|A| \leqslant k < n$.

1. *There is a unique operation* $f\colon A^n \to A$ *with* $f_\alpha = t_\alpha^{\mathbf{A}}$*, for all* $\alpha \in \mathrm{Eq}_k(n)$.
2. *The* (n,k)*-scheme* $\mathcal{F}$ *comes from a term if and only if the operation* f *given by* (1) *above is an* n*-ary term function of* $\mathbf{A}$.

Proof. For $(a_1, \dots, a_n) \in A^n$, we want to define

$$f(a_1, \dots, a_n) := t_\alpha^{\mathbf{A}}(a_1, \dots, a_n),$$

where $\alpha \in \mathrm{Eq}_k(n)$ satisfies $(a_1, \dots, a_n) = (a_{\widehat{\alpha}(1)}, \dots, a_{\widehat{\alpha}(n)})$. Since $|A| \leqslant k$, there is always at least one choice for α. We shall show that this definition is independent of our choice of α.

Let $(a_1, \dots, a_n) \in A^n$. Define the equivalence relation θ on $\underline{n}$ by $(i,j) \in \theta \iff a_i = a_j$. Now let $\alpha, \beta \in \mathrm{Eq}_k(n)$ with $\alpha, \beta \leqslant \theta$. Then $\gamma := \alpha \vee \beta \leqslant \theta$. Since the (n,k)-scheme satisfies the consistency condition (C), we obtain

$$t_\alpha^{\mathbf{A}}(a_1, \dots, a_n) = t_\alpha^{\mathbf{A}}(a_{\widehat{\gamma}(1)}, \dots, a_{\widehat{\gamma}(n)}) = t_\beta^{\mathbf{A}}(a_{\widehat{\gamma}(1)}, \dots, a_{\widehat{\gamma}(n)}) = t_\beta^{\mathbf{A}}(a_1, \dots, a_n).$$

Thus f is well defined. Using the dependency condition (D), it is easy to see that (1) holds. The proof that (2) follows from (1) is straightforward. ☐

Notation 2.7. We use a special notation in the case that $k = n-1$. Let $i < j \leqslant n$, and let α denote the equivalence relation in $\mathrm{Eq}_{n-1}(n)$ that identifies i and j. For an n-ary term t, we shall write t_α as t_{ij}; so that t_{ij} is simply the term t but with the variable x_i replaced everywhere by x_j. Similarly, we write f_α as f_{ij}.

Example 2.8. The two-element implication algebra $\mathbf{I} = \langle \{0,1\}; \to \rangle$, defined in Figure 1, has infinite degree. To illustrate the definitions, we give an example of an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{I})$ that does not come from a term, for any $n > 2$.

We shall use the following well-known characterisation of the term functions of $\mathbf{I}$. For all $n \in \mathbb{N}$, all $f\colon \{0,1\}^n \to \{0,1\}$ and all $j \in \underline{n}$, the following are equivalent:

(a) there is a term $t(x_1, \dots, x_n)$, with x_j the final variable, such that $f = t^{\mathbf{I}}$;
(b) for all $a_1, \dots, a_n \in \{0,1\}$, we have $a_j = 1 \implies f(a_1, \dots, a_n) = 1$.

(See [19, Lemma 1.1], for example.)

Now define the operation $f\colon \{0,1\}^n \to \{0,1\}$ by

$$f(a_1, \dots, a_n) := \bigwedge_{k \in \underline{n}} \bigl(\bigvee_{\ell \in \underline{n} \setminus \{k\}} a_\ell \bigr).$$

Then f is a near-unanimity operation on $\{0,1\}$. So f fails condition (b) above, for all $j \in \underline{n}$, and thus f is not a term function of $\mathbf{I}$. We will check that, for all $i < j \leqslant n$, there is an n-ary term t_{ij} with $f_{ij} = t_{ij}^{\mathbf{I}}$. It will then follow by Lemma 2.6 that $\mathcal{F} := \{\, t_{ij} \mid i < j \leqslant n \,\}$ is an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{I})$ that does not come from a term.

Let $i < j \leqslant n$. For all $a_1, \dots, a_n \in \{0,1\}$, we have

$$\begin{aligned} f_{ij}(a_1, \dots, a_n) &= f(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n) \\ &= \bigwedge_{k \in \underline{n}} \bigl(a_j \vee \bigvee_{\ell \in \underline{n} \setminus \{k,i\}} a_\ell \bigr). \end{aligned}$$

It follows that f_{ij} satisfies condition (b) above, and therefore f_{ij} is a term function of $\mathbf{I}$, as required.

The following is a modification of a well-known result in clone theory: condition (3) is due to Jablonskiĭ [12] and Romov [22]; condition (4) is due to Rosenberg and Szendrei [23]. The interpretation given in (1) and (2) appears to be new.

Theorem 2.9. *Let* $\mathbf{A}$ *be a finite algebra. Then* $\mathbf{A}$ *has finite degree if and only if there exists* $k \geqslant |A|$ *such that the following equivalent conditions hold:*

(1) *for all* $n > k$*, every* (n, k)*-scheme for* $\mathrm{V}(\mathbf{A})$ *comes from a term;*

(2) *for all* $n > k$*, every* $(n, n-1)$*-scheme for* $\mathrm{V}(\mathbf{A})$ *comes from a term;*

(3) *for all* $n > k$*, an operation* $f \colon A^n \to A$ *is a term function of* $\mathbf{A}$ *provided* $f_\alpha \colon A^n \to A$ *is a term function of* $\mathbf{A}$*, for all* $\alpha \in \mathrm{Eq}_k(n)$*;*

(4) *for all* $n > k$*, an operation* $f \colon A^n \to A$ *is a term function of* $\mathbf{A}$ *provided* $f_{ij} \colon A^n \to A$ *is a term function of* $\mathbf{A}$*, for all* $i < j \leqslant n$*.*

Moreover, if $\mathbf{A}$ *has relational degree* d*, then the conditions hold with* $k = |A|^d$*, and if the conditions hold for some* $k \geqslant |A|$*, then* $\mathbf{A}$ *has relational degree at most* $|A|^k$*.*

Proof. Jablonskiĭ [12] has shown that, if $\mathbf{A}$ has relational degree d, then (3) holds with $k = |A|^d$, and that if (3) holds for any natural number k, then $\mathbf{A}$ has relational degree at most $|A|^k$. Rosenberg and Szendrei [23] observed that (3) ⇔ (4). We shall show that (1) ⇔ (3), under the assumption that $k \geqslant |A|$. The proof that (2) ⇔ (4) is completely analogous.

(1) ⇒ (3): Assume that (1) holds and consider an operation $f \colon A^n \to A$. Assume that, for each $\alpha \in \mathrm{Eq}_k(n)$, there is an n-ary term $t_\alpha(x_1, \dots, x_n)$ with $f_\alpha = t_\alpha^{\mathbf{A}}$. It is easy to check that $\mathcal{F} := \{\, t_\alpha \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ is an (n, k)-scheme for $\mathrm{V}(\mathbf{A})$. So $\mathcal{F}$ comes from an n-ary term t, by (2). It now follows by the 'uniqueness' part of Lemma 2.6(1) that $t^{\mathbf{A}} = f$. So f is a term function of $\mathbf{A}$, whence (3) holds.

(3) ⇒ (1): Assume that (3) holds. Let $n > k$ and consider an (n, k)-scheme $\mathcal{F} = \{\, t_\alpha \mid \alpha \in \mathrm{Eq}_k(n) \,\}$ for $\mathrm{V}(\mathbf{A})$. Let $f \colon A^n \to A$ be the associated operation, as given by Lemma 2.6. Then $f_\alpha = t_\alpha^{\mathbf{A}}$, for all $\alpha \in \mathrm{Eq}_k(n)$. Thus f is a term function of $\mathbf{A}$, by (3). Hence $\mathcal{F}$ comes from a term, by Lemma 2.6, whence (1) holds. ☐

Definition 2.10. For a finite algebra $\mathbf{A}$ that has finite degree, we shall define the *term-degree* of $\mathbf{A}$ to be the least $k \geqslant |A|$ that fulfills the conditions of the previous theorem.

The utility of the new 'term scheme' condition 2.9(2) is illustrated by the following proof that finite degree is a varietal property.

Theorem 2.11. *Given two finite algebras that generate the same variety, if one has finite degree, then the other does too.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras with $\mathrm{V}(\mathbf{A}) = \mathrm{V}(\mathbf{B})$, and assume that $\mathbf{A}$ has finite degree. Then condition 2.9(2) holds for some $k \geqslant |A|$. So it also holds for $k' := \max(k, |B|)$. Thus $\mathbf{B}$ has finite degree. ☐

We shall further illustrate the 'term scheme' approach in a general context by showing that every finite commutative semigroup has finite degree (Section 3) and in a concrete context by showing that the semigroup $\mathbf{A}_2$ has finite degree (Section 5).

Remark 2.12. When we are checking that a finite algebra $\mathbf{A}$ satisfies condition 2.9(4), we can assume that $f \colon A^n \to A$ depends on all n coordinates. To see why, assume that $f_{ij} \colon A^n \to A$ is a term function, for all $i < j \leqslant n$. If f does not depend on the ith coordinate, for some $i < n$, then $f = f_{i,i+1}$ and so f is a term function. If f does not depend on the nth coordinate, then

$$f(a_1, \dots, a_n) = f(a_1, \dots, a_{n-2}, a_{n-1}, a_{n-1}) = f_{n-1,n}(a_1, \dots, a_{n-2}, a_n, a_{n-1}),$$

for all $a_1, \dots, a_n \in A$, and so again f is a term function.

3. Commutative semigroups

Our main aim in this section is to prove that every finite commutative semigroup has finite degree. Every finite commutative semigroup is a subdirect product of a nilpotent semigroup and a commutative monoid. So we first prove that finite nilpotent semigroups and commutative monoids all have finite degree. We then give sufficient conditions for finite degree to be preserved under taking subdirect products. (We shall see in Section 6 that, in general, finite degree is not preserved under taking products.)

Recall that a semigroup **S** is *ℓ-nilpotent*, for some $\ell \in \mathbb{N}$, if it satisfies the equation $x_1 \cdots x_\ell \approx y_1 \cdots y_\ell$. In this case, each term function $f\colon S^n \to S$ of **S** depends on at most $\ell - 1$ coordinates. So the following lemma applies.

Lemma 3.1. *Let* **A** *be a finite algebra and let $m \in \mathbb{N}$. Assume that each term function of* **A** *depends on at most m coordinates. Then the term-degree of* **A** *is at most* $\max(|A|, m+2)$.

Proof. We shall check that condition 2.9(4) holds. Let $n > |A|$ and let $f\colon A^n \to A$ be such that f_{ij} is a term function of **A**, for all $i < j \leqslant n$. We can complete the proof by showing that either f is a term function of **A** or $n \leqslant m + 2$. So we can assume that f depends on all its coordinates; see Remark 2.12. By a result of Willard [26, Lemma 1.2], there are $i < j \leqslant n$ such that f_{ij} depends on at least $n - 2$ coordinates. But f_{ij} is a term function of **A**. So $n - 2 \leqslant m$, as desired. □

The previous result applies to all finite algebras with an eventually zero p_n-sequence. We generalise this to bounded p_n-sequence in Section 4. (The definition of a p_n-sequence is given at the start of Section 4.)

Remark 3.2. We can use the previous lemma to show that the term-degree of a non-trivial finite unary algebra **A** is $\max(|A|, 3)$. Each term function of **A** depends on at most one coordinate. Since the term-degree of **A** is at least $|A|$, by definition, it suffices to check that a two-element unary algebra **A** cannot have term-degree 2. But this is witnessed by the ternary majority operation on A.

Using Lemma 3.1 together with the following basic fact, we can give an upper bound on the term-degree of a nilpotent semigroup of size n.

Lemma 3.3. *Let* **S** *be a finite nilpotent semigroup. Then* **S** *is $|S|$-nilpotent.*

Proof. Define $n := |S|$. Let 0 denote the zero element of the nilpotent semigroup **S**, and let $a_1, \dots, a_n \in S$. We want to show that $a_1 a_2 \cdots a_n = 0$.

Consider the following sequence of elements of S:

$$b_1 := a_1, \quad b_2 := a_1 a_2, \quad b_3 := a_1 a_2 a_3, \quad \dots, \quad b_n := a_1 a_2 \cdots a_n.$$

If one of the b_i's is 0, then $a_1 a_2 \cdots a_n = 0$, as required. Suppose, by way of contradiction, that none of the b_i's is 0. As $n = |S|$, there must be $i < j \leqslant n$ with $b_i = b_j$. This gives us

$$b_j = b_i a_{i+1} a_{i+2} \cdots a_n = b_j a_{i+1} a_{i+2} \cdots a_n.$$

So the element b_j can be expressed as arbitrarily long products in **S**. Since **S** is nilpotent, it follows that $b_j = 0$, which is a contradiction. □

Theorem 3.4. *Let* **S** *be a finite nilpotent semigroup. Then* **S** *has term-degree at most* $|S| + 1$.

The following example shows that there is no bound on the relational degree of finite nilpotent semigroups (or finite commutative semigroups).

Example 3.5. *For each $n \in \mathbb{N}$, the one-generated semigroup $\mathbf{C}_n = \langle\, c \mid c^{n+1} = 0 \,\rangle$ has relational degree greater than n.*

Proof. As $\mathbf{C}_n$ satisfies $x_1 \cdots x_{n+1} \approx y_1 \cdots y_{n+1}$, it has finite degree by Theorem 3.4. We shall use Lemma 2.1 to show that $\mathbf{C}_n$ fails to have relational degree at most n.

Define the constant tuple $\tilde{c} := (c, \dots, c) \in (C_n)^{n+1}$. Now define the operation $f \colon (C_n)^{n+1} \to C_n$ by $f(\tilde{c}) := c^n$ and $f(v) := 0$, for all $v \in (C_n)^{n+1} \backslash \{\tilde{c}\}$.

We claim that f is not a term function of $\mathbf{C}_n$. To see this, suppose that $f = t^{\mathbf{C}_n}$, for some semigroup word t in the variables $x_1, \dots, x_{n+1}$. Since $f(\tilde{c}) = c^n$, there must be some variable x_i not occurring in t. Now choose a tuple $v \in (C_n)^{n+1}$ such that each coordinate of v is equal to c, except for the ith coordinate. Then we have $0 = f(v) = t^{\mathbf{C}_n}(v) = t^{\mathbf{C}_n}(\tilde{c}) = f(\tilde{c}) = c^n$, which is a contradiction. So f is not a term function.

Using Lemma 2.1, it remains to show that f agrees with a term function of $\mathbf{C}_n$ on each subset of $(C_n)^{n+1}$ with at most n elements. So let $v_1, \dots, v_n \in (C_n)^{n+1}$ and define $S := \{v_1, \dots, v_n\}$. We can assume that $v_1, \dots, v_\ell$ all equal $\tilde{c}$ and none of $v_{\ell+1}, \dots, v_n$ equals $\tilde{c}$, where $\ell \in \{0, 1, \dots, n\}$. If $\ell = n$, then $S = \{\tilde{c}\}$, and f agrees with the semigroup word x_1^n on S. So we can assume that $\ell < n$. For each $k \geqslant \ell+1$, choose a coordinate i_k such that $v_k(i_k) \neq c$. Then f agrees with $x_{i_{\ell+1}}^{\ell+1} x_{i_{\ell+2}} \cdots x_{i_n}$ on S. $\square$

We next check that every finite commutative semigroup that happens to be a monoid has finite degree.

Theorem 3.6. *Let $\mathbf{M} = \langle M; * \rangle$ be a finite commutative monoid (viewed as a semigroup). Then $\mathbf{M}$ has term-degree at most* $\max(|M|, 4)$.

Proof. We shall establish condition 2.9(4). Let $n > \max(|M|, 4)$ and consider an operation $f \colon M^n \to M$ such that f_{ij} is a term function of $\mathbf{M}$, for all $i < j \leqslant n$. We want to show that f is a term function of $\mathbf{M}$.

Denote the identity element of $\mathbf{M}$ by 1. Let $k \leqslant n$. Choose any $i < j$ in $\underline{n} \backslash \{k\}$. Since f_{ij} is a term function of $\mathbf{M}$, there is some $p_k \in \mathbb{N} \cup \{0\}$ such that

$$f(1, 1, \dots, 1, \overset{k}{a}, 1, 1, \dots, 1) = f_{ij}(1, 1, \dots, 1, \overset{k}{a}, 1, 1, \dots, 1) = a^{p_k},$$

for all $a \in M$. Note that the same power p_k must work for any choice of $i < j$ in $\underline{n} \backslash \{k\}$.

Now let $(a_1, \dots, a_n) \in A^n$. We shall prove that f is a term function of $\mathbf{M}$ by showing that $f(a_1, \dots, a_n) = a_1^{p_1} \cdots a_n^{p_n}$. Since $n > |M|$, the elements $a_1, \dots, a_n$ cannot be pairwise distinct. For notational convenience, we assume that $a_1 = a_2$. As both f_{12} and f_{34} are term functions of $\mathbf{M}$, we get

$$\begin{aligned}
f(a_1, \dots, a_n) &= f_{12}(1, a_2, a_3, \dots, a_n) \\
&= f_{12}(1, a_2, 1, \dots, 1) * f_{12}(1, 1, a_3, 1, \dots, 1) * \cdots * f_{12}(1, \dots, 1, a_n) \\
&= f(a_2, a_2, 1, \dots, 1) * a_3^{p_3} a_4^{p_4} \cdots a_n^{p_n} \\
&= f_{34}(a_2, a_2, 1, \dots, 1) * a_3^{p_3} a_4^{p_4} \cdots a_n^{p_n} \\
&= f_{34}(a_2, 1, 1, \dots, 1) * f_{34}(1, a_2, 1, \dots, 1) * a_3^{p_3} a_4^{p_4} \cdots a_n^{p_n} \\
&= a_2^{p_1} a_2^{p_2} a_3^{p_3} \cdots a_n^{p_n} = a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}.
\end{aligned}$$

$\square$

It is also easy to show that finite commutative monoids (viewed as monoids) have finite degree.

Theorem 3.7. *Let* $\langle A; *, 1\rangle$ *be a finite commutative monoid, and let* F *be a set of compatible operations on the associated semigroup* $\langle A; *\rangle$*. Then the enriched monoid* $\mathbf{A} = \langle A; *, F, 1\rangle$ *has finite degree.*

Proof. We will show directly that $\mathrm{Clo}(\mathbf{A})$ is determined by a finite set of relations. Let $F_1 \subseteq A^{|A|}$ be the relation corresponding to the set of all unary term functions of $\mathbf{A}$. Assume that $f\colon A^n \to A$ preserves the two relations F_1 and $\mathrm{graph}(*)$, both of which are compatible with $\mathbf{A}$. We want to show that f is a term function of $\mathbf{A}$.

Let $i \in \underline{n}$, and define $f_i : A \to A$ by

$$f_i(a) := f(1, \dots, 1, \overset{i}{a}, 1, \dots, 1).$$

Then f_i preserves F_1 and it follows that f_i is a term function of $\mathbf{A}$. Now, for all $a_1, \dots, a_n \in A$, we have

$$\begin{aligned} f(a_1, \dots, a_n) &= f(a_1, 1, 1, \dots, 1) * \cdots * f(1, 1, \dots, 1, a_n) \\ &= f_1(a_1) * \cdots * f_n(a_n). \end{aligned}$$

So f is a term function of $\mathbf{A}$, as required. □

Remark 3.8. Even though we did not appear to need the associativity or commutativity of $*$ in the above proof, it follows from the assumption that $*$ is compatible with itself and has an identity element 1.

Note that, in the proof of Theorem 3.7, we only used the $|A|$-ary relation F_1 because, if a unary operation $u\colon A \to A$ preserves F_1, then u is a term function of $\mathbf{A}$. Let $\mathbf{D}_n = \langle C_n \cup \{1\}; *, 1\rangle$ be the monoid obtained by adjoining an identity element to the semigroup $\mathbf{C}_n$ of Example 3.5. It is easy to check that, if a unary operation $u\colon D_n \to D_n$ preserves the two relations $\mathrm{graph}(*)$ and $\{1\}$, then u is a term function of $\mathbf{D}_n$. So the proof of Theorem 3.7 can be adapted to show that the monoid $\mathbf{D}_n$ has relational degree at most 3, even though the semigroup $\mathbf{C}_n$ has relational degree greater than n.

We turn now to the preservation of finite degree by subdirect products. First we state two technical lemmas.

Lemma 3.9. *Let* $f\colon A^n \to A$*, for some finite set* A *and some* $n \in \mathbb{N}$*. Assume that* $S \subseteq \underline{n}$ *with* $|S| \geqslant |A| + 2$*. If* f *depends on its* k*th coordinate, for some* $k \in \underline{n}$*, then there exist* $i < j$ *in* $S\backslash\{k\}$ *such that* f_{ij} *depends on its* k*th coordinate.*

Proof. For notational convenience, we will consider the case $k = 1$. Assume that f depends on its first coordinate. Then there exist $b, c, a_2, \dots, a_n \in A$ such that $f(b, a_2, \dots, a_n) \neq f(c, a_2, \dots, a_n)$. Since $|S\backslash\{1\}| \geqslant |A| + 1$, there must be $i < j$ in $S\backslash\{1\}$ with $a_i = a_j$. So we get

$$f_{ij}(b, a_2, \dots, a_n) = f(b, a_2, \dots, a_n) \neq f(c, a_2, \dots, a_n) = f_{ij}(c, a_2, \dots, a_n).$$

Thus f_{ij} depends on its first coordinate. □

Recall that an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{A})$ determines an n-ary operation on A; see Lemma 2.6.

Lemma 3.10. *Consider a finite subdirect product* $\mathbf{A} \leqslant \mathbf{B} \times \mathbf{C}$*. Let* $n > |A|$ *and let* $\mathcal{F}$ *be an* $(n, n-1)$*-scheme for* $\mathrm{V}(\mathbf{A})$*, determining the operation* $f\colon A^n \to A$*. Assume that* $\mathcal{F}$ *comes from the* n*-ary terms* p *and* q *in* $\mathrm{V}(\mathbf{B})$ *and* $\mathrm{V}(\mathbf{C})$*, respectively. Then*

1. f *is the restriction of* $p^{\mathbf{B}} \times q^{\mathbf{C}}$ *to* A^n*, and*
2. f *is a term function of* $\mathbf{A}$ *if and only if* p *and* q *can be chosen to be the same term.*

We shall say that a finite algebra $\mathbf{A}$ is *ℓ-nilpotent*, for some $\ell \in \mathbb{N}$, if all terms with at least ℓ variables occurring (not necessarily distinct variables) are equivalent in $\mathrm{V}(\mathbf{A})$. Note that this generalises the usual definition of nilpotence for semigroups, but not for groups.

Lemma 3.11. *Consider a finite subdirect product* $\mathbf{A} \leqslant \mathbf{B} \times \mathbf{C}$*. Assume that* $\mathbf{B}$ *is* ℓ*-nilpotent and that both* $\mathbf{B}$ *and* $\mathbf{C}$ *have term-degree at most* k*, for some* $k, \ell \in \mathbb{N}$*. Then* $\mathbf{A}$ *has term-degree at most* $\max(k, |A| + \ell)$*.*

Proof. By Remark 3.2, we can assume $\mathbf{A}$ has a fundamental operation of arity at least 2. We shall establish condition 2.9(2). Let $n > \max(k, |A| + \ell)$ and let $\mathcal{F} = \{\, t_{ij}(x_1, \dots, x_n) \mid i < j \leqslant n \,\}$ be an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{A})$. By Lemma 2.6, this scheme determines an operation $f\colon A^n \to A$, and it suffices to show that f is a term function of $\mathbf{A}$. Since $f_{ij} = t_{ij}^{\mathbf{A}}$, for all $i < j \leqslant n$, we may assume that f depends on all its coordinates; see Remark 2.12.

The definition of an $(n, n-1)$-scheme is equational. So $\mathcal{F}$ is also an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{B})$ and $\mathrm{V}(\mathbf{C})$. Our assumptions and Theorem 2.9 now guarantee that $\mathcal{F}$ comes from terms p and q in $\mathrm{V}(\mathbf{B})$ and $\mathrm{V}(\mathbf{C})$, respectively. As $\mathbf{B}$ is ℓ-nilpotent, the term p is equivalent in $\mathrm{V}(\mathbf{B})$ to a term involving at most $\ell - 1$ distinct variables. But $n \geqslant |A| + \ell + 1$. So we can assume there is a set $S \subseteq \underline{n}$ with $|S| \geqslant |A| + 2$ such that the variable x_s does not occur in p, for each $s \in S$.

Choose $s \in S$. By Lemma 3.9, there are $i < j$ in $S \backslash \{s\}$ such that $f_{ij}\colon A^n \to A$ depends on its sth coordinate. Since $f_{ij} = t_{ij}^{\mathbf{A}}$, the variable x_s must occur in the term t_{ij}. As t_{ij} is part of an $(n, n-1)$-scheme coming from p in $\mathrm{V}(\mathbf{B})$, we have

$$\mathbf{B} \models p_{ij}(x_1, \dots, x_n) \approx t_{ij}(x_1, \dots, x_n).$$

Since $i \in S$, the variable x_i does not occur in p. It follows that

$$\mathbf{B} \models p(x_1, \dots, x_n) \approx t_{ij}(x_1, \dots, x_n).$$

Choose 0 to be some term in which all of the variables $x_1, \dots, x_\ell$ occur. (Recall that ℓ is the nilpotency length of $\mathbf{B}$, and that there is a fundamental operation of arity at least 2.) Since x_s does not occur in p but does occur in t_{ij}, it now follows that $p \approx 0$ in $\mathrm{V}(\mathbf{B})$. (Substitute 0 for x_s in the equation displayed above.)

Using Lemma 3.10 and the fact that f depends on all its coordinates, it now follows that $q^{\mathbf{C}}$ depends on all its coordinates. So every variable $x_1, \dots, x_n$ occurs in q. Since n is greater than the nilpotency length ℓ of $\mathbf{B}$, it follows that $\mathbf{B} \models p \approx q$. So $\mathcal{F}$ also comes from q in $\mathrm{V}(\mathbf{B})$. By Lemma 3.10, we conclude that f is a term function of $\mathbf{A}$, as required. ☐

We now have all of the tools that we need to show that finite commutative semigroups have finite degree.

Theorem 3.12. *Let* $\mathbf{C}$ *be a finite commutative semigroup. Then* $\mathbf{C}$ *has term-degree at most* $\max(2|C|, 4)$*.*

Proof. The finite commutative semigroup $\mathbf{C}$ is a subdirect product of a nilpotent semigroup $\mathbf{S}$ and a commutative monoid $\mathbf{M}$. (See [10, §5], for example.) But $\mathbf{S}$ must be $|S|$-nilpotent and have term-degree at most $|S| + 1$, by Lemma 3.3 and Theorem 3.4, and $\mathbf{M}$ must have term-degree at most $\max(|M|, 4)$, by Theorem 3.6. So the result follows from Lemma 3.11 with $k := \max(|C| + 1, 4)$ and $\ell := |S|$. ☐

4. Algebras with a bounded p_n-sequence

The *p_n-sequence* of a finite algebra $\mathbf{A}$ is the sequence $p_1(\mathbf{A}), p_2(\mathbf{A}), p_3(\mathbf{A}), \dots$, where $p_n(\mathbf{A})$ denotes the number of n-ary term functions $t\colon A^n \to A$ of $\mathbf{A}$ that depend on all n coordinates. As already noted, it follows from Lemma 3.1 that every finite algebra with an eventually zero p_n-sequence has finite degree. In this section, we shall see that any finite algebra that has an upper bound on its p_n-sequence also has finite degree.

The finite semigroups with bounded p_n-sequence have been completely described by Crvenković, Dolinka and Ruškuc [6]: they are precisely the ideal extensions by a nilpotent semigroup of either a semilattice, a Boolean group or a rectangular band. We will be using a general characterisation of the finite algebras with bounded p_n-sequence, which we state after a collection of definitions.

Fix a finite set A and an operation $f\colon A^n \to A$, for some $n \in \mathbb{N}$. Denote the set of coordinates that f depends on by $S \subseteq \underline{n}$. Then we say that f is *essentially $|S|$-ary*. We shall say that f is *symmetric* if, for each permutation $\pi\colon \underline{n} \to \underline{n}$ that preserves S, we have

$$(\forall a_1, \dots, a_n \in A) \quad f(a_1, \dots, a_n) = f(a_{\pi(1)}, \dots, a_{\pi(n)}).$$

The operation f is *totally symmetric* if it is symmetric and depends on all coordinates. An algebra $\mathbf{A}$ is said to be *eventually totally symmetric* if there exists $\sigma \in \mathbb{N}$ such that, for all $n \geqslant \sigma$, each essentially n-ary term function $t\colon A^n \to A$ of $\mathbf{A}$ is totally symmetric.

We will be using the fact that a finite algebra has a bounded p_n-sequence if and only if it is eventually totally symmetric [6, Lemma 3.4]. We will also be using two key facts about totally symmetric operations that come out of work by Berman and Kisielewicz [4] and Willard [26]. Again, we require some definitions.

Let $\mathcal{P}(A)$ denote the set of all subsets of A. Now define the two functions $\operatorname{supp}\colon A^n \to \mathcal{P}(A)$ and $\operatorname{oddsupp}\colon A^n \to \mathcal{P}(A)$ by

$$\begin{aligned} \operatorname{supp}(\vec{a}) &:= \{\, b \in A \mid b \text{ appears in the tuple } \vec{a} \,\}, \\ \operatorname{oddsupp}(\vec{a}) &:= \{\, b \in A \mid b \text{ appears an odd number of times in the tuple } \vec{a} \,\}. \end{aligned}$$

An operation $f\colon A^n \to A$ is *determined by supp* if $\operatorname{supp}(\vec{a}) = \operatorname{supp}(\vec{b})$ implies $f(\vec{a}) = f(\vec{b})$, for all $\vec{a}, \vec{b} \in A^n$; in other words, if the value of f at a tuple $\vec{a}$ is determined only by the set of elements appearing in $\vec{a}$. Similarly, an operation f is *determined by oddsupp* if $\operatorname{oddsupp}(\vec{a}) = \operatorname{oddsupp}(\vec{b})$ implies $f(\vec{a}) = f(\vec{b})$. Clearly, an operation determined by supp or by oddsupp must be totally symmetric. However, something similar to a converse holds.

The following facts are presented in Willard [26]. Given an operation $f\colon A^n \to A$ and $i < j \leqslant n$, we refer to the induced operation $f_{ij}\colon A^n \to A$ as a *collapse* of f.

Facts 4.1. *Let $f\colon A^n \to A$ be an essentially n-ary operation, for some $n \in \mathbb{N}$.*

(1) *Assume that* $n \geqslant \max(|A|, 3)+2$*, that every essentially* $(n-1)$*-ary collapse of* f *is symmetric, and that* f *has at least one essentially* $(n-1)$*-ary collapse. Then* f *is determined by* supp*.*
(2) *Assume that* $n \geqslant \max(|A|, 3)+1$ *and that* f *has no essentially* $(n-1)$*-ary collapse. Then* f *is determined by* oddsupp*. Moreover, for all* $i < j \leqslant n$*, the operation* f_{ij} *depends on all coordinates except* i, j*.*

Proof. For (1), we use [26, Theorem 2.6], which states that, if $n \geqslant \max(|A|, 3) + 2$ and every essentially $(n-1)$-ary collapse of f is symmetric, then f is determined by supp or by oddsupp. However, if f_{ij} is essentially $(n-1)$-ary, for some $i < j \leqslant n$, then f cannot be determined by oddsupp.

The first part of (2), which is essentially due to Berman and Kisielewicz, is [26, Corollary 2.3]. The second part is [26, Theorem 2.1]. □

Now let $E_n(\mathbf{A})$ denote the set of all essentially n-ary term functions $t\colon A^n \to A$ of $\mathbf{A}$. Let $E_n^{\mathrm{s}}(\mathbf{A})$ and $E_n^{\mathrm{os}}(\mathbf{A})$ denote those members of $E_n(\mathbf{A})$ determined by supp and by oddsupp, respectively.

Lemma 4.2. *Let* $\mathbf{A}$ *be a finite algebra. Then, for each* $n \geqslant |A| + 3$*,*

(1) *there is a one-to-one map* $\varphi_n\colon E_n^{\mathrm{s}}(\mathbf{A}) \to E_{n-1}^{\mathrm{s}}(\mathbf{A})$*, given by*

$$\varphi_n t(x_1, \dots, x_{n-1}) = t(x_1, \dots, x_{n-1}, x_{n-1}),$$

(2) *there is a one-to-one map* $\psi_n\colon E_n^{\mathrm{os}}(\mathbf{A}) \to E_{n-2}^{\mathrm{os}}(\mathbf{A})$*, given by*

$$\psi_n t(x_1, \dots, x_{n-2}) = t(x_1, \dots, x_{n-2}, x_{n-2}, x_{n-2}).$$

Furthermore, there is some $\gamma \geqslant |A| + 3$ *such that both* φ_n *and* ψ_n *are bijections, for all* $n \geqslant \gamma$*.*

Proof. We shall sketch proofs of the 'oddsupp' claims. (The 'supp' claims are even easier.) Let $n \geqslant |A|+3$. To see that ψ_n is well defined, let $t \in E_n^{\mathrm{os}}(\mathbf{A})$. Clearly $\psi_n t$ is determined by oddsupp. We want to check that $\psi_n t$ is essentially $(n-2)$-ary. As t depends on its first coordinate, there are $b, c, a_2, \dots, a_n \in A$ such that

$$t(b, a_2, a_3, \dots, a_n) \neq t(c, a_2, a_3, \dots, a_n).$$

Since t is determined by oddsupp and since $n - 1 \geqslant |A| + 2$, we can assume that $a_{n-2} = a_{n-1} = a_n$. This gives $\psi_n t(b, a_2, a_3, \dots, a_{n-2}) \neq \psi_n t(c, a_2, a_3, \dots, a_{n-2})$. As the operation $\psi_n t$ is determined by oddsupp, it now follows that $\psi_n t$ depends on all $n-2$ coordinates. To prove that ψ_n is one-to-one, consider $s, t \in E_n^{\mathrm{os}}(\mathbf{A})$ with $s \neq t$; it is easy to check that $\psi_n s \neq \psi_n t$. Thus (2) holds.

Define $\ell := |A| + 3$. We now have two infinite non-increasing sequences:

$$|E_{\ell-2}^{\mathrm{os}}(\mathbf{A})| \geqslant |E_{\ell}^{\mathrm{os}}(\mathbf{A})| \geqslant |E_{\ell+2}^{\mathrm{os}}(\mathbf{A})| \geqslant \cdots$$
$$\text{and} \quad |E_{\ell-1}^{\mathrm{os}}(\mathbf{A})| \geqslant |E_{\ell+1}^{\mathrm{os}}(\mathbf{A})| \geqslant |E_{\ell+3}^{\mathrm{os}}(\mathbf{A})| \geqslant \cdots .$$

Both of these sequences must eventually be constant. So we can choose $\gamma_{\mathrm{os}} \geqslant \ell$ such that ψ_n is a bijection, for all $n \geqslant \gamma_{\mathrm{os}}$. □

Theorem 4.3. *Let* $\mathbf{A}$ *be a finite algebra with bounded* p_n*-sequence. Then* $\mathbf{A}$ *has finite degree.*

Proof. There is some $\sigma \in \mathbb{N}$ such that, for all $n \geqslant \sigma$, every essentially n-ary term function $t\colon A^n \to A$ of $\mathbf{A}$ is totally symmetric [6]. Let $\gamma \geqslant |A| + 3$ be given by Lemma 4.2. Now let $n > \max(\sigma, \gamma)$ and consider $f\colon A^n \to A$ such that f_{ij} is a term function of $\mathbf{A}$, for all $i < j \leqslant n$. By Theorem 2.9, it suffices to show that f is a term function of $\mathbf{A}$. So we can assume that f depends on all n coordinates; see Remark 2.12.

First assume that f has an essentially $(n-1)$-ary collapse. Then Fact 4.1(1) implies that f is determined by supp, since $n - 1 \geqslant \sigma$. For notational convenience, assume that $f_{n-1,n}$ is essentially $(n-1)$-ary. It now follows that the term function $g\colon A^{n-1} \to A$ of $\mathbf{A}$, given by

$$g(a_1, \dots, a_{n-1}) := f(a_1, \dots, a_{n-1}, a_{n-1}),$$

belongs to $E^{\mathrm{s}}_{n-1}(\mathbf{A})$. Using Lemma 4.2, we get $t \in E^{\mathrm{s}}_n(\mathbf{A})$ with $g = \varphi_n t$. Since both t and f are determined by supp and since $n > |A|$, it is easy to check that $f = t$. So f is a term function of $\mathbf{A}$, as required.

A similar argument works for the case where f has no essentially $(n-1)$-ary collapse. ☐

5. A UNARY SEMIGROUP THAT HAS INFINITE DEGREE

Natural examples of finite algebras of finite type that do not have finite degree are surprisingly difficult to find. In particular, there are no known examples of finite semigroups that have infinite degree. In this section, we give a natural example of a five-element unary semigroup that has infinite degree.

The semigroup $\mathbf{A}_2$ is the Reese matrix semigroup over $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$: its underlying set is $A_2 := \{0\} \cup \{1,2\}^2$ and the multiplication is defined so that 0 is a zero element and

$$(i,j)(k,\ell) = \begin{cases} 0 & \text{if } j = k = 1, \\ (i,\ell) & \text{otherwise,} \end{cases}$$

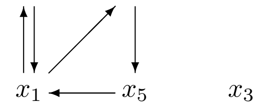
for all $i, j, k, \ell \in \{1,2\}$. We shall define $\mathbf{A}_2'$ to be the unary semigroup obtained by enriching $\mathbf{A}_2$ with the involution given by $0' := 0$ and $(i,j)' := (j,i)$. Note that $\mathbf{A}_2'$ satisfies the equation $(xy)' \approx y'x'$.

We first prove that the semigroup $\mathbf{A}_2$ has finite degree. This illustrates the use of the 'term scheme' condition 2.9(2) in a concrete context. We then prove that the unary semigroup $\mathbf{A}_2'$ has infinite degree.

We start by explaining how terms of $\mathbf{A}_2$ can be represented by graphs. Define an *in-out graph* to be a finite directed graph with distinguished 'in' and 'out' vertices. Given any semigroup word $t(x_1, \dots, x_n)$, define the in-out graph $\mathbf{G}_t$ as follows:

- the set of vertices is $\{x_1, \dots, x_n\}$;
- the 'in' and 'out' vertices are the first and last variable of the word t;
- for all $i, j \in \{1, \dots, n\}$, there is an edge from x_i to x_j if and only if $x_i x_j$ is a subword of t.

For example, the graph corresponding to the word $t(x_1, \dots, x_5) = x_2x_4x_5x_1x_2x_1x_4$ is shown below.



An in-out graph comes from a semigroup word if and only if there is a directed path from ‘in’ to ‘out’ that traverses every edge in the graph at least once; we shall refer to such a path as a *word path.*

Note that, for any word $t(x_1, \dots, x_n)$ and any equivalence relation γ on $\underline{n}$, the graph for the word $t(x_{\widehat{\gamma}(1)}, \dots, x_{\widehat{\gamma}(n)})$ can be obtained as the image of the graph $\mathbf{G}_t$ under the map $x_m \mapsto x_{\widehat{\gamma}(m)}$.

We use the fact that two semigroup words $s(x_1, \dots, x_n)$ and $t(x_1, \dots, x_n)$ are equivalent in $\mathbf{A}_2$ if and only if they have the same graphs (Trahtman [25]).

Example 5.1. *The semigroup* $\mathbf{A}_2$ *has term-degree* 5.

Proof. We shall check condition 2.9(2). Let $n > 5$ and consider an $(n, n-1)$-scheme $\mathcal{F} = \{\, t_{ij}(x_1, \dots, x_n) \mid i < j \leqslant n \,\}$ for $\mathrm{V}(\mathbf{A}_2)$.

As described above, each word $t_{ij}(x_1, \dots, x_n)$ can be represented by an in-out graph $\mathbf{G}_{ij}$ with the set of vertices $\{x_1, \dots, x_n\} \backslash \{x_i\}$. The fact that $\mathcal{F}$ satisfies the compatibility condition (C) ensures that the graphs $\mathbf{G}_{ij}$ satisfy the following compatibility condition:

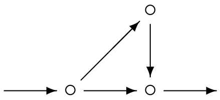
(Γ) for $i < j \leqslant n$ and $k < \ell \leqslant n$, let γ denote the least equivalence relation on $\underline{n}$ containing (i, j) and (k, ℓ), then the image of $\mathbf{G}_{ij}$ under $x_m \mapsto x_{\widehat{\gamma}(m)}$ is the same as the image of $\mathbf{G}_{k\ell}$ under $x_m \mapsto x_{\widehat{\gamma}(m)}$.

Since $n > 5$, it is fairly easy to see that condition (Γ) above ensures the existence of an in-out graph $\mathbf{G}$ on the set of vertices $\{x_1, \dots, x_n\}$ such that, for all $i < j \leqslant n$, the in-out graph obtained from $\mathbf{G}$ by collapsing x_i onto x_j is $\mathbf{G}_{ij}$.

The following general claim establishes that $\mathbf{G}$ has a word path and so comes from a semigroup word $t(x_1, \dots, x_n)$. Then $\mathcal{F}$ and $\mathcal{F}_t$ are equivalent, as required.

Claim. *Let* $\mathbf{G}$ *be any in-out graph with* $|G| \geqslant 4$. *Assume that every proper quotient of* $\mathbf{G}$ *has a word path. Then* $\mathbf{G}$ *has a word path.*

Proof of Claim. The in-out graph below shows why we need $|G| \geqslant 4$.



For distinct $u, v \in G$, we will write $\mathbf{G}_{uv}$ to denote the in-out graph obtained as a quotient of $\mathbf{G}$ by collapsing u and v.

First assume that $\mathbf{G}$ has a vertex z of degree 0. Choose any $v \in G \backslash \{z\}$. Then a word path in $\mathbf{G}_{vz}$ clearly yields a word path in $\mathbf{G}$. Next assume that $\mathbf{G}$ has a circuit C. (By a ‘circuit’, we mean a directed circuit that is not just a loop.) There is an edge $u \to v$ in C with $u \neq v$. Using the circuit C, a word path in $\mathbf{G}_{uv}$ can easily be expanded into a word path in $\mathbf{G}$.

We can now assume that $\mathbf{G}$ has no circuits and no vertices of degree 0. Construct the directed graph $\mathbf{G}'$ from $\mathbf{G}$ by taking the reflexive, transitive closure of the edge relation. Then $\mathbf{G}'$ is an ordered set. Let 0 and 1 denote the ‘in’ and ‘out’ elements of $\mathbf{G}$, respectively.

Subclaim 1: The elements 0 *and* 1 *are the bottom and top of* $\mathbf{G}'$*, respectively.* Let $v \in G \backslash \{0\}$. We want to find a (directed) path from 0 to v in $\mathbf{G}$. First choose some $u \in G \backslash \{0, v\}$. As u does not have degree 0, the word path in $\mathbf{G}_{uv}$ must yield an edge $0 \to w$ from $\mathbf{G}$, for some $w \neq 0$. So we can assume that $v \neq w$. Now the word path in $\mathbf{G}_{0w}$ yields a path from 0 to v in $\mathbf{G}$, as required. Similarly, we can check that 1 is the top element of $\mathbf{G}'$.

Subclaim 2: If $u \prec v$ is a cover in $\mathbf{G}'$, then $\mathbf{G}_{uv}$ has no circuits.
Note that $u \to v$ is an edge in $\mathbf{G}$. Suppose that C is a circuit in $\mathbf{G}_{uv}$. We can assume that C does not contain the loop $u \to u$ in $\mathbf{G}_{uv}$. The circuit C does not induce a circuit in $\mathbf{G}$. So C must contain subpaths of the form $a \to v = u \to b$, where $a \to v$ and $u \to b$ are edges in $\mathbf{G}$. But this implies that there is a path from u to v in $\mathbf{G}$ of length at least 2. But this is a contradiction, as $u \prec v$ is a cover in $\mathbf{G}'$.

Subclaim 3: The graph $\mathbf{G}$ has a word path.
We shall say that a quotient of $\mathbf{G}$ is 'linear' if it is a directed path from 0 to 1 possibly with loops. For a cover $u \prec v$ in $\mathbf{G}'$, we know that the graph $\mathbf{G}_{uv}$ has a word path but no circuits, and so it must be linear.

Since $\mathbf{G}'$ is a bounded ordered set with $|G| \geqslant 4$, we can choose $a, c \in G$ with $0 \prec a \neq c \prec 1$. As both $\mathbf{G}_{0a}$ and $\mathbf{G}_{c1}$ are linear, it follows fairly easily that $\mathbf{G}$ is also linear. Thus $\mathbf{G}$ has a word path, as required. □

Example 5.2. *The unary semigroup $\mathbf{A}_2'$ has infinite degree.*

Proof. Fix $n > 5$. We shall define an $(n, n-1)$-scheme $\mathcal{F} = \{\, w_{ij} \mid i < j \leqslant n \,\}$ for $\mathrm{V}(\mathbf{A}_2')$ that does not come from a term. By Theorem 2.9, it will follow that $\mathbf{A}_2'$ has infinite degree.

We construct the $(n, n-1)$-scheme from a directed graph. Consider the set $X := \{x_1, \dots, x_n\}$ of variables and define $X' := \{x_1', \dots, x_n'\}$. Let $\mathbf{G}$ be the directed graph on $X \mathbin{\dot\cup} X'$ with all possible edges except for those connecting vertices with the same subscript value. Note that $\mathbf{G}$ is strongly connected and every vertex has equal in-degree and out-degree. So $\mathbf{G}$ is Eulerian.

For each $j \leqslant n$, let w_j be the sequence of vertices of some Eulerian circuit in $\mathbf{G}$ that starts and ends at x_j; we can view w_j as a unary semigroup word in the variables X. Now, for all $i < j \leqslant n$, construct the word w_{ij} from the word w_j by replacing every occurrence of the variable x_i by x_j.

Define the operation $f\colon A_2^n \to A_2$ by

$$f(\vec{a}) = \begin{cases} (2,2) & \text{if there exists } \ell \leqslant n \text{ with } a_\ell \neq 0 \text{ and } a_k = (2,2), \text{ for all } k \neq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 2.9, it follows directly from the two claims below that $\mathbf{A}_2'$ has infinite degree. Alternatively, using Lemma 2.6 as well, it follows that $\mathcal{F} := \{\, w_{ij} \mid i < j \leqslant n \,\}$ is an $(n, n-1)$-scheme that does not come from a term.

Claim 1: f is not a term function of $\mathbf{A}_2'$.

Let t be any n-ary term of $\mathbf{A}_2'$. We will show that $f \neq t^{\mathbf{A}_2'}$. Since $\mathbf{A}_2'$ satisfies $(xy)' \approx y'x'$, we can assume that t is a semigroup word in the alphabet $X \mathbin{\dot\cup} X'$. The word t finishes with either x_ℓ or x_ℓ', for some $\ell \leqslant n$. Define $\vec{b} \in A_2^n$ by $b_\ell = (1,1)$ and $b_k = (2,2)$, for all $k \neq \ell$. Then $f(\vec{b}) = (2,2) \neq t^{\mathbf{A}_2'}(\vec{b})$, as required.

Claim 2: f_{ij} is the term function $w_{ij}^{\mathbf{A}_2'}$, for all $i < j \leqslant n$.

Fix $i < j \leqslant n$ and let $\vec{a} \in A^n$. First assume that $f_{ij}(\vec{a}) = (2,2)$. Then $a_j = (2,2)$ and there exists $\ell \leqslant n$ with $a_\ell \neq 0$ and $a_k = (2,2)$, for all $k \notin \{i, \ell\}$. Note that $(2,2) \cdot b \cdot (2,2) = (2,2)$ in $\mathbf{A}_2'$, for all $b \in A_2 \backslash \{0\}$. It follows that $w_{ij}^{\mathbf{A}_2'}(\vec{a}) = (2,2) = f_{ij}(\vec{a})$.

Now we can assume that $f_{ij}(\vec{a}) = 0$. So one of the following three cases applies.

Case 1: $a_k = 0$, for some $k \neq i$. Clearly $w_{ij}^{\mathbf{A}_2'}(\vec{a}) = 0 = f_{ij}(\vec{a})$, as required.

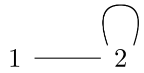


FIGURE 2. A non-dualisable graph

$\cdot$	0	1	2
0	0	0	0
1	0	0	2
2	0	1	2

FIGURE 3. Murskiĭ's groupoid

Case 2: $a_k \neq (2,2)$ and $a_\ell \neq (2,2)$, for distinct $k, \ell \in \underline{n}\backslash\{i\}$. At least one of the products $a_k a_\ell$, $a'_k a_\ell$, $a_k a'_\ell$, $a'_k a'_\ell$ in $\mathbf{A}'_2$ is 0. Since all of the words $x_k x_\ell$, $x'_k x_\ell$, $x_k x'_\ell$, $x'_k x'_\ell$ appear in w_{ij}, it follows that $w_{ij}^{\mathbf{A}'_2}(\vec{a}) = 0 = f_{ij}(\vec{a})$, as required.

Case 3: $a_j \neq (2,2)$. At least one of the two products $a_j a'_j$, $a'_j a_j$ in $\mathbf{A}'_2$ is 0. Since both of the words $x_i x'_j$, $x'_i x_j$ appear in w_j, it follows that both of the words $x_j x'_j$, $x'_j x_j$ appear in w_{ij}. Therefore $w_{ij}^{\mathbf{A}'_2}(\vec{a}) = 0 = f_{ij}(\vec{a})$, as required. □

Remark 5.3. There is a connection here with the theory of natural dualities. Jackson and Volkov [13] have a method for constructing a unary semigroup from a directed graph, such that certain quasivariety properties of directed graphs translate into variety properties of unary semigroups. Using their construction, the unary semigroup $\mathbf{A}'_2$ comes from the graph shown in Figure 2. Johansen [14, Lemma 2.7] has shown that the quasivariety generated by this graph does not admit a natural duality. The proof of Example 5.2 is loosely based around an attempt to use the method of [13] to translate the non-dualisability proof for the graph in Figure 2 into an infinite-degree proof for $\mathbf{A}'_2$.

Remark 5.4. Murskiĭ's groupoid $\mathbf{M} = \langle\{0,1,2\}; \cdot\rangle$, shown in Figure 3, is the three-element graph algebra of the graph in Figure 2. The proof of Example 5.2 can be adapted to show that $\mathbf{M}$ has infinite degree; we sketch the details below.

Fix $n > 3$. We construct a family of groupoid terms based on a complete graph. This time, let w_j denote the sequence of vertices of an Eulerian circuit in the complete graph $\mathbf{K}_n$ (treated as a symmetric directed graph) that starts and ends at vertex x_j. This can be viewed as a groupoid term by bracketing from the left. (So $wxyz$ means $((wx)y)z$, for example.) For all $i < j \leqslant n$, let w_{ij} denote the term obtained from w_j by replacing every occurrence of x_i by x_j.

Now define $f \colon M^n \to M$ by

$$f(\vec{a}) = \begin{cases} 2 & \text{if there exists } \ell \leqslant n \text{ with } a_\ell \neq 0 \text{ and } a_k = 2, \text{ for all } k \neq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Example 5.2, we can show that f is not a term function of $\mathbf{M}$ but that f_{ij} is the term function $w_{ij}^{\mathbf{M}}$, for all $i < j \leqslant n$.

$*$	0	1	2
0	1	1	2
1	0	1	2
2	0	1	2

FIGURE 4. A finite-degree groupoid containing $\mathbf{I} = \langle\{0,1\};\to\rangle$

6. NON-PRESERVATION OF FINITE DEGREE

In this final section, we show that finite degree is not preserved by taking subalgebras, homomorphic images, direct products or subdirect factors. First we give an example of a three-element groupoid $\mathbf{A}$ that demonstrates all of the following:

- a subalgebra of a finite-degree algebra can have infinite degree;
- a homomorphic image of a finite-degree algebra can have infinite degree;
- a subdirect factor of a finite-degree algebra can have infinite degree.

Then we give an example of a pair of two-element algebras $\mathbf{B}$ and $\mathbf{C}$, both having finite degree, such that $\mathbf{B} \times \mathbf{C}$ has infinite degree. Both these two examples are built around the two-element implication algebra $\mathbf{I} = \langle\{0,1\};\to\rangle$; see Figure 1.

Our first example is the three-element groupoid $\mathbf{A} = \langle\{0,1,2\};*\rangle$ defined in Figure 4. We start by characterising the term functions of $\mathbf{A}$.

Lemma 6.1. *Define the groupoid* $\mathbf{A} = \langle\{0,1,2\};*\rangle$ *as in Figure 4. Let* $n \in \mathbb{N}$ *and let* $f\colon A^n \to A$. *Then* f *is a term function of* $\mathbf{A}$ *if and only if*

(1) f *preserves the congruence* $\{0\,|\,1,2\}$ *of* $\mathbf{A}$, *and*
(2) *there is* $j \in \underline{n}$ *such that, for all* $\vec{a} \in A^n$, *we have* $f(\vec{a}) = 2 \iff a_j = 2$.

Proof. First assume that f is a term function of $\mathbf{A}$. It is easy to see that the equivalence relation $\theta := \{0\,|\,1,2\}$ on A is indeed a congruence of $\mathbf{A}$. So the term function f must preserve θ. Let $t(x_1,\dots,x_n)$ be a term corresponding to f, and let x_j be the final variable of t. Then $f(\vec{a}) = t^{\mathbf{A}}(\vec{a}) = 2 \iff a_j = 2$, for all $\vec{a} \in A^n$. So f satisfies conditions (1) and (2).

Now assume that $f\colon A^n \to A$ satisfies (1) and (2). The two-element implication algebra $\mathbf{I} = \langle\{0,1\};*\rangle$ is a subalgebra of $\mathbf{A}$. It follows straight from (2) that f preserves $I = \{0,1\}$. We shall show that the restriction $f{\restriction}_{I^n}\colon I^n \to I$ is a term function of $\mathbf{I}$, by checking that it satisfies condition (b) from Example 2.8.

Let $\vec{b} \in \{0,1\}^n$ with $b_j = 1$, where j is given by our assumption (2). Define $\vec{c} \in \{0,2\}^n$ such that $c_i = 0 \iff b_i = 0$. Then $c_j = 2$ and so $f(\vec{c}) = 2$, by (2). As $\vec{b}\ \theta\ \vec{c}$, we get $f(\vec{b}) = 1$, as required. So it follows that there is a groupoid term $t(x_1,\dots,x_n)$, with x_j as the final variable, such that $f{\restriction}_{I^n} = t^{\mathbf{I}}$. We shall finish by proving that $f = t^{\mathbf{A}}$.

Let $\vec{a} \in A^n$. Define $\vec{b} \in I^n$ such that $b_i = 0 \iff a_i = 0$. We want to show that $f(\vec{a}) = t^{\mathbf{A}}(\vec{a})$. There are three cases to consider.

Case: $a_j = 0$. We have $f(\vec{a})\ \theta\ f(\vec{b}) = t^{\mathbf{I}}(\vec{b})\ \theta\ t^{\mathbf{A}}(\vec{a})$. Since $f(\vec{a}) \neq 2$ (by (2)) and $t^{\mathbf{A}}(\vec{a}) \neq 2$ (as x_j is the final variable in t), it follows that $f(\vec{a}) = t^{\mathbf{A}}(\vec{a})$.

Case: $a_j = 1$. We have $f(\vec{a})\ \theta\ f(\vec{b}) = t^{\mathbf{I}}(\vec{b}) = 1$ and $t^{\mathbf{A}}(\vec{a}) = 1$, as x_j is the final variable in t. Since $f(\vec{a}) \neq 2$, by (2), this gives $f(\vec{a}) = 1 = t^{\mathbf{A}}(\vec{a})$.

Case: $a_j = 2$. We have $f(\vec{a}) = 2$, by (2), and $t^{\mathbf{A}}(\vec{a}) = 2$, as x_j is last. □

Example 6.2. *The groupoid* $\mathbf{A} = \langle \{0,1,2\}; *\rangle$*, defined in Figure 4, has relational degree* 3*. But* $\mathbf{A}$ *has the infinite-degree algebra* $\mathbf{I}$ *as both a retract and a subdirect factor.*

Proof. Define the congruence $\theta := \{0 \,|\, 1,2\}$ on $\mathbf{A}$, and define the endomorphism $e\colon \mathbf{A} \to \mathbf{A}$ by $e(0) = e(1) = 1$ and $e(2) = 2$. The two-element set $\mathbf{S} = \langle \{1,2\}; *\rangle$ is a subalgebra of $\mathbf{A}$, where $*$ is the 2nd projection on $S = \{1,2\}$. The clone of $\mathbf{S}$ is determined by a single ternary relation s on S. (For example, we can take s to be the graph of the Sheffer stroke.) We shall prove that $\mathrm{Clo}(\mathbf{A})$ is determined by the set of relations $R := \{s, \theta, \mathrm{graph}(e)\}$.

Let $n \in \mathbb{N}$ and assume that $f\colon A^n \to A$ preserves the relations in R. We shall show that f is a term function of $\mathbf{A}$ by checking conditions (1) and (2) from Lemma 6.1. Condition (1) is clear, so we turn straight to condition (2).

First enumerate the elements of A^n as $\vec{b}_1, \dots, \vec{b}_k$, where $k := 3^n$. Define the k-ary relation r on A by

$$r := A^k \backslash \{\, \vec{a} \in A^k \mid (\forall i \in \underline{k})\ a_i = 2 \iff f(\vec{b}_i) = 2 \,\}.$$

Note that $r_S := r \cap \{1,2\}^k$ is a compatible relation on $\mathbf{S}$. As the ternary relation s determines $\mathrm{Clo}(\mathbf{S})$, it follows that r_S is primitive-positive definable from s (see [20], for example). We have

$$r = \{\, \vec{a} \in A^k \mid e(\vec{a}) \in r_S \,\},$$

whence r is primitive-positive definable from the relations in R. It now follows that r is a compatible relation on $\mathbf{A}$ and that the operation f preserves r.

We can now show that condition (2) of Lemma 6.1 holds. By construction, we have $(f(\vec{b}_1), \dots, f(\vec{b}_k)) \notin r$. Since f preserves r, this implies that $(\vec{b}_1, \dots, \vec{b}_k) \notin r$. There is some $j \in \underline{n}$ such that $(\vec{b}_1(j), \dots, \vec{b}_k(j)) \notin r$. So, for all $\vec{a} \in A^n$, we have $f(\vec{a}) = 2 \iff a_j = 2$. Thus condition 6.1(2) holds. Hence f is a term function of $\mathbf{A}$, as required.

We can show that $\mathbf{A}$ fails to have relational degree at most 2 using Lemma 2.1. Define $f\colon A^3 \to A$ by $f(a,b,c) := \mathrm{maj}(e(a), e(b), e(c))$. Note that e is a term function of $\mathbf{A}$. So f agrees with a term function of $\mathbf{A}$ on all 2-element subsets of A^3. But f is not a term function of $\mathbf{A}$, as it fails condition 6.1(2). ☐

We close with an example of a pair of algebras of finite degree (in fact of relational degree 0) whose product is not of finite degree. We create this example by enriching the implication algebra $\mathbf{I} = \langle \{0,1\}; \to\rangle$ in two different ways by adding nullary operations. For $a, b \in I$, define $\mathbf{I}_{a,b} := \langle \{0,1\}; \to, a, b\rangle$.

Example 6.3. *Both* $\mathbf{I}_{0,1}$ *and* $\mathbf{I}_{1,0}$ *have finite degree, but* $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ *does not.*

Proof. First note that both $\mathbf{I}_{0,1}$ and $\mathbf{I}_{1,0}$ are term equivalent to the two-element Boolean algebra, and hence have finite degree.

We will use Lemma 2.6 and Theorem 2.9 to check that $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ does not have finite degree. Fix $n > 4$. There is an $(n, n-1)$-scheme for $\mathrm{V}(\mathbf{I})$ that determines an n-ary near-unanimity operation on I; see Example 2.8. Hence this is also true for $\mathbf{I} \times \mathbf{I}$, which is a reduct of $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$. So it suffices to show that $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ does not have an n-ary near-unanimity term function.

We define an n-ary relation $r := (I \times I)^n \backslash \{z\}$, where the n-tuple z has every coordinate equal to $(0,0)$. It is straightforward to check that r forms a subalgebra of $(\mathbf{I}_{0,1} \times \mathbf{I}_{1,0})^n$, but that r is not closed under any n-ary near-unanimity operation on $I \times I$. ☐

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