Algebra univers. 45 (2001) 149 – 153
0002–5240/01/030149 – 05 $ 1.50 + 0.20/0

Algebra Universalis

Dualisability of p-semilattices

B. A. DAVEY AND J. G. PITKETHLY

Abstract. We give a ghost element proof that every finite non-boolean p-semilattice is inherently non-dualisable.

A bounded meet semilattice $\mathbf{A} = \langle A; \wedge, {}^*, 0, 1\rangle$ with an extra unary operation * is called a **pseudocomplemented semilattice**, or **p-semilattice**, if

$$a^* = \max\{b \in A \mid a \wedge b = 0\}$$

for all $a \in A$. Each p-semilattice $\mathbf{A}$ satisfies $x^* \wedge x \approx 0$ and $x^{**} \wedge x \approx x$ as well as $0^* \approx 1$ and $1^* \approx 0$. We say that a p-semilattice is **boolean** if it is term equivalent to a Boolean algebra, or equivalently, if it satisfies the equation $x \approx x^{**}$. The class of all p-semilattices forms a variety. G. T. Jones [2] showed that a p-semilattice $\mathbf{A}$ is subdirectly irreducible if and only if $A\backslash\{1\}$ forms a boolean lattice under the induced order. For each $n \in \mathbb{N}$, let $\underline{\mathbf{S}}_n$ denote the subdirectly irreducible p-semilattice obtained by adding a new top element to the boolean lattice with n atoms.

We will characterise the dualisable finite p-semilattices. For a comprehensive introduction to natural dualities, dualisability and non-dualisability, we refer the reader to D. M. Clark and B. A. Davey [1]. Every finite boolean p-semilattice is (strongly) dualisable since, by definition, it is term equivalent to a finite Boolean algebra. D. M. Clark has shown (see Theorem 10.6.8 in [1]) that the subdirectly irreducible p-semilattice $\underline{\mathbf{S}}_n$ is non-dualisable, for each $n \in \mathbb{N}$. His proof, using the term-closed subset method, relies on the Third Duality Theorem (see Theorem 3.1.6 in [1]).

Let $\underline{\mathbf{M}}$ be a finite algebra. To show that $\underline{\mathbf{M}}$ is non-dualisable, via the term-closed subset method, we must find a term-closed subset X of an infinite power of M and a brute-force morphism $\alpha : X \to M$ which does not extend to a term function of $\underline{\mathbf{M}}$. The term-closed subset method is not very easy to apply. Indeed, we do not know of any non-dualisability

---

Presented by Professor Lászlo Zádori.

Received October 29, 1998; accepted in final form November 20, 1998.

2000 *Mathematics Subject Classification*: Primary: 08A35, 08A40; Secondary: 08C05.

Key words and phrases: Pseudocomplemented semilattices, p-semilattices, natural duality, inherently non-dualisable, ghost element method.

This research was carried out when both authors were in Oxford, England and they would like to thank Oxford University for its generous support.

proofs, other than Clark's, which use this method. A more popular way to prove that an algebra is non-dualisable is to use the ghost element method. This method is adapted straight from the definition of the dualisability of an algebra, and is often easy to apply. In addition, some ghost element proofs can be extended directly to a proof of a much stronger condition than non-dualisability. We say that an algebra $\underline{\mathbf{M}}$ is **inherently non-dualisable** if each finite algebra $\underline{\mathbf{N}}$, such that $\underline{\mathbf{M}} \in \mathbb{ISP}\,\underline{\mathbf{N}}$, is non-dualisable. The following theorem comes from [3]: see also Theorem 10.5.5 in [1]. Recall that if $\mathbf{A}$ is a subalgebra of $\underline{\mathbf{M}}^S and\ s \in S$, then $\rho_s := \pi_s{\restriction}_{\mathbf{A}} : \mathbf{A} \to \underline{\mathbf{M}}$ denotes the natural projection homomorphism.

THEOREM 1. *Assume that* $\underline{\mathbf{M}}$ *is a finite algebra, that* $\mathbf{A}$ *is a subalgebra of* $\underline{\mathbf{M}}^S$ *and that* B *is an infinite subset of* A *such that*

(i) *there is a function* $u : \mathbb{N} \to \mathbb{N}$ *such that if* θ *is a congruence on* $\mathbf{A}$ *of finite index at most* n, *then* $\theta{\restriction}_B$ *has only one class with more than* $u(n)$ *elements,*

(ii) $g \notin A$ *where* g *is the element of* M^S *such that* $g(s) := \rho_s(b)$, *for each* $s \in S$, *with* b *any element of the block of* $\ker(\rho_s){\restriction}_B$ *which has size greater than* $u(|M|)$.

Then $\underline{\mathbf{M}}$ *is inherently non-dualisable.*

This theorem enables us to extend Clark's result using a more elementary proof.

THEOREM 2. *Every finite non-boolean p-semilattice is inherently non-dualisable.*

Proof. Let $\underline{\mathbf{M}} = \langle M; \wedge, {}^*, 0, 1\rangle$ be a finite non-boolean p-semilattice. There is some $a \in M$ such that $a \neq a^{**}$. For each $n \in \mathbb{N}$, define $b_n \in M^{\mathbb{N}}$ by

$$b_n(i) = \begin{cases} a & \text{if } i = 1, \\ 0 & \text{if } i = n+1, \\ 1 & \text{otherwise,} \end{cases}$$

and define $g \in M^{\mathbb{N}}$ by

$$g(i) = \begin{cases} a & \text{if } i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathbf{A}$ denote the subalgebra of $\underline{\mathbf{M}}^{\mathbb{N}}$ generated by $B := \{b_n \mid n \in \mathbb{N}\}$. We will show that if θ is a congruence on $\mathbf{A}$, then $\theta{\restriction}_B$ has at most one nontrivial block.

Let $\theta \in \operatorname{Con}\mathbf{A}$. Suppose that $m, n, k, l \in \mathbb{N}$ with $m \neq n$ and $k \neq l$ such that $b_m \equiv b_n \quad (\theta)$ and $b_k \equiv b_l \quad (\theta)$. We have

$$b_n^* = b_n^* \wedge b_n^* \equiv b_n^* \wedge b_m^* = g^* \quad (\theta)$$

and, similarly, $b_k^* \equiv g^* \quad (\theta)$. This gives us $b_n = b_n^{**} \wedge b_n \equiv b_k^{**} \wedge b_n \quad (\theta)$. By symmetry, we have $b_k \equiv b_n^{**} \wedge b_k \quad (\theta)$. Thus, since $b_k^{**} \wedge b_n = b_n^{**} \wedge b_k$, we have $b_m \equiv b_n \equiv b_k \equiv b_l \quad (\theta)$.

Since $\{x \in \{0, 1, a, a^*, a^{**}\}^{\mathbb{N}} \mid x(1) \neq a \text{ or } x^{-1}(0) \neq \varnothing\}$ is closed under $\wedge$ and *, it follows that $g \notin A$. The only block of $\ker(\rho_1)\restriction_B$ is B, and we have $g(1) = a = \rho_1(b_1)$. Now let $n \in \mathbb{N}\backslash\{1\}$. The only nontrivial block of $\ker(\rho_n)\restriction_B$ is $B\backslash\{b_{n-1}\}$, and $g(n) = 1 = \rho_n(b_n)$.

It now follows by Theorem 1 that $\underline{\mathbf{M}}$ is inherently non-dualisable. ☐

The proof of Theorem 2 does not only apply to p-semilattices. A careful reading of the proof of Theorem 2 shows that we can use it to establish condition (i) of Theorem 1 provided $\underline{\mathbf{M}}$ is a finite algebra of type $\langle 2, 1\rangle$ containing elements a, 0, 1 which satisfy the equations given below:

$$\begin{aligned} x^* \wedge x^* &= x^* \text{ and } x^{**} \wedge x = x \text{ for all } x \in \{a, 0, 1\}, \\ 0^* \wedge 1^* &= 1^* = 1^* \wedge 0^* \text{ and } 0^{**} \wedge 1 = 1^{**} \wedge 0. \end{aligned} \tag{$*$}$$

To prove that $\underline{\mathbf{M}}$ is inherently non-dualisable, we also need to show that condition (ii) holds. We now give some simple conditions on $\underline{\mathbf{M}}$ which guarantee this.

THEOREM 3. *Let* $\underline{\mathbf{M}} = \langle M; \wedge, {}^*\rangle$ *be a finite algebra such that* $\langle M; \wedge\rangle$ *is a semilattice and* * *is a unary operation. Assume that* $a, 0, 1 \in M$ *with* $0 \neq 1$ *and that* $\underline{\mathbf{M}}$ *satisfies*

($\mathrm{C_i}$) $a \leqslant a^{**}$, $0 \leqslant 0^{**}$, $1 \leqslant 1^{**}$, $1^* \leqslant 0^*$, $0^{**} \wedge 1 = 1^{**} \wedge 0$, *and*

($\mathrm{C_{ii}}$) 0 *is the least element of* $\underline{\mathbf{M}}$, a *is meet-irreducible in* $\mathrm{sg}(\{a\})$ *and* $a \neq b^*$ *for all* $b \in \mathrm{sg}(\{a\})$.

Then $\underline{\mathbf{M}}$ *is inherently non-dualisable.*

Proof. Since $\wedge$ is a semilattice operation on M, the equations in $(*)$ above reduce to the relations given in ($\mathrm{C_i}$). Hence, the proof given in Theorem 2 shows that condition (i) of Theorem 1 holds. We shall now prove that ($\mathrm{C_{ii}}$) implies that condition (ii) of Theorem 1 also holds.

First, we show that if $t(x_1, \dots, x_n)$ is an n-ary term and $\underline{\mathbf{M}}$ satisfies the relation $t(a, \dots, a) = a$, then, up to commutativity and associativity of $\wedge$, we have $t(x_1, \dots, x_n) = x_i \wedge s(x_1, \dots, x_n)$ for some i and some (possibly 'empty') n-ary term $s(x_1, \dots, x_n)$. The proof is a simple induction on the complexity of t. Assume that $t(a, \dots, a) = a$. If t is a variable, there is nothing to do. Otherwise,

(a) $t(x_1, \dots, x_n) = t_1(x_1, \dots, x_n)^*$, or

(b) $t(x_1, \dots, x_n) = t_1(x_1, \dots, x_n) \wedge t_2(x_1, \dots, x_n)$.

Since $t(a, \dots, a) = a$ and $a \neq b^*$ for all $b \in \mathrm{sg}(\{a\})$, case (a) does not occur. Thus case (b) applies and hence, since a is meet-irreducible in $\mathrm{sg}(\{a\})$, we have, without loss of generality,

$t_1(a, \dots, a) = a$. By our inductive hypothesis, we conclude that $t_1(x_1, \dots, x_n) = x_i \wedge s_1(x_1, \dots, x_n)$ for some i and some (possibly 'empty') n-ary term $s_1(x_1, \dots, x_n)$, whence

$$t(x_1, \dots, x_n) = x_i \wedge s_1(x_1, \dots, x_n) \wedge t_2(x_1, \dots, x_n) = x_i \wedge s(x_1, \dots, x_n),$$

as required.

Now suppose that $g \in A$. Then $g = t(b_1, \dots, b_n)$ for some $n \in \mathbb{N}$ and n-ary term t. We have $a = g(1) = t(b_1(1), \dots, b_n(1)) = t(a, \dots, a)$. Hence, we must have $t(x_1, \dots, x_n) = x_i \wedge s(x_1, \dots, x_n)$. But this implies that

$$g(i+1) = t(b_1, \dots, b_n)(i+1) = b_i(i+1) \wedge s(b_1, \dots, b_n)(i+1) = 0,$$

since 0 is the least element of $\underline{\mathbf{M}}$, which is a contradiction. Thus condition (ii) of Theorem 1 holds. ☐

The semilattice-based algebras given in Figure 1 satisfy conditions ($\mathrm{C_i}$) and ($\mathrm{C_{ii}}$), and are therefore inherently non-dualisable.

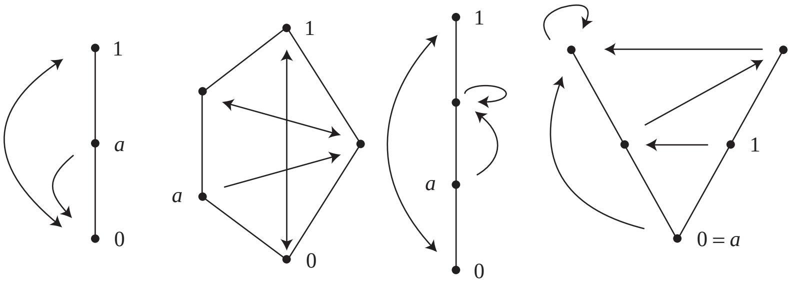

Figure 1 Some inherently non-dualisable semilattice-based algebras

In [4], a weaker notion of dualisability and correspondingly stronger notion of non-dualisability is introduced. For an infinite cardinal, κ, a finite algebra $\underline{\mathbf{M}}$ is **κ-dualisable** if some set R of algebraic relations of arity less than κ dualises $\underline{\mathbf{M}}$. Thus dualisability equals ω-dualisability. The algebra $\underline{\mathbf{M}}$ is **inherently non-κ-dualisable** if a finite algebra $\underline{\mathbf{N}}$ is non-κ-dualisable whenever $\underline{\mathbf{M}} \in \mathbb{ISP}\,\underline{\mathbf{N}}$. In [3], it is shown that Theorem 1 lifts up to κ: if we are able to choose the set B of cardinality κ, then $\underline{\mathbf{M}}$ is inherently non-κ-dualisable.

There is a simple κ-modification of the construction in the proof of Theorem 2: just use κ-sequences rather than ω-sequences in the definitions of b_n, for $n < \kappa$, and in the definition of g. Thus, we have in fact proved that every finite non-boolean p-semilattice and also each semilattice-based algebra satisfying conditions ($\mathrm{C_i}$) and ($\mathrm{C_{ii}}$) is inherently non-κ-dualisable for every infinite cardinal κ.

REFERENCES

[1] CLARK, D. M. and DAVEY, B. A., *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, 1998.

[2] JONES, G. T., *Pseudo Complemented Semi-Lattices*, PhD thesis, UCLA, 1972.

[3] DAVEY, B. A., IDZIAK, P. M., LAMPE, W. A. and MCNULTY, G. F., *Dualizability and graph algebras*, Discrete Math., *214* (2000), 145–172.

[4] CLARK, D. M., IDZIAK, P. M., SABOURIN, L., SZABÓ, CS. and WILLARD, R., *Natural dualities for quasi-varieties generated by a finite commutative ring*, Algebra Universalis (to appear).

B. A. Davey, J. G. Pitkethly
La Trobe University
Bundoora
Victoria 3083
Australia
e-mail: B.Davey@latrobe.edu.au
e-mail: J.Pitkethly@latrobe.edu.au