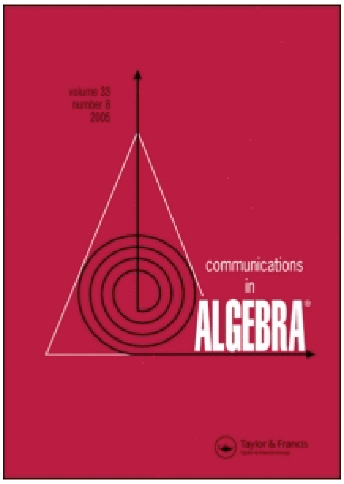

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THE DIVISION RELATION: CONGRUENCE CONDITIONS AND AXIOMATISABILITY

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We examine a universal algebraic abstraction of the semigroup theoretic concept of "divides:" a divides b in an algebra* $\mathbf{A}$ *if for some* $n \in \omega$*, there is a term* $t(x, y_1, \ldots, y_n)$ *involving all of the listed variables, and elements* $c_1, \ldots, c_n$ *such that* $t^{\mathbf{A}}(a, c_1, \ldots, c_n) = b$*. The first order definability of this relation is shown to be a very broad generalisation of some familiar congruence properties, such as definability of principal congruences. The algorithmic problem of deciding when a finitely generated variety has this relation definable is shown to be equivalent to an open problem concerning flat algebras. We also use the relation as a framework for establishing some results concerning the finite axiomatisability of finitely generated varieties.

Key Words: Definable principal congruences; Finite axiomatisability; Flat algebras; Semigroups and universal algebra.

2000 Mathematics Subject Classification: 08A30; 08C15; 20M07.

1. INTRODUCTION

Let $\mathbf{S}$ be a semigroup and $\mathbf{S}^1$ the result of adjoining an identity element to $\mathbf{S}$ if it does not already have one ($\mathbf{S}^1 = \mathbf{S}$ otherwise). The binary relation $\mid$ on $\mathbf{S}$ is defined by $a \mid b$ ("a divides b") if there exists $c, d \in S^1$ such that $cad = b$. The concept is fundamental in semigroup theory, where it is very closely associated with Green's $\mathscr{J}$-relation and the ideals of $\mathbf{S}$; see any semigroup theory text (Howie [16] for example). In this article we examine an obvious universal algebraic generalisation of the relation "divides". A precise definition is given below. While this concept is naturally motivated from ideas in semigroup theory, it also emerges when investigating the definable principal congruence (DPC) property on certain varieties. Indeed we show (Corollary 9.7) that the algorithmic question of deciding when the division relation is first order definable in the universal Horn class of an arbitrary finite *partial algebra* is polynomially equivalent to the algorithmic problem of deciding the definability of the relation on total algebras (that is, non-partial algebras) and to deciding which varieties generated by a flat algebra with absorbing bottom (in the sense of the M-algebras of Willard [35] or the flat graph algebras of, say, Delić [12]) have DPC. The problem of characterising the DPC property for

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flat algebras with absorbing bottom was proposed by Delić in Special Session on Semigroups, Algorithms, and Universal Algebra in Louisville, 1998.

The generalised division relation also provides a new framework for proving some results relating the finite axiomatisability of varieties to the axiomatisability properties of classes of subdirectly irreducible algebras. We prove (Theorem 8.3) that under the assumption of a natural finiteness condition on the definition of certain congruences, the finite axiomatisability of a variety in which the division relation is a *partial order* is equivalent to the finite axiomatisability of the subdirectly irreducible algebras in the variety. Finite semigroups in which division is a partial order correspond to members of the pseudovariety of $\mathcal{J}$-trivial semigroups, which have received substantial attention in the semigroup theory literature. We also make some easy observations demonstrating that the same result holds for the "complementary" class of finite semigroups in which the division relation is the universal relation (Theorem 8.9), as well as for locally finite commutative semigroup varieties, and varieties of nil-semigroups (Theorem 8.7). In particular, Theorem 8.9 is an extension of recent work by McNulty and Wang [25] who showed the same for the variety generated by a finite group (our observation depends heavily on theirs).

We begin with some preliminary sections that give precise definitions to the concepts alluded to above. It turns out that the study of certain properties on algebras is intimately related to the study of the division relation on partial algebras. For this reason it is pertinent to present some of our results in the language of partial algebras. All of these concepts will particularise to the subclass of (total) algebras.

We then give some discussion of congruence properties on (total) algebras (Section 3). This section is intended to provide necessary background and notation; however, it contains a number of relevant facts that have been overlooked in the literature, so the section contains some new, if elementary results.

2. PRELIMINARIES: ALGEBRAS, PARTIAL ALGEBRAS AND BASIC NOTATION

In this section we establish some notation and basic facts concerning partial and total algebras. The detour into partial algebras is necessary for at least one of the total algebraic applications of this article (Section 9).

The definition of a partial algebra $\mathbf{A} = \langle A; F\rangle$ is the same as for an algebra, except that each fundamental operation $f^{\mathbf{A}} \in F$ (of arity n, say) has a domain equal to some subset of A^n. Throughout the article, we use the word *partial algebra* to indicate an algebraic structure whose operations may be partially defined, and *algebra* to denote a total algebra.

Substructures of partial algebras are defined as for algebras, as are direct products (operations defined pointwise). A *homomorphism* between partial algebras $\phi : \mathbf{A} \to \mathbf{B}$ is a map preserving atomic formulæ; observe however that if $\mathbf{A}$ is not a total algebra, the image of $\mathbf{A}$ under ϕ does not in general have to be a subalgebra of $\mathbf{B}$. Ultraproducts and reduced products are defined in the essentially the same way as the conventional version; see Burmeister [5] for details (less detailed accounts in notation similar to that used here can be found in the introductory sections of Jackson [19] and in §1.4 of Clark and Davey [7]). We also use the convention on satisfaction of universal Horn sentences given in these references: a variable assignment θ into a partial algebra $\mathbf{A}$ gives an atomic expression $u \approx v$ the value true if $\theta(u)$ and $\theta(v)$ are both defined and equal. This notion of truth then extends to define satisfaction of identities and universal Horn sentences in the usual way. In both Burmeister [5] and Jackson [19] the empty algebra is allowed as a partial algebra in any type. In this article it is more convenient to take the opposite approach and exclude the empty algebra, although the alternative choice makes little difference.

We use the symbols I, S, P, P^+, P_u, and P_r to denote class operators on algebras or partial algebras corresponding to the taking of isomorphic copies, substructures, direct products, nonempty direct products, ultraproducts, and reduced products, respectively. We use H for the class operator corresponding to homomorphic images, but only apply it to total algebras. The symbol V abbreviates the composite HSP, and is well known to produce the variety (or equational class) generated by a class of algebras. We also frequently encounter *universal Horn classes* and *quasivarieties*, which are classes closed under $\mathsf{ISP}^+\mathsf{P}_\mathsf{u} = \mathsf{ISP}_\mathsf{r}$ and ISPP_u, respectively. We frequently use the following lemma without mention. The easy proof is omitted (see 1.4.4 of Clark and Davey [7] for something almost the same, however).

Lemma 2.1. *Let $\mathbf{A}$ be a partial algebra and $\mathscr{K}$ be a class of partial algebras, all of the same type. The following are equivalent*:

i) $\mathbf{A} \in \mathsf{ISP}^+(\mathscr{K})$;
ii) – *for each $a, b \in A$ with $a \neq b$, there is some $\mathbf{B} \in \mathscr{K}$ and a homomorphism ϕ : $\mathbf{A} \to \mathbf{B}$ with $\phi(a) \neq \phi(b)$ and*
   – *whenever $\vec{a}$ in A has $\vec{a} \notin \mathrm{dom}(f^{\mathbf{A}})$ for some fundamental operation f, then there is some $\mathbf{C} \in \mathscr{K}$ and a homomorphism $\phi : \mathbf{A} \to \mathbf{C}$ with $\phi(\vec{a}) \notin \mathrm{dom}(f^{\mathbf{C}})$.*

The definition of relatively free algebras for a class of partial algebras is reasonably clear; however, it is not easily found in the literature. The concept is used later in the article, so we sketch some details here.

Fix a class $\mathscr{K}$ of partial algebras of some type $\mathscr{F}$, and let $T(X)$ be the $\mathscr{F}$-term algebra (which is a total algebra) in the signature of $\mathscr{K}$. Now let $R(X)$ denote the subset of $T(X)$ consisting of all terms that give rise to total term functions on all members of $\mathscr{K}$. Define an equivalence $\sim$ on $R(X)$ by $s \sim t$, if $\mathscr{K} \models s \approx t$, and define $\mathbf{F}_\mathscr{K}(X)$, the *partial algebra of total term functions* for $\mathscr{K}$, on the set quotient $R(X)/{\sim}$ by defining the operations $f \in \mathscr{F}$ by $(s_1/{\sim}, \ldots, s_n/{\sim}) \in \mathrm{dom}(f^{\mathbf{F}_\mathscr{K}(X)})$ if and only if $f(s_1, \ldots, s_2) \in R(X)$, in which case

$$f^{\mathbf{F}_\mathscr{K}(X)}(s_1/{\sim}, \ldots, s_n/{\sim}) := f(s_1, \ldots, s_n)/{\sim}.$$

It is routinely verified that $\mathbf{F}_\mathscr{K}(X)$ is well defined, and that when $\mathscr{K}$ consists of total algebras, $\mathbf{F}_\mathscr{K}(X)$ is nothing other than the usual X-generated relatively free algebra for $\mathscr{K}$. More generally, even for partial algebras it remains true that $\mathbf{F}_\mathscr{K}(X)$ satisfies the universal mapping property and hence is the natural candidate for the notion of a relatively free algebra for $\mathscr{K}$ (this observation is made in Davey [9] for example). If $\mathbf{A}$ is a partial algebra, we use the notation $\mathbf{F}_\mathbf{A}(X)$ in place of $\mathbf{F}_{\{\mathbf{A}\}}(X)$.

The following facts are necessary later in the article. They correspond to basic universal algebraic facts; however, the proofs are slightly different at some points.

Proposition 2.2. *Let $\mathcal{K}$ be a class of partial algebras of similarity type $\mathcal{F}$.*

(1) $\mathbf{F}_{\mathcal{K}}(X) \in \mathsf{ISP}^{+}(\mathcal{K})$.
(2) *If $\mathcal{K} = \{\mathbf{A}\}$ where $\mathbf{A}$ is finite, then if X and $\mathcal{F}$ are finite,*

   (a) $\mathbf{F}_{\mathcal{K}}(X) \in \mathsf{S}(\mathbf{A}^{|A|^{|X|}})$,
   (b) $\mathbf{F}_{\mathcal{K}}(X)$ *can be effectively constructed from* $\mathbf{A}$.

Proof. (Sketch.) (1) (Using Lemma 2.1.) Say that $s, t \in R(X)$ have $\mathcal{K} \not\models s \approx t$. Let $\theta : X \to \mathbf{A} \in \mathcal{K}$ be a failing assignment for $s \approx t$. By the universal mapping property, θ extends to a homomorphism from $\mathbf{F}_{\mathcal{K}}(X)$ to $\mathbf{A}$ that separates $s/{\sim}$ and $t/{\sim}$. Now consider some tuple $(s_1/{\sim}, \dots, s_n/{\sim})$ not contained in the domain of some fundamental operation $f^{\mathbf{F}_{\mathcal{K}}(X)}$ on $\mathbf{F}_{\mathcal{K}}(X)$. Then the term $f(s_1, \dots, s_n)$ is not a total term function on some $\mathbf{A} \in \mathcal{K}$. That is, there is an assignment $\theta : X \to A$ making $f(s_1, \dots, s_n)$ undefined (but $\theta(s_1), \dots, \theta(s_n)$ are defined, since $s_1, \dots, s_n \in R(X)$). Using the universal mapping property again, we find that θ becomes a homomorphism mapping $(s_1/{\sim}, \dots, s_n/{\sim})$ outside of the domain of $f^{\mathbf{A}}$.

(2a) This is the same as the total algebra version: the property $\mathbf{F}_{\mathbf{A}}(X) \in \mathsf{ISP}(\mathbf{A})$ is equivalent to $\mathbf{F}_{\mathbf{A}}(X) \in \mathsf{IS}(\mathbf{A}^{L})$, where L is the homset $\hom(\mathbf{F}_{\mathbf{A}}(X), \mathbf{A})$, but as each homomorphism from $\mathbf{F}_{\mathbf{A}}(X)$ is determined by its action on the generators X, we have $|L| \leq |A|^{|X|}$.

(2b) For each $n \in \omega$, let $T_n(X)$ be the subset of $T(X)$ consisting of all terms whose height as a term tree is at most n. It is not hard to calculate an upper bound for the size of $|T_n(X)|$ (it depends on $\mathcal{F}$, however). Choose $n := |A|^{|A|^{|X|}}$, and let $R_n(X)$ denote the set of terms $t \in T_n(X)$ for which $\mathbf{A} \models t \approx t$. Clearly, $R_n(X)$ can be effectively constructed and consists of all members of $R(X)$ that are of height at most n. Let $\mathbf{G}_{\mathcal{K}}(X)$ be defined on $R_n(X)/{\sim}$ by $(s_1/{\sim}, \dots, s_n/{\sim}) \in \operatorname{dom}(f^{\mathbf{G}_{\mathcal{K}}(X)})$ if and only if there is $t \in R_n(X)$ with $\mathbf{A} \models f(s_1, \dots, s_2) \approx t$, in which case $f^{\mathbf{G}_{\mathcal{K}}(X)}(s_1/{\sim}, \dots, s_n/{\sim}) := t/{\sim}$. Again, $\mathbf{G}_{\mathcal{K}}(X)$ can be effectively constructed from $\mathbf{A}$ and X.

We claim that $R_n(X)$ contains a representative from each $\sim$-class of $R(X)$, from which it easily follows that $\mathbf{G}_{\mathcal{K}}(X) \cong \mathbf{F}_{\mathcal{K}}(X)$, as required. To prove this claim, observe that if there is a number m such that each height m term s in $R(X)$ is $\sim$-equivalent to some term t (depending on s) of height less than m, then an elementary induction argument shows that all terms in $R(X)$ of height greater than or equal to m are $\sim$-equivalent to a term in $R_{m-1}(X)$. Now, since there are at most $|A|^{|A|^{|X|}}$ distinct $\sim$-classes, it follows that such an m exists, and moreover, that $m \leq |A|^{|A|^{|X|}}$. Hence every term in $R(X)$ is $\sim$-equivalent to a member of $R_n(X)$, as required. □

Note that when $\mathbf{A}$ is a total algebra, $\mathbf{F}_{\mathbf{A}}(X)$ can be identified with the largest $|X|$-generated subalgebra of $\mathbf{A}^{|A|^{|X|}}$, but this fails for partial algebras: for example, there may be no nontrivial total term functions on $\mathbf{A}$, in which case $\mathbf{F}_{\mathbf{A}}(X)$ is the partial algebra on X with all operations completely undefined.

2.1. Flat Algebras

A *flat algebra* is a total algebra $\mathbf{F}$ whose operations include a semilattice $\wedge$ making $\mathbf{F}$ of height 1. We say that a flat algebra is a *sink algebra* (Jackson [19]) if the bottom element—say 0—with respect to the usual $\leq$ order of $\wedge$ is absorbing with respect to all operations (sink algebras are called *M-algebras* in Willard [35]). As demonstrated in Willard [34], and strengthened in Jackson [19], equational properties of sink algebras are intimately related to universal Horn properties of partial algebras, and this is the central reason that we consider partial algebras in some sections of this article.

Any partial algebra $\mathbf{A}$ can be extended to a sink algebra $\flat(\mathbf{A})$ in the following trivial way: the universe of $\flat(\mathbf{A})$ is the disjoint union $A \dot{\cup} \{0\}$; each (partial) operation g of $\mathbf{A}$ is extended to a total operation on $A \cup \{0\}$ by letting all undefined values take the value 0; introduce a new meet semilattice operation $\wedge$ making 0 the bottom element of a height 1 semilattice. It is also obvious that any sink algebra $\mathbf{B}$ is of the form $\flat(\mathbf{A})$ for some partial algebra $\mathbf{A}$: namely, remove 0 and $\wedge$ and restrict all operations in domain and codomain to appropriate powers of $B \backslash \{0\}$.

2.2. A Definability Lemma

The following model theoretic folklore fact underlies many definability proofs and is used frequently in this article.

Definability Lemma 2.3. *Consider an infinite set* $\{\Phi_i(x_1, \dots, x_k) \mid i \in I\}$ *of first order formulæ in free variables* $x_1, \dots, x_k$ *and in some similarity type* $\mathscr{F}$ *(possibly including partial operation symbols). Let D denote the k-ary relation defined on* $\mathscr{F}$ *structures by the solution set of the infinite disjunction* $\bigvee_{i \in I} \Phi_i(x_1, \dots, x_k)$.

The following are equivalent for a class $\mathscr{K}$ *of* $\mathscr{F}$*-structures closed under ultraproducts*:

(1) *The relation D is first-order definable in* $\mathscr{K}$;
(2) *There is a finite subset* $J \subseteq I$ *such that the relation D is first-order definable in* $\mathscr{K}$ *by the formula* $\bigvee_{j \in J} \Phi_j(x_1, \dots, x_k)$.

A dual statement to (2) *holds if D is instead defined by the infinite conjunction* $\&_{i \in I} \Phi_i(\vec{x})$.

Proof. (Sketch of $\neg(2) \Rightarrow \neg(1)$.) Add k new constants $a_1, \dots, a_k$ to the type. For each finite nonempty $J \subseteq I$, we can find a member $\mathbf{M}_J \in \mathscr{K}$ and interpret the a_i in M_J such that $(a_1, \dots, a_k) \in D^{\mathbf{M}_J}$ and $\mathbf{M}_J \models \neg \bigvee_{j \in J} \Phi_j(a_1, \dots, a_k)$. Taking an ultraproduct of the $\mathbf{M}_J$ over the usual nonprincipal ultrafilter on the set of all nonempty finite subsets of I (see proof of V.2.12 in Burris and Sankappanavar [6] for example) produces a member $\mathbf{M} \in \mathscr{K}$ in which $\mathbf{M} \models \neg \bigvee_{i \in I} \Phi_i(a_1, \dots, a_k)$, showing that membership of $(a_1, \dots, a_k)$ in D is not closed under ultraproducts, hence D is not first order definable. □

3. PRELIMINARIES: PRINCIPAL CONGRUENCES

Here we prove some results about the relationships between various conditions describing the definability of congruences. All structures in this section are total algebras.

Fix a signature $\mathscr{L}$. We adopt the following notation for $\mathscr{L}$-terms: the expression $t(\underline{x}_1, \ldots, \underline{x}_n, y_1, \ldots, y_m)$ means that the variables appearing in the term t include each of $x_1, \ldots, x_n$ and are amongst $x_1, \ldots, x_n, y_1, \ldots, y_m$.

Let us fix some countably infinite set of variables $Z := \{x, z_0, z_1, \ldots\}$, and let T_x denote the set of all $\mathscr{L}$-terms $t(\underline{x}, \ldots)$ in the variables Z. We also inductively define a subset T_{1x} of T_x as follows: $x \in T_{1x}$; if $t(\underline{x}, z_0, \ldots, z_{n-1}) \in T_{1x}$ and f is a k-ary fundamental operation symbol in $\mathscr{L}$, then, for all $i \in \{0, 1, \ldots, k-1\}$,

$$f(z_n, \ldots, z_{n+i-1}, t(\underline{x}, z_0, \ldots, z_{n-1}), z_{n+i}, \ldots, z_{n+k-2}) \in T_{1x}.$$

The terms in T_{1x} each contain precisely one occurrence of x, and moreover, each subterm of a member of T_{1x} is either a single variable or is in T_{1x} itself. We mention that this notation is essentially the same as that of Clark et al. [8]; however, our definition of T_{1x} is slightly stricter, in an unimportant way. If t is a term in T_x or T_{1x}, we usually just write $t(x, \vec{z})$ instead of $t(\underline{x}, \vec{z})$, since our convention on the naming of variables makes it implicit that the first named variable x appears in every term of T_x and T_{1x}.

Let $\mathbf{A}$ be an algebra and $a, b \in A$. We let $\mathrm{cg}^{\mathbf{A}}(a, b)$ denote the congruence generated by the pair (a, b). A *translation* on an algebra $\mathbf{A}$ is a function $\lambda : A \to A : x \mapsto t^{\mathbf{A}}(x, a_1, \ldots, a_n)$ where $t \in T_x$ is a term and $a_1, \ldots, a_n \in A$. A *basic* translation is a translation of the form $x \mapsto f^{\mathbf{A}}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$, where f is a fundamental operation of $\mathbf{A}$ with arity n strictly more than 0.

Let $F \subseteq T_x$. Following Baker et al. [3] for example, we write $\{a, b\} \looparrowright^1_F \{c, d\}$ if $c = d$ or there is $t(x, y_1, \ldots) \in F$ and $e_1, e_2, \ldots$ in A such that $\{c, d\} = \{t^{\mathbf{A}}(a, e_1, \ldots), t^{\mathbf{A}}(b, e_1, \ldots)\}$. If F is a singleton $\{t\}$, we write $\looparrowright^1_t$ in place of $\looparrowright^1_F$, while we write $\looparrowright^1_m$ to denote $\looparrowright^1_F$ where F is the set of all terms in T_x whose height as a term tree is at most m. We write $\{a, b\} \looparrowright^n_F \{c, d\}$ if there are $c_0, c_1, \ldots, c_k$ such that $k \le n$, $c_0 = c$, $c_k = d$ and for each $i = 0, 1, \ldots, k-1$ we have $\{a, b\} \looparrowright^1_F \{c_i, c_{i+1}\}$. We also use the following abbreviations:

a) $\{a, b\} \looparrowright_F \{c, d\}$ abbreviates $(\exists n \in \mathbb{N})\{a, b\} \looparrowright^n_F \{c, d\}$;
b) $\{a, b\} \looparrowright^n \{c, d\}$ abbreviates $(\exists F \subseteq T_x)\{a, b\} \looparrowright^n_F \{c, d\}$;
c) $\{a, b\} \looparrowright \{c, d\}$ abbreviates $(\exists n \in \mathbb{N})(\exists F \subseteq T_x)\{a, b\} \looparrowright^n_F \{c, d\}$.

An important observation is that when F is finite, the property $\{u, v\} \looparrowright^n_F \{x, y\}$ is expressible as a first order formula with free variables u, v, x, y. Such a formula is called a *principal congruence formula*.

The $\looparrowright$ construction can also be composed in a natural way. For example,

$$\{a, b\} \looparrowright^n_F \circ \looparrowright^m_G \{c, d\}$$

abbreviates

$$\exists e \exists f\big(\{a, b\} \looparrowright^n_F \{e, f\} \ \&\ \{e, f\} \looparrowright^m_G \{c, d\}\big).$$

Notice that $\{a, b\} \looparrowright^n_F \circ \looparrowright^m_G \{c, d\}$ implies $\{a, b\} \looparrowright^{nm}_H \{c, d\}$, where H is the subset of T_x of the form $\{s(t(x, \vec{y}), \vec{z}) \mid s \in G, t \in F\}$, which is finite if F and G are finite. In this case, $\{u, v\} \looparrowright^n_F \circ \looparrowright^m_G \{x, y\}$ can be viewed as a first order formula. We similarly have $\{a,b\} \looparrowright^n_k \circ \looparrowright^m_\ell \{c,d\}$ implies $\{a,b\} \looparrowright^{nm}_{k\ell} \{c,d\}$, which is also first order if the similarity type is finite.

The following lemma is due to Mal'cev.

Lemma 3.1. *Let* $\mathbf{A}$ *be an algebra and* $a,b,c,d \in A$. *Then* $(c,d) \in \mathrm{cg}^{\mathbf{A}}(a,b)$ *if and only if* $\{a,b\} \looparrowright \{c,d\}$.

The system of equalities corresponding to the statement $\{a,b\} \looparrowright \{c,d\}$ is often called a *Mal'cev scheme*.

Recall that a class $\mathscr{C}$ of algebras has *definable principal congruences* (DPC) if the 4-ary relation $\looparrowright$ is first order definable within $\mathscr{C}$; by the Definability Lemma 2.3 (or by a standard exercise; Exercise V.3.5 of Burris and Sankappanavar [6] for example) the DPC condition for an ultraproduct closed class $\mathscr{C}$ corresponds to the equivalence in all members of $\mathscr{C}$ of the relation $\looparrowright$ with the relation $\looparrowright^n_F$ for some fixed finite F and n. This fact suggests some weakenings of the DPC notion, which we now list. We say that $\mathscr{C}$ has (for some $F \subseteq T_x$ and $n \in \mathbb{N}$):

i) PC_F if $\{a,b\} \looparrowright \{c,d\}$ implies $\{a,b\} \looparrowright_F \{c,d\}$ for every $\mathbf{A} \in \mathscr{C}$ and $a,b,c,d \in A$ (in words, F *determines principal congruences*);
ii) PC^n if $\{a,b\} \looparrowright \{c,d\}$ implies $\{a,b\} \looparrowright^n \{c,d\}$ for every $\mathbf{A} \in \mathscr{C}$ and $a,b,c,d \in A$ (in words, *principal congruences have bounded length* n);
iii) PC^n_F if $\looparrowright$ is equivalent to $\looparrowright^n_F$ in every member of $\mathscr{C}$. We use PC^n_k to denote PC^n_F in the case that F is the set of all terms in T_x of height at most k.

For example, we have seen that in a variety, the DPC condition is equivalent to PC^n_F for some finite $F \subseteq T_x$.

The property PC_F for finite $F \subseteq T_x$ was examined by Clark et al. [8] where it is referred to as *term finite principal congruences* (TFPC). In the finite type case, the property is also known as *finite Mal'cev depth* and *finite principal length*; see Wang [33]. In Clark et al. [8], the TFPC property is shown to be tightly related to another congruence property, which we now describe. (A more detailed version of the next paragraph can be found in Clark et al. [8].)

Let $\mathbf{A}$ be an algebra, θ an equivalence relation on $\mathbf{A}$ and $F \subseteq T_x$. We write θ_F to denote the equivalence relation

$$\{(a,b) \mid (\forall t(x,\vec{y}) \in F)(\forall \vec{c} \text{ in } A)t^{\mathbf{A}}(a,\vec{c})\theta t^{\mathbf{A}}(b,\vec{c})\}.$$

Elementary algebra shows that there is a largest congruence of $\mathbf{A}$ contained in θ. This congruence is often referred to as the *syntactic congruence* and denoted $\mathrm{syn}(\theta)$. When $B \subseteq A$, we also use $\mathrm{syn}(B)$ to denote the congruence $\mathrm{syn}(B^2 \cup (A\backslash B)^2)$. The relation $\mathrm{syn}(\theta)$ is in fact equal to θ_{T_x}; however, in general we have $\mathrm{syn}(\theta) \subseteq \theta_F$, and sometimes (for example, if the term x is in F), we also have $\theta_F \subseteq \theta$. In particular, if there is $F \subseteq T_x$ such that θ_F is a congruence, then $\theta_F = \theta_{T_x} = \mathrm{syn}(\theta)$. We say that $F \subseteq T_x$ *determines syntactic congruences* on $\mathbf{A}$ if $\theta_F = \mathrm{syn}(\theta)$ for every equivalence relation θ on $\mathbf{A}$. We say that $\mathbf{A}$ has *finitely determined syntactic congruences* if there is a finite subset $F \subseteq T_x$ determining syntactic congruences on $\mathbf{A}$. These concepts extend to classes of algebras: F determines syntactic congruences in the class $\mathscr{C}$ if it does on each member of $\mathscr{C}$, while $\mathscr{C}$ has finitely determined syntactic congruences

(FDSC) if there is a finite set $F \subseteq T_x$ determining syntactic congruences on $\mathscr{C}$. We use the notation SC_F to denote the property that F determines syntactic congruences in a class (or on an algebra). The notation SC_k denotes SC_F in the case that F is the subset of T_x of terms with height at most k.

Remark 3.2. In an ultraproduct closed class $\mathscr{C}$, the FDSC condition is equivalent to the definability of syntactic congruences in members of $\mathscr{C}$; that is, the existence of a formula $\pi(x, y, \sigma)$ in two free variables and one unary predicate variable σ such that for any $\mathbf{A} \in \mathscr{C}$ and equivalence relation θ on $\mathbf{A}$ we have $(a, b) \in \mathrm{syn}(\theta)$ if and only if $\mathbf{A} \models \pi(a, b, \theta)$.

Proof. This follows from the Definability Lemma 2.3 because $\mathrm{syn}(\theta)$ is the solution set to the infinite conjunction $(\&_{t\in T_x} \forall z_1 \dots \forall z_{n_t} (t(x, \vec{z}), t(y, \vec{z})) \in \theta$. □

It turns out that the property of F determining syntactic congruences can be described equationally, by a scheme called *shadowing* in Clark et al. [8]. If t is a term in T_x, $n \geq 0$ and $F \subseteq T_x$, then we say F *syntactically shadows* t (in length n) in a class $\mathscr{C}$ and write $F \vdash^n t$ if, there exist terms $t_1, \dots, t_n \in F$ and terms $w_{i,j}(x, y, z_1, \dots, z_m)$ such that the following identities hold in $\mathscr{C}$:

$$\begin{aligned}
t(x, z_1, \dots, z_n) &\approx t_1(v_1, w_{1,1}, \dots, w_{1,m}),\\
t_1(v_1', w_{1,1}, \dots, w_{1,m}) &\approx t_2(v_2, w_{2,1}, \dots, w_{2,m}),\\
t_2(v_2', w_{2,1}, \dots, w_{2,m}) &\approx t_3(v_3, w_{3,1}, \dots, w_{3,m}),\\
&\vdots\\
t_n(v_k', w_{k,1}, \dots, w_{k,m}) &\approx t(y, z_1, \dots, z_n),
\end{aligned}$$

where $\{v_i, v_i'\} = \{x, y\}$, for all $i \in \{1, \dots, n\}$ (and where the $n = 0$ case means $t(x, \vec{z}) \approx t(y, \vec{z})$. We say that F *syntactically shadows* t if $F \vdash^n t$ for some n. (The reader will verify that $F \vdash^n t(x, \vec{z})$ is precisely the property $\{x, y\} \looparrowright^n_F \{t(x, \vec{z}), t(y, \vec{z})\}$ as interpreted in the relevant free algebra for $\mathscr{C}$.) We extend this notation to sets of terms as follows: for $F, G \subseteq T_x$ we write $F \vdash^n G$ if for each $g \in G$ there is $k \leq n$ such that $F \vdash^k g$. We write $F \vdash G$ if for each $t \in G$, there is k such that $F \vdash^k t$.

Lemma 3.3 (§3 of Clark et al. [8]). *Let $\mathscr{C}$ be a class containing its free algebras. Then $\mathscr{C}$ has SC_F if and only if $F \vdash T_x$.*

The definition of shadowing and Lemma 3.3 show that the FDSC property is an equational one. An important fact used below is that it is expressible by only finitely many equations. (The notation $\mathrm{Mod}(\Sigma)$ means the class of all models of Σ.)

Lemma 3.4. *Let $\mathscr{C}$ be a class of algebras of some finite type and containing its free algebras. If $\mathscr{C}$ has FDSC then there is a finite set of identities Σ, satisfied by $\mathscr{C}$, and such that $\mathrm{Mod}(\Sigma)$ has FDSC.*

Proof. Say that the finite set F determines syntactic congruences in $\mathscr{C}$. By Theorem 2.5.iv of Clark et al. [8], there is a finite set of terms $G \subseteq T_x$ such that $F \vdash T_x$ if and only if $F \vdash G$. The shadowing of G by F is a finite set of identities. □

We now state the fundamental relationship between the principal congruence properties and syntactic congruence properties.

Theorem 3.5 (Lemma 2.3 of Clark et al. [8]). *An algebra* $\mathbf{A}$ *has* PC_F *if and only if it has* SC_F.

In analogy with the notation PC_F^n (introduced after Lemma 3.1), the notation SC_F (and SC_k) has an obvious refinement: $\mathscr{C}$ has SC_F^n if $F \vdash^n T_x$ in $\mathscr{C}$. (The notation SC^n is not interesting, since $T_x \vdash^1 T_x$ always, so that every class has SC^1.) It is tempting to speculate that a similar relationship to that of Theorem 3.5 might hold between PC_F^n and SC_F^n in classes containing their free algebras. One direction is easy.

Lemma 3.6. *Let* $\mathscr{C}$ *be a class of algebras containing its free algebras. If* PC_F^n *holds (that is,* $\mathscr{C}$ *has* DPC*), then* SC_F^n *holds.*

Proof. This follows because syntactic shadowing of a term $t(x, \vec{z})$ corresponds to the Mal'cev scheme for membership of the pair $(t(x, \vec{z}), t(y, \vec{z}))$ in $\mathrm{cg}(x, y)$, as interpreted in a suitably large relatively free algebra of $\mathscr{C}$. □

As the following example demonstrates, the converse implication is false in general.

Example 3.7. The variety $\mathbf{V}$ of semigroups has SC_2^1 but does not have DPC.

Proof. Let $F := \{x, xz_1, z_1x, z_1(xz_2)\}$. Then the variety of semigroups has SC_F^1. For n even, consider the semigroup $\mathbf{S}_n$ with elements $\{0, a, b, c_1, \ldots, c_n, d_1, \ldots, d_{n+1}\}$ and multiplication $\cdot$ such that all products are equal to 0, except those shown in the following table:

$\cdot$	c_1	c_2	c_3	$\ldots$	c_{n-1}	c_n
a	d_1	d_3	d_3	$\ldots$	d_{n-1}	d_{n+1}
b	d_2	d_2	d_4	$\ldots$	d_n	d_n

Then $\{a, b\} \looparrowright_1^n \{d_1, d_n\}$, but for any $m < n$ and any k, it is not the case that $\{a, b\} \looparrowright_k^m \{d_1, d_n\}$. Thus $\mathbf{V}$ does not have DPC. □

Lemma 3.8. *Assume that a class* $\mathscr{C}$ *of algebras has* SC_F^k. *Then, for all* $\mathbf{A} \in \mathscr{C}$ *and all* $a, b, c, d \in \mathbf{A}$, *we have*

$$\{a, b\} \looparrowright^1 \{c, d\} \implies \{a, b\} \looparrowright_F^k \{c, d\}.$$

Proof. Let $\mathscr{C}$ be a class of algebras with SC_m^k. Let $\mathbf{A} \in \mathscr{C}$, and let $a, b, c, d \in A$. Assume that $\{a, b\} \looparrowright^1 \{c, d\}$, that is, there exists a term t and elements $e_1, \ldots, e_n$ and u, u', where $\{u, u'\} = \{a, b\}$ such that

$$\begin{aligned} c &= t^{\mathbf{A}}(u, e_1, \ldots, e_n) \\ t^{\mathbf{A}}(u', e_1, \ldots, e_n) &= d. \end{aligned}$$

As $\mathscr{C}$ has SC_F^k, there are terms $f_1, \ldots, f_\ell \in F$, with $\ell \leq k$, and terms $w_{i,j}$ with variables amongst $x, y, z_1, \ldots, z_n$ such that

$$\begin{aligned} t(x, z_1, \ldots, z_n) &\approx f_1(v_1, w_{1,1}, \ldots, w_{1,m}) \\ f_1(v_1', w_{1,1}, \ldots, w_{1,m}) &\approx f_2(v_2, w_{2,1}, \ldots, w_{2,m}) \\ &\vdots \\ f_\ell(v_\ell', w_{\ell,1}, \ldots, w_{\ell,m}) &\approx t(y, z_1, \ldots, z_n), \end{aligned}$$

where $\{v_i, v_i'\} = \{x, y\}$.

Let $\overline{w_{i,j}}$ denote the value of the term $w_{i,j}$ in $\mathbf{A}$ under the map $x \mapsto u$, $y \mapsto u'$, $z_1 \mapsto e_1, \ldots, z_n \mapsto e_n$. Also let $\overline{x} = u$ and $\overline{y} = u'$, so that $\{\overline{v_i}, \overline{v_i'}\} = \{u, u'\} = \{a, b\}$. We have

$$\begin{aligned} c &= f_1^{\mathbf{A}}(\overline{v_1}, \overline{w_{1,1}}, \ldots, \overline{w_{1,m}}) \\ f_1^{\mathbf{A}}(\overline{v_1'}, \overline{w_{1,1}}, \ldots, \overline{w_{1,m}}) &= f_2^{\mathbf{A}}(\overline{v_2}, \overline{w_{2,1}}, \ldots, \overline{w_{2,m}}) \\ &\vdots \\ f_\ell^{\mathbf{A}}(\overline{v_\ell'}, \overline{w_{\ell,1}}, \ldots, \overline{w_{\ell,m}}) &= d. \end{aligned}$$

Thus $\{a, b\} \looparrowright_F^\ell \{c, d\}$ with $\ell \leq k$. □

The following proposition is close to a converse of Lemma 3.6, despite Example 3.7. It is an immediate consequence of Lemma 3.8.

Proposition 3.9. *Let $\mathscr{C}$ be a class of algebras. If $\mathscr{C}$ has SC_m^k and PC^p, then $\mathscr{C}$ has PC_m^{kp}.*

Lemma 3.10. *Let $\mathscr{C}$ be a class of algebras, and let $F, G, H \subseteq T_x$. If $F \vdash^p G \vdash^q H$, then $F \vdash^{pq} H$.*

Proof. Let $t(x, z_1, \ldots) \in H$. Each of the p variable switches in the syntactic shadowing of t by G—say, $t_i(v_i, w_{i,1}, \ldots)$ becomes $t_i(v_i', w_{i,1}, \ldots)$—can be replaced by the syntactic shadowing of $t_i(x, w_1, \ldots)$ by F of length q. □

4. DIVISION

Let $\mathbf{A}$ be a partial algebra and $a, b \in A$. For each subset $F \subseteq T_x$, we say that *a divides b by way of F* and write $a \rightsquigarrow_F b$ if there is a term $t(x, \vec{z}) \in F$ and a string of elements $\vec{c}$ from A such that $t^{\mathbf{A}}(a, \vec{c}) = b$. If $F = \{t(x, \vec{z})\}$, we abbreviate $\rightsquigarrow_{\{t(x,\vec{z})\}}$ to $\rightsquigarrow_{t(x,\vec{z})}$, while if $F = T_x$, then we write $a \rightsquigarrow b$ in place of $a \rightsquigarrow_F b$ and say that a divides b. The following fact is easily verified.

Lemma 4.1. *The relation* $\rightsquigarrow$ *is reflexive and transitive* (*that is, it is a preorder*).

A set $F \subseteq T_x$ *determines division* in a class $\mathscr{K}$ if, for every $\mathbf{A} \in \mathscr{K}$ and every $a, b \in \mathbf{A}$, if $a \rightsquigarrow b$ we also have $a \rightsquigarrow_F b$. The class $\mathscr{K}$ has *finitely determined division* (FDD) if there is a finite set F that determines division in $\mathscr{K}$.

As an example, observe that in the term algebra $T(X)$, we have $s \rightsquigarrow t$ if and only if s is a subterm of t.

The reader will also easily verify that on the multiplicative semigroup of positive integers $\langle \mathbb{N}; \cdot \rangle$ we have $n \rightsquigarrow m$ if and only if $n \mid m$. More generally, the relation $\mid$ that we defined on an arbitrary semigroup in the introduction corresponds to $\rightsquigarrow_F$, where $F = \{x, z_0x, xz_0, z_0xz_1\}$[1]. It is not hard to verify that this choice of F determines division on any semigroup. In other words $\rightsquigarrow_F$ coincides with $\rightsquigarrow$ (and $\mid$) in the class of semigroups. In general, for a subset $F \subseteq T_x$ and a term $t = t(\underline{x}, \vec{z}) \in T_x$ we write $F \Vdash\!\models_{\mathscr{K}} t(\underline{x}, \vec{z})$ if the relation $\rightsquigarrow_F$ contains $\rightsquigarrow_t$ on every algebra in $\mathscr{K}$. For a subset $G \subseteq T_x$, write $F \Vdash\!\models_{\mathscr{K}} G$ if $F \Vdash\!\models_{\mathscr{K}} t$ for every $t \in G$. Thus F determines division in $\mathscr{K}$ if and only if $F \Vdash\!\models_{\mathscr{K}} T_x$.

For $t(x, \vec{z}) \in T_x$, we write $F \Vdash_{\mathscr{K}} t(x, \vec{z})$ if there is a term $s(x, \vec{z}) \in F$ and a string $\vec{u}$ of terms whose variables are amongst $x, z_0, \ldots$ such that $\mathscr{K}$ satisfies

$$t(x, \vec{z}) \approx t(x, \vec{z}) \to t(x, \vec{z}) \approx s(x, \vec{u}). \qquad (*)$$

For $G \subseteq T_x$ we write $F \Vdash_{\mathscr{K}} G$ if $F \Vdash_{\mathscr{K}} t$ for every $t \in G$. Notice that when $\mathscr{K}$ consists of total algebras, the quasi-identity $*$ is logically equivalent to the identity $t(x, \vec{z}) \approx s(x, \vec{u})$. The following yields a completeness theorem for $\Vdash$ relative to $\Vdash\!\models$.

Theorem 4.2. *Let* $\mathscr{K}$ *be a class of similar partial algebras closed under taking direct products and subalgebras. The following are equivalent for a subset* F *of* T_x:

(1) F *determines division in* $\mathscr{K}$ (*in symbols*: $F \Vdash\!\models_{\mathscr{K}} T_x$);
(2) $F \Vdash_{\mathscr{K}} T_x$;
(3) $F \Vdash_{\mathscr{K}} T_{1x}$.

Proof. We prove (1) $\Rightarrow$ (2) by the contrapositive. Suppose there is a term $t(x, \vec{z}) \in T_x$ such that, for every $s(x, \vec{z}) \in F$, and every string of terms $\vec{u}$, the quasi-identity $*$ fails on some member $\mathbf{A}_{s(x,\vec{u})}$ of $\mathscr{K}$ under some interpretation

$$x \mapsto a_{s(x,\vec{u})}, z_0 \mapsto b_{s(x,\vec{u}),0}, z_1 \mapsto b_{s(x,\vec{u}),1}, \ldots.$$

Let $\mathbf{A}$ be the direct product of the $\mathbf{A}_{s(x,\vec{u})}$ over all choices of $s(x, \vec{z}) \in F$ and the terms $\vec{u}$.

[1]Strictly, z_0xz_1 should be either $z_0(xz_1)$ or $(z_0x)z_1$, but we can work with semigroup words instead of terms.

Let **B** be the subalgebra of **A** generated by the elements $a : s(x, \vec{u}) \mapsto a_{s(x,\vec{u})}$ and $b_i : s(x, \vec{u}) \mapsto b_{s(x,\vec{u}),i}$. We have $a \rightsquigarrow t^{\mathbf{A}}(a, \vec{b}) =: \{c\}$. However, for every term $s(x, \vec{z}) \in F$ and every tuple of elements $\{\vec{d}\}$ in B, there are terms $u_0(x, \vec{z}), u_1(x, \vec{z}), \ldots$ such that

$$u_i^{\mathbf{B}}(a, \vec{b}) = d_i \qquad \text{for } i = 0, 1, \ldots.$$

But then c differs from $s^{\mathbf{B}}(a, \vec{d})$ on the coordinate $s(x, \vec{u})$ (because either $s^{\mathbf{B}}(a, \vec{d})$ is undefined, or it is defined but takes a different value to c on the coordinate $s(x, \vec{u})$). So $a \not\rightsquigarrow_F c$, as required.

As (2) $\Rightarrow$ (3) is trivial, we turn to (3) $\Rightarrow$ (1). Assume that $a \rightsquigarrow b$ for some a, b in some member $\mathbf{A} \in \mathscr{K}$. So there is a term $t(x, \vec{z}) \in T_x$ and elements $\vec{c}$ in A with $t^{\mathbf{A}}(a, \vec{c}) = b$. Let t_1 be a term in T_{1x} such that there are terms $u_0 = u_0(x, \vec{z}), u_1 = u_1(x, \vec{z}), \ldots$ such that $t(x, \vec{z}) = t_1(x, \vec{u})$ (that is, the terms are identical). Since $t^{\mathbf{A}}(a, \vec{c})$ is defined, so are each $u_i^{\mathbf{A}}(a, \vec{c}) =: d_i$, and $t_1(a, \vec{d}) = b$. By (3), there is a term $s(x, \vec{z}) \in F$ and terms $\vec{v}$ in variables amongst $x, z_0, \ldots$ such that

$$\mathbf{A} \models t_1(x, \vec{z}) \approx t_1(x, \vec{z}) \rightarrow t_1(x, \vec{z}) \approx s(x, \vec{v}).$$

We then have $(s^{\mathbf{A}}(a, v_0(a, \vec{d}), v_1(a, \vec{d}), \ldots) = b)$ so that $a \rightsquigarrow_F b$, as required. □

Example 4.3. Let $\mathscr{K}$ be a universal Horn class of total algebras of signature $\mathscr{F}$ and let F determine syntactic congruences in $\mathscr{K}$. Then the set

$$F \cup \{t(f(y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_k), \vec{z}) \mid t(x, \vec{z}) \in F \text{ and } f \in \mathscr{F}\}$$

determines division in $\mathscr{K}$.

Proof. Consider a term $s(x, \vec{z}) \in T_{1x}$. As a first case, assume that $\mathscr{K} \not\models s(x, \vec{z}) \approx s(y, \vec{z})$. After applying a substitution $y \mapsto x$ if necessary, the first equation in the shadowing of s by F yields a term $t_1 \in F$ and an equation of the form

$$\mathscr{K} \models s(x, \vec{z}) \approx t_1(x, w_{1,1}, \ldots, w_{1,m}),$$

showing that $F \Vdash_{\mathscr{K}} s$.

Now assume that $\mathscr{K} \models s(x, \vec{z}) \approx s(y, \vec{z})$. Inductively define $s_0(x, \vec{z}) := s(x, \vec{z})$ and $s_{i+1}(x, \vec{z})$ to be the term obtained from $s_i(x, \vec{z})$ by replacing the smallest nontrivial subterm containing x by x. For example, if f is a ternary symbol, and $s(x, \vec{z})$ is $f(z_2, f(z_0, z_1, x), z_3)$, then $s_1(x, \vec{z})$ is $f(z_2, x, z_3)$.

Let ℓ be the smallest integer for which the identity $s_\ell(x, \vec{z}) \approx s_\ell(y, \vec{z})$ fails in $\mathscr{K}$. So $s_{\ell-1}(x, \vec{z})$ is the term $s_\ell(f(z_k, \ldots, x, \ldots, z_{k'}), z_{k'+1}, \ldots)$ for some operation symbol f (and some k, k'). Note also that minimality of ℓ ensures $\mathscr{K} \models s_{\ell-1}(x, \vec{z}) \approx s_{\ell-1}(y, \vec{z})$, which implies $\mathscr{K} \models s_{\ell-1}(x, \vec{z}) \approx s(x, \vec{z})$. Then by the first case considered, there is $t(x, \vec{z}) \in F$ with $\mathscr{K} \models s_\ell(x, \vec{z}) \approx t(x, \vec{w})$ (where the w_i are terms) which gives

$$\mathscr{K} \models s(x, \vec{z}) \approx s_{\ell-1}(x, \vec{z}) \equiv s_\ell(f(\ldots, x, \ldots), z_{k'+1}, \ldots) \approx t(f(\ldots, x, \ldots), \vec{w}'),$$

where each term w_i' is obtained from w_i by replacing x with $f(z_k, \ldots, x, \ldots, z_{k'})$.

□

Example 4.4. Let $\mathscr{K}$ be a class of similar total algebras, and let $\mathscr{L}$ have $\mathsf{ISP}^+(\mathscr{K}) \subseteq \mathscr{L} \subseteq \mathsf{V}(\mathscr{K})$. A set $F \subseteq T_x$ determines division in $\mathscr{L}$ if and only if it determines division in $\mathsf{ISP}^+(\mathscr{K})$.

Proof. Both $\mathsf{V}(\mathscr{K})$ and $\mathsf{ISP}^+(\mathscr{K})$ (whence $\mathscr{L}$) satisfy the same identities and are closed under taking subalgebras and direct products. The claim now follows from Theorem 4.2 because the property $F \Vdash T_x$ is equational for total algebras. □

Define an *absorbing ideal* of a partial algebra $\mathbf{A}$ to be a subset $I \subseteq A$ (possibly empty) such that if f is a fundamental operation of $\mathbf{A}$ and $\vec{a}$ is a string of elements from A containing at least one element of I, then $f(\vec{a})$ is either undefined or contained in I. The notion of an absorbing ideal agrees with the notion of an ideal in semigroup theory, but is different from the ring-theoretic notion of an ideal (which are absorbing only for the multiplicative semigroup).

If θ is an equivalence relation on a partial algebra $\mathbf{A}$, then we say θ is a congruence on $\mathbf{A}$ if, for every n-ary (partial) operation F, if $(a_1, b_1), \dots, (a_n, b_n) \in \theta$, then $(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta$ whenever both $f(a_1, \dots, a_n)$ and $f(b_1, \dots, b_n)$ are defined. The quotient $\mathbf{A}/\theta$ is defined on A/θ by

$$f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f(b_1, \dots, b_n)/\theta$$

provided $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ and $f^{\mathbf{A}}(b_1, \dots, b_n)$ is defined; $f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta)$ is undefined otherwise. The following lemma is obvious.

Lemma 4.5. *Let $\mathbf{A}$ be a partial algebra and I a nonempty absorbing ideal. The equivalence $\theta_I := \Delta \cup (I \times I)$ is a congruence.*

We write $\mathbf{A}/I$ in place of $\mathbf{A}/\theta_I$. It is clear that the factor partial algebra $\mathbf{A}/I$ has an element 0 (corresponding to the congruence class I) that is absorbing in any defined term function, that is, if $t^{\mathbf{A}/I}(\vec{x})$ is a term function and 0 appears in the tuple $\vec{a}$, then $t^{\mathbf{A}/I}(\vec{a})$ is either undefined or equal to 0.

For any element a of an algebra $\mathbf{A}$, we define two subsets

$$I_a := \{b \in A \mid a \rightsquigarrow b\} \qquad \text{and} \qquad J_a := \{b \in A \mid b \not\rightsquigarrow a\},$$

and refer to them as the *principal absorbing ideal* of a and the *principal absorbing complement* of a in $\mathbf{A}$, respectively.

5. DEFINABLE DIVISION

If a class $\mathscr{K}$ has FDD then it is not hard to see that the relation $\rightsquigarrow$ is first order definable within $\mathscr{K}$. In fact these two properties are equivalent.

Theorem 5.1. *Let $\mathscr{K}$ be a class of similar partial algebras closed under ultraproducts. The following are equivalent:*

(1) *There is a finite set of terms determining division in $\mathscr{K}$* (FDD);

(2) *There is a finite set $F \subseteq T_x$ such that, for every $\mathbf{A} \in \mathcal{K}$, we have $a \rightsquigarrow b$ if and only if*

$$\mathbf{A} \models \bigvee_{t(x,\vec{z}) \in F} \exists \vec{c}(t(a, \vec{c}) \approx b);$$

(3) *The relation $\rightsquigarrow$ is first order definable in $\mathcal{K}$;*
(4) *Principal absorbing ideals I_a are first-order definable in $\mathcal{K}$;*
(5) *Principal absorbing complements J_a are first-order definable in $\mathcal{K}$.*

Proof. (1) ⇔ (2) ⇒ (3) ⇒ (4) and (3) ⇒ (5) are all trivial. (4) ⇔ (5) holds because a sentence defines principal absorbing ideals if and only if its negation defines principal absorbing complements.[2] For (4) ⇒ (3), we can use $a \rightsquigarrow b$ if and only if $b \in I_a$. The condition (3) ⇒ (2) follows from the Definability Lemma 2.3. □

Lemma 5.2. *Let F and G be two subsets of T_x, both determining division in a universal Horn class of partial algebras $\mathcal{K}$. If F is finite, then there is a finite subset of G determining division in $\mathcal{K}$.*

Proof. For each $t := t(x, \vec{z})$ in F, choose a term $s_t := s_t(x, \vec{z}) \in G$ such that $\rightsquigarrow_t \subseteq \rightsquigarrow_{s_t}$, as is guaranteed by Theorem 4.2. We have

$$\rightsquigarrow \; = \; \rightsquigarrow_F \; = \bigcup_{t \in F} \rightsquigarrow_t \; \subseteq \bigcup_{t \in F} \rightsquigarrow_{s_t} \; \subseteq \; \rightsquigarrow,$$

giving equality throughout. So $\{s_t \mid t \in F\} \Vdash_{\mathcal{K}} T_x$. □

In the following theorem, the *height* of a term t is the height of the term tree of t. The last statement is the FDD analogue of Lemma 3.4.

Theorem 5.3. *Let $\mathcal{K}$ be a universal Horn class of partial algebras. Let F be a finite set of terms from T_x, and let n be the maximal height of any term in F. The following are equivalent:*

(1) *F determines division in $\mathcal{K}$;*
(2) *For each term $s(x, \vec{z})$ of T_{1x} with height $n + 1$, $\mathcal{K}$ satisfies an implication of the form*

$$s(x, \vec{z}) \approx s(x, \vec{z}) \rightarrow s(x, \vec{z}) \approx t(x, w_0(x, \vec{z}), w_1(x, \vec{z}), \dots) \quad \textit{for some } t \in F.$$

In particular, if $\mathcal{K}$ is of finite type and has FDD, then there is a finite set of implications Σ such that $\mathcal{K} \models \Sigma$ and $\mathrm{Mod}(\Sigma)$ has FDD. If $\mathcal{K}$ consists of total algebras, then Σ can be chosen to consist of identities.

[2]Note also, that the definability of I_a and J_a are obviously equivalent to the definability of the corresponding congruences. For example, if θ_a denotes the Rees congruence corresponding to I_a and $\Phi(x, y)$ defines principal ideals, then the sentence $\Phi'(x, y, z) := (x \approx y) \vee \Phi(z, x) \;\&\; \Phi(z, y)$ defines θ in the sense that $(c, d) \in \theta_a$ if and only if $\mathbf{A} \models \Phi'(c, d, a)$.

Proof. The final statement follows from (1) $\Leftrightarrow$ (2) since finite type ensures that the height $n+1$ terms of T_{1x} are a finite subset of T_{1x}. Now we prove the equivalence (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2) is part of Theorem 4.2. Now assume that (2) holds. We show that $F \Vdash_{\mathcal{K}} s(x, \vec{z})$ for every term in T_{1x}; then (3) $\Rightarrow$ (1) from Theorem 4.2 gives condition (1) of the present theorem.

The proof is by induction on the height of s. The base case is height $n+1$, which holds by (2). Suppose we have proved that the claim is true for height $k \geq n+1$, and let $s(x, \vec{z})$ be of height $k+1$. There is a fundamental operation f such that $s(x, \vec{z}) = f(z_i, \ldots, r(x, \vec{z}), \ldots, z_{i+\ell})$, for some height k term $r(x, \vec{z})$. By the induction hypothesis and the definition of $\Vdash$, we have

$$\mathcal{K} \models r(x, \vec{z}) \approx r(x, \vec{z}) \to r(x, \vec{z}) \approx t(x, \vec{w}),$$

for some term $t \in F$, where $w_i = w_i(x, \vec{z})$. So

$$\mathcal{K} \models s(x, \vec{z}) \approx s(x, \vec{z}) \to s(x, \vec{z}) \approx f(z_i, \ldots, t(x, \vec{w}), \ldots, z_{i+\ell}).$$

Let $r'(x, \vec{z}, \vec{y})$ denote the term $f(z_i, \ldots, t(x, \vec{y}), \ldots, z_{i+\ell})$, where each y_j is a new variable. This term is of height at most $n+1$, and so by (2) we have that there is a term $t'(x, \vec{z}) \in F$ with

$$\mathcal{K} \models r'(x, \vec{z}, \vec{y}) \approx r'(x, \vec{z}, \vec{y}) \to r'(x, \vec{z}, \vec{y}) \approx t'(x, \vec{v})$$

for some terms $v_i = v_i(x, \vec{z}, \vec{y})$.

Applying the substitution that fixes all variables except the y_j and has $y_j \mapsto w_j$, we obtain

$$\mathcal{K} \models r'(x, \vec{z}, \vec{w}) \approx r'(x, \vec{z}, \vec{w}) \to r'(x, \vec{z}, \vec{w}) \approx t'(x, v_0(x, \vec{z}, \vec{w}), \ldots).$$

However, $s(x, \vec{z}) \approx s(x, \vec{z}) \to s(x, \vec{z}) \approx r'(x, \vec{z}, \vec{w})$ holds, and so we obtain

$$s(x, \vec{z}) \approx s(x, \vec{z}) \to s(x, \vec{z}) \approx r'(x, \vec{z}, \vec{w}) \approx t'(x, v_0(x, \vec{z}, \vec{w}), \ldots),$$

as required. □

It is easy to see that for any similarity type $\mathcal{F}$ involving a non-nullary operation symbol, the class of term (total) algebras of type $\mathcal{F}$ fails to have FDD. We finish this section with a finitely generated variety without FDD.

Example 5.4. Let **F** be the following groupoid:

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	0

Then **F** generates a finitely based variety but fails to have FDD.

Proof. It is convenient to give 0 the status of a nullary operation (or to use it to abbreviate the term xx). This algebra is presented in Davey et al. [10] as an example of a smallest possible algebra generating a variety without FDSC; however, in fact it has the stronger property that it is without FDD. As explained in Davey et al. [10], it is not hard to show that the following set is a basis for the identities of $\mathbf{F}$:

$$\Sigma_{\mathbf{F}} := \{xx \approx 0, x(yz) \approx 0, (xy)z \approx (xz)y, (xy)y \approx xy\}.$$

Let θ denote the fully invariant congruence on the groupoid term algebra $T(X)$ (where X is, say, $\{x_0, x_1, \dots\}$) corresponding to the countably generated free algebra in $\mathsf{V}(\mathbf{F})$. Using $\Sigma_{\mathbf{F}}$ it is easy to verify that: every term in the variables $\{x_0, x_1, \dots\}$ reduces to either 0 or one of the form $(\cdots(x_{i_0}x_{i_1})x_{i_2}\cdots)x_{i_{n-1}}$, where $i_0 \notin \{i_1, \dots, i_{n-1}\}$; and $i_j < i_{j+1}$ (for $j > 0$); and that the equivalence class of $(\cdots(x_{i_0}x_{i_1})x_{i_2}\cdots)x_{i_{n-1}}$ modulo θ is

$$\{(\cdots(x_{j_0}x_{j_1})x_{j_2}\cdots)x_{j_{m-1}} \mid j_0 = i_0 \ \&\ \{j_1, \dots, j_{m-1}\} = \{i_1, \dots, i_{n-1}\}\}.$$

Now we show that $\mathsf{V}(\mathbf{F})$ fails to have FDD. First observe that, for each $n \in \mathbb{N}$, we have $x_0 \rightsquigarrow (\cdots(x_0x_1)x_2\cdots)x_{n-1}$. Let $t(x, \vec{z})$ be a term such that there is a string of terms $\vec{u}$ with $\mathbf{F} \models t(x_0, \vec{u}) \approx (\cdots(x_0x_1)x_2\cdots)x_{n-1}$. Since x appears in $t(x, \vec{z})$, the observations above show that we have that $t(x, \vec{z}) = (\cdots(xz_{j_1})z_{j_2}\cdots)z_{j_{m-1}}$, where $m \geq n$. Hence no finite set of terms can determine division in $\mathsf{V}(\mathbf{F})$. □

6. UNIVERSAL DIVISION

We say that a class of similar partial algebras has *universal division* if the relation $\rightsquigarrow$ is the universal relation on every member of the class. The following lemma is obvious.

Lemma 6.1. *Let* $\mathbf{A}$ *be a partial algebra for which there is a term* $p(\underline{x}, y)$ *such that* $\mathbf{A} \models p(\underline{x}, y) \approx y$. *Then the universal Horn class generated by* $\mathbf{A}$ *has universal division, determined by the term* $p(x, z_0)$. *If* $\mathbf{A}$ *is a total algebra, the variety generated by* $\mathbf{A}$ *has universal division.*

Proof. For any term $s(x, \vec{z})$, we have $\mathbf{A} \models s(x, \vec{z}) \approx s(x, \vec{z}) \rightarrow s(x, \vec{z}) \approx p(x, s(x, \vec{z}))$. (Note that this holds even if y does not appear in $p(\underline{x}, y)$, since in that case $\mathbf{A}$ satisfies the equation $x \approx y$.) □

Classes $\mathscr{K}$ for which there is a single term $p(\underline{x}, y)$ with $\mathscr{K} \models p(x, y) \approx y$ (as in Lemma 6.1) were examined by Jackson [19] in the context of flat algebras. It was shown that when such a term is present, the variety generated by the flat extensions of the members of $\mathscr{K}$ is finitely axiomatised if and only if the universal Horn class of $\mathscr{K}$ is finitely axiomatised. The following example shows that this phenomenon is quite widespread amongst familiar algebras.

Example 6.2. (In this example, all algebras are total.) Any congruence-modular variety has universal division.

Proof. Let **V** be a congruence modular variety. Using a well-known result of Day [11], congruence modularity of **V** can be characterised by the existence of 4-ary terms $m_0(x, y, z, u), \dots, m_n(x, y, z, u)$ such that **A** satisfies the following identities:

$$m_0(x, y, z, u) \approx x, \qquad m_n(x, y, z, u) \approx u, \tag{D$_1$}$$

$$m_i(x, y, y, x) \approx x \quad \text{for } 1 \leq i \leq n, \tag{D$_2$}$$

$$m_i(x, x, y, y) \approx m_{i+1}(x, x, y, y), \quad \text{for even } i, \tag{D$_3$}$$

$$m_i(x, y, y, z) \approx m_{i+1}(x, y, y, z), \quad \text{for odd } i. \tag{D$_4$}$$

If one of the Day terms supplies a term $p(\underline{x}, y)$ with $\mathbf{V} \models p(\underline{x}, y) \approx y$, then we are done by Lemma 6.1.

Now, if y or z appears in one of the terms $m_i(x, y, z, u)$ we are done by (D$_2$). But otherwise (D$_1$), (D$_3$), and (D$_4$) easily yield $x \approx y$, so we can choose $p(\underline{x}, y) := x$. □

The converse of the first statement of Lemma 6.1 is also true.

Proposition 6.3. *The following are equivalent for a class of partial algebras $\mathcal{K}$ containing its two-generated free algebra*:

(1) *$\mathcal{K}$ has universal division*;
(2) *There is a term $p(\underline{x}, y)$ such that $\mathcal{K} \models p(\underline{x}, y) \approx y$.*

Also, the universal division property is decidable for the universal Horn class generated by a finite partial algebra of finite type.

Proof. (1) $\Rightarrow$ (2) Assume $\mathcal{K}$ has universal division and consider the two-generated relatively free algebra $\mathbf{K} := \mathbf{F}_{\mathcal{K}}(u, v)$ in $\mathcal{K}$, with free generators u, v. By assumption we have $u \rightsquigarrow v$. Let $s(\underline{x}, \vec{z}) \in T_x$ be the term witnessing this. There are terms $w_i = w_i(x, y)$, for $i = 0, 1, \dots$ with $s^{\mathbf{K}}(u, w_0^{\mathbf{K}}(u, v), \dots) = v$, so $\mathcal{K} \models p(\underline{x}, y) \approx y$ where $p(x, y) := s(x, w_0(x, y), \dots)$. The reverse implication is trivial (Lemma 6.1).

For the final claim, let **A** be a finite partial algebra of finite type. Proposition 2.2 part 2(b) shows that we can effectively construct the two-generated relatively free algebra for $\mathsf{ISP}(\mathbf{A})$, and then we can check if there is an element corresponding to a term $p(\underline{x}, y)$ with $\mathsf{ISP}(\mathbf{A}) \models p(\underline{x}, y) \approx y$. □

It is not hard to verify that every partial algebra whose signature contains an operation symbol of arity more than 1 can be embedded in an algebra (total if the original is total) with universal division, and so a (total) algebra with universal division need not generate a variety with universal division.

Corollary 6.4. *A class $\mathcal{K}$ with universal division has finitely determined division provided that the two-generated relatively free algebra for $\mathcal{K}$ is contained in $\mathcal{K}$, or $\mathcal{K}$ is closed under ultraproducts.*

Proof. The first case follows from Proposition 6.3, while in the second case, observe that principal ideals are trivially definable, so Theorem 5.1 shows that $\mathcal{K}$ has FDD. □

The following result was announced without proof in Jackson [19], and follows immediately from Example 6.2, Proposition 6.3 and the results of Jackson [19] discussed above.

Corollary 6.5. *Let $\mathcal{K}$ be a class of partial algebras generating a universal Horn class with universal division. Then the variety generated by the flat extensions of members of $\mathcal{K}$ is finitely axiomatised if and only if the universal Horn class of $\mathcal{K}$ is finitely axiomatised. In particular, this is true if $\mathcal{K}$ is a class of* (*total*) *algebras generating a congruence modular variety.*

We recall that there are many known finite total algebras generating congruence modular varieties but whose universal Horn theory is not finitely axiomatised, including examples from amongst groups (Ol'shanskii [28]), rings, and lattices (Belkin [4]).

7. DIVISION-ORDERED ALGEBRAS

Let us say that a class $\mathcal{K}$ of partial algebras is *division ordered* if the division relation is an order in every member of $\mathcal{K}$. A partial algebra **A** is division ordered if the class $\{\mathbf{A}\}$ is. The (total) term algebras are division ordered, so every (total) algebra is a quotient of a division-ordered algebra.

Lemma 7.1. *The class of all division-ordered partial algebras of a given type is a quasivariety, and is not finitely axiomatisable provided that the signature contains at least one non-nullary symbol.*

Proof. The following quasi-identities are obviously an axiomatisation:

$$\{t(x, \vec{z}) \approx y \ \& \ s(y, \vec{w}) \approx x \to x \approx y \mid s, t \in T_x\}. \tag{DO}$$

To prove that there is no finite basis we initially work within the class of (total) unary algebras. Let $\mathbf{C}_n$ denote the algebra on $n = \{0, 1, \ldots, n-1\}$ elements whose unary operations are all equal to the operation of successor modulo n. These have universal division. However, any ultraproduct of $\{\mathbf{C}_n \mid n \in \omega\}$ over a nonprincipal ultrafilter is division ordered. Hence the class of partial (or total) unary algebras that are *not* division ordered is not closed under ultraproducts. Therefore, the class of division ordered partial (or total) algebras is not finitely axiomatisable.

For arbitrary type, simulate the above idea, but using algebras $\mathbf{D}_n$ on the set $C_n \cup \{\infty\}$, where ∞ is the value of any nullary, and each fundamental f of arity $n \geq 1$ is defined by

$$f^{\mathbf{D}_n}(i_1, i_2, \ldots, i_n) = \begin{cases} i+1 \bmod n & \text{if } i_1 = i_2 = \cdots = i_n = i \in \{0, \ldots, n-1\}, \\ \infty & \text{otherwise.} \end{cases}$$

We leave the details to the reader. □

The class of finite division-ordered semigroups is a well-studied class of semigroups, called the $\mathcal{J}$*-trivial* semigroups. It coincides, for example, with the class of finite semigroups lying in the variety generated by some semigroup of reflexive binary relations on a finite set (under the operation of composition); see §4.1 of Pin [30]. This class is well known to form a pseudovariety, that is, it is closed under the formation of homomorphic images, subsemigroups, and finitary direct products. In this section, we prove an analogous fact: any locally finite variety generated by a division-ordered (total) algebra of arbitrary type consists of division-ordered algebras. This contrasts the fact that every algebra is the quotient of a division-ordered algebra (the term algebras).

For a total algebra $\mathbf{A}$, we define a labelled directed graph $\mathrm{G}(\mathbf{A})$ as follows. The vertices are the elements of A. The edges are labelled by basic translations (see Section 3): a translation $\lambda(x)$ labels an edge $a \to b$ if and only if $\lambda(a) = b$. It is easy to see that $\rightsquigarrow$ is the reflexive transitive closure of the edge relation, and consequently, that $\mathbf{A}$ is division ordered if and only if $\mathrm{G}(\mathbf{A})$ has no directed cycles other than loops.

Lemma 7.2. *Let* $\mathbf{B}$ *be a locally finite algebra and* $\lambda_0(x), \ldots, \lambda_{n-1}(x)$ *be a finite sequence of basic translations of* $\mathbf{B}$ (*this sequence possibly repeats and does not necessarily exhaust the set of all translations of* $\mathbf{B}$, *which may be infinite if* B *is infinite*). *For each* $k \in \omega$, *write* k *as* $i + nj$ *for* $i, j \in \omega$ *and* $i < n$, *and let* γ_k *denote the translation*

$$\lambda_i \lambda_{i-1} \cdots \lambda_0 (\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0)^j (x).$$

If $\mathbf{B}$ *is division ordered and* $a \in B$ *then the sequence* $(\gamma_i(a))_{i \in \omega}$ *is eventually constant.*

Proof. We show that if the sequence is not eventually constant, then $\mathbf{B}$ is not division ordered.

Let $b_1, \ldots, b_m$ be the elements of B used to build the translations $\lambda_i(x)$. Let $c_i := \gamma_i(a)$. So the sequence $(c_i)_{i \in \omega}$ is a sequence in the subalgebra of $\mathbf{B}$ generated by $\{a, b_1, \ldots, b_m\}$, which is finite since $\mathbf{B}$ is locally finite. So $(c_i)_{i \in \omega}$ is an infinite but not eventually constant sequence in a finite set, so there is a number i, j, k such that $i < j < k$ and $c_i = c_k \neq c_j$. Then $c_i \rightsquigarrow c_j \rightsquigarrow c_k = c_i$ so that $\mathbf{B}$ is not division ordered. □

Theorem 7.3. *Let* $\mathbf{A}$ *be an algebra that is division ordered. If the variety* $\mathsf{V}(\mathbf{A})$ *is locally finite then it is division ordered.*

Proof. Consider an algebra $\mathbf{S} \in \mathsf{V}(\mathbf{A})$. In particular, there is an algebra $\mathbf{B} \in \mathsf{SP}(\mathbf{A})$ and a homomorphism $\phi : \mathbf{B} \twoheadrightarrow \mathbf{S}$. Now as $\mathbf{B}$ lies in the quasivariety generated by $\mathbf{A}$, we have that $\mathbf{B}$ is also locally finite and division ordered. Consider the graph $\mathrm{G}(\mathbf{S})$. Assume that $s_0, \ldots, s_{n-1}, s_0$ is a cycle in the graph (with no repeats amongst $s_0, \ldots, s_{n-1}$), with associated translations $\eta_0, \ldots, \eta_{n-1}$ (so $\eta_i(s_i) = s_{i+1 \bmod n}$). We will show that $n = 1$, that is, we have a loop. Suppose that $\eta_i(x) = f_i^{\mathbf{S}}(a_{i,0}, \ldots, x, \ldots, a_{i,m_i})$, for some basic operations f_i and elements $a_{i,0}, \ldots, a_{i,m_i}$. For each $i \leq n-1$ and $j_i \leq m_i$, choose $b_{i,j_i} \in \phi^{-1}(a_{i,j_i})$. Let $\lambda_i(x) := f_i^{\mathbf{B}}(b_{i,0}, \ldots, x, \ldots, b_{i,m_i})$, and choose $t_0 \in \phi^{-1}(s_0)$. Define $t_{i+1} := \lambda_i(t_i)$. Evidently, $\phi(t_i) = s_{i \bmod n}$. But also, by Lemma 7.2, there exists k such that $t_i = t_j$ whenever $i, j \geq k$. Choose $i \geq k$ such that $i = 0 \bmod n$. Then $s_0 = \phi(t_i) = \phi(t_{i+1}) = s_{1 \bmod n}$, which shows that $n = 1$, by the assumption that $s_0, \ldots, s_{n-1}$ does not repeat. Hence every cycle in $\mathrm{G}(\mathbf{S})$ is a loop, and $\mathbf{S}$ is division ordered. □

If, in the above proof, we choose **B** to be a relatively free algebra of the variety $\mathsf{V}(\mathbf{A})$, we see that the equalities guaranteed by Lemma 7.2 are expressible as identities satisfied by the variety. So for locally finite varieties, the division-order property is equational. The list of identities one obtains is infinite (since we get an identity for each possible finite sequence of translations in a free algebra), but in many cases these all follow from some finite list of laws. For example, it is known that the division-order property of the variety generated by a finite division-ordered semigroup is equivalent to the satisfaction of the identities $x^n \approx x^{n+1}$, $(xy)^n \approx (yx)^n$ for some n (see Exercise 5.1.5 in Almeida [1] for example).

Let us say that the division-order property of a class $\mathscr{K}$ is *finitely defined* if there is a first order sentence Ψ such that any model of Ψ is division ordered and $\mathscr{K} \models \Psi$. It is trivial that the division-order property is finitely defined in any finitely axiomatised division-ordered class.

Proposition 7.4. *The division-order property is finitely defined in any division-ordered variety of finite type and with* FDD.

Proof. Assume $F \subseteq T_x$ determines division in a division-ordered variety $\mathscr{V}$ of finite type. Let Σ be the finite set of identities guaranteed by Theorem 5.3. Now let Ξ denote the set of quasi-identities in (DO) (see proof of Lemma 7.1) restricted to terms from F. Then $\Sigma \cup \Xi$ is a finite set of quasi-identities satisfied by $\mathscr{V}$, any model of which is division ordered. □

Example 7.5. Let **F** be the finite division-ordered total algebra of type $\langle 2 \rangle$ introduced in Example 5.4. Then $\mathsf{V}(\mathbf{F})$ does *not* have FDD but the division order property of $\mathsf{V}(\mathbf{F})$ is finitely defined.

Proof. In Example 5.4, we showed that $\mathsf{V}(\mathbf{F})$ fails to have FDD. In the proof of this fact we used a known finite basis $\Sigma_{\mathbf{F}}$ for the identities of **F**. Since **F** is division ordered, Theorem 7.3 shows that $\mathsf{V}(\mathbf{F})$ is division ordered, and hence the division-order property is finitely defined by $\Sigma_{\mathbf{F}}$. As we now show, we do not need to know that $\Sigma_{\mathbf{F}}$ is a basis for the identities of **F** in order to show that $\mathrm{Mod}(\Sigma_{\mathbf{F}})$ is division ordered.

Assume $\mathbf{G} \models \Sigma_{\mathbf{F}}$ and that we have $a, b \in G$ with $a \rightsquigarrow b$ and $b \rightsquigarrow a$. We show that $a = b$. Now either $a = b$ or both a and b are composite elements of **G** in the sense that they are in the range of the multiplication operation. Also if one of a or b is 0, then they both are (as 0 is absorbing). Using the identity $x(yz) \approx 0$, we can deduce that any product involving a that leads to b is of the form $(\cdots(ac_1)\cdots)c_n = b$. Likewise we can assume that $(\cdots(bd_1)\cdots)d_m = a$. We assume a left-bracketing convention, so that these equalities can be written $ac_1c_2\ldots c_n = b$ and $bd_1d_2\ldots d_m = a$. Then

$$\begin{aligned} b &= ac_1c_2\ldots c_n = bd_1\ldots d_mc_1\ldots c_n \\ &= bd_1\ldots d_md_1\ldots d_mc_1\ldots c_n = ad_1\ldots d_mc_1\ldots c_n \\ &= ac_1\ldots c_md_1\ldots d_m = bd_1\ldots d_m = a, \end{aligned}$$

as required.

So the division-order property of $\mathsf{V}(\mathbf{F})$ is finitely defined by $\Sigma_{\mathbf{F}}$. □

For another proof that the division-order property is finitely defined for $\mathsf{V}(\mathbf{F})$, observe that $\mathbf{F}$ is isomorphic to a subalgebra of the division-ordered algebra $\mathbf{B}$ in Bajusz et al. [2], which is shown to have a finite identity basis.

8. THE FINITE BASIS PROBLEM

We consider only total algebras in this section, and assume finite type throughout. The division relation provides a new framework for examining the axiomatisability properties of finitely generated varieties. We present a result relating the axiomatisability of division-ordered varieties to the axiomatisability properties of the class of subdirectly irreducible algebras in the variety. Before giving this result, we recall some background facts.

We begin with a folklore lemma (for a proof see Baker et al. [3]). Recall that an algebra $\mathbf{A}$ is subdirectly irreducible if it has a unique minimal congruence.[3] In other words, there is a pair of distinct elements $a, b \in A$ such that every nontrivial congruence on $\mathbf{A}$ contains $\mathrm{cg}^{\mathbf{A}}(a, b)$. Such a pair is said to be a *critical pair*. When $\mathscr{K}$ is a class of algebras, we use the notation $\mathscr{K}_{\mathrm{s.i.}}$ to denote the subdirectly irreducible members of $\mathscr{K}$. We use $\mathsf{V}_{\mathrm{s.i.}}(\mathscr{K})$ to abbreviate $[\mathsf{V}(\mathscr{K})]_{\mathrm{s.i.}}$.

Lemma 8.1. *The following are equivalent for an elementary class $\mathscr{K}$:*

(1) *$\mathscr{K}_{\mathrm{s.i.}}$ is closed under taking ultraproducts;*
(2) *$\mathscr{K}_{\mathrm{s.i.}}$ is an elementary class;*
(3) *$\mathscr{K}_{\mathrm{s.i.}}$ is finitely axiomatisable within $\mathscr{K}$;*
(4) *There is a number n and finite set $F \subseteq T_{1x}$ such that $\mathscr{K}_{\mathrm{s.i.}}$ satisfies*

$$\exists c \exists d \big(c \not\approx d \ \& \ [\forall a \forall b \ (a \not\approx b \to \{a, b\} \looparrowright^n_F \{c, d\})]\big); \qquad (\Gamma^n_F)$$

(5) *There is a number m such that $\mathscr{K}_{\mathrm{s.i.}}$ satisfies*

$$\exists c \exists d \big(c \not\approx d \ \& \ [\forall a \forall b \ (a \not\approx b \to \{a, b\} \looparrowright^m_m \{c, d\})]\big). \qquad (\Gamma^m_m)$$

(As explained in Section 3, both Γ^n_F and Γ^m_m are first order sentences since the property $\{a, b\} \looparrowright^n_F \{c, d\}$ can be written as a first order formula.)

A famous conjecture due to Park, and often attributed to Jónsson states that a finite algebra $\mathbf{A}$ has a finite axiomatisation for its identities provided its variety contains only finitely many distinct subdirectly irreducibles. In this case, the local finiteness of $\mathsf{V}(\mathbf{A})$ and the theorem of Quackenbush (see Theorem V.3.8 of Burris and Sankappanavar [6]) ensure that the subdirectly irreducibles in $\mathsf{V}(\mathbf{A})$ will all be finite, and hence form a strictly elementary class. A strengthening of this "Jónsson-Park conjecture" (see McNulty and Wang [25] for example) states that $\mathbf{A}$ has a finite identity basis provided that $\mathsf{V}_{\mathrm{s.i.}}(\mathbf{A})$ is a strictly elementary class. To clarify these relationships, we mention that all known examples so far point toward the situation described in Table 1. The table compares the possible axiomatisability properties of a finitely generated variety $\mathscr{V}$ to the axiomatisability properties of $\mathscr{V}_{\mathrm{s.i.}}$, and is to be read as an observed implication: *there are no known* examples contradicting the stated implication.

[3]In this article we do not consider the one element algebra to be subdirectly irreducible.

Table 1 Axiomatisability for finitely generated varieties: relationships supported by all currently known examples

	$\mathscr{V}_{\text{s.i.}}$ is:		$\mathscr{V}$ is:
1.	finite	⇒	finitely axiomatised
2.	finitely axiomatised	⇒	finitely axiomatised
3.	first order but not finitely axiomatisable	⇒	not finitely axiomatisable
4.	not first order	⇒	not finitely axiomatisable

The first row corresponds to the Jónsson–Park conjecture, while the second corresponds to the strengthening just described (in particular, it implies row 1). There are many examples supporting these two implications (a number of which are supplied by the results below). The third row is the only implication currently known to be true: by Lemma 8.1, if $\mathscr{V}$ is finitely axiomatised and $\mathscr{V}_{\text{s.i.}}$ is axiomatisable, then $\mathscr{V}_{\text{s.i.}}$ is finitely axiomatisable (the contrapositive of row 3). The fourth row is also supported by numerous examples: McKenzie's algebra **A** in McKenzie [24] is an example (as is the 6 element division-ordered groupoid underlying **A**). If all of these implications are true, then $\mathscr{V}$ is finitely axiomatised if and only if $\mathscr{V}_{\text{s.i.}}$ is finitely axiomatised. The results in this section confirm this relationship for various classes of algebras. We mention that (to the authors' knowledge) none of the implications are known to be contradicted by locally finite varieties either.

The following lemma is due to Jónsson and is frequently used below.

Lemma 8.2. *Let $\mathscr{U}$ be a variety contained within a finitely axiomatised class $\mathscr{K}$ in which $\mathscr{K}_{\text{s.i.}}$ is axiomatisable. Then $\mathscr{U}$ is finitely axiomatisable if and only if $\mathscr{U}_{\text{s.i.}}$ is finitely axiomatisable.*

This lemma also reveals a connection between rows 2 and 4 of Table 1. Recall that a locally finite variety $\mathscr{V}$ is *inherently nonfinitely based* if every locally finite variety containing $\mathscr{V}$ is without a finite identity basis. Lemma 8.2 shows that *if* the implication of row 4 is true, then any locally finite variety contradicting row 2 (or row 1) must be inherently nonfinitely based. Indeed, if a variety $\mathscr{V}$ is finitely axiomatised and locally finite then row 4 implies that $\mathscr{V}_{\text{s.i.}}$ is axiomatisable, whence Lemma 8.2 would show that every subvariety of $\mathscr{V}$ is finitely axiomatisable if and only if its subdirectly irreducibles are a strictly elementary class.

8.1. A Finite Basis Theorem for Division-Ordered Finite Algebras

Division-ordered finite algebras have been a frequent source of counterexamples in the study of the finite basis property. Lyndon's algebra in Lyndon [23] (we denote it by **L**) is division ordered as is the finitely based algebra **B** used in Bajusz et al. [2] to prove that **L** is not inherently nonfinitely based. The 6-element groupoid underlying McKenzie's algebra **A** in McKenzie [24] is division ordered (although **A** itself has universal division). Finite division-ordered semigroups (equivalently, finite $\mathscr{J}$-trivial semigroups) are a particularly abundant source of counterexamples, including the "equal first" example of a nonfinitely based finite semigroup (the semigroup $\mathbf{H}_1$ of Perkins [29]), the first examples of finite algebras with infinite irredundant identity bases (Jackson [17]), and the (uniformly locally finite) semigroup $\mathbf{S}_\infty$ that is minimally inherently nonfinitely based (Sapir [31]). See also Jackson and Sapir [21] and Jackson [18] for numerous other $\mathcal{J}$-trivial counterexamples concerning the finite basis property.

In this subsection, we prove the following result.

Theorem 8.3. *Let $\mathcal{V}$ be a division-ordered variety with FDSC and of finite type. Then $\mathcal{V}_{\text{s.i.}}$ is finitely axiomatised relative to $\mathcal{V}$, and $\mathcal{V}$ is finitely axiomatised if and only if $\mathcal{V}_{\text{s.i.}}$ is.*

The proof covers most of this subsection.

We begin with a definition. Following Baker et al. [3], we say that a class $\mathcal{K}$ has *term finite critical depth* (TFCD) if there is a finite set of terms $F \subseteq T_x$ such that whenever $\mathbf{A} \in \mathcal{K}_{\text{s.i.}}$ and $\{c, d\}$ is a critical pair of $\mathbf{A}$, then $\{a, b\} \looparrowright_F \{c, d\}$ for any distinct $a, b \in A$. The set F is said to *witness the TFCD property* in $\mathcal{K}$. Clearly, the TFCD property is a particularisation of the TFPC property defined in Section 3.

Lemma 8.4. *Let $\mathbf{S}$ be a division-ordered subdirectly irreducible algebra. The following hold*:

i) *There is an absorbing element, $0 \in S$;*
ii) *There is an element $p \in S\backslash\{0\}$ such that every element of $S\backslash\{0\}$ divides p;*
iii) *The pair $\{0, p\}$ is the unique critical pair for* $\mathbf{S}$.

Proof. Let $\{p, q\}$ be a critical pair in $\mathbf{S}$. As $\mathbf{S}$ is division ordered, we cannot have both $p \rightsquigarrow q$ and $q \rightsquigarrow p$, so without loss of generality, we may assume that $q \not\rightsquigarrow p$. Thus $q \in J_p$, so the assumption that $\mathbf{S}$ is subdirectly irreducible ensures that $\mathbf{S}/J_p = \mathbf{S}$. That is, $J_p = \{q\}$ and q is an absorbing element; we denote it by 0. This also shows that every element of $S\backslash\{q\}$ divides p, which for a division-ordered algebra can be satisfied by at most one pair (p, q). Hence the pair $\{0, p\}$ is the unique critical pair. □

We mention here that it is already possible to obtain the restriction of Theorem 8.3 to the class of $\mathcal{J}$-trivial semigroups by using Lemma 8.4 above and Theorem 3.7 of Schein [32].

Lemma 8.5. *Let $\mathcal{K}$ be a division-ordered elementary class. The following are equivalent*:

(1) *$\mathcal{K}$ has term finite critical depth;*
(2) *There is a finite set of terms $G \subseteq T_{1x}$ such that $\mathcal{K}_{\text{s.i.}}$ satisfies Γ^1_G;*
(3) *$\mathcal{K}_{\text{s.i.}}$ is axiomatisable.*

Proof. (1) ⇒ (2). Let $F \subseteq T_{1x}$ witness the TFCD property in $\mathcal{K}_{\text{s.i.}}$. We claim that the following sentence axiomatises $\mathcal{K}_{\text{s.i.}}$ within $\mathcal{K}$:

$$\exists x \exists y \forall u \forall v \; (x \not\approx y \;\&\; (u \not\approx v \rightarrow \{u, v\} \looparrowright^1_F \circ \looparrowright^1_F \{x, y\})). \qquad (\Phi_F)$$

Certainly, any model of this sentence is subdirectly irreducible. Note also that Φ_F implies Γ^1_G, where $G = \{s(t(x, \vec{z}), \vec{w}) \mid s, t \in F\}$.

For any $\mathbf{S} \in \mathscr{K}_{\text{s.i.}}$, there is by Lemma 8.4 a unique critical pair $\{0, p\}$ (where 0 is absorbing). Pick any pair of distinct elements a, b in $\mathbf{S}$. By assumption we have $\{a, b\} \looparrowright_F \{0, p\}$. Using the first step of the Mal'cev scheme witnessing this, we find a translation $\lambda(x)$ built from F such that $\{\lambda(a), \lambda(b)\} = \{0, q\}$ for some q. If $q = p$ we are done. Otherwise, we have $\{q, 0\} \looparrowright_F \{0, p\}$. Again, using the first step of the Mal'cev scheme, we have a translation γ built from F with $\{\gamma(q), \gamma(0)\} = \{p, r\}$ for some $r \neq p$. However 0 is an absorbing element, so we have $\gamma(q) = p$ and $r = 0$. Then $\{\gamma(\lambda(a)), \gamma(\lambda(b))\} = \{\gamma(q), \gamma(0)\} = \{0, p\}$. This is the statement $\{a, b\} \looparrowright^1_F \circ \looparrowright^1_F \{0, p\}$. Hence $\mathbf{S}$ satisfies Φ_F, whence Γ^1_G.

(2) $\Rightarrow$ (3) is (4) $\Rightarrow$ (2) of Lemma 8.1.

(3) $\Rightarrow$ (1). First observe that implication (2) $\Rightarrow$ (4) of Lemma 8.1 shows that there is a finite set of terms F such that for any $\mathbf{S} \in \mathscr{K}_{\text{s.i.}}$, there is *some* critical pair $\{c, d\}$ for $\mathbf{S}$ with F determining $\{a, b\} \looparrowright \{c, d\}$ for any distinct $a, b \in S$. As Lemma 8.4 shows that $\{c, d\}$ is unique, we find that F witnesses the TFCD property. □

Theorem 8.6. *Let $\mathscr{V}$ be a division-ordered variety of finite type. Assume there is a sentence Δ satisfied by $\mathscr{V}$ and such that* Mod(Δ) *is division ordered and has TFCD. Then $\mathscr{V}_{\text{s.i.}}$ is finitely axiomatised relative to $\mathscr{V}$, and $\mathscr{V}$ has a finite basis for its identities if and only if the class of subdirectly irreducible members of $\mathscr{V}$ is strictly elementary.*

Proof. Since $\mathscr{V}$ is division ordered and has TFCD, $\mathscr{V}_{\text{s.i.}}$ is axiomatisable by Lemma 8.5, and therefore finitely axiomatisable relative to $\mathscr{V}$ by Lemma 8.1.

Lemma 8.5 also shows that $[\text{Mod}(\Delta)]_{\text{s.i.}}$ is axiomatisable, so by Lemma 8.2, $\mathscr{V}$ is finitely axiomatisable if and only if $\mathscr{V}_{\text{s.i.}}$ is finitely axiomatisable. □

We may now prove the main theorem of this section (Theorem 8.3), which is in fact a restriction of Theorem 8.6.

Proof. By Theorem 8.6, it suffices to find a sentence Δ satisfied by $\mathscr{V}$ such that Mod(Δ) is division ordered and has TFCD. By Lemma 3.4, there is a sentence Φ satisfied by $\mathscr{V}$ such that Mod(Φ) has FDSC (and therefore TFCD). By Example 4.3, $\mathscr{V}$ has FDD, so Proposition 7.4 shows that there is a sentence Ψ satisfied by $\mathscr{V}$ such that Mod(Ψ) is division ordered. Let $\Delta := \Phi \,\&\, \Psi$. □

Theorem 8.3 shows that if there is a division-ordered counterexample to the Jónsson–Park conjecture (or even row 2 of Table 1), then it cannot generate a variety with FDSC. The algebra $\mathbf{F}$ of Examples 5.4 and 7.5 is particularly interesting in this context since it is division ordered, but its variety fails to have FDD (let alone FDSC). Furthermore the authors can show that it generates a residually very finite variety (in fact the only subdirectly irreducibles in its variety are $\mathbf{F}$ and its subalgebras), however we have already observed that it has a finite identity basis, so it is not a counterexample to the Jónsson–Park conjecture.

There is another division-related property of a variety that implies the TFCD property on subdirectly irreducible division ordered algebras and hence, provided the division-order property is finitely determined, can replace the assumption of FDSC in Theorem 8.3. Say that a variety $\mathscr{V}$ has *finitely determined nonconstant division* (FDND) if there is a finite set of terms $F \subseteq T_x$ such that for every algebra $\mathbf{A} \in \mathscr{V}$, if $a, b, c \in A$ have $\{c\} \subsetneq \{\lambda(a), \lambda(b)\}$ for some translation λ, then there is a translation $\lambda'(x)$ built from a term in F with $\{c\} \subsetneq \{\lambda'(a), \lambda'(b)\}$. This property can be used in the proof of Lemma 8.5 to yield the sentence Φ_F of Lemma 8.5. It is easily seen to be implied by TFPC, but moreover it is a proper generalisation: the 4 element linear algebra of Davey et al. [10] has an underlying additive group structure, and so has FDND, but it fails to have TFPC. The FDND concept can be given a development along the lines of that for FDD in Section 5 above; in particular, it is easily seen to be equivalent to the existence of a first order formula $\Xi(x, y, z)$ defining syntactic congruences of singleton subsets (that is $(a, b) \in \mathrm{syn}^{\mathbf{A}}(\{c\}) \Leftrightarrow \mathbf{A} \vDash \Xi(a, b, c)$); cf. Remark 3.2. However we were unable to find any finite division-ordered algebras with FDND that did not also have TFPC, hence we have no examples distinguishing Theorem 8.3 from its nonconstant division analogue.

8.2. Some Further Finite Axiomatisability Theorems for Finite Semigroups

A recent article by McNulty and Wang [25] verified that the subdirectly irreducible groups in a finitely generated group variety form an axiomatisable class; hence by Lemma 8.1, and the Oates–Powell Theorem (every finitely generated group variety is finitely axiomatisable, Oates and Powell [27]), the subdirectly irreducible groups in a finitely generated group variety are a strictly elementary class. This confirms row 2 of Table 1 for groups (rows 3 and 4 hold vacuously). This result encouraged the authors to look for similar results amongst other classes of semigroups, which in turn led to Theorem 8.3.

In this subsection we observe some further finite basis results of this style using known characterisations of certain subdirectly irreducible semigroups. The only result requiring detailed discussion turns out to be a natural complement to Theorem 8.3; see Theorem 8.9 below.

First recall that a semigroup or variety of semigroups is *periodic* if there are natural numbers $i, p > 0$ such that $x^i \approx x^{i+p}$ holds. A nonperiodic variety contains the one-generated free semigroup, which generates the variety of all commutative semigroups.

We now recall some known descriptions of subdirectly irreducible semigroups from any of the following classes:

a) Nilpotent semigroups (meaning, there is a number k such that all products of length k coincide; Theorem 4.6 of Schein [32]);
b) Nil-semigroups (meaning, every element has a power equal to 0; Theorem 4.5 of Schein [32]);
c) Commutative semigroups (§5 of Schein [32]; see also Grillet [15], Demlová and Koubek [13], and Theorem 2.4 of Demlová and Koubek [14]);
d) Completely simple semigroups (defined below; Demlová and Koubek [14]).

Theorem 8.7. *Let $\mathscr{V}$ be a locally finite semigroup variety contained within one of the classes: nilpotent semigroups; nil-semigroups; commutative semigroups. Then $\mathscr{V}_{\text{s.i.}}$ is finitely axiomatised relative to $\mathscr{V}$, and $\mathscr{V}$ is finitely axiomatised if and only if $\mathscr{V}_{\text{s.i.}}$ is.*

Proof. In the nilpotent case, $\mathscr{V}$ must be contained within the semigroup variety $\mathscr{V}_1$ defined by $x_1 \dots x_n \approx y_1 \dots y_n$ for some n. Since Schein's description of subdirectly irreducible nilpotent semigroups is trivially verified to be first order, we can use Lemma 8.2 with $\mathscr{K} := \mathscr{V}_1$. In the nil-semigroup case, $\mathscr{V}$ is contained within the semigroup variety $\mathscr{V}_2$ defined by $x^n y \approx yx^n \approx y$ for some n. Again, Schein's description is trivially seen to be first order, so we can use Lemma 8.2 with $\mathscr{K} := \mathscr{V}_2$.

In the commutative semigroup case, the local finiteness of $\mathscr{V}$ implies that there are numbers i, d such that $\mathscr{V} \models x^i \approx x^{i+d}$. The characterisation (as presented in, say, Theorem 2.4 of Demlová and Koubek [14]) of subdirectly irreducibles in the variety of commutative semigroups is routinely seen to be first order when restricted to the subvariety $\mathscr{V}_3$ satisfying $x^i \approx x^{i+d}$ (we leave this to the reader to verify). So we can use Lemma 8.2 with $\mathscr{K} := \mathscr{V}_3$. □

Local finiteness is not required for the nilpotent semigroup or nil-semigroup case (only periodicity, which is already necessary to ensure that $\mathscr{V}$ consists entirely of the desired kind of semigroup). The variety of all commutative semigroups is the only non-locally finite variety of commutative semigroups, but its subdirectly irreducible members are not axiomatisable in first order logic. Indeed, the Prüfer groups are the only subdirectly irreducible infinite commutative groups (see Theorem 2.4 of Demlová and Koubek [14] for example), and they have cardinality $2^{\aleph_0}$, while one may find an ultrapower of a Prüfer group of arbitrary cardinality (which evidently is not itself a Prüfer group and hence not subdirectly irreducible). We mention that Perkins [29] has proved that every variety of commutative semigroups is finitely based, and observed that this is trivially true for varieties of nilpotent semigroups. On the other hand, there are many nonfinitely based varieties of nil-semigroups, including the inherently nonfinitely based example found by Sapir [31] (which is also division ordered).

Now we can turn to completely simple semigroups. A *simple semigroup* is a semigroup containing no proper, nontrivial ideals (that is, has universal division). A *completely simple semigroup* is a simple semigroup satisfying: $(x^2 \approx x \ \& \ y^2 \approx y \ \& \ xy \approx yx \approx x) \rightarrow x \approx y$. This quasi-identity holds in every simple periodic semigroup.

Proposition 8.8. *The following are equivalent for a universal Horn class $\mathscr{H}$ of semigroups:*

(1) *$\mathscr{H}$ has universal division;*
(2) *$\mathscr{H}$ consists of completely simple semigroups;*
(3) *There is a number d such that $\mathscr{H}$ consists of completely simple semigroups of period dividing d;*
(4) $\mathscr{H} \models \{x^{d+1} \approx x, (xyx)^d x \approx x\}$.

Proof. (1) ⇒ (2). Certainly, $\mathscr{H}$ consists of simple semigroups. However, if a member of $\mathscr{H}$ contains a failure of $(x^2 \approx x \ \& \ y^2 \approx y \ \& \ xy \approx yx \approx x) \rightarrow x \approx y$, at a pair $x = e$ and $y = f$, then $\{e, f\}$ forms a subsemigroup that fails to have universal division (it is a semilattice).

(2) $\Rightarrow$ (3). Otherwise, the 1-generated relatively free semigroup in $\mathscr{H}$ is the 1-generated free semigroup, which is not completely simple.

(3) $\Rightarrow$ (4). Follows from the Rees–Sushkevich Theorem; see Theorem 3.3.1 of Howie [16] for example.

(4) $\Rightarrow$ (1). Follows using the term $p(x, y) := (yxy)^d y$ and Proposition 6.3. $\square$

Now we can state the following result, which confirms Table 1 for another class of semigroups, forms a natural complement to Theorem 8.3 and is an extension of the result of McNulty and Wang [25] (the proof depends heavily on their result).

Theorem 8.9. *Let* $\mathbf{S}$ *be a finite semigroup with universal division. The class of subdirectly irreducible algebras in* $\mathsf{V}(\mathbf{S})$ *is finitely axiomatised relative to* $\mathsf{V}(\mathbf{S})$, *and* $\mathsf{V}(\mathbf{S})$ *has a finite axiomatisation if and only if* $\mathsf{V}_{\text{s.i.}}(\mathbf{S})$ *has.*

The proof of this theorem will cover the remainder of this subsection. We again direct the reader to a semigroup theoretic text such as Howie [16] for background information on semigroups.

Recall that in a completely simple semigroup, Green's equivalence relations $\mathscr{L}$ and $\mathscr{R}$ are the solution sets to

$$L(x, y) := x \approx y \vee \exists \ell_1 \exists \ell_2 (\ell_1 x \approx y \ \& \ \ell_2 y \approx x)$$

and

$$R(x, y) := x \approx y \vee \exists r_1 \exists r_2 (x r_1 \approx y \ \& \ y r_2 \approx x),$$

respectively.

Green's relation $\mathscr{H}$ is given by $\mathscr{R} \cap \mathscr{L}$; that is, the solution set to $H(x, y) := L(x, y) \ \& \ R(x, y)$. The $\mathscr{H}$-classes of a completely simple semigroup $\mathbf{S}$ are the maximal subgroups, and all are isomorphic. This group is called the *structure group* of $\mathbf{S}$.

Here is the characterisation of subdirectly irreducible completely simple semigroups, taken from the proof of Corollary 2.8 of Demlová and Koubek [14] (rather than their actual statement).[4]

Theorem 8.10 (Corollary 2.8 of Demlová and Koubek [14]). *A completely simple semigroup* $\mathbf{S}$ *is subdirectly irreducible if and only if its structure group is subdirectly irreducible and the following property holds*:

($\heartsuit$) *for every pair of distinct elements* x, y, *if* $x \mathscr{L} y$ *then there is* s *such that* $xs \mathscr{H} ys$ *but* $xs \neq ys$, *while if* $x \mathscr{R} y$, *then there is* s *such that* $sx \mathscr{H} sy$ *but* $sx \neq sy$.

[4]The statement of Corollary 2.8 of Demlová and Koubek [14] requires the phrase "where P is in normal form" to be added for it to read correctly.

This formulation (and the definability of the relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$) reveals that subdirect irreducibility is first order amongst completely simple semigroups with subdirectly irreducible structure groups.

Now, the main result of McNulty and Wang [25] is that if $\mathbf{G}$ is a finite group then $\mathsf{V}_{\text{s.i.}}(\mathbf{G})$ can be axiomatised within $\mathsf{V}(\mathbf{G})$ by the sentence Γ_n^n of Lemma 8.1:

$$\exists c \exists d \big(c \not\approx d \ \&\ \forall a \forall b\, (a \not\approx b \rightarrow \{a, b\} \looparrowright_n^n \{c, d\})\big) \qquad (\Gamma_n^n)$$

for some n. As the $\mathscr{H}$ relation on a semigroup is first order definable, we can make a new sentence $\overline{\Gamma}_n^n$ from Γ_n^n by asking that each variable appearing in Γ_n^n lie in the same $\mathscr{H}$-class (simply insert—in suitable places—formulæ of the form $H(z, c)$ for each variable z appearing in Γ_n^n). Then the completely simple semigroups satisfying $\overline{\Gamma}_n^n$ are precisely those whose maximal subgroups satisfy Γ_n^n.

By a well known result of Oates and Powell [27], every finite group generates a semigroup variety that is finitely axiomatised by the laws of the form $\{x^d y \approx yx^d \approx y, u \approx x^d\}$, where d is the exponent, $u = u(x_1, \ldots, x_n)$ is some semigroup word and x is a single variable not appearing in $u = u(x_1, \ldots, x_n)$. For a finite group $\mathbf{G}$, let us denote by $\mathsf{CS}_\mathbf{G}$, the class of all completely simple semigroups whose subgroups lie in the semigroup variety $\mathsf{V}(\mathbf{G})$.

The following "folklore" fact follows easily from the Rees–Sushkevich Theorem. We omit the proof.

Lemma 8.11. *Let* $\mathbf{G}$ *be a finite group, and* $\{u \approx x^d, x^d y \approx yx^d \approx y\}$ *axiomatise the semigroup variety* $\mathsf{V}(\mathbf{G})$. *Then* $\mathsf{CS}_\mathbf{G}$ *is a variety axiomatised by*

$$\Sigma_\mathbf{G} := \{x^{d+1} \approx x,\, (xyx)^d \approx x^d,\, u(x^d x_1 x^d, \ldots, x^d x_n x^d) \approx x^d\}.$$

Lemma 8.12. *Let* $\mathbf{G}$ *be a finite group. The class of subdirectly irreducible semigroups in* $\mathsf{CS}_\mathbf{G}$ *is finitely axiomatised within* $\mathsf{CS}_\mathbf{G}$.

Proof. Let n be such that $\mathsf{V}_{\text{s.i.}}(\mathbf{G}) \models \Gamma_n^n$, which exists by McNulty and Wang [25]. The completely simple semigroups whose maximal subgroups satisfy Γ_n^n are those satisfying $\overline{\Gamma}_n^n$. Then, by Theorem 8.10, the class of subdirectly irreducible semigroups in $\mathsf{CS}_\mathbf{G}$ is axiomatised by the property (♡) and $\overline{\Gamma}_n^n$. □

Finally, we may prove Theorem 8.9.

Proof. Since $\mathbf{S}$ is finite, it is periodic and therefore completely simple. Let $\mathbf{G}$ be a maximal subgroup of $\mathbf{S}$. So $\mathsf{V}(\mathbf{S}) \subseteq \mathsf{CS}_\mathbf{G}$. Now apply Lemma 8.2 using $\mathscr{K} := \mathsf{CS}_\mathbf{G}$ (by Lemmas 8.11 and 8.12) and $\mathscr{U} := \mathsf{V}(\mathbf{S})$. □

We mention that—unlike for finite groups—not every finite completely simple semigroup has a finite identity basis; see Mashevitzky [26].

9. FLAT ALGEBRAS

In this section we show the definability of $\rightsquigarrow$ in the members of a universal Horn class is intimately related to the DPC and TFPC properties for varieties generated by sink algebras (flat algebras with absorbing bottom). Delić [12] classified the so-called flat graph algebras (which are sink algebras) with a finite identity basis, and showed that this is equivalent to the generated variety having the DPC property (the two-way implication is not explicitly stated in Delić [12]; see the Math Review by Bill Lampe for explanation of the claim). In general the DPC property is not equivalent to the finite basis property amongst sink algebras; see Jackson [19] (other examples are provided by combining Corollaries 6.4, 6.5 and Theorem 9.6 of the present article). The TFPC property (or rather, its failure) for varieties generated by flat algebras, and sink algebras in particular, has also arisen in the ostensibly unrelated study of compact, totally disconnected algebraic structures; see Example 7.7 of Clark et al. [8] or §5 of Jackson [20] for example. As we show below, all of the properties DPC, TFPC, and FDD are equivalent for varieties generated by sink algebras. Moreover, we show that the algorithmic problem of deciding when a sink algebra generates a variety with DPC is equivalent to the problem of deciding when an arbitrary partial algebra or an arbitrary total algebra generates a universal Horn class with FDD.

We first prove that if $\mathbf{A}$ is a partial algebra that generates a universal Horn class with FDD, then $\mathsf{V}(\flat(\mathbf{A}))$ has SC^2_m. We will then combine this with a result in Delić [12] to show that $\mathsf{V}(\flat(\mathbf{A}))$ has DPC.

Lemma 9.1. *Let $\mathbf{A}$ be a partial algebra. Let $s(x, \vec{z})$, $t(x, \vec{z})$ be terms in the language of $\mathbf{A}$, and suppose that the variables occurring in $t(x, \vec{z})$ are among those occurring in $s(x, \vec{z})$. Then we have*

$$\mathbf{A} \models s(x, \vec{z}) \approx s(x, \vec{z}) \rightarrow s(x, \vec{z}) \approx t(x, \vec{z}) \iff \flat(\mathbf{A}) \models s(x, \vec{z}) \leq t(x, \vec{z}).$$

Proof. First suppose $\mathbf{A} \models s(x, \vec{z}) \approx s(x, \vec{z}) \rightarrow s(x, \vec{z}) \approx t(x, \vec{z})$. Let $a, \vec{c}$ be elements of $\flat(A)$. If $s^{\flat(\mathbf{A})}(a, \vec{c}) = 0$, then we are done. Otherwise, $s^{\flat(\mathbf{A})}(a, \vec{c}) \neq 0$, so all the variables that appear in s are elements of A, and therefore, all the variables appearing in t are elements of A. So without loss of generality, we may assume that $a, \vec{c} \in A$. Therefore $s^{\flat(\mathbf{A})}(a, \vec{c}) = s^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(a, \vec{c})$, and hence $s^{\flat(\mathbf{A})}(a, \vec{c}) = t^{\flat(\mathbf{A})}(a, \vec{c})$.

Now suppose $\flat(\mathbf{A}) \models s(x, \vec{z}) \leq t(x, \vec{z})$. Let $a, \vec{c}$ be elements of A such that $s^{\mathbf{A}}(a, \vec{c})$ is defined. Then $s^{\mathbf{A}}(a, \vec{c}) = s^{\flat(\mathbf{A})}(a, \vec{c}) \neq 0$ and as $s^{\flat(\mathbf{A})}(a, \vec{c}) \leq t^{\flat(\mathbf{A})}(a, \vec{c})$, we have $s^{\mathbf{A}}(a, \vec{c}) = s^{\flat(\mathbf{A})}(a, \vec{c}) = t^{\flat(\mathbf{A})}(a, \vec{c}) = t^{\mathbf{A}}(a, \vec{c})$. □

Lemma 9.2. *Let $\mathbf{A}$ be a partial algebra, and let $f(\underline{x}, y, z)$ be a term in the language of $\flat(\mathbf{A})$. Then $\flat(\mathbf{A}) \models f(x \wedge y, x, \vec{z}) \approx f(x \wedge y, y, \vec{z})$.*

Proof. If $a \neq b$ in $\flat(A)$, then $a \wedge b = 0$, and then for any $\vec{c}$ in $\flat(A)$, we have $f^{\flat(\mathbf{A})}(a \wedge b, a, \vec{c}) = 0 = f^{\flat(\mathbf{A})}(a \wedge b, b, \vec{c})$. Otherwise, $a = b$, so that again, $f^{\flat(\mathbf{A})}(a \wedge b, a, \vec{c}) = f^{\flat(\mathbf{A})}(a \wedge b, b, \vec{c})$.v □

The following lemma is proved by an elementary induction argument on the height of a term. We omit the details.

Lemma 9.3. *Let $\mathbf{B} = \langle B; \mathscr{F} \cup \{\wedge\}\rangle$ be an algebra in which all operations in $\mathscr{F}$ preserve the order of the semilattice operation $\wedge$. Let $t(x, \vec{z}) \in T_1(x)$ be a term in the language of $\mathbf{B}$. Then there exists a term $t'(\underline{x}, \vec{z})$ such that $t'(x, \vec{z})$ contains no $\wedge$ operations and $\mathbf{B} \models t(x, \vec{z}) \approx t(x, \vec{z}) \wedge t'(x, \vec{z})$.*

This lemma can be used to prove that if $\mathsf{ISP_r}(\langle B; \mathscr{F}\rangle)$ has FDD and all operations in $\mathscr{F}$ preserve the order of some semilattice operation $\wedge$ on B then $\mathsf{ISP_r}(\langle B; \mathscr{F} \cup \{\wedge\}\rangle)$ has FDD. We omit the details, but the idea is essentially the same as the first part of the next proof.

Lemma 9.4. *Let* $\mathbf{A}$ *be a partial algebra and* $F \subseteq T_x$ *be a set of terms in the language of* $\mathbf{A}$ *determining division in* $\mathsf{ISP_r}(\mathbf{A})$. *Then* $\mathsf{V}(\flat(\mathbf{A}))$ *has* SC^2_G, *where*

$$G := \{x_1 \wedge f(x \wedge x_2, \vec{z}) \mid f \in F\}.$$

Proof. Let $t(x, \vec{z}) \in T_{1x}$ be a term in the language of $\flat(\mathbf{A})$, and let $t'(\underline{x}, \vec{z})$ be the $\wedge$-free term guaranteed by Lemma 9.3; so $\flat(\mathbf{A}) \vDash t(x, \vec{z}) \approx t(x, \vec{z}) \wedge t'(x, \vec{z})$. As t' is a term in the language of $\mathbf{A}$, there is $h(x, \vec{z}) \in F$ and terms $w_1(x, \vec{z}), \dots, w_m(x, \vec{z})$ such that $\mathbf{A} \vDash t'(x, \vec{z}) \approx t'(x, \vec{z}) \to t'(x, \vec{z}) \approx h(x, w_1(x, \vec{z}), \dots)$, or by Lemma 9.1, such that $\flat(\mathbf{A}) \vDash t'(x, \vec{z}) \leq h(x, w_1(x, \vec{z}), \dots)$. So

$$\flat(\mathbf{A}) \vDash t(x, \vec{z}) \approx t(x, \vec{z}) \wedge t'(x, \vec{z}) \approx t(x, \vec{z}) \wedge h(x, w_1, \dots, w_m).$$

Therefore, we have the following (two step) shadowing of $t(x, \vec{z})$ by G:

$$t(x, \vec{z}) \approx t(x, \vec{z}) \wedge h(x \wedge x, w_1(x, \vec{z}), \dots),$$

$$t(x, \vec{z}) \wedge h(x \wedge y, w_1(x, \vec{z}), \dots, w_m(x, \vec{z})) \approx t(y, \vec{z}) \wedge h(x \wedge y, w_1(y, \vec{z}), \dots) \quad \text{(by Lemma 9.2)},$$

$$t(y, \vec{z}) \wedge h(y \wedge y, w_1(y, \vec{z}), \dots) \approx t(y, \vec{z}).$$

So $\mathsf{V}(\flat(\mathbf{A}))$ has SC^2_G. □

The following lemma is a slight strengthening of Theorem 13 in Delić [12].

Lemma 9.5. *Let* $\mathbf{A}$ *be a partial algebra. Then* $\mathsf{V}(\flat(\mathbf{A}))$ *has* PC^2.

Proof. Let $\mathbf{B} \in V(\flat(\mathbf{A}))$, and let $a, b, c, d \in \mathbf{B}$ with $\{a, b\} \looparrowright \{c, d\}$. So there are terms $t_1(\underline{x}, \vec{z}), \dots, t_n(\underline{x}, \vec{z})$, elements $\vec{f}_i$ of $\mathbf{B}$, and choices of $\{e_i, e_i'\} = \{a, b\}$ such that

$$\begin{aligned} c &= t_1^{\mathbf{B}}(e_1, \vec{f}_1) \\ t_1^{\mathbf{B}}(e_1', \vec{f}_1) &= t_2^{\mathbf{B}}(e_2, \vec{f}_2) \\ &\vdots \\ t_n^{\mathbf{B}}(e_n', \vec{f}_n) &= d. \end{aligned}$$

By Lemma 9.2 (and $\{e_1, e_1'\} = \{a, b\}$), we have

$$\begin{aligned} t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1) &= t_1^{\mathbf{B}}(e_1', \vec{f}_1) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1) \\ &= t_2^{\mathbf{B}}(e_2, \vec{f}_2) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1) = t_2^{\mathbf{B}}(e_2', \vec{f}_2) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1) \\ &\vdots \\ &= t_n^{\mathbf{B}}(e_n, \vec{f}_n) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1) = t_n^{\mathbf{B}}(e_n', \vec{f}_n) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1', \vec{f}_1). \end{aligned}$$

So in particular,

$$t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_1^{\mathbf{B}}(e_1 \wedge e'_1, \vec{f}_1) = t_n^{\mathbf{B}}(e'_n, \vec{f}_n) \wedge t_1^{\mathbf{B}}(e_1 \wedge e'_1, \vec{f}_1).$$

By a symmetric argument,

$$t_n^{\mathbf{B}}(e'_n, \vec{f}_n) \wedge t_n^{\mathbf{B}}(e_n \wedge e'_n, \vec{f}_n) = t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_n^{\mathbf{B}}(e_n \wedge e'_n, \vec{f}_n).$$

But then, using Lemma 9.2 twice, we have

$$t_n^{\mathbf{B}}(e'_n, \vec{f}_n) \wedge t_1^{\mathbf{B}}(e_1 \wedge e'_1, \vec{f}_1) = t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_n^{\mathbf{B}}(e_n \wedge e'_n, \vec{f}_n)$$

so that the following 2-step Mal'cev scheme witnesses $\{a, b\} \looparrowright \{c, d\}$:

$$\begin{aligned} c &= t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_1^{\mathbf{B}}(e_1 \wedge e_1, \vec{f}_1) \\ t_1^{\mathbf{B}}(e_1, \vec{f}_1) \wedge t_1^{\mathbf{B}}(e_1 \wedge e'_1, \vec{f}_1) &= t_n^{\mathbf{B}}(e'_n, \vec{f}_n) \wedge t_n^{\mathbf{B}}(e_n \wedge e'_n, \vec{f}_n) \\ t_n^{\mathbf{B}}(e'_n, \vec{f}_n) \wedge t_n^{\mathbf{B}}(e'_n \wedge e'_n, \vec{f}_n) &= d. \end{aligned}$$

□

Theorem 9.6. *Let* $\mathbf{A}$ *be a partial algebra of finite type. The following are equivalent*:

(1) $\mathsf{ISP_r}(\mathbf{A})$ *has* FDD;
(2) $\mathsf{ISP_r}(\flat(\mathbf{A}))$ *has* FDD;
(3) $\mathsf{V}(\flat(\mathbf{A}))$ *has* FDD;
(4) $\mathsf{V}(\flat(\mathbf{A}))$ *has* TFPC;
(5) $\mathsf{V}(\flat(\mathbf{A}))$ *has* DPC.

Proof. (1) ⇒ (5). By Lemma 9.4, we know that $\mathsf{V}(\flat(\mathbf{A}))$ has SC^2_G for some finite G. Moreover, $\mathsf{V}(\flat(\mathbf{A}))$ has PC^2 by Lemma 9.5. Hence $\mathsf{V}(\flat(\mathbf{A}))$ has PC^4_G by Proposition 3.9.

(5) ⇒ (4) trivially, and (4) ⇒ (3) by Example 4.3 and Theorem 3.5.

(3) ⇒ (1). Let H be a finite set of terms that determines division in $\mathsf{V}(\flat(\mathbf{A}))$. Let $t(\underline{x}, \vec{z})$ be a term in the language of $\mathbf{A}$. Then $t(x, \vec{z})$ is also a term in the language of $\flat(\mathbf{A})$, so there exists a term $h(x, \vec{z}) \in H$ and terms $\vec{u}(x, \vec{z})$ such that $t(x, \vec{z}) \approx h(x, \vec{u})$. By Lemma 9.3, there exists a term $h'(x, \vec{z})$ with no $\wedge$ operations (so that $h'(x, \vec{z})$ is a term in the language of $\mathbf{A}$) such that $h(x, \vec{z}) \leq h'(x, \vec{z})$ and therefore $h(x, \vec{u}) \leq h'(x, \vec{u})$. Thus $\flat(\mathbf{A}) \models t(x, \vec{z}) \leq h'(x, \vec{u})$ and therefore by Lemma 9.1, we have $\mathbf{A} \models t(x, \vec{z}) \approx t(x, \vec{z}) \to t(x, \vec{z}) \approx h'(x, \vec{u})$. Therefore, the set $\{h'(x, \vec{z}) \mid h(x, \vec{z}) \in H\}$ determines division in $\mathsf{ISP_r}(\mathbf{A})$.

Finally, (2) ⇔ (3) by Example 4.4. □

The *FDD problem* for a finite partial algebra is the problem of deciding when the universal Horn class of a partial algebra has FDD. The *DPC problem* (*TFPC problem*) of a (total) algebra is the problem of deciding when a given finite algebra generates a variety with DPC (TFPC, respectively).

Corollary 9.7. *The following algorithmic problems are polynomially equivalent*:

(a) *The FDD problem for finite partial algebras*;
(b) *The FDD problem for finite total algebras*;
(c) *The FDD problem for finite sink algebras*;
(d) *The TFPC problem for finite sink algebras*;
(e) *The DPC problem for finite sink algebras.*

Proof. The reduction of (a) to (b) and of (b) to (c) are supplied by taking the flat extension construction and using (1) $\Leftrightarrow$ (2) in Theorem 9.6. The properties (c), (d), (e) are equivalent by (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) in Theorem 9.6, which also provides a trivial reduction from (e) to (a). □

We mention that Kiss has shown that the DPC property is decidable for finitely generated congruence distributive varieties (Kiss [22]). The problem of determining decidability of the TFPC problem for finite algebras of finite type is Problem 9.3 of Clark et al. [8].

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