# Abstracts for Workshop on Differential Geometry & Applications La Trobe University 13th – 16th June 2006

## Lectures

**Thomas Ivey**

**Outline**

1. **Basic notions**:

   Differential ideals, Pfaff & Frobenius theorems, jet spaces and standard contact systems

2. **Involutivity**:

   Integral elements, the Cartan-Kahler theorem, Cartan's Test

3. **Geometric examples**:

   The Euclidean frame bundle, Frenet equations & vortex filament flow, surfaces in $R^3$ (CMC, Weingarten, canal surfaces); triply orthogonal systems; minimal hypersurfaces in $S^4$

4. **Prolongation**:

   General theory; linear Pfaffian systems & tableaux; more geometric examples, including Hopf hypersurfaces

5. **Characteristics**:

   The characteristic variety, Darboux's method, Hopf hypersurfaces in $CH^2$

**Keizo Yamaguchi – Geometry of linear differential systems – Towards the contact geometry of second order**

**Outline**

1. **Geometry of jet spaces**:

   Linear differential systems (Pfaffian systems); Grassmannian construction of geometric jet spaces; Strong and weak derived systems; Cauchy characteristic systems; Backlund Theorem

2. **Tanaka theory of linear differential systems**:

   Symbol algebras of regular differential systems; (algebraic) Prolongation of symbol algebras; Proof of Backlund Theorem

3. **PD manifolds of second order**:

   Realization Lemma; Characterization of submanifolds in second order jet spaces; Reduction Theorems; Higher order contact manifolds; Goursat flags

4. **Differential systems associated with simple graded Lie algebras**:

   Semi-simple graded Lie algebras; Gradations in terms of root space decompositions; Gradation in terms of matix representations; Theorem on prolongations

5. **G2 geometry of overdetermined systems of second order**:

   Low dimensional (local) classification of regular differential systems; Cartans G2-model on 5-dimensional space; G2-model of overdetermined systems of second order; Other examples

# Short Talks

### Robert Bartnik – Recognising static metrics

**Abstract**
With Paul Tod we have recently [cqg2006] given a criterion which determines whether or not a given 3-metric $g$ admits a static potential $N$, ie such that $D^2N = NRic(g)$. This talk will review the construction and its generalisation, and outline possible applications.

### Ian Benn – Killing Tensors and Symmetry Operators

**Abstract**
I will briefly review some well known tensor generalisations of the conformal Killing equation, and some situations where they arise. One such situation is in the symmetry operators for various differential equations of mathematical physics. In particular, the anti-symmetric conformal Killing tensors (the conformal Killing-Yano tensors) give rise to first-order symmetry operators for the massless Dirac equation, and all such symmetries are obtained this way. I will show how (at least in conformally flat space) one can also obtain all the first-order symmetry operators for the twistor equation from conformal Killing-Yano tensors.

### Michael Crampin – Some remarks on the geometry of systems of higher-order ordinary differential equations

**Abstract**
'Higher-order' means 'of order greater than 2'. I shall discuss the adaptation of the geometric methods used to analyse 2nd-order systems, based on non-linear and Berwald-type connections, to higher-order systems. The talk will have a mildly historical flavour, and in particular it will pay attention to the pioneering work carried out by the Indian mathematician D.D. Kosambi in 1936. The subject may sound a bit tedious, but the talk will contain one or two surprises (or to be more careful, one or two points will be covered that I found surprising).

### Jonathan Kress – Quadratic algebras of Killing tensors

**Abstract**
A classical or quantum superintegrable system is a Hamiltonian system that admits the maximum number of functionally independent constants or symmetries that are polynomial in the momenta. In a few examples, such as the Hydrogen atom or harmonic oscillator, commutators of the symmetries, given by Killing vectors of the underlying manifold, close to form a finite dimensional Lie algebra. In this talk I will discuss a class of superintegrable systems in which commutators of symmetries can be expressed as quadratic expressions in a basic set of symmetries. These symmetries are given in terms of second rank Killing tensors on the underlying manifold. The possibility of classifying a manifold according to the symmetry algebra of its Killing tensors will be discussed.

### Demeter Krupka – Variational sequences and the total divergence equation

**Abstract**
It is well known that the exactness of variational bicomplexes and variational sequences on fibred spaces is based on the solvability of the higher order total divergence equation. We study this equation by means of the methods used in the variational sequence theory. We find the integrability condition for this equation, and give an explicit description of its solutions. We also describe the solutions as certain differential forms on jet spaces.

**Olga Krupkova – Symmetries associated with differential equations**

**Abstract**
This talk is devoted to different kinds of symmetries, which can be associated with differential equations, their relationships and meaning for integration. In addition to known symmetries, I will introduce and discuss the very recently discovered Helmholtz symmetries. In particular, I will focus on variational equations, and ODEs.

**Tom Mestdag – Reduction aspects of second-order systems with symmetry**

**Abstract**
In this talk, I consider dynamical systems that are invariant under the action of a Lie group. I will review first for first-order systems how the dynamics can be reduced and how, with the aid of a principal connection, the integral curves can be reconstructed. Things become more complicated for second-order systems. I will show that a principal connection generates two associated connections on two relevant fibre bundles and that these two connections can be used in the processes of reduction and reconstruction for second-order systems.

**Yuri Nikolayevsky – On Einstein solvable Lie algebras**

**Abstract**
Our main object of study are noncompact homogeneous Einstein manifolds. According to the Alekseevski Conjecture (which is still open), any such manifold is a solvmanifold, a solvable Lie group with a left-invariant Riemannian metric. All the examples of Einstein solvmanifolds known so far are "standard", which means that the orthogonal complement to the nilradical is abelian. The structure of standard Einstein manifolds is extensively studied and is reasonably well understood. In the talk, we show that for some classes of nilpotent Lie algebras (abelian, some two-step nilpotent, filiforms, nilpotent algebras of dimension 6 or less), any Einstein solvable Lie algebra having such a nilradical must be standard.

**Geoff Prince – EDS and the Inverse Problem in the Calculus of Variations**

**Abstract**
Exterior differential systems theory (EDS) provides a means of describing the uniqueness and existence of Lagrangians whose Euler-Lagrange equations are solution-equivalent to a given system of second order ODEs on a manifold. This is done by finding all closed 2-forms in a particular submodule. In this talk we discuss the differential ideal stage of the EDS process and show how this initial step may lead immediately to a solution of the inverse problem.

**Willy Sarlet – Recursion operators on a tangent bundle**

**Abstract**
In the context of the study of ODEs, the term 'recursion operator' is used for a type (1,1) tensor field which is invariant under the flow of a given dynamics, but also to designate the tensor field which links two compatible Poisson structures, i.e. gives rise to a Poisson-Nijenhuis structure. On the tangent bundle $TQ$ of a manifold, equipped with a symplectic and Poisson structure coming from a regular Lagrangian $L$, we explore how to obtain a Poisson-Nijenhuis structure from a given type (1,1) tensor $J$ on the base manifold $Q$. It is argued that a natural tangent bundle construction leads to a tensor field $R$ which is the pullback under the Legendre transform of the complete lift of $J$ to the cotangent bundle. For the special case of a Lagrangian of kinetic energy type, we thus obtain a tangent bundle picture of the theory of so-called special conformal Killing tensors on a Riemannian manifold, which has applications in the field of Hamilton-Jacobi separability. There is a natural generalization of the construction of such $R$-tensors, when $J$ is allowed to be a tensor field along the tangent bundle projection. We study under what circumstances such an $R$ is invariant under the flow of the Euler-Lagrange field of the given $L$ and look at the problem of vanishing Nijenhuis torsion. Open questions are the possible application of these generalized

$R$-tensors to cases of separability or Liouville integrability with non-quadratic integrals, and to a meaningful generalization of special conformal Killing tensors from Riemannian to Finsler spaces.

**David Saunders – Homogeneous variational sequences**

**Abstract**
We consider homogeneous variational problems defined on bundles of higher-order velocities, and construct an analogue of the variational bicomplex by using spaces of vector-valued forms rather than of scalar forms. The vertical and horizontal differentials are replaced, respectively, by the de Rham differential and a version of the total time derivative. We define a global homotopy operator in order to show that the variational sequences using the total time derivative are, apart from the first, globally exact. As an application, we use the homotopy operator to constrict the Hilbert forms, the analogues in the homogeneous case of the Poincare-Cartan forms. We also propose analogues of the Krupka form, which is closed precisely when the variational problem is trivial. In the fibred case, the Krupka form has been defined only for first-order problems, whereas the homogeneous definition proposed here applies to problems of arbitrary order. For these new forms, the equivalence of closure and triviality is straightforward in the first order case, and we give a proof of equivalence in the special second-order case where there are two independent variables. This is therefore a definite advance over the fibred case. The proof of equivalence in the general second-order (or higher-order) case remains open, and is likely to involve considerable computational complexity.

**Gerard Thompson – The Lie group inverse problem**

**Abstract**
We will discuss recent progress on the inverse problem of Lagrangian dynamics for the canonical symmetric connection on a Lie group. A number of pure Lie theoretic results obtained as consequences of the general theory will also be considered.

**Peter Vassiliou – Weak Darboux integrability and wave maps**

**Abstract**
I'll discuss the method of Darboux for solving a single PDE in one dependent and two independent variables. For *systems* of PDE such as those that arise for wave maps into a Riemannian manifold it turns out to be effective to weaken the definition of Darboux integrability in order that the method encapass a broader class of differential systems than is possible within the classical theory. I'll give an example of a PDE system that is weakly Darboux integrable but not Darboux integrable. Time permitting, I'll describe some preliminary results on applications of these considerations to the explicit construction of 1+1-wave maps into some 2-metrics. All this may be viewed as an application of ideas to be covered in Tom Ivey's lectures.