Algebra univers. **47** (2002) 51–54
0002-5240/02/010051 – 04 \$1.50 + 0.20/0

Algebra Universalis

A non-dualisable entropic algebra

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Abstract. We give an example of a five-element unary algebra which is entropic but does not admit a natural duality.

An algebra $\mathbf{A} = \langle A; F\rangle$ is called entropic if every operation $f \in F$ is a homomorphism $f : \mathbf{A}^n \to \mathbf{A}$, where n is the arity of f. Some of the first known examples of dualisable algebras are entropic. (See Clark and Davey [1] for an introduction to duality theory.) For instance, the cyclic group $\underline{\mathbf{A}}_m = \langle\{0, \dots, m-1\}; +, ^-, 0\rangle$ is dualised by the discrete cyclic group $\underset{\sim}{\mathbf{A}}_m = \langle\{0, \dots, m-1\}; +, ^-, 0, \mathcal{T}\rangle$, for all $m \geqslant 1$ (Davey and Werner [6]), and the two-element semilattice $\underline{\mathbf{S}} = \langle\{0, 1\}; \vee\rangle$ is dualised by $\underset{\sim}{\mathbf{S}} = \langle\{0, 1\}; \vee, 0, 1, \mathcal{T}\rangle$ (Hofmann, Mislove and Stralka [7]). These results were extended by Davey [3]: every finite abelian group is entropic and dualisable; and every finite semilattice is entropic and dualisable. More recently, it was proved by Davey, Idziak, Lampe and McNulty [4] that a finite graph algebra is dualisable if and only if it is entropic. Similarly, the dualisable finite flat graph algebras are precisely the entropic ones [5].

The results mentioned above led the authors of [5] to remark that “it does not seem unreasonable to speculate that every finite entropic algebra is dualizable”. Given a finite entropic algebra $\underline{\mathbf{M}} = \langle M; F\rangle$, we know we can include all the operations in F in the type of a potential dualising structure for $\underline{\mathbf{M}}$. We then aim to find a set R of algebraic relations on $\underline{\mathbf{M}}$ such that $\underset{\sim}{\mathbf{M}} = \langle M; F, R, \mathcal{T}\rangle$ dualises $\underline{\mathbf{M}}$. We shall show that having F as a starting point for $\underset{\sim}{\mathbf{M}}$ is not enough. There is a finite entropic algebra which is not dualisable.

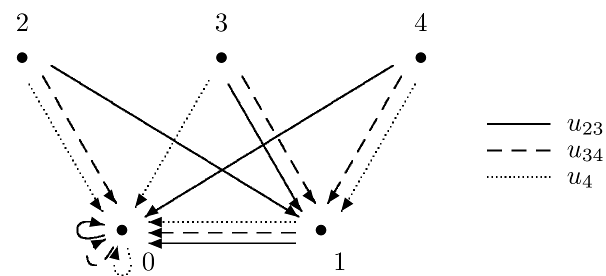
Our example is a five-element unary algebra. A unary algebra is entropic if and only if its monoid of unary term functions is commutative. It is straightforward (but tedious) to show that, up to isomorphism and term-equivalence, there are exactly eleven three-element entropic unary algebras. It follows from Clark, Davey and Pitkethly [2] that these algebras are all dualisable. We do not know whether every four-element entropic unary algebra is dualisable.

Presented by R. W. Quackenbush.
Received September 26, 2000; accepted in final form April 4, 2001.
2000 *Mathematics Subject Classification*: 08A60, 08C15, 18A40.
Key words and phrases: Natural duality, entropic algebra, unary algebra, ghost element method.

Let $M = \{0, 1, 2, 3, 4\}$ and, for each $S \subseteq M$, define the operation $u_S : M \to M$ by

$$u_S(m) = \begin{cases} 1 & \text{if } m \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that the algebra $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3, 4\}; u_{23}, u_{34}, u_4 \rangle$, illustrated below, is not dualisable. The algebra $\underline{\mathbf{M}}$ is entropic, as $u_S \circ u_T = u_{\varnothing}$, for all $S, T \subseteq M \backslash \{1\}$.



$\underline{\mathbf{M}} = \langle \{0, 1, 2, 3, 4\}; u_{23}, u_{34}, u_4 \rangle$

Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\,\underline{\mathbf{M}}$. To show that $\underline{\mathbf{M}}$ is not dualisable, it is enough to find a set S, an algebra $\mathbf{A} \leqslant \underline{\mathbf{M}}^S$ and a continuous map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ such that α preserves all the finitary algebraic relations on $\underline{\mathbf{M}}$ but α is not given by evaluation. The map α is continuous provided it has finite support (that is, there is a finite set $B \subseteq A$ such that $\alpha(x) = \alpha(y)$, for all $x, y \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $x{\restriction}_B = y{\restriction}_B$). The map α preserves every algebraic relation on $\underline{\mathbf{M}}$ as long as it is locally an evaluation (that is, for each finite set $X \subseteq \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$, there is some $a \in A$ such that $\alpha(x) = x(a)$, for all $x \in X$). For every $s \in S$, let $\rho_s := \pi_s{\restriction}_A : \mathbf{A} \to \underline{\mathbf{M}}$ denote the natural projection homomorphism, and define $a_\alpha \in M^S$ by $a_\alpha(s) := \alpha(\rho_s)$. To see that α is not an evaluation, it suffices to show that $a_\alpha \notin A$. (If α is given by evaluation at some $a \in A$, then $a(s) = \rho_s(a) = \alpha(\rho_s) = a_\alpha(s)$, for all $s \in S$.) The element a_α is referred to as a ghost element of $\mathbf{A}$. The discussion in this paragraph is summarised by the following theorem.

Ghost Element Theorem (Clark and Davey [1])**.** *Let $\underline{\mathbf{M}}$ be a finite algebra, let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^S$, for some set S, and let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$. Assume that*

(i) $a_\alpha \notin A$,
(ii) *α has finite support,*
(iii) *α is locally an evaluation.*

Then $\underline{\mathbf{M}}$ is not dualisable.

Theorem. *The entropic algebra $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3, 4\}; u_{23}, u_{34}, u_4 \rangle$ is not dualisable.*

Proof. For each $n \in \omega$, define $a_n \in M^\omega$ by

$$a_n(i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

and define $A_0 := \{\, a_n \mid n > 1 \,\}$. For all $m, n \in \omega$, with $m, n > 1$ and $m \neq n$, define $b_{mn} \in M^\omega$ by

$$b_{mn}(i) = \begin{cases} 3 & \text{if } i = 1, \\ 2 & \text{if } i = m, \\ 4 & \text{if } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

and define $B := \{\, b_{mn} \mid m, n > 1 \text{ and } m \neq n \,\}$. Let $\mathbf{A}$ denote the subalgebra of $\underline{\mathbf{M}}^\omega$ generated by $A_0 \cup B$. Note that $a_1 \notin A$.

Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\,\underline{\mathbf{M}}$ and let $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. We want to show that $x{\restriction}_{A_0}$ is the restriction of a projection. Note that the constant map $\underline{0} := u_\varnothing$ is a term function of $\underline{\mathbf{M}}$. For all $n > 1$, we have

$$u_{23}(x(a_n)) = x(u_{23}(a_n)) = x(\underline{0}(a_n)) = \underline{0}(x(a_n)) = 0$$

and

$$u_4(x(a_n)) = x(u_4(a_n)) = x(\underline{0}(a_n)) = \underline{0}(x(a_n)) = 0,$$

and consequently $x(a_n) \in \{0, 1\}$. Therefore $x(A_0) \subseteq \{0, 1\}$. If $x(A_0) = \{0\}$ or $x(A_0) = \{1\}$, then $x{\restriction}_{A_0} = \pi_0{\restriction}_{A_0}$ or $x{\restriction}_{A_0} = \pi_1{\restriction}_{A_0}$, respectively. So we can assume that $x(A_0) = \{0, 1\}$. Thus, there exist $m, n > 1$ with $x(a_m) = 0$ and $x(a_n) = 1$.

Let $k > 1$ such that $k \neq n$. We will show that $x(a_k) = 0$ and it will then follow that $x{\restriction}_{A_0} = \pi_n{\restriction}_{A_0}$. As

$$u_{23}(x(b_{mn})) = x(u_{23}(b_{mn})) = x(a_m) = 0$$

and

$$u_{34}(x(b_{mn})) = x(u_{34}(b_{mn})) = x(a_n) = 1,$$

we have $x(b_{mn}) = 4$. So

$$u_4(x(b_{kn})) = x(u_4(b_{kn})) = x(u_4(b_{mn})) = u_4(x(b_{mn})) = u_4(4) = 1,$$

which implies that $x(b_{kn}) = 4$ and therefore

$$x(a_k) = x(u_{23}(b_{kn})) = u_{23}(x(b_{kn})) = u_{23}(4) = 0.$$

Now define the map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x(a_2) = x(a_3) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For all $n \in \omega$, we have $\alpha(\rho_n) = a_1(n)$, and so $a_\alpha = a_1 \notin A$. As $\{a_2, a_3\}$ is a finite support for α, it remains to check that α is locally an evaluation. Let $X \subseteq \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$

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be finite. For each $x \in X$, there is some $n_x \in \omega$ with $x{\restriction}_{A_0} = \pi_{n_x}{\restriction}_{A_0}$. Choose $\ell \in \omega$ such that $\ell > n_x$, for all $x \in X$. Then, for each $x \in X$, we have

$$\alpha(x) = \alpha(\rho_{n_x}) = a_1(n_x) = a_\ell(n_x) = \rho_{n_x}(a_\ell) = x(a_\ell).$$

It follows by the Ghost Element Theorem that $\underline{\mathbf{M}}$ is not dualisable. □

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