

ON THE LATTICE OF SUBVARIETIES

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Let  $A$  be a finite set of finite algebras and assume that  $K = Var(A)$ , the variety generated by  $A$ , is congruence-distributive. Then by Jónsson's Lemma, the lattice  $\Lambda(K)$  of subvarieties of  $K$  is a finite distributive lattice. Hence  $\Lambda(K)$  is isomorphic to the lattice  $O(P)$  of order-ideals of the poset  $P$  of join-irreducible elements of  $\Lambda(K)$ . Again by Jónsson's Lemma,  $K'$  is join-irreducible in  $\Lambda(K)$  if and only if  $K' = Var(\{A\})$  for some (necessarily finite) subdirectly irreducible algebra  $A$ ; furthermore, if  $A$  and  $B$  are subdirectly irreducible algebras from  $K$ , then  $Var(\{A\}) \subseteq Var(\{B\})$  if and only if  $A \in HS(\{B\})$ . We have established the following simple but useful result. Let  $Si(K)$  consist of precisely one algebra from each of the isomorphism classes of the subdirectly irreducible algebras in  $K$ .

**THEOREM.** *Let  $K = Var(A)$  be a congruence-distributive variety generated by a finite set  $A$  of finite algebras, and order  $Si(K)$  by  $A \leq B$  if and only if  $A \in HS(\{B\})$ . Then the lattice  $\Lambda(K)$  is a finite distributive lattice and is isomorphic to  $O(Si(K))$ .*

For example, consider the variety  $K$  generated by the distributive double p-algebra,  $A_1$ ,

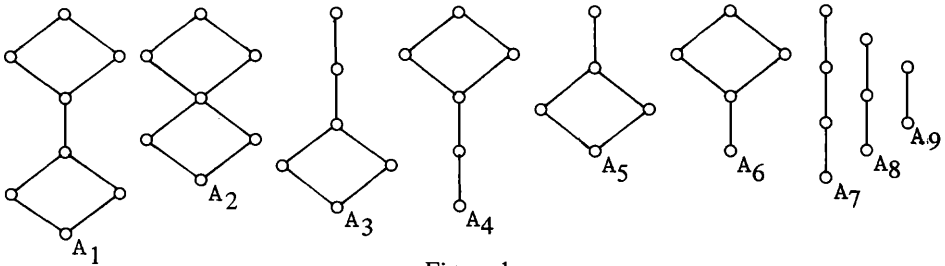


Figure 1

of Figure 1. By Jónsson's Lemma and the results of B. A. Davey [9], the subdirectly irreducibles in  $K$  are the algebras listed in Figure 1; the poset  $Si(K)$  is shown in Figure 2. Since  $O(2^3)$

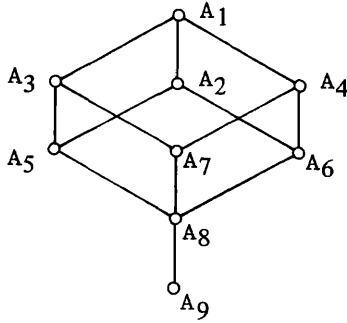


Figure 2

is isomorphic to  $F_{\mathcal{D}}(3)$ , the free bounded distributive lattice on three generators, it follows that  $\Lambda(K)$  is isomorphic to  $F_{\mathcal{D}}(3)$  with a new zero adjoined. This example is generalized in Section 4; in the notation of that section,  $K$  is  $\mathcal{B}(2,2,2)$ .

The aim of this paper is to consider extensions of the theorem above. The main results are Theorem 3.3 which gives an extension to locally finite congruence-distributive varieties, and Theorem 3.5 which gives an extension to congruence distributive varieties with the property that every finitely generated subalgebra of a subdirectly irreducible is finite. The paper closes with some applications to polyadic algebras, Brouwerian algebras and Heyting algebras, distributive p-algebras, and distributive double p-algebras.

1. **Complete rings of sets.** We commence by recalling some terminology. A complete lattice  $L$  satisfies the *join-infinite distributive law*, (JID), if for each  $x \in L$  and every subset  $\{y_i | i \in I\}$  of  $L$ ,

$$x \wedge \vee \{y_i | i \in I\} = \vee \{x \wedge y_i | i \in I\} .$$

The dual of (JID) is the *meet-infinite distributive law* (MID); these laws are independent. The lattice  $L$  is *completely distributive* if for all  $x_{ij} \in L$ ,

$$\wedge (\vee \{x_{ij} | j \in J\} | i \in I) = \vee (\wedge \{x_{i\phi(i)} | i \in I\} | \phi: I \rightarrow J) ;$$

this law is self-dual (see G. N. Raney [24]). A complete sublattice of the lattice of all subsets of a set is known as a *complete ring of sets*; of course, a complete ring of sets is completely distributive. An element  $x$  of a complete lattice  $L$  is a *completely join irreducible*, (CJI), if

$$x = \vee \{y_i | i \in I\} \text{ implies } x = y_i \text{ for some } i \in I ;$$

and  $x$  is a *completely join prime*, (CJP), if

$$x \leq \vee(y_i | i \in I) \text{ implies } x \leq y_i, \text{ for some } i \in I.$$

The poset of all CJI elements of a complete lattice  $L$  is denoted by  $J_i(L)$ , and  $J_p(L)$  denotes the poset of CJP elements of  $L$ . A subset  $P$  of a complete lattice  $L$  is *join-dense* if every element of  $L$  is a join of elements from  $P$ . Finally, a subset  $U$  of a poset  $P$  is an *order-ideal* if  $x \in U$  and  $y \leq x$  imply  $y \in U$ ; the lattice of all order-ideals of  $P$  is denoted by  $O(P)$ . For all other lattice-theoretic notions, in particular for the definition of *compact element* and *algebraic lattice*, we refer to [2], [8], or [13].

The following result dates back to the early fifties to G. N. Raney [24], V. K. Balachandran [1], and J. R. Büchi [7] (see also. G. Bruns [4,5,6], J. Schmidt [27], G. Grätzer and E. T. Schmidt [12], and Chapter 12 of R. Balbes and Ph. Dwinger [2]).

1.1. PROPOSITION. *Let  $L$  be a lattice. Then the following are equivalent:*

- (i)  $L$  is isomorphic to  $O(P)$  for some poset  $P$ ;
- (ii)  $L$  is isomorphic to a complete ring of sets;
- (iii)  $L$  is distributive, algebraic, and dually algebraic;
- (iv)  $L$  is dually algebraic and completely distributive;
- (v)  $L$  is complete, the CJI elements of  $L$  are join-dense, and  $L$  satisfies (JID);
- (vi)  $L$  is dually algebraic and every CJI is a CJP;
- (vii)  $L$  is complete and the CJP elements are join-dense;
- (viii) the map  $\eta: L \rightarrow O(J_p(L))$ , given by  $a\eta = (a] \cap J_p(L)$ , is an isomorphism.

PROOF. The following facts are both well known and easily established; (a) a family of subsets of a set  $P$  which is closed under arbitrary intersections and directed unions is an algebraic lattice; (b) every distributive algebraic lattice satisfies (JID); (c) in every dually algebraic lattice the CJI elements are join-dense; (d) in every complete lattice satisfying (JID), each CJI is a CJP. Using these results a locally trivial proof of the proposition may be obtained by establishing the following chains of implications: (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (vi)  $\Rightarrow$  (viii)  $\Rightarrow$  (i) and (ii)  $\Rightarrow$  (iv)  $\Leftrightarrow$  (vi)  $\Leftrightarrow$  (vii).

1.2. REMARKS. (a) For a categorical formulation of the relationship between the lattice  $L$  and the poset  $P$  in Proposition 1.1 the reader is referred to L. Geissinger and W. Graves [11].

(b) Apart from the trivial dual formulations, there are two further nontrivial equivalents; the first is due to G. Bruns [5] and the second to K. H. Hofmann, M. Mislove, and A. Stralka [16]:

(ix)  $L$  is complete, satisfies (JID) and (MID), and every proper interval of  $L$  contains a prime interval (i.e. a two-element closed interval);

(x)  $L$  may be endowed with a compact totally disconnected topology with respect to which it is a topological distributive lattice.

These two conditions will not be used here.

(c) Of course, we could have replace  $J_p(L)$  by  $J_i(L)$  in Condition (viii). In fact, it is easily seen that if  $P \subseteq L$  and the map  $\eta: L \rightarrow \mathbf{O}(P)$  given by  $a\eta = (a) \cap P$ , is an isomorphism, then  $P = J_i(L) = J_p(L)$ .

**2. The lattice of subvarieties.** For any variety of algebras,  $K$ , the lattice  $\Lambda(K)$  of subvarieties is dually isomorphic to the lattice of fully invariant congruences on the countably generated  $K$ -free algebra, and hence  $\Lambda(K)$  is dually algebraic. Hence we have the following trivial consequence of Proposition 1.1.

2.1. PROPOSITION. *Let  $K$  be a variety. Then the following are equivalent:*

(i)  $\Lambda(K)$  is isomorphic to  $\mathbf{O}(P)$  for some poset  $P$ ;

(ii)  $\Lambda(K)$  is isomorphic to a complete ring of sets;

(iii)  $\Lambda(K)$  is distributive and algebraic;

(iv)  $\Lambda(K)$  is completely distributive;

(v)  $\Lambda(K)$  satisfies (JID);

(vi) every CJI element of  $\Lambda(K)$  is a CJP;

(vii) the CJP elements of  $\Lambda(K)$  are join-dense;

(viii) the map  $\eta: \Lambda(K) \rightarrow \mathbf{O}(J_p(\Lambda(K)))$ , given by  $K'\eta = (K') \cap J_p(\Lambda(K))$ , is an isomorphism.

Let  $P_F(K)$  be the poset consisting of those subvarieties of  $K$  which are generated by a finite subdirectly irreducible algebra. In [20], R. McKenzie observes that if  $K$  is congruence distributive, then  $P_F(K) \subseteq J_i(\Lambda(K))$ ; and if  $K$  is also generated by its finite members, then  $J_p(\Lambda(K)) \subseteq P_F(K)$ . This, and Proposition 2.1, suggest that we should consider the question: *When is  $\Lambda(K)$  isomorphic to  $\mathbf{O}(P_F(K))$ ?* The answer requires the notion of a splitting algebra as was introduced in [20].

A finite subdirectly irreducible algebra  $A \in K$  is a *splitting algebra* in  $K$  if  $\text{Var}(\{A\})$ , the variety generated by  $A$ , is a CJP in  $\Lambda(K)$ . Equivalently, a finite subdirectly irreducible algebra  $A$  is splitting if there is a largest subvariety  $K'$  of  $K$  which does not contain  $A$ ; whence  $\Lambda(K)$  splits into the disjoint union of the dual ideal generated by  $\text{Var}(\{A\})$  and the ideal generated by  $K'$ . We refer to [20] for further details.

2.2. PROPOSITION. *The natural map from  $\Lambda(K)$  to  $\mathbf{O}(\mathbf{P}_F(K))$  is an isomorphism if and only if every subvariety of  $K$  is generated by its finite members and every finite subdirectly algebra in  $K$  is a splitting algebra.*

PROOF. Assume that the natural map  $\eta: \Lambda(K) \rightarrow \mathbf{O}(\mathbf{P}_F(K))$  is an isomorphism. By Remark 1.2(c),  $\mathbf{P}_F(K) = \text{Ji}(\Lambda(K)) = \text{Jp}(\Lambda(K))$ ; whence every finite subdirectly irreducible algebra in  $K$  is splitting. Furthermore, by Proposition 2.1(vii),  $\mathbf{P}_F(K)$  is join-dense in  $K$ ; i.e. every subvariety of  $K$  is generated by its finite members.

Conversely, if every subvariety of  $K$  is generated by its finite members, then  $\mathbf{P}_F(K)$  is join-dense in  $\Lambda(K)$  and  $\text{Jp}(\Lambda(K)) \subseteq \text{Ji}(\Lambda(K)) \subseteq \mathbf{P}_F(K)$ ; but if every finite subdirectly irreducible is splitting, then  $\mathbf{P}_F(K) \subseteq \text{Jp}(\Lambda(K))$ , and so  $\mathbf{P}_F(K) = \text{Jp}(\Lambda(K))$  is join-dense in  $\Lambda(K)$ , whence  $\eta$  is an isomorphism, by Proposition 2.1(viii).

Recall that  $K$  is *locally finite* if every finitely generated algebra in  $K$  is finite.

2.3. COROLLARY. *If every finitely generated subdirectly irreducible algebra in  $K$  is finite, in particular, if  $K$  is locally finite, then the natural map from  $\Lambda(K)$  to  $\mathbf{O}(\mathbf{P}_F(K))$  is an isomorphism if and only if every finite subdirectly irreducible algebra in  $K$  is a splitting algebra.*

3. **The congruence-distributive case.** In the application of Proposition 2.2 and Corollary 2.3 we are faced with two problems:

- (I) *find a convenient description of the poset  $\mathbf{P}_F(K)$ ;*
- (II) *find reasonable conditions under which every finite subdirectly irreducible algebra in  $K$  is a splitting algebra.*

In the congruence-distributive case, both of these problems are tractable.

Let  $\text{Si}_F(K)$  consist of precisely one algebra from each of the isomorphism classes of the finite subdirectly irreducible algebras in  $K$ ; define a partial order on  $\text{Si}_F(K)$  by

$$A \leq B \text{ if and only if } A \in \text{HS}(\{B\}).$$

The following lemma is an immediate consequence of Jónsson's Lemma [17].

3.1. LEMMA. *Let  $K$  be a congruence-distributive variety. Then the map  $A \mapsto \text{Var}(\{A\})$  is an order-isomorphism between the posets  $\text{Si}_{\mathcal{F}}(K)$  and  $\text{P}_{\mathcal{F}}(K)$ .*

In [10], A. Day introduced finitely projected algebras in order to give sufficient conditions for every finite subdirectly irreducible algebra in a congruence-distributive variety to be a splitting algebra.

A finite algebra  $A$  in  $K$  is *finitely projected* if for each epimorphism  $\phi: B \rightarrow A$  in  $K$  there is a finite subalgebra of  $B$  whose image under  $\phi$  is  $A$ .

Day proved that  $A$  is finitely projected if and only if it is a homomorphic image of a finite projective algebra in  $K$ ; he also proved that if  $K$  is congruence distributive, then every finitely projected subdirectly irreducible algebra in  $K$  is a splitting algebra. Of course, if  $K$  is locally finite, then every finite algebra in  $K$  is a homomorphic image of a finite projective algebra (in fact, of a finitely generated  $K$ -free algebra), and hence every finite algebra is finitely projected.

3.2. PROPOSITION. (A. Day [10]). *If  $K$  is a locally finite congruence-distributive variety, then every finite subdirectly irreducible algebra in  $K$  is a splitting algebra.*

We now combine Corollary 2.3, Lemma 3.1, and Proposition 3.2.

3.3. THEOREM. *Let  $K$  be a locally finite congruence-distributive variety. Then  $\Lambda(K)$  is a completely distributive lattice and is isomorphic to  $\mathbf{O}(\text{Si}_{\mathcal{F}}(K))$ .*

The observation with which we commenced this paper follows immediately.

3.4. COROLLARY. *Let  $A$  be a finite set of finite algebras and assume that  $K = \text{Var}(A)$  is congruence distributive. Then  $\Lambda(K)$  is a finite distributive lattice and is isomorphic to  $\mathbf{O}(\text{Si}(K))$ .*

Congruence distributivity will also guarantee that every finite subdirectly irreducible is splitting under slightly different conditions.

3.5. THEOREM. *Let  $K$  be a congruence-distributive variety of finite type and assume that the class consisting of all subdirectly irreducible algebras in  $K$  is axiomatic. If every finitely generated subalgebra of a subdirectly irreducible algebra in  $K$  is finite, then  $\Lambda(K)$  is a completely distributive lattice and is isomorphic to  $\mathbf{O}(\text{Si}_{\mathcal{F}}(K))$ .*

PROOF. By Corollary 2.3 and Lemma 3.1 it is sufficient to show that every finite subdirectly irreducible algebra  $A$  is splitting. Assume that  $\text{Var}(\{A_i\}) \subseteq \vee(\mathcal{K}_i | i \in I)$ , where  $\mathcal{K}_i \in \Lambda(\mathcal{K})$  for all  $i \in I$ ; we must show that  $A \in \mathcal{K}_i$  for some  $i \in I$ . Since  $A$  is subdirectly irreducible, by Jónsson's Lemma [17] there is a family  $(A_j | j \in J)$  of subdirectly irreducible algebras from  $\cup(\mathcal{K}_i | i \in I)$  such that  $A$  is a homomorphic image of a subalgebra  $B$  of an ultraproduct  $C$  of the family  $(A_j | j \in J)$ . Since  $A$  is finite we can choose  $B$  to be finitely generated. But the class of subdirectly irreducible members of  $\mathcal{K}$  is axiomatic, and so is closed under ultraproducts; whence  $C$  is subdirectly irreducible. Hence, by assumption,  $B$  is finite. Since  $\mathcal{K}$  is of finite type and  $B$  is finite, "has a subalgebra isomorphic to  $B$ " is a first-order property. Thus, since  $B$  is a subalgebra of an ultraproduct of the family  $(A_j | j \in J)$ , there exists  $j \in J$  such that  $A_j$  has a subalgebra isomorphic to  $B$ . There exists  $i \in I$  such that  $A_j \in \mathcal{K}_i$ , and hence  $A \in \mathcal{K}_i$ , as required.

**4. Applications.** The most natural classes to which our results apply are varieties of lattice-ordered algebras.

(A) POLYADIC ALGEBRAS. We refer to P. R. Halmos [14] for the basic results of the theory. It follows from Lemma 8 on page 47 and Theorem 7 on page 50 of [14] that every subdirectly irreducible monadic algebra is simple, the subdirectly irreducible monadic algebras form an axiomatic class, and every finitely generated subalgebra of a subdirectly irreducible monadic algebra is finite; hence Theorem 3.5 applies. In fact, it is easily seen that the variety  $PA_1$  of monadic algebras is locally finite (see H. Bass [3]), and so Theorem 3.3 applies also. It is clear that the poset  $\text{Si}_F(PA_1)$  is order-isomorphic to  $\omega$ ; note that  $\mathbf{O}(\omega) \cong \omega + 1$ .

4.1. THEOREM. (D. Monk [21]). *The lattice  $\Lambda(PA_1)$  of subvarieties of monadic algebras is a chain of type  $\omega + 1$ .*

As D. Monk [21] points out, the situation for the variety  $PA_2$  of two-dimensional polyadic algebras is somewhat more complicated; for example,  $PA_2$  is not locally finite. By Lemma (8.2) on page 127 and Theorem (9.3) on page 131 of [14], every subdirectly irreducible in  $PA_2$  is simple and the subdirectly irreducibles form an axiomatic class, and it follows from Theorem 2.5 and Theorem 3.1 of [21] that if  $\mathcal{K}$  is a proper subvariety of  $PA_2$ , then every finitely generated subalgebra of a

subdirectly irreducible in  $\mathbf{K}$  is finite; whence our Theorem 3.5 applies. Of course, the problem of describing the poset  $\text{Si}_{\mathbf{F}}(\mathbf{K})$  remains.

4.2. THEOREM. *Let  $\mathbf{K}$  be a proper subvariety of  $\mathbf{PA}_2$ , then  $\Lambda(\mathbf{K})$  is completely distributive and is isomorphic to  $\mathbf{O}(\text{Si}_{\mathbf{F}}(\mathbf{K}))$ .*

(B) RELATIVE STONE ALGEBRAS AND L-ALGEBRAS. Brouwerian algebras which satisfy the identity  $(x * y) \vee (y * x) = 1$  are called *relative Stone algebras* and we denote this variety by  $S_{\omega}$ , the corresponding Heyting algebras are known as *L-algebras* and the variety is denoted by  $L_{\omega}$ . It is easily seen that a member of  $S_{\omega}$  or  $L_{\omega}$  is subdirectly irreducible if and only if it is a chain with a dual atom (T. Hecht and T. Katriňák [15]); whence the subdirectly irreducibles form an axiomatic class. In  $S_{\omega}$  an  $n$ -generated subalgebra of a subdirectly irreducible has at most  $n + 1$  elements, and in  $L_{\omega}$  it has at most  $n + 2$  elements; since an  $n$ -generated free algebra is a subdirect product of  $n$ -generated subdirectly irreducibles, it follows that both  $S_{\omega}$  and  $L_{\omega}$  are locally finite. Hence both Theorem 3.3 and Theorem 3.5 apply. Clearly both  $\text{Si}_{\mathbf{F}}(S_{\omega})$  and  $\text{Si}_{\mathbf{F}}(L_{\omega})$  are order-isomorphic to  $\omega$ .

4.3. THEOREM. (T. Hecht and T. Katriňák [15]). *The lattices  $\Lambda(S_{\omega})$  and  $\Lambda(L_{\omega})$  of subvarieties of relative Stone algebras and L-algebras are both chains of type  $\omega + 1$ .*

(C) DISTRIBUTIVE  $p$ -ALGEBRAS. K. B. Lee [19] and H. Lakser [18] proved that an algebra in the variety  $B_{\omega}$  of distributive  $p$ -algebras is subdirectly irreducible if and only if it is isomorphic to a Boolean algebra with a new unit adjoined. Hence the subdirectly irreducibles form an axiomatic class and  $\text{Si}_{\mathbf{F}}(B_{\omega})$  is order-isomorphic to  $\omega$ . It also follows that an  $n$ -generated subalgebra of a subdirectly irreducible has at most  $2^{2^n} + 1$  elements, whence the variety  $B_{\omega}$  is locally finite. Again, both Theorem 3.3 and Theorem 3.5 apply.

4.4. THEOREM. (K. B. Lee [19]). *The lattice  $\Lambda(B_{\omega})$  of subvarieties of distributive  $p$ -algebras is a chain of type  $\omega + 1$ .*

(D) DISTRIBUTIVE DOUBLE  $p$ -ALGEBRAS. Subdirectly irreducible distributive double  $p$ -algebras are described topologically in B. A. Davey [9]. Although we are unable to describe the lattice of subvarieties, we can apply our results to certain principal ideals in the lattice. Let  $\mathbf{A}$  be a poset with a zero and let  $\mathbf{B}$  be a

poset with a unit; then  $A//B$  denotes the linear sum of  $B$  and  $A$ , and  $A/B$  denotes the quotient of  $A//B$  obtained by identifying the zero of  $A$  with the unit of  $B$ .

For  $1 \leq m, n \leq \omega$ , let  $B(m,n,2)$  be the variety of distributive double  $p$ -algebras generated by  $2^m//2^n$ , and let  $B(m,n,1)$  be the variety generated by  $2^m/2^n$ . Note that  $2^m/2^n$  is simple, and  $2^m//2^n$  is subdirectly irreducible; in fact, the congruence lattice of  $2^m//2^n$  is a three-element chain, its only proper nontrivial homomorphic image being  $2^m/2^n$  (see [9]). By Jónsson's Lemma, the subdirectly irreducibles of  $B(\omega,\omega,2)$  are homomorphic images of subalgebras of ultrapowers of  $2^\omega//2^\omega$ . Since "is a linear sum of two Boolean algebras" is a first-order property, it is easily seen that the subdirectly irreducibles of  $B(\omega,\omega,2)$  are the two-element chain and all algebras of the form  $B//C$  or  $B/C$  where both  $B$  and  $C$  are Boolean algebras; whence the subdirectly irreducibles of  $B(\omega,\omega,2)$  form an axiomatic class. A crude upper-bound for the cardinality of an  $n$ -generated subalgebra of a subdirectly irreducible algebra in  $B(\omega,\omega,2)$  is  $2 \cdot 2^{2^n}$ , and, as before, it follows that  $B(\omega,\omega,2)$  is locally finite. Once more, both Theorem 3.3 and Theorem 3.5 apply. It follows from the information given above that for  $1 \leq k \leq 2$  and  $1 \leq m, n \leq \omega$ , the poset  $\text{Si}_F(B(m,n,k))$  is order-isomorphic to  $(m \times n \times k)//1$ . An easy application of H. Priestley's duality for the category  $D$  of bounded distributive lattices (see [22,23]) yields that for  $1 \leq m,n,k \leq \omega$ ,  $O(m \times n \times k)$  is isomorphic to  $(m+1) * (n+1) * (k+1)$ , where  $*$  denotes coproduct in  $D$ .

4.5. THEOREM. *For  $1 \leq k \leq 2$  and for  $1 \leq m, n \leq \omega$ , the lattice  $\Lambda(B(m,n,k))$  is isomorphic to  $O(m \times n \times k)//1$ . Consequently,  $\Lambda(B(m,n,2))$  is isomorphic to  $((m+1) * (n+1) * 3)//1$ , and  $\Lambda(B(m,n,1))$  is isomorphic to  $((m+1) * (n+1))//1$ .*

This theorem extends the results of A. Romanowska [25,26], where  $\Lambda(K)$  is described for certain proper subvarieties  $K$  of  $B(\omega,\omega,2)$ .

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