Math. Z. 166, 205–223 (1979)

Mathematische Zeitschrift

Injectivity and Boolean Powers

Brian A. Davey[1]* and Heinrich Werner[2]

[1] La Trobe University, Bundoora, Victoria, Australia
[2] Fachbereich 17 Mathematik, Gesamthochschule Kassel
Heinrich-Plett-Str. 40, D-3500 Kassel, Federal Republic of Germany

Herrn Prof. Dr. Reinhold Baer zum 75. Geburtstag

For several years it has been widely appreciated that injectivity and Boolean powers are closely connected. There are a number of papers in which it is shown that the injectives, and more generally the weak injectives, of a particular variety are precisely finite products of complete Boolean powers of appropriate subdirectly irreducible algebras. Our aim here is to present a theorem which will encompass all the known results of this type. In Sect. 1 the main theorem is stated and some of its applications are discussed. Sect. 2 is devoted to principal-congruence formulae, and contains all the first-order logic which we require in the proof of the main theorem. A general discussion of injectives and weak injectives is given in Sect. 3. Finally, in Sect. 4, the proof is given. A brief excursion into sheaf theory is required, but we have attempted to make this as painless as possible for the uninitiated; in fact, the only sheaf-theoretic tool used is the representation theorem due to S. Comer.

1. The Main Theorem

We denote the usual class operators corresponding to isomorphic copies, homomorphic images, subalgebras, and direct products by $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$ respectively; and $\mathbb{S}\mathrm{i}(\mathfrak{K})$ denotes the class of subdirectly irreducible algebras in a class $\mathfrak{K}$. The lattice of congruences on an algebra A is denoted by $\mathrm{Con}(A)$ with smallest element Δ and largest element ∇; the principal congruence generated by a and b is denoted by $\Theta(a,b)$.

Recall that an algebra I is *injective [weak injective]* in a class $\mathfrak{K}$ if I is a member of $\mathfrak{K}$ and for each algebra B in $\mathfrak{K}$ and each subalgebra A of B, every homomorphism [epimorphism] $\phi: A \to I$ extends to a homomorphism $\psi: B \to I$ with $\psi \restriction A = \phi$.

For an algebra A and a Boolean algebra B, the *bounded Boolean power*, $A[B]^*$, is defined here as the algebra $C(X_B, A)$ of continuous functions from the

* Der erste Autor möchte dem Fachbereich Mathematik der Technischen Hochschule Darmstadt seinen Dank für die großzügige Unterstützung ausdrücken, die er als Gastforscher erhalten hat, als diese Arbeit entstand

0025-5874/79/0166/0205/$04.00

Boolean space X_B of prime ideals of B into the algebra A, where A is endowed with the discrete topology; we refer to Burris [6] and Banaschewski and Nelson [4] for the equivalent algebraic definition and for a detailed discussion of Boolean powers.

Let $\mathfrak{A}$ be a class of algebras. An $\exists\forall$ conjunct of equations, say

$$\alpha(a,b)=\exists\vec{x}\,\forall\vec{y}\,[\mathop{\&}_{i\leqq n} p_i(\vec{x},\vec{y},a,b)=q_i(\vec{x},\vec{y},a,b)],$$

is a *simplicity formula for* $\mathfrak{A}$ if for each A in $\mathfrak{A}$,

$$\{\Theta(a,b)\mid A\vDash\alpha(a,b)\}=\{\Delta,\nabla\}.$$

We say that $\mathfrak{A}$ has *factorizable congruences* if for all n and all $A_0,\ldots,A_n\in\mathfrak{A}$, the natural map from

$$\mathrm{Con}(A_0)\times\ldots\times\mathrm{Con}(A_n)\quad\text{to}\quad\mathrm{Con}(A_0\times\ldots\times A_n)\quad\text{is onto.}$$

Our main theorem can now be stated.

1.1. **Theorem.** *Let $\mathfrak{K}$ be a variety, let $\mathfrak{A}$ be a finite set of finite algebras from $\mathfrak{K}$, and assume that $\mathbb{S}\mathrm{i}(\mathfrak{K})\subseteq\mathbb{I}\mathbb{S}(\mathfrak{A})$. If there is a simplicity fromula for $\mathfrak{A}$, and $\mathfrak{A}$ has factorizable congruences, then the following are equivalent:*

(i) *I is a [weak] injective in $\mathfrak{K}$;*

(ii) *I is isomorphic to $A_0[B_0]^*\times\ldots\times A_n[B_n]^*$, where for all $j\leqq n$, $A_j\in\mathbb{H}(\mathfrak{A})\cap\mathbb{S}\mathrm{i}(\mathfrak{K})$, A_j is [weak] injective in $\mathfrak{K}$, and B_i is a complete Boolean algebra, and the algebras A_j are pairwise nonisomorphic.*

Weak injectives were introduced by Grätzer and Lakser in [22] as a natural intermediary between injectives and absolute subretracts. In [22], and in Balbes and Grätzer [2] (which [22] generalizes), the [weak] injectives are described indirectly in terms of bounded Boolean powers. The first explicit use of bounded Boolean powers to describe injective algebras occurs in A. Day's thesis (see Day [16]). Since [22] and [16] appeared, there has been a series of papers in which the [weak] injectives in varieties are described, either directly or indirectly, via bounded Boolean powers. For the remainder of this section we discuss the relationship between these results and the theorem above.

Recall the following two injectivity conditions. An algebra $I\in\mathfrak{K}$ is an *absolute subretract* in $\mathfrak{K}$ if for every embedding $\phi: I\rightarrowtail A$ with $A\in\mathfrak{K}$ there is an onto homomorphism $\psi: A\twoheadrightarrow I$ such that $\psi\phi=\mathrm{id}_I$; and I is *self-injective* if it is injective in the class $\{I\}$, that is, every homomorphism from a subalgebra of I into I extends to an endomorphism of I.

Firstly, we consider the congruence-distributive case. A variety $\mathfrak{K}$ is *congruence distributive* if Con(A) is distributive for all $A\in\mathfrak{K}$; for example, any variety of lattice-ordered algebras is congruence distributive. The key to congruence-distributive varieties is Jónsson's Lemma (see [23; Lemma 3.1; p. 114]) which amongst other things tells us that if $\mathfrak{K}$ is a congruence-distributive variety generated by a finite set $\mathfrak{K}'$ of finite algebras, then $\mathbb{S}\mathrm{i}(\mathfrak{K})\subseteq\mathbb{H}\mathbb{S}(\mathfrak{K})$, and hence (up to isomorphism) $\mathfrak{K}$ has only finitely many subdirectly irreducibles and they are all finite. Consequently in the congruence-distributive case, a natural choice for the class $\mathfrak{A}$ of Theorem 1.1 is $\mathbb{S}\mathrm{i}(\mathfrak{K})$ or some appropriate subclass of it.

The net effect of Theorem 1.1 is to reduce the question "Which algebras are [weak] injective in $\mathfrak{K}$?" to "Which subdirectly irreducible algebras are [weak] injective in $\mathfrak{K}$?" In the congruence-distributive case we have a satisfactory answer to the second question. A class $\mathfrak{K}$ has *enough injectives* if every algebra in $\mathfrak{K}$ has an injective extension in $\mathfrak{K}$. A variety $\mathfrak{K}$ has enough injectives if and only of $\mathbb{S}\mathrm{i}(\mathfrak{K})$ has enough injectives (see [16]); and if the variety $\mathfrak{K}$ has enough injectives, then in $\mathfrak{K}$ the concepts of injective, weak injective, and absolute subretract are equivalent (see [3]). Part (i) of the following lemma is proved in Davey [12; Corollary 2.3], part (iii) is proved in [16; Theorem 4.1; p. 212], and part (ii) is an obvious generalization of (iii) which may proved using the Jónsson-diagram technique developed in [12].

1.2 **Proposition.** (i) *Let $\mathfrak{K}$ be a congruence-distributive variety generated by a finite set of finite algebras. Then a subdirectly irreducible algebra in $\mathfrak{K}$ is weak injective in $\mathfrak{K}$ if and only if it is weak injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})$.*

(ii) *Let $\mathfrak{A}$ be a finite set of finite algebras and assume that $\mathfrak{K} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$ is congruence distributive. Let $A \in \mathfrak{A}$ and suppose that every subalgebra of A is either subdirectly irreducible or weak injective in $\mathfrak{K}$; then A is injective in $\mathfrak{K}$ if and only if it is injective in $\mathfrak{A}$.*

(iii) *Let A be a finite algebra all of whose subalgebras are either subdirectly irreducible or weak injective in $\mathfrak{K}$, and assume that $\mathfrak{K} := \mathbb{I}\mathbb{S}\mathbb{P}(A)$ is congruence distributive. Then A is injective in $\mathfrak{K}$ and $\mathfrak{K}$ has enough injectives if and only if A is self-injective.*

Let $\mathfrak{C}$ be a subclass of $\mathfrak{K}$; if the [weak] injectives in $\mathfrak{K}$ are precisely the algebras of the form $\prod(A_j[B_j]^* \mid j \leqq n)$ where the A_j are pairwise nonisomorphic members of $\mathfrak{C}$, and the B_j are complete, then for brevity, we say that the *[weak] injectives in $\mathfrak{K}$ are induced by $\mathfrak{C}$*. In this terminology our main theorem reads as follows.

1.1. **Theorem.** *Let $\mathfrak{K}$ be a variety having a finite subset $\mathfrak{A}$ of finite algebras such that $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$.*

If $\mathfrak{A}$ has a simplicity formula and factorizable congruences then the [weak] injectives in $\mathfrak{K}$ are induced by those algebras in $\mathbb{H}(\mathfrak{A}) \cap \mathbb{S}\mathrm{i}(\mathfrak{K})$ which are [weak] injective in $\mathfrak{K}$.

A subdirectly irreducible algebra A is called *maximal* if every embedding $\phi: A \rightarrowtail B$, with $B \in \mathbb{S}\mathrm{i}(\mathfrak{K})$ is an isomorphism.

1.3. **Theorem.** *Let $\mathfrak{K}$ be a congruence-distributive-variety generated by finitely many finite algebras, and assume that there is a simplicity formula for the maximal subdirectly irreducible algebras in $\mathfrak{K}$. Then the injectives [weak injectives] in $\mathfrak{K}$ are induced by the subdirectly irreducibles which are injective in $\mathfrak{K}$ [weak injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})$].*

We are left with the question: "*Under what reasonably general conditions does the class of maximal subdirectly irreducible algebras of $\mathfrak{K}$ have a simplicity formula?*"

Of course, the most obvious answer to this question is: "When every maximal subdirectly irreducible algebra in $\mathfrak{K}$ is simple"; in which case either the empty formula or "$a=a$ & $b=b$" will suffice.

1.4. **Theorem.** *Let $\mathfrak{K}$ be a congruence-distributive variety generated by finitely many finite algebras, and assume that every maximal subdirectly irreducible algebra in $\mathfrak{K}$ is simple. Then the injectives [weak injectives] in $\mathfrak{K}$ are induced by the maximal subdirectly irreducibles which are injective in $\mathfrak{K}$ [weak injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})$].*

A class $\mathfrak{K}$ satisfies the *congruence extension property* (CEP) if for each algebra A in $\mathfrak{K}$, every congruence on a subalgebra of A extends to a congruence on A. It is proved in Davey [12; Theorem 3.3] that if $\mathfrak{K}$ is a congruence-distributive variety generated by finitely many finite algebras, then $\mathfrak{K}$ satisfies CEP if and only if $\mathbb{S}\mathrm{i}(\mathfrak{K})$ satisfies CEP. Every maximal subdirectly irreducible is an absolute subretract (see [22]), and if $\mathfrak{K}$ satisfies CEP, then it is easily seen that every absolute subretract is weak injective. Hence it follows that if the assumptions of 1.4 hold and $\mathbb{S}\mathrm{i}(\mathfrak{K})$ satisfies CEP, then the weak injectives in $\mathfrak{K}$ are induced by the maximal subdirectly irreducibles; thus, with the aid of 1.2 (ii) in the injective case, we have the following corrollary.

1.5. **Corollary.** *Let $\mathfrak{K}$ be a congruence-distributive variety generated by finitely many finite algebras, and assume that every subalgebra of each subdirectly irreducible in $\mathfrak{K}$ is simple. Then the weak injectives in $\mathfrak{K}$ are induced by the maximal subdirectly irreducible algebras, and the injectives in $\mathfrak{K}$ are induced by the maximal subdirectly irreducible algebras which are injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})$.*

If $\mathfrak{K}$ is generated by a finite set $\mathfrak{A}$ of weakly independent quasi-primal algebras, then $\mathfrak{K}$ is congruence distributive, and up to isomorphism its subdirectly irreducible algebras are the subalgebras of algebras in $\mathfrak{A}$, all of which are simple; whence 1.5 applies. A quasi-primal algebra is called demi-semi-primal if it is self-injective; hence a maximal subdirectly irreducible is injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})=\mathbb{I}\mathbb{S}(\mathfrak{A})$ if and only if it is demi-semi-primal and for all $B\in\mathfrak{A}$ with $B\notin\mathbb{I}\mathbb{S}(\{A\})$ the only subalgebras of B which are isomorphic to subalgebras of A are trivial ones. Thus 1.5 yields a description of the [weak] injectives in $\mathfrak{K}$; in the case where $\mathfrak{A}=\{A\}$ this was proved in Quackenbush [27, 28], and the general case is proved in Werner [32].

It follows from 1.2 (i) that if $\mathfrak{K}$ is a congruence-distributive variety generated by a single subdirectly irreducible algebra A, then A is weak injective in $\mathfrak{K}$, irrespective of whether $\mathfrak{K}$ satisfies CEP or not. The assumption "$\mathbb{S}\mathrm{i}(\mathfrak{K})\subseteq\mathbb{I}\mathbb{S}(\mathfrak{A})$" of Theorem 1.1 is required only to show that every weak injective of $\mathfrak{K}$ is in $\mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$; see Sect. 4. But it is easily proved (see [12; Lemma 2.9]) that if $\mathfrak{K}$ is a congruence-distributive variety generated by a finite set $\mathfrak{A}$ of finite algebras, then every weak injective of $\mathfrak{K}$ lies in $\mathbb{I}\mathbb{S}\mathbb{P}\mathbb{H}(\mathfrak{A})$. Given these observations we obtain the following corollary; see [12] and Quackenbush [29].

1.6. **Corollary.** *Let $\mathfrak{K}$ be a congruence-distributive variety generated by a finite simple algebra A. Then the weak injectives in $\mathfrak{K}$ are induced by A.*

The descriptions of [weak] injectives given in [1, 7, 18, 19] follow from 1.6 and 1.2 (iii); we refer to [12] for a fuller discussion.

How does one find a simplicity formula when the maximal subdirectly irreducibles are not simple? If we restrict our attention to lattice-ordered algebras, then we can proceed as follows. It is well known that direct product decompositions of a bounded lattice L correspond bijectively with the complemented neutral elements of L (see Birkhoff [5; Theorem 12; p. 69]). Consider the following formula:

$$\eta(a) := \exists u \forall xy [a \vee u = 1 \,\&\, a \wedge u = 0 \,\&\, (x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) \\ \&\, (x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)].$$

Then $L \models \eta(a)$ if and only if a is a complemented neutral element; whence, if L is a directly irreducible bounded lattice, then $L \models \eta(a)$ if and only if $a \in \{0, 1\}$. Consequently, $\alpha(a, b) := (\eta(a) \,\&\, b = 0)$ is a simplicity formula for any class of bounded-lattice-ordered algebras which are directly irreducible as lattices. In Theorem 1.1 we assume that $\mathfrak{A}$ is a set of finite algebras; hence the restriction that 0 and 1 be nullary operations can be done away with, since we can add nullary operations to the type without killing any [weak] injectives (see Lemma 3.4).

1.7. **Theorem.** *Let $\mathfrak{A}$ be a finite set of a finite lattice-ordered algebras each of which is directly irreducible as a lattice, let $\mathfrak{K}$ be the variety generated by $\mathfrak{A}$, and assume $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{IS}(\mathfrak{A})$. Then the injectives [weak injectives] in $\mathfrak{K}$ are induced by the subdirectly irreducibles which are injective in $\mathfrak{K}$ [weak injective in $\mathbb{S}\mathrm{i}(\mathfrak{K})$].*

Consider the case where $\mathfrak{K}$ is the variety $\mathfrak{B}_n$ $(n \geqq 0)$ of distributive p-algebras; see Lee [26] or Lakser [25]. The subdirectly irreducible algebra $\bar{B}_n$ obtained by adjoining a new unit to the Boolean algebra $\mathbf{2}^n$ generates the variety $\mathfrak{B}_n$. Let B be a Boolean algebra and define

$$B^{[n+1]} = \{\langle x_0, \ldots, x_n \rangle \in B^{n+1} \mid x_0 \leqq x_1 \wedge \ldots \wedge x_n\}.$$

Let P_n be the poset obtained by adjoining a zero element to an n-element antichain; then $B^{[n+1]}$ is isomorphic to the algebra of order-preserving maps from P_n into B. Since P_n is order-isomorphic to the poset of prime filters of $\bar{B}_n$, it follows by Davey [10], that $B^{[n+1]}$ is isomorphic to the algebra of continuous maps from X_B into $\bar{B}_n$; i.e. $B^{[n+1]} \cong \bar{B}_n[B]$. (Alternatively, this may be verified by a direct calculation.) Every subalgebra of $\bar{B}_n$ is subdirectly irreducible, and so 1.2 (ii) (iii) are applicable. It is easily seen that $\bar{B}_m$ is weak injective in $\mathbb{S}\mathrm{i}(\mathfrak{B}_n)$ if and only if $m \in \{0, n\}$, $\bar{B}_n$ is self-injective if and only if $n \in \{0, 1, 2\}$, and, by 1.2 (ii), $\mathbf{2}$ is injective in $\mathfrak{B}_n$ for all n. Since $\bar{B}_n$ is directly irreducible as a lattice we obtain the description of the [weak] injectives in $\mathfrak{B}_n$ due to Balbes and Grätzer [2] and Grätzer and Lakser [22] (note that $\bar{B}_0 \cong \mathbf{2}$ and $\mathbf{2}[B] \cong B$): for $n \in \{1, 2\}$, $\mathfrak{B}_n$ has enough injectives and I is injective in $\mathfrak{B}_n$ if and only if it is isomorphic to $B_0 \times B_1^{[n+1]}$ where B_0 and B_1 are complete Boolean algebras; for $n \geqq 3$, $\mathfrak{B}_n$ does not have enough injectives, I is weak injective in $\mathfrak{B}_n$ if and only if it is isomorphic to $B_0 \times B_1^{[n+1]}$ where B_0 and B_1 are complete Boolean algebras, and I is injective in $\mathfrak{B}_n$ if and only if it is a complete Boolean algebra.

The variety $\mathscr{L}_n$ of Heyting algebras generated by an n-element chain and the variety $\mathscr{S}_n$ of Brouwerian algebras generated by an n-element chain may be analysed similarly. Every subalgebra of C_n is subdirectly irreducible, and C_n is directly irreducible as a lattice; C_m is weak injective in $\mathbb{S}\mathrm{i}(\mathscr{L}_n)$ if and only if $m \in \{0, n\}$, and C_m is weak injective in $\mathbb{S}\mathrm{i}(\mathscr{S}_n)$ if and only if $m=n$; C_n is self-injective as a Heyting algebra if and only if $n \in \{2, 3\}$, and as a Brouwerian algebra if and only if $n=2$; and by 1.2 (ii), C_2 is injective in $\mathscr{L}_n$ for all n, and is injective in $\mathscr{S}_n$ if and only if $n=2$. Hence 1.1 yields the description of the [weak] injectives in $\mathscr{L}_n$ and $\mathscr{S}_n$ given in Davey [11]: $\mathscr{L}_3$ has enough injectives and they are the algebras of the form $B_0 \times C_3[B_1]$, where B_0 and B_1 are complete Boolean algebras; for $n \geqq 4$, $\mathscr{L}_n$ does not have enough injectives, the weak injectives are the algebras of the form $B_0 \times C_n[B_1]$, where B_0 and B_1 are complete Boolean algebras, and the injectives are precisely the complete Boolean algebras; for $n \geqq 3$, $\mathscr{S}_n$ does not have enough injectives, the weak injective are the algebras of the form $C_n[B]$, where B is complete, and the injectives are trivial. To obtain the Heyting algebra C_n from the Brouwerian algebra C_n we simply add 0 as a nullary operation; hence $\mathscr{L}_n$ and $\mathscr{S}_n$ illustrate the fact that adding nullary operations to the type can increase the number of [weak] injectives.

Katriňák [24] has found the injectives in the class $\mathscr{S}$ of double Stone algebras; $\mathscr{S}$ is generated by C_4 regarded as a double p-algebra. A similar argument to that given above yields his result: $\mathscr{S}$ has enough injectives and they are induced by C_3 and C_4. A similar analysis may be carried out to find the weak injectives in other varieties of double p-algebras generated by finite algebras; a description of the subdirectly irreducible distributive double p-algebras may be found in Davey [13] (they are all directly irreducible as lattices).

Of course, 1.7 may be applied to varieties of lattices. For example, in the variety $\mathcal{N}_5$, generated by the pentagon, N_5 we find that the weak injectives are induced by N_5 and the injectives are trivial; in fact, it is proved in Day [15] that in every variety of lattices, other then the variety of distributive lattices the injectives are trivial. Let N_5^+ denote the pentagon with all its elements added as nullary operations, and let $\mathcal{N}_5{}^+$ be the variety it generates; then $\mathcal{N}_5{}^+$ has enough injectives and they are induced by N_5^+.

This completes our discussion of the congruence-distributive case. As we have already noted, the assumption "$\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$" is required in the proof of Theorem 1.1 only to establish that every weak injective in $\mathfrak{K}$ is in $\mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$; we claim that if $\mathfrak{A}$ is a finite set of finite simple algebras, $\mathfrak{A}$ has factorizable congruences, and $\mathfrak{K}$ is the variety generated by $\mathfrak{A}$, then every weak injective in $\mathfrak{K}$ is automatically in $\mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$. In order to prove this we must borrow one result from Sect. 2. Assume that I is weak injective in $\mathfrak{K}$; then since $\mathfrak{K} = \mathbb{H}\mathbb{S}\mathbb{P}(\mathfrak{A})$ we have $I \in \mathbb{H}\mathbb{P}(\mathfrak{A})$. Let $\mathbb{B}^*(\mathfrak{A})$ denote the class of all bounded Boolean powers of members of $\mathfrak{A}$ (see Sect. 2), and let $\mathbb{P}_f(\mathfrak{A})$ denote all finite products of members of $\mathfrak{A}$. Since every power of an algebra A is isomorphic to a bounded Boolean Power of A, and since $\mathfrak{A}$ is finite, it follows that $\mathbb{P}(\mathfrak{A}) \subseteq \mathbb{I}\mathbb{P}_f\mathbb{B}^*(\mathfrak{A})$. Hence $I \in \mathbb{H}\mathbb{P}(\mathfrak{A}) \subseteq \mathbb{H}\mathbb{P}_f\mathbb{B}^*(\mathfrak{A})$. By 2.6, $\mathbb{B}^*(\mathfrak{A})$ has factorizable congruences, and hence $\mathbb{H}\mathbb{P}_f\mathbb{B}^*(\mathfrak{A}) \subseteq \mathbb{P}_f\mathbb{H}\mathbb{B}^*(\mathfrak{A})$. But each member A of $\mathfrak{A}$ satisfies $\mathrm{Con}(A^n) \cong \mathbf{2}^n$, and hence by Proposition 2.5 and Corollary 3.6 of [6], $\mathbb{H}\mathbb{B}^*(\mathfrak{A}) \subseteq \mathbb{B}^*(\mathfrak{A})$. Thus $I \in \mathbb{P}_f \mathbb{B}^*(\mathfrak{A}) \subseteq \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$ and so 1.8 below is a particular case of Theorem 1.1. Not only have we proved that $I \in \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$, but we have almost proved 1.8 itself: until now this was the best (unpublished) result known to us; it was the authors' desire to generalize this result which led to Theorem 1.1.

1.8. **Theorem.** *Let* $\mathfrak{A}$ *be a finite set of finite algebras such that for all* $A_j \in \mathfrak{A} (j<m)$, $\mathrm{Con}(\prod(A_j | j<m)) \cong \mathbf{2}^m$, *and let* $\mathfrak{K}$ *be the variety generated by* $\mathfrak{A}$. *Then the [weak] injectives in* $\mathfrak{K}$ *are induced by the algebras in* $\mathfrak{A}$ *which are [weak] injective in* $\mathfrak{K}$.

A variety $\mathfrak{K}$ is congruence permutable if for every $A \in \mathfrak{K}$ and all $\Theta, \Phi \in \mathrm{Con}(A)$ we have $\Theta \circ \Phi = \Phi \circ \Theta$; for example, if the algebras in $\mathfrak{K}$ have an underlying group structure, then $\mathfrak{K}$ is congruence permutable. In Werner [31; Theorem 4; p. 100] it is proved that if $\mathfrak{A}$ is a subclass of a congruence-permutable variety, then $\mathfrak{A}$ has factorizable congruences if and only if for all $A, B \in \mathfrak{A}$ the natural map from $\mathrm{Con}(A) \times \mathrm{Con}(B)$ to $\mathrm{Con}(A \times B)$ is onto.

1.9. **Corollary.** *Let* $\mathfrak{A}$ *be a finite set of finite algebras such that* $\mathrm{Con}(A \times B) \cong \mathbf{2}^2$ *for all* $A, B \in \mathfrak{A}$, *and assume that the variety* $\mathfrak{K}$ *generated by* $\mathfrak{A}$ *is congruence permutable. Then the [weak] injectives in* $\mathfrak{K}$ *are induced by the algebras in* $\mathfrak{A}$ *which are [weak] injective in* $\mathfrak{K}$.

A finite algebra A is *functionally complete* if every map $\phi\colon A^n \to A$ is an algebraic function (i.e. a polynomial with perhaps some variables replaced by members of A). Every functionally complete algebra A satisfies $\mathrm{Con}(A^n) \cong \mathbf{2}^n$, and it is proved in Werner [31; Theorem 1; p. 91] that if $\mathfrak{K}$ is congruence permutable, then A is functionally complete if and only if $\mathrm{Con}(A^2) \cong \mathbf{2}^2$. The description of the [weak] injectives in a variety generated by a functionally complete algebra follows from 1.8; the real question becomes: "*Which functionally complete algebras are [weak] injective in the variety they generate?*" Since finite simple nonabelian groups are functionally complete, we are led to ask "*When is a finite simple nonabelian group [weak] injective in the variety it generates?*" In Banaschewski and Nelson [4], algebras with the property $\mathrm{Con}(A^n) \cong \mathbf{2}^n$ are called *finitely power simple*. The description of the injectives in a variety generated by a finitely power simple algebra which follows from 1.8 is also established in [4].

Let $\mathfrak{A}$ be a finite set of finite simple planar squags of cardinality greater than three, or finite simple planar sloops of cardinality greater than 2, and let $\mathfrak{K}$ be the variety generated by $\mathfrak{A}$; see Quackenbush [30]. In [30] it is shown that $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$, that $\mathfrak{K}$ is congruence permutable, that $\mathrm{Con}(A \times B) \cong \mathbf{2}^2$ for all, $A, B \in \mathfrak{A}$, and that each member of $\mathfrak{A}$ is weak injective, but not injective, in $\mathfrak{K}$. Quackenbush's description of the [weak] injectives in $\mathfrak{K}$ follows at once from 1.9.: the weak injectives in $\mathfrak{K}$ are induced by $\mathfrak{A}$, and the injectives in $\mathfrak{K}$ are trivial. These varieties illustrate the advantages of assuming "$\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$", rather than the more natural assumption "$\mathbb{S}\mathrm{i}(\mathfrak{K}) = \mathfrak{A}$": each nontrivial squag [sloop] has a simple subsquag [subsloop] A, with three [two] elements with the undesirable property $\mathrm{Con}(A^2) \not\cong \mathbf{2}^2$; that is, although $\mathfrak{A}$ has factorizable congruences, $\mathbb{S}\mathrm{i}(\mathfrak{A})$ does not.

2. Principal-Congruence Formulae

Recall that a formula is called *primitive positive* if it is an existential conjunct of atomic formulae; since we are working with algebras this means an existential conjunct of equations. A primitive-positive formula $\gamma(x, y, a, b)$ is a *principal-congruence-formula* if it is of the form

$$\exists \vec{z}\left[x = p_0(a_0, \vec{z})\ \&\ \left(\mathop{\&}_{i=1}^{n} p_{i-1}(b_{i-1}, \vec{z}) = p_i(a_i, \vec{z})\right)\ \&\ p_n(b_n, \vec{z}) = y\right]$$

where $\{a_i, b_i\} = \{a, b\}$ for all $i \leqq n$ and p_i is a polynomial for all $i \leqq n$. By Mal'cev's lemma (see [21; Theorem 3; p. 54]), if $\gamma(x, y, a, b)$ is a principal-congruences formula and x, y, a, b are elements of an algebra A such that $A \vDash \gamma(x, y, a, b)$, then $x \equiv y(\Theta(a, b))$. Conversely if $x \equiv y(\Theta(a, b))$ then there exists a principal congruence formula γ such that $A \vDash \gamma(x, y, a, b)$. A principal-congruence formula $\gamma(x, y, a, b)$ is a *principal-congruence* formula for a class $\mathfrak{A}$ of algebras if for all A in $\mathfrak{A}$ and all x, y, a, b in A $A \vDash \gamma(x, y, a, b)$ if and only if $x \equiv y(\Theta(a, b))$.

In [20], Fried, Grätzer, and Quackenbush study varieties which have a principal-congruence formula; they refer to a principal-congruence formula as a *uniform congruence scheme.* The following result is closely related to their Theorem 3.5.

2.1. **Lemma.** *Let γ be a principal-congruence formula for $\mathfrak{A}$.*

(i) *γ is a principal-congruence formula for $\mathbb{P}(\mathfrak{A})$.*

(ii) *For every family $(A_j | j \in J)$ of algebras in $\mathfrak{A}$, and each pair $a, b \in \prod(A_j | j \in J)$, $\Theta(a, b) = \prod(\Theta(a(j), b(j)) | j \in J)$.*

(iii) *$\mathfrak{A}$ has factorizable congruences.*

Proof. Let $(A_j | j \in J)$ be a family of algebras in $\mathfrak{A}$, define $A := \prod(A_j | j \in J)$, and for $a, b \in A$ define $\Phi = \prod(\Theta(a(j), b(j)) | j \in J)$. We prove (i) and (ii) simultaneously by establishing the following chain of implications:

$$x \equiv y(\Theta(a, b)) \Rightarrow x \equiv y(\Phi) \Rightarrow A \vDash \gamma(x, y, a, b) \Rightarrow x \equiv y(\Theta(a, b)).$$

Since $a \equiv b(\Phi)$ we have $\Theta(a, b) \leqq \Phi$ and the first implication follows. Assume $x \equiv y(\Phi)$; as γ is a principal congruence formula for $\mathfrak{A}$, for all $j \in J$ we have $A_j \vDash \gamma(x(j), y(j), a(j), b(j))$. Now since γ is primitive positive it follows that $A \vDash \gamma(x, y, a, b)$, which establishes the second implication. The third implication follows by Mal'cev's lemma, as we observed above. It is easily proved that all congruences on $A_0 \times \ldots \times A_n$ factor if and only if all principal congruences factor (see [17; Theorem 3; p. 392]), and hence (iii) follows from (ii).

If $\mathfrak{A}$ is a finite set of finite algebras, then the converse of 2.1 (iii) also holds.

2.2. **Proposition.** *Let $\mathfrak{A}$ be a finite set of finite algebras. Then there is a principal-congruence formula for $\mathfrak{A}$ if and only if $\mathfrak{A}$ has factorizable congruences.*

Proof. Assume $\mathfrak{A}$ has factorizable congruences. Let $A = A_0 \times \ldots \times A_n$ be a large enough finite product of members of $\mathfrak{A}$ such that we can find elements x, y, a, b in A with the following properties:

(i) for all $j \leqq n$, $x_j \equiv y_j(\Theta(a_j, b_j))$;

(ii) for each $C \in \mathfrak{A}$ and all $s, t, u, v \in C$ such that $s \equiv t(\Theta(u, v))$, there exists $j \leqq n$ such that $A_j = C$ and $x_j = s$, $y_j = t$, $a_j = u$, $b_j = v$.

Since $\mathfrak{A}$ is a finite set of finite algebras, such an A exists. Since $\mathfrak{A}$ has factorizable congruences $\Theta(a, b) = \prod(\Theta(a_j, b_j) | j \leqq n)$, and hence $x \equiv y(\Theta(a, b))$. By Mal'cev's lemma there is a principal-congruence formula γ such that $A \models \gamma(x, y, a, b)$. Because of the choice of A, γ is a principal-congruence formula for $\mathfrak{A}$.

It follows from 2.2 that under the assumptions of Theorem 1.1 there is a principal-congruence formula γ for $\mathfrak{A}$. In fact, we need to know that the same formula γ will provide a principal-congruence formula for every retract of a product of bounded Boolean powers of members of $\mathfrak{A}$; which we now proceed to prove.

For a subalgebra A of $\prod(A_j | j \in J)$, a formula $\beta(\vec{z})$, and $\vec{a} \in A^n$ we introduce the notation

$$[\![\beta(\vec{a})]\!] = \{j \in J \mid A_j \models \phi(\vec{a}(j))\}.$$

When applied to the formula $x = y$ and the subalgebra $A[B]^*$ of A^X, where $X = X_B$, we have

$$[\![a = b]\!] = \{x \in X \mid a(x) = b(x)\};$$

that is, $[\![a = b]\!]$ is the equalizer of the continuous functions a and b. Since continuous functions from a compact space into a discrete space have finite image and since

$$[\![a = b]\!] = U(a^{-1}(c) \cap b^{-1}(c) \mid c \in \operatorname{Im}(a) \cap \operatorname{Im}(b)),$$

it follows that $[\![a = b]\!]$ is clopen in X_B for all $a, b \in A[B]^*$.

2.3. **Lemma.** *Let $\alpha(\vec{z})$ be a positive formula, let $\beta(\vec{z})$ be a primitive-positive formula, and let B be a Boolean algebra. Define $\delta := \forall \vec{z}[\alpha(\vec{z}) \rightarrow \beta(\vec{z})]$.*

(i) *$A[B]^* \models \beta(\vec{a})$ if and only if $[\![\beta(\vec{a})]\!] = X_B$.*

(ii) *$A \models \delta$ implies $A[B]^* \models \delta$.*

(iii) *$A_j \models \delta$ for all $j \in J$ implies $\prod(A_j | j \in J) \models \delta$.*

Proof. (i) Since a positive formula is preserved by every onto homomorphism, the forward implication is trivial. Assume $[\![\beta(\vec{a})]\!] = X_B$ and that $\beta(\vec{z}) = \exists y_0 \dots y_n \rho(y_0, \dots, y_n, \vec{z})$. Then for all $x \in X$ we can find $b_{0x}, \dots, b_{nx} \in A$ such that $A \models \rho(b_{0x}, \dots, b_{nx}, \vec{a}(x))$. For each $a \in A$, let $\hat{a}$ be the constant map from X onto $\{a\}$. Since ρ is a finite conjunct of equations, $U_x := [\![\rho(\hat{b}_{0x}, \dots, \hat{b}_{nx}, \vec{a})]\!]$ is a finite intersection of clopen sets and hence is a clopen neighbourhood of x. Since X_B is compact and since each U_x is clopen there is a finite partition $\{V_k | k \leqq m\}$ of X_B into clopen sets such that for all $k \leqq m$ there exists $x_k \in X_B$ with $V_k \subseteq U_{x_k}$. For each $j \leqq n$ we paste together the appropiate $\hat{b}_{jx}$'s over this partition; that is, we define $b_j \in A[B]^*$ by $b_j \restriction V_k = \hat{b}_{jx_k}$ for all $k \leqq m$. Clearly the b_j's provide a global solution for $\exists y_0 \dots y_n \rho(y_0, \dots, y_n, \vec{a})$, and hence $A[B]^* \models \beta(\vec{a})$.

(ii) Assume $A \models \delta$ and $A[B]^* \models \alpha(\vec{a})$. Because α is positive, $[\![\alpha(\vec{a})]\!] = X_B$, and hence $[\![\beta(\vec{a})]\!] = X_B$ since $A \models \delta$. Thus, by (i), $A[B]^* \models \beta(\vec{a})$, as required.

(iii) The proof is almost identical to the proof of (ii), with X_B replaced by J. Instead of using (i) for the last step we use the fact that if β is primitive positive then $\prod A_j \models \beta(\vec{a})$ if and only if $A_j \models \beta(\vec{a}(j))$ for all $j \in J$.
We note in passing that the obvious generalizations of 2.3 (i) and 2.3 (ii) to algebras of global sections over Boolean spaces are valid; but we shall not require them here.

Recall that A is a *retract* of A' if there is an embedding $\phi: A \rightarrowtail A'$ and an onto homomorphism $\psi: A' \twoheadrightarrow A$ with $\phi\psi = \mathrm{id}_A$.

2.4. **Lemma.** *Let* $\alpha(\vec{x}, \vec{y})$ *and* $\gamma_i(\vec{x}, \vec{y}, \vec{z})$ $(i \leqq n)$ *be positive formulae, let* $\beta_i(\vec{x}, \vec{z})$ $(i \leqq n)$ *be existential formulae, and define*

$$\delta := \forall\vec{x}\,\exists\vec{y}[\alpha(\vec{x}, \vec{y})\,\&\,\forall\vec{z}[\mathop{\&}_{i \leqq n}(\beta_i(\vec{x}, \vec{z}) \rightarrow \gamma_i(\vec{x}, \vec{y}, \vec{z}))]].$$

If A *is a retract of* A' *and* $A' \models \delta$, *then* $A \models \delta$.

Proof. Let $\phi: A \rightarrowtail A'$ and $\psi: A' \twoheadrightarrow A$ be homomorphisms such that $\phi\psi = \mathrm{id}_A$, and assume $A' \models \delta$. Let $\vec{a} \in A$; then there exists $\vec{b} \in A'$ such that $A' \models \alpha(\vec{a}\phi, \vec{b})$ and $A' \models \forall\vec{z}\,[\mathop{\&}_{i \leqq n}(\beta_i(\vec{a}\phi, \vec{z}) \rightarrow \gamma_i(\vec{a}\phi, \vec{b}, \vec{z}))]$. We claim that $\vec{b}\psi$ does the same job down in A. As α is positive and $\vec{a}\phi\psi = \vec{a}$, we have $A \models \alpha(\vec{a}, \vec{b}\psi)$. Let $\vec{c} \in A$ and assume that $A \models \beta_i(\vec{a}, \vec{c})$ for some $i \leqq n$. Since β_i is existential and ϕ is an embedding, we have $A' \models \beta(\vec{a}\phi, \vec{c}\phi)$ and hence $A' \models \gamma_i(\vec{a}\phi, \vec{b}, \vec{c}\phi)$. Finally, γ_i is positive, so we get $A \models \gamma_i(\vec{a}, \vec{b}\psi, \vec{c})$.

If $\gamma(x, y, a, b)$ is a principal-congruence formula, then, as we have already observed, for an algebra A and each pair $a, b \in A$,

$$\{\langle x, y\rangle \in A^2 \mid A \models \gamma(x, y, a, b)\} \subseteq \Theta(a, b).$$

Hence γ is a principal-congruence formula for a class $\mathfrak{A}$ of algebras if and only if each algebra A in $\mathfrak{A}$ satisfies

(i) $\forall abx[\gamma(x, x, a, b)\,\&\,\gamma(a, b, a, b)]$;
(ii) $\forall abxy[\gamma(x, y, a, b) \rightarrow \gamma(y, x, y, b)]$;
(iii) $\forall abxyz[\gamma(x, y, a, b)\,\&\,\gamma(y, z, a, b) \rightarrow \gamma(x, z, a, b)]$;
(iv) for each fundamental operation f,

$$\forall ab\vec{x}\vec{y}[(\mathop{\&}_{i \leqq n}\gamma(x_i, y_i, a, b)) \rightarrow \gamma(f\vec{x}, f\vec{y}, a, b)].$$

Hence the following proposition and its corollary follow at once from 2.1, 2.2, 2.3, and 2.4.

We introduce two new class operators corresponding to bounded Boolean powers and retracts:

$$\mathbb{B}^*(\mathfrak{A}) = \{B[A]^* \mid A \in \mathfrak{A};\ B \text{ is a Boolean algebra}\};$$

$$\mathbb{R}(\mathfrak{A}) = \{A \mid A \text{ is a retract of } A';\ A' \in \mathfrak{A}\}.$$

2.5. **Proposition.** *If γ is a principal-congruence formula for a class $\mathfrak{A}$, then it is a principal-congruence formula for $\mathbb{R}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$, and hence $\mathbb{R}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$ has factorizable congruences.*

2.6. **Corollary.** *Let $\mathfrak{A}$ be a finite set of finite algebras. If $\mathfrak{A}$ has factorizable congruences, then $\mathbb{R}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$ has factorizable congruences.*

Let $\alpha(a, b)$ and $\gamma(x, y, a, b)$ be formulae and consider the following two sentences:

(P) $\forall abcdxyz[\alpha(a, b) \,\&\, \gamma(x, y, a, b) \,\&\, \gamma(y, z, c, d)$
$\rightarrow \exists w[\gamma(x, w, c, d) \,\&\, \gamma(w, z, a, b)]]$;

(D) $\forall abcdefwxyz[a, b) \,\&\, \alpha(c, d) \,\&\, \gamma(w, x, a, b) \,\&\, \gamma(x, y, e, f)$
$\&\, \gamma(w, z, c, d) \,\&\, \gamma(z, y, e, f) \rightarrow \exists v[\gamma(w, v, a, b) \,\&\, \gamma(w, v, c, d) \,\&\, \gamma(v, y, e, f)]]$.

If γ is a principal-congruence formula for A, then $A \models (\mathrm{P})$ if and only if

$$A \models \alpha(a, b) \quad \text{implies} \quad \Theta(a, b) \circ \Psi = \Psi \circ \Theta(a, b),$$

for all principal congruences Ψ; and $A \models (\mathrm{D})$ if and only if

$$A \models \alpha(a, b) \,\&\, \alpha(c, d) \quad \text{implies}$$
$$(\Theta(a, b) \circ \Psi) \cap (\Theta(c, d) \circ \Psi) \subseteq (\Theta(a, b) \cap \Theta(c, d)) \circ \Psi,$$

for all principal congruences Ψ. Thus if $A \models (\mathrm{P}) \,\&\, (\mathrm{D})$, then

$$A \models \alpha(a, b) \,\&\, \alpha(c, d) \quad \text{implies}$$
$$(\Theta(a, b) \vee \Psi) \wedge (\Theta(c, d) \vee \Psi) = (\Theta(a, b) \wedge \Theta(c, d)) \vee \Psi,$$

for every principal congruence Ψ.

Note that, by Lemmas 2.3 and 2.4, if γ is a principal-congruence formula and α is existential positive, then both (P) and (D) are preserved by the operators $\mathbb{R}$, $\mathbb{P}$, and $\mathbb{B}^*$.

For the remainder of this section let $\gamma(x, y, a, b)$ be a principal-congruence formula for $\mathfrak{A}$, let $\alpha(u, v)$ be an existential positive formula, and assume that $\mathfrak{A}$ satisfies (P) and (D). Let $A \in \mathbb{R}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$; say, $A' \in \mathbb{P}\mathbb{B}^*(\mathfrak{A})$ with homomorphisms $\phi: A \rightarrowtail A'$ and $\psi: A' \twoheadrightarrow A$ satisfying $\phi\psi = \mathrm{id}_A$. Define

$$\mathscr{L}_A := \{\Theta(a, b) \mid A \models \alpha(a, b)\},$$

and define $\mathscr{L}_{A'}$ similarly. For every binary relation Φ on A' define a binary relation $\Phi\bar{\psi}$ on A by

$$\Phi\bar{\psi} := \{\langle x\psi, y\psi \rangle \mid \langle x, y \rangle \in \Phi\}.$$

2.7. **Proposition.** (i) *If $\Theta, \Phi \in \mathscr{L}_A$ and Ψ is a principal congruence on A, then*

$$\Theta \circ \Psi = \Psi \circ \Theta \quad \text{and} \quad (\Theta \vee \Psi) \wedge (\Phi \vee \Psi) = (\Theta \wedge \Phi) \vee \Psi; \quad \text{and similarly for } A'.$$

(ii) *$A' \models \alpha(a, b)$ implies that $A \models \alpha(a\psi, b\psi)$ and $\Theta(a, b)\bar{\psi} = \Theta(a\psi, b\psi)$. Hence $\bar{\psi}$ maps $\mathscr{L}_{A'}$ into $\mathscr{L}_A$.*

(iii) $A \models \alpha(c, d)$ *implies that* $A' \models \alpha(c\phi, d\phi)$. *Hence* $\bar{\psi}$ *maps* $\mathscr{L}_{A'}$ *onto* $\mathscr{L}_A$.

(iv) *The map* $\bar{\phi}: \mathscr{L}_A \to \mathscr{L}_{A'}$, *defined by* $\Theta(c,d)\bar{\phi} = \Theta(c\phi, d\phi)$, *order-embeds* $\mathscr{L}_A$ *into* $\mathscr{L}_{A'}$ *as an order-retract.*

(v) $\Delta\bar{\psi} = \Delta$, $\nabla\bar{\psi} = \nabla$, *and for all* $\Theta, \Phi \in \mathscr{L}_{A'}$, $(\Theta \circ \Phi)\bar{\psi} = \Theta\bar{\psi} \circ \Phi\bar{\psi}$ *and* $(\Theta \wedge \Phi)\bar{\psi} = \Theta\bar{\psi} \wedge \Phi\bar{\psi}$.

Proof. Throughout the proof let $\Psi = \ker \psi$.

(i) This follows from the discussion above.

(ii) As α is positive, it is preserved by ψ. For any homomorphism μ, $\langle x, y\rangle \in \Theta(a,b)$ implies $\langle x\mu, y\mu\rangle \in \Theta(a\mu, b\mu)$; hence

$$\langle a\psi, b\psi\rangle \in \Theta(a,b)\bar{\psi} = \{\langle x\psi, y\psi\rangle \mid \langle x, y\rangle \in \Theta(a,b)\} \subseteq \Theta(a\psi, b\psi).$$

Thus, to prove that $\Theta(a,b)\bar{\psi} = \Theta(a\psi, b\psi)$, it remains to show that $\Theta(a,b)\bar{\psi}$ is a congruence. Only transitivity is nontrivial. Assume $\langle x, y\rangle, \langle y, z\rangle \in \Theta(a,b)\bar{\psi}$; then there exists $\langle s, t\rangle$, $\langle u, v\rangle \in \Theta(a,b)$ such that $\langle s\psi, t\psi\rangle = \langle x, y\rangle$ and $\langle u\psi, v\psi\rangle = \langle y, z\rangle$. Hence $\langle t, u\rangle \in \Psi$ and by (i) $\langle s, v\rangle \in \Theta(a,b) \circ \Psi \circ \Theta(a,b) = \Psi \circ \Theta(a,b) \circ \Psi$, and so we can find $\langle c, d\rangle \in \Theta(a,b)$ such that $\langle s, c\rangle, \langle d, v\rangle \in \Psi$. Thus $x = s\psi = c\psi$ and $z = v\psi = d\psi$, which yields $\langle x, z\rangle \in \Theta(a,b)\bar{\psi}$, as required.

(iii) Since α is existential it is preserved by the embedding ϕ. Trivially, by (ii), $\Theta(c\phi, d\phi)\bar{\psi} = \Theta(c\phi\psi, d\phi\psi) = \Theta(c,d)$, and so $\bar{\psi}$ maps $\mathscr{L}_{A'}$ onto $\mathscr{L}_A$.

(iv) Since $\Theta(c,d)\bar{\phi}\bar{\psi} = \Theta(c,d)$, it remains to show that $\bar{\phi}$ and $\bar{\psi}$ are order-preserving. But this is easily seen.

(v) Of course, $\Delta\bar{\psi} = \Delta$ and $\nabla\bar{\psi} = \nabla$ are trivial. Let $\langle x, y\rangle \in (\Theta \circ \Phi)\bar{\psi}$; then there exist $u, v, w \in A'$ such that $x = u\psi$, $\langle u, v\rangle \in \Theta$, $\langle v, w\rangle \in \Phi$, and $w\psi = y$. Hence $\langle x, v\psi\rangle \in \Theta\bar{\psi}$ and $\langle v\psi, y\rangle \in \Phi\bar{\psi}$, and so $\langle x, y\rangle \in \Theta\bar{\psi} \circ \Phi\bar{\psi}$. Conversely, let $\langle x, y\rangle \in \Theta\bar{\psi} \circ \Phi\bar{\psi}$; then there $z \in A$ such that $\langle x, z\rangle \in \Theta\bar{\psi}$ and $\langle z, y\rangle \in \Phi\bar{\psi}$. Hence there exist $s, t, u, v \in A$ with $\langle s, t\rangle \in \Theta$, $\langle u, v\rangle \in \Phi$, $s\psi = x$, $t\psi = z = u\psi$, and $v\psi = y$. By (i), $\langle s, v\rangle \in \Theta \circ \Psi \circ \Phi = \Theta \circ \Phi \circ \Psi$, and so there exists $w \in A$ such that $\langle s, w\rangle \in \Theta \circ \Phi$ and $w\psi = v\psi = y$. Since $s\psi = x$, it follows that $\langle x, y\rangle (\Theta \circ \Phi)\bar{\psi}$.

Let $\langle x, y\rangle \in (\Theta \wedge \Phi)\bar{\psi}$; then there exist $\langle u, v\rangle \in \Theta \wedge \Phi$ with $u\psi = x$ and $v\psi = y$, and hence $\langle x, y\rangle \in \Theta\bar{\psi} \wedge \Phi\bar{\psi}$. Conversely, assume that $\langle x, y\rangle \in \Theta\bar{\psi} \wedge \Phi\bar{\psi}$; then there exist $\langle s, t\rangle \in \Theta$ and $\langle u, v\rangle \in \Phi$ such that $s\psi = x = u\psi$ and $t\psi = y = v\psi$. By (i), $\langle s, v\rangle \in (\Theta \circ \Psi) \cap (\Psi \circ \Phi) = (\Theta \cap \Phi) \circ \Psi$, and so there exists $w \in A'$ such that $\langle s, w\rangle \in \Theta \cap \Phi$ and $w\psi = v\psi = y$. Since $s\psi = x$, we have $\langle x, y\rangle \in (\Theta \cap \Phi)\bar{\psi}$.

Recall that a lattice-homomorphic image of a Boolean algebra is a Boolean algebra, and that an order-retract of a complete lattice is complete.

2.8. **Corollary.** *If* $\mathscr{L}_{A'}$ *is a sublattice of* Con(A'), *then* $\mathscr{L}_A$ *is a sublattice of* Con(A), $\mathscr{L}_A$ *is a homomorphic image of* $\mathscr{L}_{A'}$, *and both* $\mathscr{L}_A$ *and* $\mathscr{L}_{A'}$ *are distributive lattices of permuting congruences. If* $\mathscr{L}_{A'}$ *is a [complete] Boolean sublattice of* Con(A'), *then* $\mathscr{L}_A$ *is a [complete] Boolean sublattice of* Con(A).

Finally we present a sentence ε which is preserved under the retraction ψ; note that this is not covered by Lemma 2.4. The sentence ε will play an important role in the proof of Theorem 1.1.

2.9. **Corollary.** *Let C be a finite set of nullary operation symbols in the type of A and A'. If $A' \models \varepsilon$, then $A \models \varepsilon$, where*

$$\varepsilon := \forall d (\exists a_c b_c)_{c \in C} [\mathop{\&}_{c \in C} (\alpha(a_c, b_c) \,\&\, \gamma(d, c, a_c, b_c)) \\ \&\, \forall uv [(\mathop{\&}_{c \in C} \gamma(u, v, a_c, b_c)) \rightarrow u = v]].$$

Proof. Observe that $A \models \varepsilon$ if and only if for every $d \in A$ there are elements $a_c, b_c \in A$ $(c \in C)$ such that $A \models \alpha(a_c, b_c)$ and $\langle d, c \rangle \in \Theta(a_c, b_c)$ for all $c \in C$ and $\bigcap_{c \in C} \Theta(a_c, b_c) = \Delta$; and, of course, similarly for A'. Assume that $A' \models \varepsilon$. Let $d \in A$; then there are $a_c, b_c \in A'$ $(c \in C)$ such that $A' \models \alpha(a_c, b_c)$ and $\langle d\phi, c \rangle \in \Theta(a_c, b_c)$ and $\bigcap_{c \in C} \Theta(a_c, b_c) = \Delta$. Thus since $c\psi = c$, we have $A \models \alpha(a_c\psi, b_c\psi)$. $\langle d, c \rangle \Theta \in (a_c\psi, b_c\psi)$, and

$$\bigcap_{c \in C} \Theta(a_c\psi, b_c\psi) = \bigcap_{c \in C} \Theta(a_c, b_c)\bar{\psi} = (\bigcap_{c \in C} \Theta(a_c, b_c))\bar{\psi} = \Delta\bar{\psi} = \Delta.$$

3. Injectives and Weak Injectives

Here we collect together some useful results on injectives and weak injectives which we shall need anon.

An algebra A is a *subdirect retract* of a family $(A_j | j \in J)$ if there is a homomorphism $\phi: A \rightarrowtail \prod(A_j | j \in J)$ which embeds A as a subdirect product and as a retract simultaneously.

3.1. **Proposition.** (i) *A subdirect retract of [weak] injectives is itself [weak] injective.*

(ii) *A direct product of [weak] injectives is itself [weak] injective.*

(iii) *If A is a finite algebra, then for every complete Boolean algebra B, $A[B]^*$ is a subdirect retract of copies of A.*

(iv) *If A is a finite [weak] injective in a variety $\mathfrak{K}$, then for every complete Boolean algebra B, $A[B]^*$ is a [weak] injective in $\mathfrak{K}$.*

Proof. A proof of (i) may be found in Grätzer and Lakser [22; Lemma 5; p. 477]; of course, (ii) is an immediate corollary of (i). For a proof of (iii) we turn to Davey [11; Proposition 4.4; p. 136]; and (iv) follows from (i) and (iii).

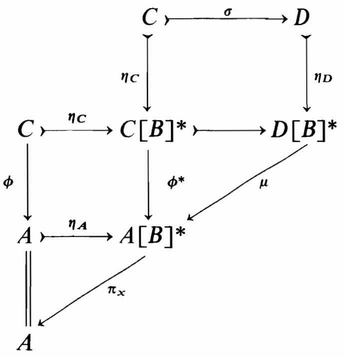
If all Hom-sets in a variety $\mathfrak{K}$ are nonempty, then the injectivity of $A \times A'$ implies the injectivity of each factor. But in general this is false; and no such result holds for weak injectives or absolute subretracts. The following result was announced in Davey and Gunn [14]; it generalizes Lemma 4.3 of [11], and its proof may be obtained by a careful disection of the proof given there.

3.2. **Proposition.** *Let $\mathfrak{K}$ be a variety and assume that $\mathfrak{K} = \mathbb{I}\mathbb{S}(\mathfrak{B})$, where $\mathfrak{B}$ is a class of algebras with factorizable congruences. If $A, A' \in \mathfrak{B}$ and $A \times A'$ is an absolute subretract [a weak injective, an injective, self-injective] in $\mathfrak{K}$, then so are both A and A'.*

If $A[B]^*$ displays some degree of injectivity then it is natural to ask what this implies about A. The following result is proved for Brouwerian algebras and Heyting algebras in [11; Lemmas 4.6, 4.9; pp. 136, 139].

3.3. **Proposition.** *If $A[B]^*$ is an absolute subretract [a weak injective, an injective, self-injective] in a variety $\mathfrak{K}$, then so is A.*

Proof. The easiest way to prove this is to observe that (a) the Hom-set functor $G: \mathfrak{K} \to \mathfrak{K}$ which maps A to $A[B]^*$ preserves injections and surjections, (b) the map $\eta_A: A \to A[B]^*$, which maps a to the constant map onto $\{a\}$, is a natural transformation from $\mathrm{id}_{\mathfrak{K}}$ to G, and (c) η_A embeds A as a retract of $A[B]^*$, indeed for each $x \in X_B$ the natural projection π_x: $A[B]^* \to A$ satisfies $\eta_A \pi_x = \mathrm{id}_A$. Hence if $\sigma: C \rightarrowtail D$ is an embedding and $\phi: C \to A$ is a homomorphism, then the diagram below results (we denote $G(\alpha)$ by α^*). The existence of μ is provided by the



appropriate injectivity condition on $A[B]^*$: for absolute subretract we let $\phi = \mathrm{id}_A$, for weak injective we assume ϕ is onto, and for self-injective we let $D = A$. Then $\psi = \eta_D \mu \pi_x$ satisfies $\sigma\psi = \phi$, as required.

In the proof of Theorem 1.1 we shall augment the type of the algebras by adding some nullary operations; of course we must know that this does not kill any [weak] injectives.

3.4. **Lemma.** *Assume that $\mathfrak{K}$ is the variety generated by $\mathfrak{A}$ and that $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \boldsymbol{\Pi}\mathbb{S}(\mathfrak{A})$. Let $\mathfrak{A}^+$ be obtained by adding some new nullary operations to the algebras in $\mathfrak{A}$, and let $\mathfrak{K}^+$ be the variety generated by $\mathfrak{A}^+$. If I is an absolute subretract [a weak injective, an injective] in $\mathfrak{K}$, then the new nullary operations can be interpreted on I in such a way that the augmented algebra, I^+, is an absolute subretract [a weak injective, an injective] in $\mathfrak{K}^+$.*

Proof. Let I be an absolute subretract [a weak injective, an injective] in $\mathfrak{K}$. Then, since $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \boldsymbol{\Pi}\mathbb{S}(\mathfrak{A})$, there is an embedding $\phi: I \to \prod(A_j | j \in J)$ with $A_j \in \mathfrak{A}$ for all $j \in J$. Since I is an absolute subretract in $\mathfrak{K}$, there is a homomorphism ψ of $\prod(A_j | j \in J)$ onto I. As $\prod(A_j | j \in J)$ can be viewed as a member of $\mathfrak{K}^+$, we can evaluate the new nullary operations on I via the map ψ. The augmented algebra, I^+, belongs to $\mathfrak{K}^+$ since it is a homomorphic image of an algebra in $\mathfrak{K}^+$. That I^+ is an absolute subretract [a weak injective, an injective] in $\mathfrak{K}^+$ follows easily from the fact that I is an absolute subretract [a weak injective, an injective] in $\mathfrak{K}$.

The last step of the proof of Theorem 1.1 requires the following observation.

3.5. **Lemma.** *Assume that* $\mathfrak{A}$ *has factorizable congruences. If* $A \in \mathbb{H}(\mathfrak{A})$ *is a finite directly indecomposable absolute subretract in the variety* $\mathfrak{K}$ *generated by* $\mathfrak{A}$, *then* A *is subdirectly irreducible.*

Proof. Assume that $\Theta, \Phi \in \mathrm{Con}(A)$ with $\Theta \wedge \Phi = \Delta$. Then A can be embedded into $A/\Theta \times A/\Phi$, and hence, since A is an absolute subretract in $\mathfrak{K}$, A is a homomorphic image of $A/\Theta \times A/\Phi$. Since $\mathfrak{A}$ has factorizable congruences, $\mathbb{H}(\mathfrak{A})$ has factorizable congruences, and so A is isomorphic to a direct product of a homomorphic image of A/Θ with a homomorphic image of A/Φ. Since A is directly irreducible we have $A \cong A/\Theta$ or $A \cong A/\Phi$, and since A is finite it follows that $\Theta = \Delta$ or $\Phi = \Delta$. Hence A is subdirectly irreducible.

4. The Proof of Theorem 1.1

Throughout this section we make the following assumptions:

(a) $\mathfrak{A}$ *is a finite set of finite algebras;*

(b) $\mathfrak{A}$ *has a simplicity formula* $\alpha(u, v)$;

(c) $\mathfrak{A}$ *has factorizable congruences;*

(d) $\mathfrak{K}$ *is the variety generated by* $\mathfrak{A}$ *and* $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$.

It follows from 3.1 that if $A_0, \ldots, A_n$ are finite [weak] injectives in a variety $\mathfrak{K}$ and $B_0, \ldots, B_n$ are complete Boolean algebras, then $\prod(A_j[B_j]^* \mid j \leqq n)$ is itself [weak] injective in $\mathfrak{K}$. Hence we must show that if $\mathfrak{K}$ satisfies the conditions above, then every [weak] injective in $\mathfrak{K}$ is of the form $\prod(A_j[B_j]^* \mid j \leqq n)$, where each $A_j \in \mathbb{H}(\mathfrak{A}) \cap \mathbb{S}\mathrm{i}(\mathfrak{K})$, each A_j is [weak] injective in $\mathfrak{K}$, and each Boolean algebra B_j is complete; that the algebras A_j can be chosen to be pairwise nonisomorphic follows trivially from the fact that $C(X, A) \times C(Y, A) \cong C(X \dot{\cup} Y, A)$. By assumption, $\mathfrak{A}$ has factorizable congruences and hence, by 2.6, $\mathbb{R}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$ has factorizable congruences, in particular $\mathbb{P}\mathbb{B}^*(\mathfrak{A})$ has factorizable congruences. Since $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{I}\mathbb{S}(\mathfrak{A})$, it follows that $\mathfrak{K} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{A})$ and so $\mathfrak{K} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{B}^*(\mathfrak{A})$. Thus by 3.2 with $\mathfrak{B} = \mathbb{P}\mathbb{B}^*(\mathfrak{A})$, if $\prod(A_j[B_j]^* \mid j \leqq n)$ is [weak] injective in $\mathfrak{K}$, then each $A_j[B_j]^*$ is also [weak] injective in $\mathfrak{K}$, and so, by 3.3, each A_j is [weak] injective in $\mathfrak{K}$. Consequently, a proof of the following result would be more than enough to complete Theorem 1.1.

4.1. **Proposition.** *Under the assumptions listed above, each absolute subretract in* $\mathfrak{K}$ *is of the form* $\prod(A_j[B_j]^* \mid j \leqq n)$, *where each* A_j *is a member of* $\mathbb{H}(\mathfrak{A}) \cap \mathbb{S}\mathrm{i}(\mathfrak{K})$ *and each* B_j *is complete.*

By 3.4 we may add nullary operations to the type of our algebras without killing any absolute subretracts. Let C be a set of nullary operation symbols, one for each element of $\prod \mathfrak{A} := \prod_{A \in \mathfrak{A}} A$, and interpret these on each algebra in $\mathfrak{A}$ via the natural projection. For the remainder of the proof we shall assume that the set of operation symbols contains C. The simplicity formula α was assumed to be an $\exists\forall$ conjunct of equations; but since we now have names for all the elements of algebras in $\mathfrak{A}$ we may assume that α is an existential conjunct of equations; that is, that α is primitive positive. By 2.2, $\mathfrak{A}$ has a principal-congruence formula say γ. Thus the theory developed in Sect. 2 is now fully applicable.

Assume that I is an absolute subretract in $\mathfrak{K}$; since $\mathbb{S}\mathrm{i}(\mathfrak{K}) \subseteq \mathbb{IS}(\mathfrak{A})$, it follows that I is a retract of a product $A' := \prod(A_j | j \in J)$ with each A_j from $\mathfrak{A}$. For each A in $\mathfrak{A}$ we have

$$A \models \alpha(a, b) \quad \text{implies} \quad \Theta(a, b) \in \{\Delta, \nabla\};$$

it follows at once that $\mathfrak{A}$ satisfies the sentences (P) and (D), and hence 2.7 and 2.8 apply. But,

$$\begin{aligned} \mathscr{L}_{A'} &= \{\Theta(a, b) | A' \models \alpha(a, b)\}, \\ &= \{\textstyle\prod \Theta(a(j), b(j)) | A' \models \alpha(a, b)\}, \quad \text{by 2.1}, \\ &= \{\textstyle\prod \Theta(a(j), b(j)) | A_j \models \alpha(a(j), b(j)) \text{ for all } j \in J\}, \end{aligned}$$

since α is primitive positive and so is preserved by both $\mathbb{H}$ and $\mathbb{P}$,

$$= \{\textstyle\prod \Theta_j | \Theta_j \in \{\Delta, \nabla\} \text{ for all } j \in J\},$$

since α is a simplicity formula for $\mathfrak{A}$. Hence it is clear that $\mathscr{L}_{A'}$ is a complete Boolean sublattice of $\mathrm{Con}(A')$; in fact, $\mathscr{L}_{A'}$ is isomorphic to $\mathbf{2}^J$. Thus, by 2.8, $\mathscr{L}_I$ is a complete Boolean sublattice of $\mathrm{Con}(I)$ whose elements permute.

We can avoid sheaf theory no longer. We assume that the reader is familiar with the rudiments of sheaf theory; the uninitiated are referred to Comer [8] or Davey [9]. We now apply Comer's representation theorem; see [8; Lemma 3.5; p. 35] or [9; Theorem 4.5; p. 288]. Let X be the Boolean space of prime ideals of the Boolean algebra $\mathscr{L}_I$; recall that $\{x \in X | \Theta \in x\}$ and $\{x \in X | \Theta \notin x\}$ are basic open (in fact clopen) sets for each $\Theta \in \mathscr{L}_I$. For each $x \in X$, define $\Theta_x = \bigcup(\Theta | \Theta \in x)$. Then there is a sheaf space $\langle S, \pi, X \rangle$ over X such that the stalk at x is I/Θ_x (i.e. $S = \bigcup(I/\Theta_x | x \in X)$) and I is isomorphic to the algebra $\Gamma(S)$ of global sections of this sheaf under the isomorphism which maps an element a of I to the global section $\hat{a}: X \to S$ given by $\hat{a}(x) = [a]\Theta_x$. Furthermore, for all $\Phi \in \mathscr{L}_I$ we have $\Phi = \bigcap(\Theta_x | \Phi \in x)$; see [8; Lemma 3.4; p. 34] or [9; Lemma (4.2); p. 286]. But, if $I \models \alpha(a, b)$, then

$$\{x \in X | \Theta(a, b) \in x\} = \{x \in X | a \equiv b(\Theta_x)\} = [\![\hat{a} = \hat{b}]\!];$$

hence we have

$$(*) \quad \text{if } I \models \alpha(a, b), \quad \text{then} \quad \Theta(a, b) = \bigcap(\Theta_x | x \in [\![\hat{a} = \hat{b}]\!]).$$

Since U is clopen in X if and only if there exists $\phi \in \mathscr{L}_I$ such that $U = \{x \in X | \Phi \in x\}$, it also follows that the clopen subsets of X are precisely the sets of the form $[\![\hat{a} = \hat{b}]\!]$ where $I \models \alpha(a, b)$.

Since $\mathfrak{K}$ is the variety generated by $\prod \mathfrak{A}$ and the elements of $\prod \mathfrak{A}$ are all nullary operations, it follows that $\prod \mathfrak{A}$ is the $\mathfrak{K}$-free algebra on 0 generators. Hence for each algebra in $\mathfrak{K}$, the nullary operations form a subalgebra which is a homomorphic image of $\prod\mathfrak{A}$. Thus, if we can show that the nullary operations exhaust the elements of each stalk of the sheaf space $\langle S, \pi, X\rangle$, then it will follow that each stalk is a member of $\mathbb{H}(\{\prod\mathfrak{A}\})$.

Since $I\in\mathbb{R}\mathbb{P}(\mathfrak{A})$, γ is a principal-congruence formula for I, by 2.5. Consequently, if

$$I\models\alpha(a, b)\,\&\,\gamma(d, c, a, b),$$

then there is a clopen subset of X, namely $[\![\hat{a}=\hat{b}]\!]$, on which $\hat{d}$ equals $\hat{c}$. And if

$$I\models(\underset{c\in C}{\&}\alpha(a_c, b_c))\,\&\,\forall uv[(\underset{c\in C}{\&}\gamma(u, v, a_c, b_c))\rightarrow u=v],$$

then for all $c\in C$, $[\![\hat{a}_c=\hat{b}_c]\!]$ is clopen, and $Y:=\bigcup_{c\in C}[\![\hat{a}_c=\hat{b}_c]\!]$ contains all those x for which the stalk I/Θ_x is nontrivial. Indeed, assume that $x\notin Y$ and that I/Θ_x is nontrivial; say $s, t\in I$ with $s\not\equiv t(\Theta_x)$. Define $\sigma\colon X\rightarrow S$ by $\sigma\restriction Y=\hat{a}$ and $\sigma\restriction(X-Y)=\hat{b}$. Then $\sigma\in\Gamma(S)$ since Y is clopen, and since $I\cong\Gamma(S)$ there exists $u\in I$ with $\hat{u}=\sigma$. Let $v=a$; then we have $\hat{u}\restriction Y=\hat{v}\restriction Y$ with $u\neq v$. Hence, for all $c\in C$, $\hat{u}$ equals $\hat{v}$ on $[\![\hat{a}_c=\hat{b}_c]\!]$ and so $u\equiv v(\Theta_x)$ for all $x\in[\![\hat{a}_c=\hat{b}_c]\!]$. It follows from (*) that $u\equiv v(\Theta(a_c, b_c))$ for all $c\in C$, and consequently $I\models\gamma(u, v, a_c, b_c)$ for all $c\in C$; which, since $u\neq v$, is a contradiction.

Consider again the sentence ε of 2.9:

$$\begin{aligned}\varepsilon:=\forall d(\exists a_c b_c)_{c\in C}[&\underset{c\in C}{\&}(\alpha(a_c, b_c)\,\&\,\gamma(d, c, a_c, b_c))\\ &\&\forall uv[(\underset{c\in C}{\&}\gamma(u, v, a_c, b_c))\rightarrow u=v]].\end{aligned}$$

By the remarks above, ε says that for every element σ of $\Gamma(S)$ there are clopen sets $U_c(c\in C)$ such that σ equals $\hat{c}$ on U_c and $\bigcup_{c\in C}U_c$ contains all x for which the stalk at x is nontrivial. It follows at once that if $I\models\varepsilon$, then the nullary operations exhaust the stalks of the sheaf space $\langle S, \pi, X\rangle$; but by 2.9, to prove that $I\models\varepsilon$ it is enough to show that $A'=\prod A_j\models\varepsilon$, which is easily seen to be true. Thus the nullary operations do exhaust the stalks, and consequently each stalk is a homomorphic image of $\prod\mathfrak{A}$; whence there are only finitely many pairwise nonisomorphic stalks.

We wish now to prove that the subset $X(A)$ of X on which a given stalk A occurs is clopen. Let A be a homomorphic image of $\prod\mathfrak{A}$ and consider the sentence

$$\delta_A:=\&(c=d\,|\,c, d\in C \text{ and } A\models c=d).$$

Then $D\models\delta_A$ if and only if D is a homomorphic image of A. Thus

$$\begin{aligned}Z_A:&=\{x\in X\,|\,I/\Theta_x\in\mathbb{H}(\{A\})\}=\{x\in X\,|\,I/\Theta_x\models\delta_A\}\\ &=\bigcap([\![\hat{c}=\hat{d}]\!]\,|\,A\models c=d).\end{aligned}$$

Hence Z_A will be clopen provided $[\![\hat{c}=\hat{d}]\!]$ is clopen for all $c, d\in C$. Since

$$X(A):=\{x\in X\,|\,I/\Theta_x\cong A\}=Z_A-\bigcup(Z_D|D\in\mathbb{H}(\{A\})-\mathbb{I}(\{A\})),$$

it will then follow that $X(A)$ is clopen for all $A\in\mathbb{H}(\{\prod\mathfrak{A}\})$.

Consider the sentence

$$\delta_{cd} := \exists ab[\alpha(a,b) \,\&\, \gamma(c,d,a,b) \\ \&\, \forall uv[(\alpha(u,v) \,\&\, \gamma(c,d,u,v)) \rightarrow \gamma(a,b,u,v)]].$$

If $I \models \delta_{cd}$, then $[\![\hat{c}=\hat{d}]\!] \supseteq [\![\hat{a}=\hat{b}]\!]$, and $[\![\hat{a}=\hat{b}]\!]$ is the largest clopen set with this property. Since $[\![\hat{c}=\hat{d}]\!]$ is the equalizer of two global sections it is open; and hence, since the clopen subsets of X form a basis, δ_{cd} says that $[\![\hat{c}=\hat{d}]\!] = [\![\hat{a}=\hat{b}]\!]$, whence δ_{cd} says that $[\![\hat{c}=\hat{d}]\!]$ is clopen. It remains to show that for all $c, d \in C$ we have $I \models \delta_{cd}$. By 2.4, δ_{cd} is preserved under retractions, and so it is sufficient to show that $A' = \prod A_j \models \delta_{cd}$; again, this is easily seen.

If $\langle S, \pi, X \rangle$ is a sheaf space over a Boolean space such that there are only finitely many pairwise nonisomorphic stalks, and each stalk occurs on a clopen set, then we have

$$\Gamma(S) \cong \prod (C(X_j, A_j) | j \leqq n) = \prod (A_j[B_j]^* | j \leqq n),$$

where $A_0, \ldots, A_n$ is a list of all the possible stalks, X_j is the clopen set on which A_j occurs, and B_j is the Boolean algebra of clopen subsets of X_j. A Boolean algebra is complete if and only if its Boolean space of prime ideals is extremally disconnected, and a Boolean space is extremally disconnected if and only if its Boolean algebra of clopen subsets is complete. Hence, since $\mathscr{L}_I$ is a complete Boolean algebra, and a clopen subset of an extremally disconnected space is extremally disconnected, it follows that we have I in the form required by Proposition 4.1 except that the A_j belong to $\mathbb{H}(\{\prod \mathfrak{A}\})$ rather than to $\mathbb{H}(\mathfrak{A}) \cap \mathbb{S}\mathrm{i}(\mathfrak{K})$; this is easily remidied. Since $\mathfrak{A}$ has factorizable congruences, it follows that each homomorphic image of $\prod \mathfrak{A}$ is a product of homomorphic images; say $A_j \cong \prod (A_j^i | i \leqq n_i)$ with $A_j^i \in \mathbb{H}(\mathfrak{A})$ for each i. But then $A_j[B_j]^* \cong \prod (A_j^i[B_j]^* | i \leqq n_j)$, and so I is isomorphic to a finite product, $\prod (A_k[B_k]^* | k \leqq m)$ with each A_k in $\mathbb{H}(\mathfrak{A})$ and each B_k complete. By precisely the same argument we may assume that each A_k is directly indecomposible. Since $\mathfrak{A}$ has factorizable congruences, $\mathbb{H}(\mathfrak{A})$ also has factorizable congruences, and so, by 3.2 and 3.3, each A_k is an absolute subretract in $\mathfrak{K}$, whence, by 3.5 each A_k is subdirectly irreducible, as required.

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Received September 5, 1977