



Functor category dualities for varieties of Heyting algebras

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Abstract

Let \mathcal{A} be a finitely generated variety of Heyting algebras and let $\text{SI}(\mathcal{A})$ be the class of subdirectly irreducible algebras in \mathcal{A} . We prove that \mathcal{A} is dually equivalent to a category of functors from $\text{SI}(\mathcal{A})$ into the category of Boolean spaces. The main tool is the theory of multisorted natural dualities.

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1. Introduction

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ of type $(2, 2, 2, 0, 0)$ is called a *Heyting algebra* if

- (a) $\langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice with smallest element 0 and largest element 1,
- (b) for all $a, b \in A$, we have

$$\{x \in A \mid x \wedge a \leq b\} = \downarrow(a \rightarrow b).$$

For a detailed account of Heyting algebras, we refer the reader to [1].

Let \mathfrak{A} be a finite set of finite Heyting algebras, define $\mathcal{A} := \text{Var}(\mathfrak{A})$ to be the variety generated by \mathfrak{A} and let $\text{SI}(\mathcal{A})$ be the class consisting of all subdirectly irreducible algebras in \mathcal{A} . Then $\mathcal{A} = \text{HSP}(\mathfrak{A})$ and $\text{SI}(\mathcal{A}) \subseteq \text{HS}(\mathfrak{A})$, by fundamental results due to Birkhoff, Tarski and Jónsson (see [2]). Since \mathfrak{A} is a finite set of finite algebras,

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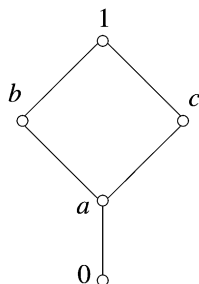
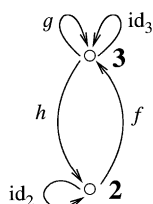


Fig. 1. The algebra \mathbf{A} .



\circ	f	g	h
f	-	-	g
h	id_2	h	-
g	f	g	-

Fig. 2. The category $\mathfrak{M} = \{2, 3\}$.

every algebra in $\mathbb{HS}(\mathfrak{A})$ is finite and consequently there is a finite subset \mathfrak{M} of $\text{SI}(\mathfrak{A})$ which contains exactly one isomorphic copy of each algebra in $\text{SI}(\mathfrak{A})$. By Birkhoff’s Subdirect Product Theorem, we have $\mathfrak{A} = \text{Var}(\mathfrak{M}) = \mathbb{ISP}(\mathfrak{M})$. We regard \mathfrak{M} as a full subcategory of \mathfrak{A} . We want to obtain a concrete dual category for the variety \mathfrak{A} . First, we give an example to illustrate how we set up the objects and morphisms of the dual category of \mathfrak{A} .

Let $\mathfrak{A} = \{\mathbf{A}\}$ where $\mathbf{A} = \mathbf{1} \oplus \mathbf{2}^2$; see Fig. 1. Then clearly, $\mathfrak{M} = \{2, 3\}$, where $\mathbf{2}$ is the two-element Heyting chain with $0 < 1$ and $\mathbf{3}$ is the three-element Heyting chain with $0 < a < 1$. The map $f : \mathbf{2} \rightarrow \mathbf{3}$ defined by $0 \mapsto 0, 1 \mapsto 1$, the map $g : \mathbf{3} \rightarrow \mathbf{3}$ defined by $0 \mapsto 0, a \mapsto 1, 1 \mapsto 1$ and the map $h : \mathbf{3} \rightarrow \mathbf{2}$ defined by $0 \mapsto 0, a \mapsto 1, 1 \mapsto 1$, along with id_3 and id_2 are only the Heyting homomorphisms between $\mathbf{2}$ and $\mathbf{3}$ and so form the morphisms of the category \mathfrak{M} ; see Fig. 2.

Let \mathfrak{B} be the category of Boolean spaces and let $\mathfrak{B}^{\mathfrak{M}}$ be the class of all functors from \mathfrak{M} into \mathfrak{B} . Then $\mathfrak{B}^{\mathfrak{M}}$ forms a category with natural transformations as morphisms. Surprisingly, there is a dual category equivalence between \mathfrak{A} and $\mathfrak{B}^{\mathfrak{M}}$.

A functor $\mathbf{X} : \mathfrak{M} \rightarrow \mathfrak{B}$ is really a pair of Boolean spaces $\mathbf{X}_2 := \mathbf{X}(\mathbf{2})$ and $\mathbf{X}_3 := \mathbf{X}(\mathbf{3})$ with continuous maps corresponding to f, g and h . For each $\mathbf{A} \in \mathfrak{A}$, define

$$D(\mathbf{A})_2 := \mathfrak{A}(\mathbf{A}, \mathbf{2}) \quad \text{and} \quad D(\mathbf{A})_3 := \mathfrak{A}(\mathbf{A}, \mathbf{3}),$$

with the topology inherited from $\mathbf{2}_\tau^A$ and $\mathbf{3}_\tau^A$, respectively, where $\mathbf{2}_\tau$ and $\mathbf{3}_\tau$ denote $\{0, 1\}$ and $\{0, a, 1\}$ endowed with the discrete topology. The map corresponding to the homomorphism h is the map $h^A : \mathfrak{A}(\mathbf{A}, \mathbf{3}) \rightarrow \mathfrak{A}(\mathbf{A}, \mathbf{2})$ given by

$$h^A(x) = h \circ x, \quad \text{for all } x \in \mathfrak{A}(\mathbf{A}, \mathbf{3}).$$

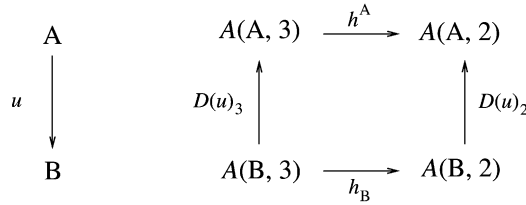


Fig. 3. The natural transformation $D(u)$.

It is a simple calculation to check that h^A is continuous. The maps f^A and g^A are defined similarly. Then $D(\mathbf{A}) : \mathfrak{M} \rightarrow \mathfrak{B}$ is a functor.

Each homomorphism $u : \mathbf{A} \rightarrow \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, induces a natural transformation $D(u) : D(\mathbf{B}) \rightarrow D(\mathbf{A})$. The function $D(u)_3 : \mathcal{A}(\mathbf{B}, 3) \rightarrow \mathcal{A}(\mathbf{A}, 3)$ is given by

$$D(u)_3(x) := x \circ u, \quad \text{for all } x \in \mathcal{A}(\mathbf{B}, 3),$$

and $D(u)_2$ is defined similarly. The commutativity of diagrams like the one in Fig. 3, which guarantee that $D(u)$ is a natural transformation, simply express the associativity of composition. It is straightforward to check that $D : \mathcal{A} \rightarrow \mathfrak{B}^{\mathfrak{M}}$ is a contravariant functor.

Let $I_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{B}$ be the natural ‘inclusion’ functor which replaces the Heyting algebras $\mathbf{3}$ and $\mathbf{2}$ with the corresponding discrete topological spaces, $\mathbf{3}_\tau$ and $\mathbf{2}_\tau$, and regards each morphism in \mathfrak{M} as a continuous map rather than as a Heyting algebra homomorphism. Thus, $I_{\mathfrak{M}}$ is an object of the category $\mathfrak{B}^{\mathfrak{M}}$. Given an object \mathbf{X} of $\mathfrak{B}^{\mathfrak{M}}$, a natural transformation $\alpha : \mathbf{X} \rightarrow I_{\mathfrak{M}}$ has two components, namely, $\alpha_3 : \mathbf{X}_3 \rightarrow \mathbf{3}_\tau$ and $\alpha_2 : \mathbf{X}_2 \rightarrow \mathbf{2}_\tau$. Define

$$E(\mathbf{X}) := \{(\alpha_3, \alpha_2) \mid \alpha : \mathbf{X} \rightarrow I_{\mathfrak{M}} \text{ is a natural transformation}\}.$$

Thus $E(\mathbf{X})$ is a subset of $\mathfrak{B}(\mathbf{X}_3, \mathbf{3}_\tau) \times \mathfrak{B}(\mathbf{X}_2, \mathbf{2}_\tau)$. It is easy to verify that $E(\mathbf{X})$ is a subalgebra of $\mathbf{3}^{\mathbf{X}_3} \times \mathbf{2}^{\mathbf{X}_2}$, whence $E(\mathbf{X}) \in \mathcal{A}$. A diagram chase shows that E has a natural extension to morphisms in $\mathfrak{B}^{\mathfrak{M}}$ and defines a contravariant functor $E : \mathfrak{B}^{\mathfrak{M}} \rightarrow \mathcal{A}$. We claim that D and E give a dual category equivalence between \mathcal{A} and $\mathfrak{B}^{\mathfrak{M}}$ and, in particular, that $\mathbf{A} \cong ED(\mathbf{A})$, for all $\mathbf{A} \in \mathcal{A}$, and $\mathbf{X} \cong DE(\mathbf{X})$, for all $\mathbf{X} \in \mathfrak{B}^{\mathfrak{M}}$. We prove this, as an application of our general results, in Theorem 6.4 in Section 6.

In fact, in this case the dual category can be greatly simplified. We can drop all the structure from each dual object, except the Boolean space \mathbf{X}_3 and the retraction $g^{\mathbf{X}}$. Indeed, \mathcal{A} is dually equivalent to the category \mathcal{X} consisting of objects $\langle X; g, \mathcal{T} \rangle$ where

- $\langle X; \mathcal{T} \rangle$ is a Boolean space, and
- $g : X \rightarrow X$ is a continuous retraction.

We prove this in Theorem 6.3 in Section 6.

We now return to the general setting with which we began: $\mathcal{A} = \text{Var}(\mathfrak{A}) = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$ is a finitely generated variety of Heyting algebras and \mathfrak{M} is a finite set of finite Heyting algebras which is a transversal of the isomorphism classes of $\text{SI}(\mathcal{A})$. It should be clear how to define the contravariant functors D and E in this general setting.

$$\begin{array}{ccc}
 \mathbf{X}(\mathbf{M}_i) & \xrightarrow{\alpha_{\mathbf{M}_i}} & \mathbf{M}_i \\
 \mathbf{X}(g) \downarrow & & \downarrow g \\
 \mathbf{X}(\mathbf{M}_j) & \xrightarrow{\alpha_{\mathbf{M}_j}} & \mathbf{M}_j
 \end{array}$$

Fig. 4. $\alpha: \mathbf{X} \rightarrow I_{\mathfrak{M}}$ is a natural transformation.

For $\mathbf{A} \in \mathcal{A}$, define $\mathbf{X}_{\mathbf{A}} \in \mathcal{B}^{\mathfrak{M}}$ as follows:

- For all $\mathbf{M} \in \mathfrak{M}$, define $\mathbf{X}_{\mathbf{A}}(\mathbf{M}) = \mathcal{A}(\mathbf{A}, \mathbf{M})$ endowed with the relative topology from the power \mathbf{M}_{τ}^A of $\mathbf{M}_{\tau} := \langle M; \mathcal{T} \rangle$, where \mathcal{T} is the discrete topology. Thus, $\mathbf{X}_{\mathbf{A}}(\mathbf{M}) \in \mathcal{B}$.
- If $g \in \mathfrak{M}(\mathbf{M}_1, \mathbf{M}_2)$, then $\mathbf{X}_{\mathbf{A}}(g): \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \rightarrow \mathcal{A}(\mathbf{A}, \mathbf{M}_2)$ is given by $\mathbf{X}_{\mathbf{A}}(g)(x) = g \circ x$, for all $x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_1)$.

Then $D: \mathcal{A} \rightarrow \mathcal{B}^{\mathfrak{M}}$ is defined (on objects) by $D(\mathbf{A}) = \mathbf{X}_{\mathbf{A}}$. As in our example above, each homomorphism $u: \mathbf{A} \rightarrow \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, yields a natural transformation $D(u): D(\mathbf{B}) \rightarrow D(\mathbf{A})$: for $\mathbf{M} \in \mathfrak{M}$, the map $D(u)_{\mathbf{M}}: \mathcal{A}(\mathbf{B}, \mathbf{M}) \rightarrow \mathcal{A}(\mathbf{A}, \mathbf{M})$ is defined by

$$D(u)_{\mathbf{M}}(x) := x \circ u, \quad \text{for all } x \in \mathcal{A}(\mathbf{B}, \mathbf{M}).$$

This defines the contravariant functor $D: \mathcal{A} \rightarrow \mathcal{B}^{\mathfrak{M}}$.

Given $\mathbf{X} \in \mathcal{B}^{\mathfrak{M}}$, we denote the underlying set of the Boolean space $\mathbf{X}(\mathbf{M})$ by $X(\mathbf{M})$. In order to define the contravariant functor $E: \mathcal{B}^{\mathfrak{M}} \rightarrow \mathcal{A}$, it is convenient to write $\mathfrak{M} = \{\mathbf{M}_1, \dots, \mathbf{M}_n\}$. As in the example, let $I_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathcal{B}$ be the natural inclusion functor and for each $\mathbf{X} \in \mathcal{B}^{\mathfrak{M}}$, define $E(\mathbf{X})$ to be the subalgebra of $\mathbf{M}_1^{X(\mathbf{M}_1)} \times \dots \times \mathbf{M}_n^{X(\mathbf{M}_n)}$ whose underlying set is

$$\{(\alpha_{\mathbf{M}_1}, \dots, \alpha_{\mathbf{M}_n}) \mid \alpha: \mathbf{X} \rightarrow I_{\mathfrak{M}} \text{ is a natural transformation}\}.$$

Thus, $E(\mathbf{X}) \in \mathcal{A}$. The fact that $\alpha: \mathbf{X} \rightarrow I_{\mathfrak{M}}$ is a natural transformation says simply that, for all $g \in \mathfrak{M}(\mathbf{M}_i, \mathbf{M}_j)$ and all $i, j \in \{1, \dots, n\}$, the diagram in Fig. 4 is commutative. It is easy to check that a natural transformation $\varphi: \mathbf{X} \rightarrow \mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathcal{B}^{\mathfrak{M}}$, induces a homomorphism $E(\varphi): E(\mathbf{Y}) \rightarrow E(\mathbf{X})$ and that $E: \mathcal{B}^{\mathfrak{M}} \rightarrow \mathcal{A}$ is a contravariant functor.

For all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{B}^{\mathfrak{M}}$, there are maps $e_{\mathbf{A}}: \mathbf{A} \rightarrow ED(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$ given in a natural way via evaluation. Let $\mathbf{A} \in \mathcal{A}$ and $a \in A$. Then $e_{\mathbf{A}}(a) \in ED(\mathbf{A})$ is defined by

$$e_{\mathbf{A}}(a) := (e_{\mathbf{A}}^{\mathbf{M}_1}(a), \dots, e_{\mathbf{A}}^{\mathbf{M}_n}(a)),$$

where $e_{\mathbf{A}}^{\mathbf{M}_i}(a): \mathcal{A}(\mathbf{A}, \mathbf{M}_i) \rightarrow M_i$ is given by $e_{\mathbf{A}}^{\mathbf{M}_i}(a)(x) := x(a)$, for all $x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_i)$ and all $i \in \{1, \dots, n\}$. For $\mathbf{X} \in \mathcal{B}^{\mathfrak{M}}$ we define a natural transformation $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$ as follows. For each $i \in \{1, \dots, n\}$, the \mathbf{M}_i -component of $\varepsilon_{\mathbf{X}}$ is the continuous map

$$(\varepsilon_{\mathbf{X}})_{\mathbf{M}_i}: \mathbf{X}(\mathbf{M}_i) \rightarrow (DE(\mathbf{X}))(\mathbf{M}_i) = \mathcal{A}(E(\mathbf{X}), \mathbf{M}_i)$$

defined by

$$((\varepsilon_{\mathbf{X}})_{\mathbf{M}_i}(x))((\alpha_{\mathbf{M}_1}, \dots, \alpha_{\mathbf{M}_n})) := \alpha_{\mathbf{M}_i}(x),$$

for all natural transformations $\alpha: \mathbf{X} \rightarrow I_{\mathfrak{M}}$ and all $x \in X(\mathbf{M}_i)$. It is routine to check that $\langle D, E, e, \varepsilon \rangle$ is a dual adjunction between \mathcal{A} and $\mathcal{B}^{\mathfrak{M}}$.

We can now state the functor-category duality which is the focus of this paper.

Theorem 1.1. *Let \mathcal{A} be a finitely generated variety of Heyting algebras and let \mathfrak{M} be a transversal of the isomorphism classes of the class $\text{SI}(\mathcal{A})$ of subdirectly irreducible algebras in \mathcal{A} . The functor $D: \mathcal{A} \rightarrow \mathcal{B}^{\mathfrak{M}}$ and $E: \mathcal{B}^{\mathfrak{M}} \rightarrow \mathcal{A}$ give a dual category equivalence between \mathcal{A} and a full subcategory of the functor category $\mathcal{B}^{\mathfrak{M}}$. In particular, $\mathbf{A} \cong ED(\mathbf{A})$, via $e_{\mathbf{A}}$, for all $\mathbf{A} \in \mathcal{A}$.*

We shall prove Theorem 1.1 as an application of the theory of multisorted natural dualities. In the following two sections, we give a brief introduction to multisorted natural dualities with a particular emphasis on applications to varieties of Heyting algebras. Theorem 1.1 follows from the results in Section 3: see Remark 3.7. In Section 4 we develop the theory of multisorted strong dualities which were treated very briefly in Clark and Davey [3] due to a lack of tractable examples. We prove in Section 5 that each finitely generated variety of Heyting algebras possesses a multisorted strong duality which in general involves the use of partial operations. It is natural to ask for which finitely generated varieties \mathcal{A} of Heyting algebras the result of Theorem 1.1 can be sharpened to a dual category equivalence between \mathcal{A} and the functor category $\mathcal{B}^{\mathfrak{M}}$ (rather than to a subcategory of $\mathcal{B}^{\mathfrak{M}}$). This true if \mathcal{A} is either the variety of Boolean algebras or $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{3})$ is the example considered earlier in this section. Both of these cases have the additional property that $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$. In the final section of the paper, we prove that if \mathcal{A} is dually equivalent to $\mathcal{B}^{\mathfrak{M}}$ and $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$, then \mathcal{A} must be the variety of Boolean algebras or $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{3})$. We believe that this is true even without the assumption that $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$ but we have not been able to prove it.

2. Multisorted piggyback dualities

We refer to Clark and Davey [3] for a detailed introduction to the general theory of multisorted natural dualities and piggyback dualities (Chapter 7) and for the theory of single-sorted natural dualities which the multisorted case generalises. We shall sketch the details.

Let \mathfrak{M} be a finite set of finite algebras $\mathbf{M} = \langle M; F \rangle$ (of the same type F) and let $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$ be the quasi-variety it generates. A candidate for a category \mathcal{X} of multisorted topological structures which is dual to \mathcal{A} is obtained as follows.

The multisorted generating structure for the topological quasi-variety $\mathcal{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ has the following form:

$$\mathfrak{M} := \langle \mathfrak{M}_0; G, H, R, \mathcal{T} \rangle,$$

where

$$(i) \mathfrak{M}_0 = \bigcup \{M \mid \mathbf{M} \in \mathfrak{M}\},$$

- (ii) G consists of homomorphisms $g: \mathbf{M}_1 \times \cdots \times \mathbf{M}_n \rightarrow \mathbf{M}_{n+1}$ for some $\mathbf{M}_1, \dots, \mathbf{M}_{n+1} \in \mathfrak{M}$ and some $n \geq 0$,
- (iii) H consists of homomorphisms $h: \mathbf{D} \rightarrow \mathbf{M}_{n+1}$ with \mathbf{D} a proper subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$ for some $\mathbf{M}_1, \dots, \mathbf{M}_{n+1} \in \mathfrak{M}$ and some $n \geq 1$,
- (iv) R consists of relations r such that \mathbf{r} is a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$ for some $\mathbf{M}_1, \dots, \mathbf{M}_n \in \mathfrak{M}$ and some $n \geq 1$,
- (v) \mathcal{T} is the discrete topology on \mathfrak{M}_0 .

By analogy, with the case in which \mathfrak{M} consists of a single algebra, we refer to the maps in G as (*multisorted*) *operations*, to the maps in H as (*multisorted*) *partial operations* and to the relations in R as (*multisorted*) *relations* on \mathfrak{M} and we summarize (ii)–(iv) by saying that the structure on \mathfrak{M} is *algebraic over* \mathfrak{M} . We now consider multisorted structures

$$\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}} \rangle$$

such that

- (i) $X = \bigcup \{X_{\mathbf{M}} \mid \mathbf{M} \in \mathfrak{M}\}$,
- (ii) for each operation $g: \mathbf{M}_1 \times \cdots \times \mathbf{M}_n \rightarrow \mathbf{M}_{n+1}$ in G there is a corresponding map $g^{\mathbf{X}}: X_{\mathbf{M}_1} \times \cdots \times X_{\mathbf{M}_n} \rightarrow X_{\mathbf{M}_{n+1}}$ in $G^{\mathbf{X}}$, and similarly for the partial operations in H and the relations in R , and
- (iii) \mathcal{T} is the union topology on X .

Then we refer to \mathbf{X} as an \mathfrak{M} -sorted structure of the same type as \mathfrak{M} and $X_{\mathbf{M}}$ is called the \mathbf{M} -sort of \mathbf{X} . In this terminology, the \mathbf{M} -sort of the structure \mathfrak{M} is M (the underlying set of the algebra \mathbf{M}); in symbols, $\mathfrak{M}_{\mathbf{M}} = M$. For any set S , let

$$\mathfrak{M}^S := \left\langle \bigcup \{M^S \mid \mathbf{M} \in \mathfrak{M}\}; G^{\mathfrak{M}^S}, H^{\mathfrak{M}^S}, R^{\mathfrak{M}^S}, \mathcal{T}^{\mathfrak{M}^S} \right\rangle,$$

where the total operations, partial operations and relations are obtained by pointwise extension of those in G , H and R , respectively, and the topology is the disjoint union of the respective product topologies on \mathbf{M}_{τ}^S , for $\mathbf{M} \in \mathfrak{M}$. (Here \mathbf{M}_{τ} denotes the set M with the discrete topology.)

Let \mathcal{X} be the category of all \mathfrak{M} -sorted structures of the same type as \mathfrak{M} which are isomorphic to a closed substructure of some power \mathfrak{M}^S of \mathfrak{M} with $S \neq \emptyset$; in symbols, $\mathcal{X} := \mathbb{I}_{\mathcal{S}_c} \mathbb{P}^+(\mathfrak{M})$. Let \mathbf{X} and \mathbf{Y} be \mathfrak{M} -sorted structures of the same type as \mathfrak{M} , then a map $\varphi: X \rightarrow Y$ is a *morphism* provided it is continuous and preserves the sorts (that is, φ maps $X_{\mathbf{M}}$ into $Y_{\mathbf{M}}$, for each $\mathbf{M} \in \mathfrak{M}$) and preserves the operations, partial operations and relations in the obvious sense. Since a morphism $\varphi: \mathbf{X} \rightarrow \mathbf{Y}$ preserves sorts, the map $\varphi_{\mathbf{M}} := \varphi \upharpoonright X_{\mathbf{M}}: X_{\mathbf{M}} \rightarrow Y_{\mathbf{M}}$ is well defined. For each $\mathbf{A} \in \mathcal{A}$, let $D(\mathbf{A})$ be the closed substructure whose underlying (\mathfrak{M} -sorted) set is

$$D(\mathbf{A})_0 := \bigcup \{\mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathfrak{M}\}.$$

For every $\mathbf{X} \in \mathcal{X}$, the homset $\mathcal{X}(\mathbf{X}, \mathfrak{M})$ forms a subalgebra of the product $\prod \{\mathbf{M}^{X_{\mathbf{M}}} \mid \mathbf{M} \in \mathfrak{M}\}$ and we denote this algebra by $E(\mathbf{X})$. This defines, at the object level, a pair of contravariant functors $D: \mathcal{A} \rightarrow \mathcal{X}$ and $E: \mathcal{X} \rightarrow \mathcal{A}$ between \mathcal{A} and \mathcal{X} for which

all the maps

$$e_A : \mathbf{A} \rightarrow ED(\mathbf{A}),$$

defined by $e_A(a)(x) := x(a)$ for all $a \in A$ and $x \in D(\mathbf{A})_0$, and

$$\varepsilon_X : \mathbf{X} \rightarrow DE(\mathbf{X}),$$

defined by $\varepsilon_X(x)(\alpha) := \alpha(x)$, for all $x \in X$ and all $\alpha \in E(\mathbf{X})$, are embeddings.

If $\mathbf{A} \cong ED(\mathbf{A})$ (via e_A) for all $\mathbf{A} \in \mathcal{A}$, then we say that \mathfrak{M} (or sometimes that $G \cup H \cup R$) yields a (natural) duality on \mathcal{A} . If, in addition, $\mathbf{X} \cong DE(\mathbf{X})$ (via ε_X) for all $\mathbf{X} \in \mathcal{X}$, then we say that \mathfrak{M} yields a full duality on \mathcal{A} .

Is it always possible to choose G , H and R so that \mathfrak{M} yields a duality on \mathcal{A} ? Unfortunately, the answer is ‘no’. It was shown in [10] that the variety generated by the 2-element implication algebra has no natural duality. Nevertheless, Davey and Priestley [8] proved the multisorted NU Duality Theorem which gives widely satisfied conditions under which the answer is ‘yes’. Unfortunately, the set $G \cup H \cup R$ produced by the NU Duality Theorem is in general extremely large and highly redundant. For example, if \mathbf{M} is the pentagonal lattice \mathbf{N}_5 , then the NU Duality Theorem gives us a dualising structure with 5896 binary relations [18]. A useful tool for producing more efficient dualising sets $G \cup H \cup R$ is provided by the Piggyback Duality Theorem (see [11] or [12]) and its multisorted variant (see [8] or [17]) which we now state.

Let $\mathbf{D} := \langle \{0, 1\}; \wedge, \vee, 0, 1 \rangle$ be the two-element distributive lattice. Then $\mathcal{D} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{D})$ is the variety of bounded distributive lattices. We shall say that the class \mathfrak{M} has a term-reduct in \mathcal{D} if there exist binary terms \wedge and \vee and constant unary terms u and z such that, for all $\mathbf{M} \in \mathfrak{M}$, the algebra $\mathbf{M}^p := \langle M; \wedge^{\mathbf{M}}, \vee^{\mathbf{M}}, 0^{\mathbf{M}}, 1^{\mathbf{M}} \rangle$ is a bounded distributive lattice, where $0^{\mathbf{M}}$ and $1^{\mathbf{M}}$ are the values in M of the constant unary term functions $z^{\mathbf{M}}$ and $u^{\mathbf{M}}$, respectively.

Theorem 2.1 (Multisorted Piggyback Duality Theorem [8] or Priestley [17]). *Assume that \mathfrak{M} is a finite set of finite algebras which has a term-reduct in \mathcal{D} and let $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. For each $\mathbf{M} \in \mathfrak{M}$, let $\Omega_{\mathbf{M}}$ be a subset of $\mathcal{D}(\mathbf{M}^p, \mathbf{2})$. Let $\mathfrak{M} := \langle \mathfrak{M}_0; G, R, \mathcal{T} \rangle$, where*

(i) R is the set of \mathcal{A} -subalgebras of $\mathbf{M}_1 \times \mathbf{M}_2$ which are maximal in

$$(\omega_1, \omega_2)^{-1}(\leq) := \{(a, b) \in M_1 \times M_2 \mid \omega_1(a) \leq \omega_2(b)\},$$

for some $\omega_1 \in \Omega_{\mathbf{M}_1}$, $\omega_2 \in \Omega_{\mathbf{M}_2}$ and $\mathbf{M}_1, \mathbf{M}_2 \in \mathfrak{M}$,

(ii) $G \subseteq \bigcup \{ \mathcal{A}(\mathbf{M}_1, \mathbf{M}_2) \mid \mathbf{M}_1, \mathbf{M}_2 \in \mathfrak{M} \}$ satisfies the separation condition:

(S) for all $\mathbf{M}_1 \in \mathfrak{M}$ and all $a \neq b$ in M_1 , we have $\omega(a) \neq \omega(b)$, for some $\omega \in \Omega_{\mathbf{M}_1}$, or $\omega(g(a)) \neq \omega(g(b))$, where $\omega \in \Omega_{\mathbf{M}_2}$ for some $\mathbf{M}_2 \in \mathfrak{M}$ and $g \in \mathcal{A}(\mathbf{M}_1, \mathbf{M}_2)$ is a composite of a finite number of maps from G ,

(iii) \mathcal{T} is the discrete topology on \mathfrak{M}_0 .

Then \mathfrak{M} yields a duality on \mathcal{A} .

The maps in $\Omega_{\mathbf{M}}$, for each $\mathbf{M} \in \mathfrak{M}$, are called carriers and the relations in the set R are referred to as the piggyback relations.

3. Multisorted piggyback dualities for Heyting algebras

Let \mathcal{A} be a finitely generated variety of Heyting algebras, that is, assume \mathcal{A} is generated by a finite set of finite algebras. By Jónsson's Lemma, there is a finite set \mathfrak{M} of finite subdirectly irreducible algebras from \mathcal{A} such that $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$. (Indeed, a set \mathfrak{N} of finite subdirectly irreducible algebras from \mathcal{A} satisfies $\mathcal{A} = \mathbb{ISP}(\mathfrak{N})$ if and only if \mathfrak{N} includes (an isomorphic copy of) each maximal subdirectly irreducible algebras in \mathcal{A} .) We claim that the Multisorted Piggyback Duality Theorem may be applied to produce a duality for \mathcal{A} based on the set \mathfrak{M} . We shall define the carriers, describe the piggyback relations and show that condition (S) of Theorem 2.1(ii) holds.

Let \mathbf{M} be a finite, subdirectly irreducible Heyting algebra. Since $\text{Con}(\mathbf{M})$ is isomorphic to the lattice of filters of \mathbf{M} , it follows that $\mathbf{M}^p = \mathbf{L} \oplus 1$ can be obtained by adding a new top, 1, to a finite distributive lattice \mathbf{L} . Hence $\omega_{\mathbf{M}} := \chi_{\{1\}} : \mathbf{M}^p \rightarrow \mathbf{D}$, the characteristic function of $\{1\}$, is a \mathcal{D} homomorphism. We require only one carrier for each algebra $\mathbf{M} \in \mathfrak{M}$, namely the map $\omega_{\mathbf{M}}$. To simplify the notation, whenever we have a pair $\mathbf{M}_1, \mathbf{M}_2$ of finite subdirectly irreducible Heyting algebras, we shall denote the maps $\omega_{\mathbf{M}_1}$ and $\omega_{\mathbf{M}_2}$ simply by ω_1 and ω_2 , respectively.

The Heyting subalgebras of $\mathbf{M}_1 \times \mathbf{M}_2$ which are contained in $(\omega_1, \omega_2)^{-1}(\leq)$ have a very simple structure.

Lemma 3.1. *Let \mathbf{M}_1 and \mathbf{M}_2 be finite subdirectly irreducible Heyting algebras. A subalgebra \mathbf{r} of $\mathbf{M}_1 \times \mathbf{M}_2$ is contained in $(\omega_1, \omega_2)^{-1}(\leq)$ if and only if \mathbf{r} is the graph of a (partial) homomorphism from \mathbf{M}_1 into \mathbf{M}_2 .*

Proof. Note that

$$\begin{aligned} (\omega_1, \omega_2)^{-1}(\leq) &= ((M_1 \setminus \{1\}) \times (M_2 \setminus \{1\})) \cup (M_1 \times \{1\}) \\ &= (M_1 \times M_2) \setminus (\{1\} \times (M_2 \setminus \{1\})). \end{aligned}$$

Since a (partial) homomorphism h from \mathbf{M}_1 into \mathbf{M}_2 maps 1 to 1, we certainly have $r := \text{graph}(h) \subseteq (\omega_1, \omega_2)^{-1}(\leq)$.

Conversely, let \mathbf{r} be a subalgebra of $\mathbf{M}_1 \times \mathbf{M}_2$ with $r \subseteq (\omega_1, \omega_2)^{-1}(\leq)$. Then for all $x \in M_1$ and $y, z \in M_2$,

$$\begin{aligned} (x, y), (x, z) \in r &\Rightarrow (x, y) \rightarrow (x, z) \in r \\ &\Rightarrow (x \rightarrow x, y \rightarrow z) \in r \\ &\Rightarrow (1, y \rightarrow z) \in r \\ &\Rightarrow (1, y \rightarrow z) \in (\omega_1, \omega_2)^{-1}(\leq) \text{ as } r \subseteq (\omega_1, \omega_2)^{-1}(\leq) \\ &\Rightarrow \omega_1(1) \leq \omega_2(y \rightarrow z) \\ &\Rightarrow \omega_2(y \rightarrow z) = 1 \\ &\Rightarrow y \rightarrow z = 1 \\ &\Rightarrow y \leq z. \end{aligned}$$

By symmetry, we have $z \leq y$. Hence $y = z$. Therefore, r is the graph of a (partial) homomorphism from \mathbf{M}_1 into \mathbf{M}_2 . \square

Hence, a subalgebra of $\mathbf{M}_1 \times \mathbf{M}_2$ which is maximal in $(\omega_1, \omega_2)^{-1}(\leq)$ is either the graph of a homomorphism from \mathbf{M}_1 into \mathbf{M}_2 or the graph of a non-extendable, proper partial homomorphism from \mathbf{M}_1 into \mathbf{M}_2 . Now we want to show that, for any $\mathbf{M}_1 \in \mathfrak{M}$ and $a \neq b$ in M_1 , there is an $\mathbf{M}_2 \in \mathfrak{M}$ and a homomorphism g from \mathbf{M}_1 into \mathbf{M}_2 such that g composed with ω_2 separates a and b . This will establish condition (S) of Theorem 2.1. We prove slightly more.

Lemma 3.2. *Let \mathcal{A} be a finitely generated variety of Heyting algebras and let \mathfrak{M} be a set of (necessarily finite) subdirectly irreducible Heyting algebras such that $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. For each $\mathbf{A} \in \mathcal{A}$ and all $a, b \in A$ with $a \neq b$ there exists an algebra $\mathbf{M} \in \mathfrak{M}$ and a homomorphism $g : \mathbf{A} \rightarrow \mathbf{M}$ such that $\omega_{\mathbf{M}}(g(a)) \neq \omega_{\mathbf{M}}(g(b))$.*

Proof. Let $\mathbf{A} \in \mathcal{A}$ and let $a, b \in A$ with $a \neq b$. Without loss of generality, we can assume that $a \not\leq b$. Define the filter $F := \uparrow a$ of \mathbf{A} and let $\varphi : \mathbf{A} \rightarrow \mathbf{A}/F$ be the natural homomorphism. Then $\varphi(a) = 1$ and $\varphi(b) < 1$. Now, by Birkhoff's Theorem, there exists a subdirectly irreducible algebra \mathbf{N} and a surjective homomorphism $\psi : \mathbf{A}/F \rightarrow \mathbf{N}$ such that $1 = \psi(\varphi(a)) \neq \psi(\varphi(b))$. Since $\mathbf{N} \in \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$ and \mathbf{N} is subdirectly irreducible, there exists $\mathbf{M} \in \mathfrak{M}$ and an embedding $\eta : \mathbf{N} \rightarrow \mathbf{M}$. Let $g = \eta \circ \psi \circ \varphi$. Then $\omega_{\mathbf{M}}(g(a)) = 1$ and $\omega_{\mathbf{M}}(g(b)) = 0$. \square

Since every maximal subalgebra of $\mathbf{M}_1 \times \mathbf{M}_2$ in $(\omega_1, \omega_2)^{-1}(\leq)$ is the graph of a (partial) homomorphism from \mathbf{M}_1 into \mathbf{M}_2 , we can replace every piggyback relation r by the (partial) homomorphism h satisfying $\text{graph}(h) = r$. We use the notation $h : \mathbf{M}_1 \dashrightarrow \mathbf{M}_2$ to denote a partial homomorphism from \mathbf{M}_1 into \mathbf{M}_2 . Define $\text{hom}(\mathfrak{M})$ to be the set of all homomorphisms between algebras in \mathfrak{M} , that is, $\text{hom}(\mathfrak{M})$ is the set of all morphisms of the category \mathfrak{M} . Define $\text{hom}_p(\mathfrak{M})$ to consist of all non-extendable, proper partial homomorphisms between algebras in \mathfrak{M} . Lemma 3.2 ensures that we can apply Theorem 2.1 to obtain the following result.

Theorem 3.3 (Heyting Multisorted Duality Theorem). *Assume that \mathcal{A} is a finitely generated variety of Heyting algebras and let \mathfrak{M} be a finite set of finite subdirectly irreducible algebras in \mathcal{A} such that $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Let \mathcal{T} be the discrete topology on \mathfrak{M}_0 and define*

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \text{hom}_p(\mathfrak{M}), \mathcal{T} \rangle.$$

Then \mathfrak{M} yields a duality on \mathcal{A} .

Given \mathcal{A} , there are two natural extreme choices of \mathfrak{M} to which this theorem can be applied. The most efficient would be to take \mathfrak{M} to consist of one algebra from each isomorphism class of the maximal subdirectly irreducible algebras in \mathcal{A} . This has the disadvantage that, in general, it requires us to use partial operations in the type of \mathfrak{M} . At the other end of the spectrum, we may choose \mathfrak{M} to consist of a transversal of the

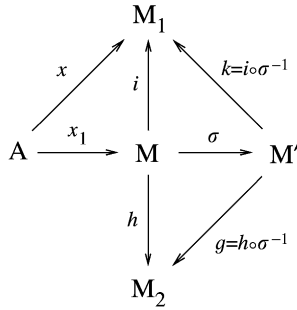


Fig. 5.

isomorphism classes of the subdirectly irreducible algebras in \mathcal{A} . As we now prove, this has the advantage of allowing us to dispense with all partial operations.

A class \mathfrak{A} of finite algebras (of the same type) is *rich* (in total unary maps) if for all $\mathbf{M}_1, \mathbf{M}_2 \in \mathfrak{A}$ and each maximal member h of the set of partial homomorphisms from \mathbf{M}_1 to \mathbf{M}_2 we have, up to isomorphism, $\text{dom}(h) \in \mathfrak{A}$. Thus, \mathfrak{A} is rich if every partial homomorphism between algebras in \mathfrak{A} has (up to isomorphism) an extension in \mathfrak{A} . We need the following general result.

Lemma 3.4. *Let $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$ for some finite set \mathfrak{M} of finite algebras and assume that \mathfrak{M} is rich. Let $\mathbf{A} \in \mathcal{A}$. If $\alpha : D(\mathbf{A})_0 \rightarrow \mathfrak{M}_0$ preserves the sorts and preserves every operation in $\text{hom}(\mathfrak{M})$, then α preserves every operation in $\text{hom}_p(\mathfrak{M})$.*

Proof. Let $\mathbf{A} \in \mathcal{A}$ and assume that $\alpha : D(\mathbf{A})_0 \rightarrow \mathfrak{M}_0$ preserves the sorts and preserves every operation in $\text{hom}(\mathfrak{M})$. Let $h : \mathbf{M}_1 \dashrightarrow \mathbf{M}_2$ be a non-extendable, proper partial homomorphism and denote by \mathbf{M} the subalgebra of \mathbf{M}_1 whose underlying set is $\text{dom}(h)$. We must prove that α preserves h . Since \mathfrak{M} is rich, there is some $\mathbf{M}' \in \mathfrak{M}$ for which there is an isomorphism $\sigma : \mathbf{M} \rightarrow \mathbf{M}'$. The partial operation h acts on $D(\mathbf{A})$ as a partial operation h from $\mathcal{A}(\mathbf{A}, \mathbf{M}_1)$ to $\mathcal{A}(\mathbf{A}, \mathbf{M}_2)$. The domain of h is $\text{dom}(h) = \{x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_1) \mid x(A) \subseteq M\}$. Let $x \in \text{dom}(h)$. Since $x(A) \subseteq M$, there is a unique homomorphism $x_1 : \mathbf{A} \rightarrow \mathbf{M}$ (restrict the codomain of x down to \mathbf{M}) such that $x = i \circ x_1$, where $i : \mathbf{M} \rightarrow \mathbf{M}_1$ is the inclusion map: see Fig. 5.

We define $h(x) := h \circ x_1$. (It is usual to write simply $h(x) := h \circ x$ and to remark that $h(x)$ is well defined since $x(A) \subseteq M = \text{dom}(h)$.) We must prove that $\alpha(\text{dom}(h)) \subseteq \text{dom}(h)$ and that $\alpha(h(x)) = h(\alpha(x))$, for all $x \in \text{dom}(h)$. Define $k := i \circ \sigma^{-1} : \mathbf{M}' \rightarrow \mathbf{M}_1$ and note that

$$k(\sigma \circ x_1) = k \circ \sigma \circ x_1 = i \circ \sigma^{-1} \circ \sigma \circ x_1 = i \circ x_1 = x.$$

Since $k \in \text{hom}(\mathfrak{M})$, the map α preserves k , and hence

$$\begin{aligned}
 \alpha(x) &= \alpha(k(\sigma \circ x_1)) = k(\alpha(\sigma \circ x_1)) = i(\sigma^{-1}(\alpha(\sigma \circ x_1))) \\
 &= \sigma^{-1}(\alpha(\sigma \circ x_1)) \in M = \text{dom}(h),
 \end{aligned}$$

whence $\alpha(\text{dom}(\mathbf{h})) \subseteq \text{dom}(h)$. Now define $g := h \circ \sigma^{-1} : \mathbf{M}' \rightarrow \mathbf{M}_2$. Since α preserves k and g , as $g \in \text{hom}(\mathfrak{M})$, we have

$$\begin{aligned} \alpha(\mathbf{h}(x)) &= \alpha(h \circ x_1) = \alpha(h \circ \sigma^{-1} \circ \sigma \circ x_1) = \alpha(g(\sigma \circ x_1)) \\ &= g(\alpha(\sigma \circ x_1)) = h(\sigma^{-1}(\alpha(\sigma \circ x_1))) \\ &= h(i(\sigma^{-1}(\alpha(\sigma \circ x_1)))) = h(k(\alpha(\sigma \circ x_1))) \\ &= h(\alpha(k(\sigma \circ x_1))) = h(\alpha(i \circ \sigma^{-1} \circ \sigma \circ x_1)) = h(\alpha(i \circ x_1)) \\ &= h(\alpha(x)). \end{aligned}$$

Thus, $\alpha(\mathbf{h}(x)) = h(\alpha(x))$, as required. \square

Our next theorem is a total version of Theorem 3.3 and is an immediate corollary of Lemma 3.4.

Theorem 3.5. *Let \mathfrak{M} be a finite set of finite subdirectly irreducible Heyting algebras. Assume that \mathfrak{M} is rich and define $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Let \mathcal{T} be the discrete topology on \mathfrak{M}_0 and define*

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \mathcal{T} \rangle.$$

Then \mathfrak{M} yields a duality on \mathcal{A} .

Our next theorem gives a functor-category interpretation of the previous theorem.

Theorem 3.6. *Let \mathfrak{M} be a finite set of finite subdirectly irreducible Heyting algebras. Assume that \mathfrak{M} is rich and define $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Then \mathcal{A} is dually equivalent to a full subcategory of the functor category $\mathcal{B}^{\mathfrak{M}}$.*

Proof. Let \mathfrak{M} be a finite set of finite algebras and let

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \mathcal{T} \rangle.$$

It is easy to show that $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ is isomorphic to a full subcategory of the functor category $\mathcal{B}^{\mathfrak{M}}$. Indeed, every structure in \mathcal{X} is a collection of Boolean spaces indexed by \mathfrak{M} with continuous maps between them indexed by $\text{hom}(\mathfrak{M})$. Moreover, if $\mathbf{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ and $g, h, k \in \text{hom}(\mathfrak{M})$ with $k = h \circ g$, then $k^{\mathbf{X}} = h^{\mathbf{X}} \circ g^{\mathbf{X}}$. Thus, every structure in \mathcal{X} gives rise to a functor from the category \mathfrak{M} into \mathcal{B} . Now let \mathfrak{M} be a finite set of finite subdirectly irreducible Heyting algebras which is rich and let $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Since \mathfrak{M} is rich, Theorem 3.5 implies that $\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \mathcal{T} \rangle$ yields a duality on \mathcal{A} . Consequently, \mathcal{A} is dually equivalent to a full subcategory of \mathcal{X} . Since \mathcal{X} is isomorphic to a full subcategory of $\mathcal{B}^{\mathfrak{M}}$, we conclude that \mathcal{A} is dually equivalent to a full subcategory of $\mathcal{B}^{\mathfrak{M}}$, as required. \square

Remark 3.7 (Theorem 1.1 revisited). Since a transversal \mathfrak{M} of the isomorphism classes of the subdirectly irreducible algebras in a finitely generated variety of Heyting algebras is rich, Theorem 1.1 can be obtained as an immediate application of Theorem 3.6.

If $\mathfrak{M} = \{\mathbf{M}\}$, then $\text{hom}(\mathfrak{M})$ is simply the endomorphism monoid $\text{End}(\mathbf{M})$ of \mathbf{M} . An algebra \mathbf{M} is called *endodualisable* if the structure $\mathfrak{M} = \langle \mathbf{M}; \text{End}(\mathbf{M}), \mathcal{T} \rangle$ yields a duality on $\mathbb{S}\mathbb{P}(\mathbf{M})$. The proof above shows that if \mathbf{M} is endodualisable, then $\mathbb{S}\mathbb{P}(\mathbf{M})$ is dually equivalent to a full subcategory of the functor category $\mathcal{B}^{\mathfrak{M}}$, where $\mathfrak{M} = \{\mathbf{M}\}$ is a single-object category. There are many finite algebras which are known to be endodualisable: see, for example, [5–7,14,15]. The only finite subdirectly irreducible Heyting algebras which are endodualisable are the finite chains and the algebra $\mathbf{2}^2 \oplus \mathbf{1}$: see [11,12,9]. Theorem 1.1 may be viewed as establishing a multisorted generalization of endodualisability which applies to every finitely generated variety of Heyting algebras.

4. Multisorted strong dualities

When are the dualities, described in the previous section, full? Experience tells us that the best way to approach this question is via a stronger condition. Below we introduce the multisorted version of strong duality. As in the single sorted case, it turns out that \mathfrak{M} yields a strong duality on $\mathcal{A} := \mathbb{S}\mathbb{P}(\mathfrak{M})$ if and only if \mathfrak{M} yields a full duality on \mathcal{A} and \mathfrak{M} is injective in $\mathcal{X} := \mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ (see Theorem 4.1). Every full natural duality known is, in fact, strong. We shall prove in Section 5 that the duality given in Theorem 3.3 is in fact a strong duality (for every finitely generated variety of Heyting algebras) and that the duality given in Theorem 3.5 is strong only if \mathcal{A} is generated by $\mathbf{2}$, $\mathbf{3}$ or $\mathbf{2}^2 \oplus \mathbf{1}$. Then, in Section 6, we shall show that the functor-category duality from Theorem 1.1 is full and $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$ if and only if \mathcal{A} is generated by $\mathbf{2}$ or $\mathbf{3}$. In order to do this, we need to look carefully at multisorted strong dualities. (These were alluded to in Chapter 7 of [3] but few details and no proofs were given.)

Let \mathfrak{M} be a finite set of finite algebras and define $\mathcal{A} := \mathbb{S}\mathbb{P}(\mathfrak{M})$. Let I be an arbitrary set, let $\mathbf{M}_i \in \mathfrak{M}$, for all $i \in I$, let $\mathbf{B} \leq \prod_{i \in I} \mathbf{M}_i$ and let $h: \mathbf{B} \rightarrow \mathbf{M}$ be a homomorphism for some $\mathbf{M} \in \mathfrak{M}$. Then we say that h is an *algebraic I-ary partial operation on \mathfrak{M}* . We may extend the map h pointwise to an I -ary partial operation \mathbf{h} on any multisorted power

$$\mathfrak{M}_0^S := \bigcup \{M^S \mid \mathbf{M} \in \mathfrak{M}\},$$

of \mathfrak{M}_0 . For each $s \in S$, let $\pi_s: \mathfrak{M}_0^S \rightarrow \mathfrak{M}_0$ denote the s th projection given by $\pi_s(y) = y(s)$ for each $y \in \mathfrak{M}_0^S$. Then the domain of the extension \mathbf{h} is

$$\text{dom}(\mathbf{h}) = \{x \in \prod_{i \in I} M_i^S \mid \pi_s \circ x \in B \text{ for all } s \in S\} \subseteq \prod_{i \in I} M_i^S$$

and $\mathbf{h}: \text{dom}(\mathbf{h}) \rightarrow M^S$ is defined by

$$(\mathbf{h}(x))(s) = h(\pi_s \circ x) \text{ for } x \in \text{dom}(\mathbf{h}).$$

We say that a subset X of \mathfrak{M}_0^S is *closed under h* provided $\mathbf{h}(x) \in X$ whenever $x \in \text{dom}(\mathbf{h})$ and $x(i) \in X$ for each $i \in I$. We shall say that X is *hom-closed (in \mathfrak{M}_0^S)* if, for each set I , the set X is closed under every algebraic I -ary partial operation h on \mathfrak{M} .

We say that \mathfrak{M} yields a *strong duality on \mathcal{A}* if \mathfrak{M} yields a duality on \mathcal{A} and every closed substructure of a power of \mathfrak{M} is hom-closed. In this section, we will show that if \mathfrak{M} yields a duality on \mathcal{A} and \mathfrak{M} generates a congruence-distributive variety, then the duality can be upgraded to a multisorted strong duality by adding finitely many (partial) operations to \mathfrak{M} .

The following result shows the close link between strong dualities and full dualities. We omit the proof of this theorem as it can be obtained by straightforward modifications of the single-sorted case (Theorem 3.2.4 [3]).

Theorem 4.1 (Multisorted Strong Duality Theorem). *Let \mathfrak{M} be a finite set of finite algebras and define $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Assume that \mathfrak{M} is algebraic over \mathfrak{M} . Then \mathfrak{M} yields a strong duality on \mathcal{A} if and only if \mathfrak{M} yields a full duality on \mathcal{A} and is injective in $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$.*

Our aim now is to impose a condition on \mathfrak{M} which will ensure that every closed substructure of a power of \mathfrak{M} is hom-closed. This was done in [3] (Lemma 3.3.6) in the single sorted case. We will do it here for the multisorted case.

Let \mathfrak{M} be a finite set of finite algebras, and let I be an arbitrary non-empty set. Let $\mathbf{M}_i \in \mathfrak{M}$ for all $i \in I$ and let \mathcal{U} be an ultrafilter on I . Define a map $f_{\mathcal{U}} : \prod_{i \in I} M_i \rightarrow \mathfrak{M}_0$ by

$$f_{\mathcal{U}}(\beta) = a \text{ if and only if } \beta^{-1}(a) \in \mathcal{U}.$$

Since \mathcal{U} is an ultrafilter, $f_{\mathcal{U}}$ is well defined. We claim that there exists $\mathbf{M} \in \mathfrak{M}$ such that $f_{\mathcal{U}}(\beta) \in M$ for all $\beta \in \prod_{i \in I} M_i$. Assume $\beta, \gamma \in \prod_{i \in I} M_i$ with

$$f_{\mathcal{U}}(\beta) = a \in M \quad \text{and} \quad f_{\mathcal{U}}(\gamma) = b \in M', \text{ for some } \mathbf{M}, \mathbf{M}' \in \mathfrak{M}.$$

Then $\beta^{-1}(a), \gamma^{-1}(b) \in \mathcal{U}$. This implies $J := \beta^{-1}(a) \cap \gamma^{-1}(b) \in \mathcal{U}$, as \mathcal{U} is a filter. Thus $J \neq \emptyset$. Let $j \in J$ be a fixed element; then $\beta(j) = a$ and $\gamma(j) = b$. This implies $M_j = M$ and $M_j = M'$, and hence $M = M'$. Thus, there exists $\mathbf{M} \in \mathfrak{M}$ such that the map $f_{\mathcal{U}} : \prod_{i \in I} M_i \rightarrow M$, given by

$$f_{\mathcal{U}}(\beta) = a \text{ if and only if } \beta^{-1}(a) \in \mathcal{U},$$

is well defined.

The following lemma is the multisorted version of Lemma 3.3.5 [3].

Lemma 4.2. *Let \mathfrak{M} be a finite set of finite algebras, let $\mathbf{M}_i \in \mathfrak{M}$ for all $i \in I$, let \mathcal{U} be an ultrafilter on I and let $f_{\mathcal{U}} : \prod_{i \in I} M_i \rightarrow M$ be defined as above. Then, for all $S \neq \emptyset$, every topologically closed subspace X of the multisorted product space $\mathfrak{M}_t^S := \bigcup \{\mathbf{M}_t^S \mid \mathbf{M} \in \mathfrak{M}\}$ is closed under $f_{\mathcal{U}}$.*

Proof. Let $f_{\mathcal{U}}$ denote the pointwise extension of $f_{\mathcal{U}}$ to \mathfrak{M}_0^S . Let $\mathbf{x} \in \text{dom}(f_{\mathcal{U}})$ with $\mathbf{x}(i) \in X$, for all $i \in I$. To show that $f_{\mathcal{U}}(\mathbf{x}) \in X$ we let F be a finite subset of S and check that $f_{\mathcal{U}}(\mathbf{x})$ agrees with a member of X on F . For each $s \in F$, denote $f_{\mathcal{U}}(\pi_s \circ \mathbf{x}) \in M$

by a_s . Then $(\pi_s \circ \mathbf{x})^{-1}(a_s) \in \mathcal{U}$. Now since F is finite and \mathcal{U} is an ultrafilter,

$$J := \bigcap \{(\pi_s \circ \mathbf{x})^{-1}(a_s) \mid s \in F\} \in \mathcal{U}.$$

Then, for any $j \in J$, we find that $f_{\mathcal{U}}(\mathbf{x})$ agrees with $\mathbf{x}(j)$ on F since, for all $s \in F$,

$$f_{\mathcal{U}}(\mathbf{x})(s) = f_{\mathcal{U}}(\pi_s \circ \mathbf{x}) = a_s = (\pi_s \circ \mathbf{x})(j) = \mathbf{x}(j)(s). \quad \square$$

For any finite algebra \mathbf{Q} , let $\text{irr}(\mathbf{Q})$ be the least n such that the zero congruence $\mathbf{0}^{\mathbf{Q}}$ on \mathbf{Q} is a meet of n meet-irreducible congruences. We define the *irreducibility index* of a finite algebra \mathbf{M} by

$$\text{Irr}(\mathbf{M}) = \max\{\text{irr}(\mathbf{Q}) \mid \mathbf{Q} \text{ is a subalgebra of } \mathbf{M}\}.$$

The irreducibility index of \mathfrak{M} is defined by

$$\text{Irr}(\mathfrak{M}) = \max\{\text{Irr}(\mathbf{M}) \mid \mathbf{M} \in \mathfrak{M}\}.$$

The following lemma is the multisorted version of Lemma 3.3.6 [3].

Lemma 4.3. *Let \mathfrak{M} be a finite set of finite algebras and let $\mathfrak{M} = \langle \mathfrak{M}_0; G, H, R, \mathcal{F} \rangle$ be algebraic over \mathfrak{M} . Assume that the variety generated by \mathfrak{M} is congruence distributive and that, for $0 \leq k \leq \text{Irr}(\mathfrak{M})$, the set $G \cup H$ includes all homomorphisms $h: \mathbf{D} \rightarrow \mathbf{M}_{k+1}$, where \mathbf{D} is a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_k$ for some $\mathbf{M}_1, \dots, \mathbf{M}_{k+1} \in \mathfrak{M}$. Then every closed substructure of a non-zero power of \mathfrak{M} is hom-closed.*

Proof. Let \mathbf{X} be a closed substructure of \mathfrak{M}^S for some non-empty set S , let $\mathbf{M}_i \in \mathfrak{M}$ for all $i \in I$, let $\mathbf{B} \leq \prod_{i \in I} \mathbf{M}_i$ and let $g: \mathbf{B} \rightarrow \mathbf{M}$ be a homomorphism where $\mathbf{M} \in \mathfrak{M}$. First assume that $g(\mathbf{B})$ is trivial. Then $g(\mathbf{B}) = \{a\}$, for some $a \in \mathbf{M}$. Fix $i \in I$ and define $g_a: \mathbf{M}_i \rightarrow \mathbf{M}$ with $g_a(M_i) = \{a\}$. As \mathbf{X} is closed under g_a (by assumption), it follows that \mathbf{X} is closed under g .

Now assume that $g(\mathbf{B})$ is non-trivial. There are $k \leq \text{Irr}(\mathfrak{M})$ meet-irreducible congruences ψ_1, \dots, ψ_k on \mathbf{B} with $\bigcap_{j=1}^k \psi_j = \ker(g)$. Since the variety generated by \mathfrak{M} is congruence distributive, Jónsson’s Lemma [16] yields ultrafilters $\mathcal{U}_1, \dots, \mathcal{U}_k$ on I such that $\theta_{\mathcal{U}_1} \subseteq \psi_1, \dots, \theta_{\mathcal{U}_k} \subseteq \psi_k$, where $\theta_{\mathcal{U}_j}$ is determined by the ultrafilter \mathcal{U}_j for each j . For each $j = 1, \dots, k$, let $f_{\mathcal{U}_j}: \prod_{i \in I} \mathbf{M}_i \rightarrow \mathbf{M}_{j_0}$ be the homomorphism defined by

$$f_{\mathcal{U}_j}(\beta) = a \text{ if and only if } \beta^{-1}(a) \in \mathcal{U}_j$$

and let

$$f = f_{\mathcal{U}_1} \sqcap \cdots \sqcap f_{\mathcal{U}_k} : \prod_{i \in I} \mathbf{M}_i \rightarrow \prod_{j=1}^k \mathbf{M}_{j_0}$$

so that

$$f(\beta) = (f_{\mathcal{U}_1}(\beta), \dots, f_{\mathcal{U}_k}(\beta))$$

for $\beta \in \prod_{i \in I} \mathbf{M}_i$. Let $\mathbf{D} = f(\mathbf{B}) \leq \prod_{j=1}^k \mathbf{M}_{j_0}$. We claim that g can be factored through $f \upharpoonright_{\mathbf{B}}$, that is, there is a homomorphism $h: \mathbf{D} \rightarrow \mathbf{M}$ such that $g = h \circ f \upharpoonright_{\mathbf{B}}$. To prove this, it is sufficient to show that $\ker(f \upharpoonright_{\mathbf{B}}) \subseteq \ker(g)$. Let $(\beta, \gamma) \in \ker(f \upharpoonright_{\mathbf{B}})$. Then

$$(f_{\mathcal{U}_1}(\beta), \dots, f_{\mathcal{U}_k}(\beta)) = (f_{\mathcal{U}_1}(\gamma), \dots, f_{\mathcal{U}_k}(\gamma)).$$

So, for $j = 1, 2, \dots, k$, we have $f_{\mathcal{U}_j}(\beta) = f_{\mathcal{U}_j}(\gamma)$
 $\Rightarrow f_{\mathcal{U}_j}(\beta) = f_{\mathcal{U}_j}(\gamma) = a_j$ for some $a_j \in M_{j_0}$
 $\Rightarrow \beta^{-1}(a_j), \gamma^{-1}(a_j) \in \mathcal{U}_j$
 $\Rightarrow \beta^{-1}(a_j) \cap \gamma^{-1}(a_j) \in \mathcal{U}_j$
 $\Rightarrow \text{eq}(\beta, \gamma) \in \mathcal{U}_j$ (since $\beta^{-1}(a_j) \cap \gamma^{-1}(a_j) \subseteq \text{eq}(\beta, \gamma)$),

where $\text{eq}(\beta, \gamma) := \{i \in I \mid \beta(i) = \gamma(i)\}$. This implies $(\beta, \gamma) \in \theta_{\mathcal{U}_j}$ for all $j = 1, 2, \dots, k$. So we have

$$(\beta, \gamma) \in \bigcap_{j=1}^k \theta_{\mathcal{U}_j} \subseteq \bigcap_{j=1}^k \psi_j = \ker(g).$$

Thus $\ker(f \upharpoonright_B) \subseteq \ker(g)$.

By Lemma 4.2, for each $j \in \{1, 2, \dots, k\}$, the structure \mathbf{X} is closed under $f_{\mathcal{U}_j}$ and is closed under h by assumption. Assume that \mathbf{x} belongs to the domain of g on \mathbf{X} , that is, $\mathbf{x}(i) \in X$, for all $i \in I$, and $\pi_s \circ \mathbf{x} \in B$, for each $s \in S$. Then

$$\begin{aligned} g(\mathbf{x})(s) &= g(\pi_s \circ \mathbf{x}) = (h \circ f)(\pi_s \circ \mathbf{x}) = h(f(\pi_s \circ \mathbf{x})) \\ &= h(f_{\mathcal{U}_1}(\pi_s \circ \mathbf{x}), \dots, f_{\mathcal{U}_k}(\pi_s \circ \mathbf{x})) \\ &= h(f_{\mathcal{U}_1}(\mathbf{x}), \dots, f_{\mathcal{U}_k}(\mathbf{x}))(s). \end{aligned}$$

Now, since \mathbf{X} is closed under h and each $f_{\mathcal{U}_j}$, it follows that \mathbf{X} is hom-closed. \square

This result yields the following multisorted version of Theorem 3.3.7 [3].

Theorem 4.4. *Let \mathfrak{M} be a finite set of finite algebras and let $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Assume that \mathfrak{M} generates a congruence distributive variety and that $\mathfrak{M} = \langle \mathfrak{M}_0; G, H, R, \mathcal{F} \rangle$ yields a duality on \mathcal{A} . If \mathfrak{M}' is obtained from \mathfrak{M} by adding all n -ary non-extendable algebraic partial operations to $G \cup H$, for all $0 \leq n \leq \text{Irr}(\mathfrak{M})$, then \mathfrak{M}' yields a strong duality on \mathcal{A} .*

Define K to be the set of all elements which form a one-element subalgebra of some $\mathbf{M} \in \mathfrak{M}$; then K will determine a substructure \mathbf{K} of \mathfrak{M} . The structure \mathbf{K} plays a special role in (multisorted) full dualities. Note that if G contains no nullary operations, then the empty structure \emptyset belongs to \mathcal{X} . Let $\mathbf{1}$ denote the one-element algebra in \mathcal{A} . Our next lemma is the multisorted analogue of Lemma 3.1.2 [3].

Lemma 4.5. *Let \mathfrak{M} be a finite set of finite algebras and assume that \mathfrak{M} yields a full duality on $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$.*

- (i) \mathbf{K} and $\mathbf{1}$ are dual to one another: $E(\mathbf{K}) \cong \mathbf{1}$ and $D(\mathbf{1}) \cong \mathbf{K}$.
- (ii) \mathbf{K} is the substructure of \mathfrak{M} generated by the distinguished elements.
- (iii) For every $\mathbf{X} \in \mathcal{X}$ there is a unique embedding of \mathbf{K} into \mathbf{X} .

- (iv) \mathbf{K} is an initial object in \mathcal{X} while $\mathbf{1}$ is a final object in \mathcal{A} .
- (v) $\emptyset \in \mathcal{X}$ if and only if $\mathbf{K} = \emptyset$ if and only if \mathfrak{M} has no nullary operation if and only if $\mathbf{1} \notin \mathbb{S}(\mathfrak{M})$.

Proof. Since the embeddings of $\mathbf{1}$ into \mathbf{M} for some $\mathbf{M} \in \mathfrak{M}$ correspond exactly to the elements of K , we have $D(\mathbf{1}) \cong \mathbf{K}$. Now since we have a duality, $E(\mathbf{K}) \cong ED(\mathbf{1}) \cong \mathbf{1}$. Hence, (i) holds. Consider (ii). Let \mathbf{C} denote the substructure of \mathfrak{M} generated by the distinguished elements of \mathfrak{M} . Then, clearly, $E(\mathbf{C}) \cong \mathbf{1} \cong E(\mathbf{K})$ and, since the duality is full, $\mathbf{C} \cong DE(\mathbf{C}) \cong DE(\mathbf{K}) \cong \mathbf{K}$. From the definition of \mathbf{C} , this isomorphism must be the identity, that is $\mathbf{C} = \mathbf{K}$. Now (iii)–(v) follow immediately from (ii). \square

We wish to apply Theorem 4.4 to upgrade a piggyback duality, obtained via Theorem 2.1, to a strong duality. This can always be done, but we shall concentrate on the special case in which $\text{Irr}(\mathfrak{M}) = 1$ as this applies whenever \mathfrak{M} is a finite set of finite subdirectly irreducible Heyting algebras.

Theorem 4.6. *Let \mathfrak{M} be a finite set of finite algebras. Assume that every subalgebra of each $\mathbf{M} \in \mathfrak{M}$ is subdirectly irreducible and assume that the conditions of the Multisorted Piggyback Duality Theorem hold. In particular, let R be the set of all piggyback relations and assume that $\text{hom}(\mathfrak{M})$ satisfies condition (S) of Theorem 2.1. Let \mathcal{T} be the discrete topology on \mathfrak{M}_0 . Then*

$$\mathfrak{M} = \langle \mathfrak{M}_0; K \cup \text{hom}(\mathfrak{M}), \text{hom}_p(\mathfrak{M}), R, \mathcal{T} \rangle,$$

yields a strong duality on $\mathcal{A} := \mathbb{S}\mathbb{P}(\mathfrak{M})$.

Proof. Since we have assumed that the conditions of the Multisorted Piggyback Duality Theorem hold, $\text{hom}(\mathfrak{M}) \cup R$ yields a duality on \mathcal{A} . Furthermore, as \mathfrak{M} has a term-reduct in \mathcal{D} , the algebras in \mathfrak{M} have a definable lattice structure and hence \mathfrak{M} generates a congruence distributive variety. The assumption that each subalgebra of every $\mathbf{M} \in \mathfrak{M}$ is subdirectly irreducible, gives $\text{Irr}(\mathfrak{M}) = 1$. Consequently, $K \cup \text{hom}(\mathfrak{M}) \cup \text{hom}_p(\mathfrak{M})$ contains every n -ary algebraic (partial) operation on \mathfrak{M} with $0 \leq n \leq \text{Irr}(\mathfrak{M})$. The result now follows at once from Theorem 4.4. \square

Since we wish to relate the strong duality given by this theorem to dualities via functor categories, it is natural to ask when the set $\text{hom}_p(\mathfrak{M})$ can be deleted from the structure on \mathfrak{M} without destroying the strong duality. We claim that this is possible precisely when every algebra in \mathfrak{M} is injective in \mathcal{A} . The following result is true much more generally but we state it for multisorted structures whose operations and partial operations are at most unary as this is all that we require here. (See Sections 3.2 and 6.1 of Clark and Davey [3] for a detailed discussion in the single-sorted case.)

Lemma 4.7. *Let \mathfrak{M} be a finite set of finite algebras. Define $\mathcal{A} := \mathbb{S}\mathbb{P}(\mathfrak{M})$. Consider the following conditions:*

- (i) *the total structure $\mathfrak{M} = \langle \mathfrak{M}_0; K \cup \text{hom}(\mathfrak{M}), R, \mathcal{T} \rangle$ yields a strong duality on \mathcal{A} ;*

(ii) the structure $\mathfrak{M}' = \langle \mathfrak{M}_0; K \cup \text{hom}(\mathfrak{M}), \text{hom}_p(\mathfrak{M}), R, \mathcal{T} \rangle$ yields a strong duality on \mathcal{A} and every algebra $\mathbf{M} \in \mathfrak{M}$ is injective in \mathcal{A} .

Then (i) implies (ii), and, if \mathfrak{M} is rich, the conditions are equivalent.

Proof. First assume that \mathfrak{M} yields a strong duality on \mathcal{A} and define $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$. Since adding extra algebraic partial operations to the type of \mathfrak{M} cannot destroy a strong duality, the structure \mathfrak{M}' also yields a strong duality on \mathcal{A} . The claim that each algebra in \mathfrak{M} is injective in \mathcal{A} is equivalent to the claim that, for each embedding $u: \mathbf{A} \rightarrow \mathbf{B}$ in \mathcal{A} , the dual map $D(u)$ is surjective. Let $u: \mathbf{A} \rightarrow \mathbf{B}$ be an embedding in \mathcal{A} . Since the type of \mathfrak{M} includes no partial operations, the image of $D(u): D(\mathbf{B}) \rightarrow D(\mathbf{A})$ is an \mathfrak{M} -substructure of $D(\mathbf{A})$ and hence $D(u)$ can be factored in \mathcal{X} as $D(u) = \varphi \circ \psi$ where $\psi: D(\mathbf{B}) \rightarrow \mathbf{X}$ is surjective for some $\mathbf{X} \in \mathcal{X}$ and $\varphi: \mathbf{X} \rightarrow D(\mathbf{A})$ is an embedding. Since \mathfrak{M} yields a duality on \mathcal{A} , the double dual $ED(u)$ of the embedding u is also an embedding. Since $ED(u) = E(\psi) \circ E(\varphi)$, it follows that $E(\varphi)$ is an embedding. By Theorem 4.1, \mathfrak{M} is injective in the dual category \mathcal{X} and hence the dual $E(\varphi)$ of the embedding φ is surjective. Thus, we have proved that $E(\varphi)$ is an isomorphism. Since \mathfrak{M} yields a full duality on \mathcal{A} , by Theorem 4.1, it follows that φ is an isomorphism. Consequently, $D(u)$, which equals $\psi \circ \varphi$, is surjective as ψ is surjective. Hence, every algebra $\mathbf{M} \in \mathfrak{M}$ is injective in \mathcal{A} .

For the converse, assume that \mathfrak{M} is rich. Hence, by Lemma 3.4, the set $\text{hom}_p(\mathfrak{M})$ may be removed from the type of \mathfrak{M}' without destroying the duality. Now assume that every algebra in \mathfrak{M} is injective in \mathcal{A} . Since every substructure of a power of \mathfrak{M}' is hom-closed, to show that every substructure of a power of \mathfrak{M} is hom-closed it suffices to show that every substructure \mathbf{X} of a power of \mathfrak{M} is closed under each partial operation in $h \in \text{hom}_p(\mathfrak{M})$. Let $h: \mathbf{M}_1 \dashrightarrow \mathbf{M}_2$, with $\mathbf{M}_1, \mathbf{M}_2 \in \mathfrak{M}$, and let \mathbf{X} be a substructure of a power of \mathfrak{M} . Since \mathbf{M}_2 is injective in \mathcal{A} , the partial map h has an extension $g: \mathbf{M}_1 \rightarrow \mathbf{M}_2$ in $\text{hom}(\mathfrak{M})$. Since \mathbf{X} is closed under g , by assumption, it follows easily that \mathbf{X} is also closed under h . Thus, \mathfrak{M} yields a strong duality on \mathcal{A} . □

5. Multisorted strong dualities for varieties of Heyting algebras

It is a simple matter to combined our earlier results to give several different multisorted strong dualities for finitely generated varieties of Heyting algebras. Our first theorem shows that the duality established in Theorem 3.3 is in fact strong.

Theorem 5.1 (Heyting Multisorted Strong Duality Theorem). *Assume that \mathcal{A} is a finitely generated variety of Heyting algebras and let \mathfrak{M} be a finite set of finite subdirectly irreducible algebras in \mathcal{A} such that $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Let \mathcal{T} be the discrete topology on \mathfrak{M}_0 and define*

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \text{hom}_p(\mathfrak{M}), \mathcal{T} \rangle.$$

Then \mathfrak{M} yields a strong duality on \mathcal{A} .

Proof. This follows immediately from Theorems 3.3 and 4.4. □

By Lemma 4.7, if we wish to dispense with the partial operations without destroying the strong duality, then every algebra in \mathfrak{M} must be injective in \mathcal{A} . Unfortunately, this occurs only rarely.

Theorem 5.2. *Let \mathcal{A} be a finitely generated variety of Heyting algebras, let \mathfrak{M} be a finite set of subdirectly irreducible Heyting algebras such that $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$ and consider the structure*

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \mathcal{T} \rangle.$$

The following are equivalent:

- (i) \mathfrak{M} yields a strong duality on \mathcal{A} ;
- (ii) one of the following three conditions holds:
 - (a) \mathfrak{M} is $\{\mathbf{2}\}$ (and \mathcal{A} is the variety of Boolean algebras),
 - (b) \mathfrak{M} is either $\{\mathbf{3}\}$ or $\{\mathbf{3}, \mathbf{2}\}$ (and \mathcal{A} is the variety generated by $\mathbf{3}$),
 - (c) \mathfrak{M} is either $\{\mathbf{2}^2 \oplus \mathbf{1}\}$ or $\{\mathbf{2}^2 \oplus \mathbf{1}, \mathbf{2}\}$ (and \mathcal{A} is the variety generated by $\mathbf{2}^2 \oplus \mathbf{1}$).

Proof. Assume that \mathfrak{M} yields a strong duality on \mathcal{A} . By Lemma 4.7, every algebra in \mathfrak{M} is injective in \mathcal{A} . Thus, since $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$, every algebra in the variety \mathcal{A} can be embedded into an algebra which is injective in \mathcal{A} . By an unpublished result of A. Day (see [13]), the only varieties \mathcal{A} of Heyting algebras (whether finitely generated or not) which have this property are the varieties generated by $\mathbf{2}$, $\mathbf{3}$ or $\mathbf{2}^2 \oplus \mathbf{1}$. We leave it to the reader to verify that, (1) up to isomorphism, the only subdirectly irreducible algebras in the variety generated by $\mathbf{2}^2 \oplus \mathbf{1}$ are $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{2}^2 \oplus \mathbf{1}$, (2) $\mathbf{2}$ and $\mathbf{3}$ are injective in the variety generated by $\mathbf{3}$, and (3) $\mathbf{2}$ and $\mathbf{2}^2 \oplus \mathbf{1}$ are injective in the variety generated by $\mathbf{2}^2 \oplus \mathbf{1}$ while $\mathbf{3}$ is not. (The only general algebraic tools required are Jónsson’s Lemma and the results from [4] on injectives in congruence-distributive varieties.) Thus, (i) implies (ii). The converse holds by Lemma 4.7 and Theorem 5.1 and facts (1)–(3) listed above. \square

6. Strong dualities via functor categories

Let \mathcal{A} be a finitely generated variety of Heyting algebras. In this section, we address the question: ‘When is the duality between \mathcal{A} and the functor category $\mathcal{B}^{\mathfrak{M}}$ a dual category equivalence?’ While we do not know the answer to this question, we can prove the following result. We believe that the injectivity assumption can be dropped but have not been able to prove this.

Theorem 6.1. *Let \mathcal{A} be a finitely generated variety of Heyting algebras and let \mathfrak{M} be a transversal of the isomorphism classes of the class $SI(\mathcal{A})$ of subdirectly irreducible algebras in \mathcal{A} . The functors $D: \mathcal{A} \rightarrow \mathcal{B}^{\mathfrak{M}}$ and $E: \mathcal{B}^{\mathfrak{M}} \rightarrow \mathcal{A}$ give a dual category equivalence between \mathcal{A} and the functor category $\mathcal{B}^{\mathfrak{M}}$ and $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$ if and only if \mathcal{A} is either Boolean algebras or the variety generated by $\mathbf{3}$.*

The remainder of this section is devoted to proving this theorem.

6.1. The variety of Boolean algebras

For the variety $\text{Var}(\mathbf{2})$ of Boolean algebras we have $\mathfrak{M} = \{\mathbf{2}\}$ and $\text{hom}(\mathfrak{M}) = \{\text{id}_{\mathbf{2}}\}$. Thus, $\mathcal{B}^{\mathfrak{M}}$ is isomorphic to \mathcal{B} and consequently the dual equivalence between $\text{Var}(\mathbf{2})$ and $\mathcal{B}^{\mathfrak{M}}$ amounts to Stone’s duality for Boolean algebras.

6.2. The variety $\text{Var}(\mathbf{3})$

We now return to the example with which we commenced the paper: $\mathcal{A} = \text{Var}(\mathbf{3})$. By Theorem 5.2, we may obtain a single-sorted strong duality by choosing $\mathfrak{M} = \{\mathbf{3}\}$ or a two-sorted strong duality by choosing $\mathfrak{M} = \{\mathbf{3}, \mathbf{2}\}$.

Since the only non-identity endomorphism of $\mathbf{3}$ is the retraction g given by $0 \mapsto 0$, $a \mapsto 1$, $1 \mapsto 1$, the structure $\mathfrak{Z} := \langle \{0, a, 1\}; g, \mathcal{T} \rangle$ yields a strong duality on $\text{Var}(\mathbf{3})$. Our first task is to axiomatize the topological quasi-variety $\mathbb{S}_c\mathbb{P}^+(\mathfrak{Z})$.

Lemma 6.2. *Let X be a set and let $g: X \rightarrow X$ satisfy $g \circ g = g$. Let U be a subset of X and define*

$$V = g^{-1}(U) \quad \text{and} \quad W = X \setminus g^{-1}(U).$$

Then $g(V) \subseteq V$ and $g(W) \subseteq W$.

Proof. Let U be a subset of X . Then, since $g \circ g = g$,

$$\begin{aligned} v \in V = g^{-1}(U) &\Rightarrow g(v) \in U \\ &\Rightarrow gg(v) \in U \Rightarrow g(v) \in g^{-1}(U) = V. \end{aligned}$$

Hence $g(V) \subseteq V$. Similarly,

$$\begin{aligned} w \in W = X \setminus g^{-1}(U) &\Rightarrow w \notin g^{-1}(U) \Rightarrow g(w) \notin U \\ &\Rightarrow gg(w) \notin U \Rightarrow g(w) \notin g^{-1}(U) \Rightarrow g(w) \in W. \end{aligned}$$

Hence $g(W) \subseteq W$. \square

Theorem 6.3. (i) *Let $\mathbf{X} = \langle X; g, \mathcal{T} \rangle$ be a Boolean space with a continuous map g satisfying $g \circ g = g$. Then the continuous g -preserving maps from \mathbf{X} into \mathfrak{Z} separate the points of X .*

(ii) *The dual category $\mathcal{X} := \mathbb{S}_c\mathbb{P}^+(\mathfrak{Z})$ is exactly the category of structures $\mathbf{X} = \langle X; g, \mathcal{T} \rangle$, where $\langle X; \mathcal{T} \rangle$ is a Boolean space and $g: X \rightarrow X$ is a continuous map which satisfies $g \circ g = g$.*

(iii) *The variety $\text{Var}(\mathbf{3})$ is dually equivalent to the functor category $\mathcal{B}^{\mathfrak{M}}$, where $\mathfrak{M} = \{\mathbf{3}\}$.*

Proof. (i) Let $\mathbf{X} = \langle X; g, \mathcal{T} \rangle$ be a Boolean space with a continuous map g satisfying $g \circ g = g$ and let $x, y \in X$ with $x \neq y$.

Case 1: $g(x) \neq g(y)$. Since X is a Boolean space, there exists a clopen subset U of X such that $g(x) \in U$ and $g(y) \notin U$. Let $V = g^{-1}(U)$. Then, clearly, V is clopen, $x \in V$, $y \notin V$ and by Lemma 6.2, $g(V) \subseteq V$. Let $W = X \setminus V$. Then, again by Lemma 6.2, $g(W) \subseteq W$. So we can define a morphism $\alpha : \mathbf{X} \rightarrow \underline{\mathbf{3}}$ such that $\alpha(V) = \{1\}$ and $\alpha(W) = \{0\}$. We then have $\alpha(x) \neq \alpha(y)$.

Case 2: $g(x) = g(y)$. We can choose a clopen subset U' of X such that $x, g(x) \in U'$ and $y \notin U'$. Let

$$U := U' \cap g^{-1}(U').$$

Then U is clopen, $x \in U$ and $y \notin U$. We now use the fact that $g \circ g = g$ to prove that $g(U) \subseteq U$.

$$\begin{aligned} U \subseteq g^{-1}(U') &\Rightarrow g(U) \subseteq U' \\ &\Rightarrow g(g(U)) \subseteq U' \Rightarrow g(U) \subseteq g^{-1}(U'). \end{aligned}$$

Hence $g(U) \subseteq U' \cap g^{-1}(U') = U$. Let

$$V = g^{-1}(U) \setminus U.$$

Then clearly $g(V) \subseteq U$. Let

$$W = X \setminus g^{-1}(U).$$

Then, by Lemma 6.2, $g(W) \subseteq W$. Thus, we can define a morphism $\alpha : \mathbf{X} \rightarrow \underline{\mathbf{3}}$ such that $\alpha(U) = \{1\}$, $\alpha(V) = \{a\}$ and, $\alpha(W) = \{1\}$. We then have $\alpha(x) \neq \alpha(y)$.

(ii) Let $\mathbf{X} \in \mathcal{X}$. Since $\underline{\mathbf{3}} := \langle \{0, a, 1\}; g, \mathcal{T} \rangle$ is a Boolean space with a continuous map g satisfying $g \circ g = g$ and $\mathcal{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underline{\mathbf{3}})$, it follows easily that \mathbf{X} is also a Boolean space with continuous map g satisfying $g \circ g = g$ (see the Preservation Theorem 1.4.3 in [3]). Conversely, assume that $\mathbf{X} = \langle X; g, \mathcal{T} \rangle$ is a Boolean space with continuous map g satisfying $g \circ g = g$. It follows immediately from (i) that $\mathbf{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underline{\mathbf{3}}) = \mathcal{X}$ (see the Separation Theorem 1.4.4 [3]).

(iii) Let $\mathfrak{M} = \{\mathbf{3}\}$. Since $\text{hom}(\mathbf{3}) = \{g, \text{id}\}$, the category $\mathcal{B}^{\mathfrak{M}}$ is obviously isomorphic to the category \mathcal{Y} of all Boolean topological structures $\langle X; g, \mathcal{T} \rangle$, where $g \circ g = g$. By (ii), \mathcal{Y} is dually equivalent to $\text{Var}(\mathbf{3})$ and hence $\mathcal{B}^{\mathfrak{M}}$ is dually equivalent to $\text{Var}(\mathbf{3})$. \square

Now let $\mathfrak{M} = \{\mathbf{3}, \mathbf{2}\}$. The non-identity homomorphisms f , g and h in $\text{hom}(\mathfrak{M})$ are described in Section 1. By Theorem 5.2,

$$\mathfrak{M} = \langle 3 \dot{\cup} 2; f, g, h, \mathcal{T} \rangle \tag{1}$$

yields a strong duality on $\mathcal{A} = \text{Var}(\mathbf{3}) = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$. Part (ii) of the following theorem is the functor-category duality announced in Section 1.

Theorem 6.4. (i) Let $\mathfrak{M} = \langle 3 \dot{\cup} 2; f, g, h, \mathcal{T} \rangle$. Then the dual category $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ is exactly the category of \mathfrak{M} -sorted Boolean spaces

$$\mathbf{X} = \langle X_3 \dot{\cup} X_2; f, g, h, \mathcal{T} \rangle$$

with continuous maps f, g and h which satisfy the equations given in the table in Fig. 2.

(ii) The variety $\text{Var}(\mathbf{3})$ is dually equivalent to the functor category $\mathcal{B}^{\mathfrak{M}}$, where $\mathfrak{M} = \{\mathbf{3}, \mathbf{2}\}$.

Proof. Let $\mathbf{X} \in \mathcal{X}$. Since the operations f, g, h on \mathfrak{M}^S are the pointwise extensions of the operations f, g, h in $\text{hom}(\mathfrak{M})$ and since the equations in the table in Fig. 2 hold for $\text{hom}(\mathfrak{M})$, it is clear that the same equations hold in \mathfrak{M}^S and therefore in any substructure of \mathfrak{M}^S .

Conversely, let $\mathbf{X} = \langle X_3 \dot{\cup} X_2; f, g, h, \mathcal{F} \rangle$ be a Boolean space with continuous maps f, g and h satisfying the equations given in the table in Fig. 2. Let $\alpha_3 : X_3 \rightarrow 3$ be a continuous map which preserves the operation g . Define $\alpha_2 : X_2 \rightarrow 2$ via

$$\alpha_2(x) = h(\alpha_3(f(x)))$$

for each $x \in X_2$. We claim that the extension $\alpha = (\alpha_3, \alpha_2)$ of α_3 is a morphism from \mathbf{X} to \mathfrak{M} . It is enough to show that α preserves f and h . For $x \in X$, we have

$$\alpha_2(h(x)) = h(\alpha_3(f(h(x)))) = h(\alpha_3(g(x))) = h(g(\alpha_3(x))) = h(\alpha_3(x))$$

and

$$\alpha_3(f(x)) = \alpha_3(g(f(x))) = g(\alpha_3(f(x))) = f(h(\alpha_3(f(x)))) = f(\alpha_2(x)).$$

This implies α preserves h and f and hence α is a morphism.

We can now prove that the morphisms from \mathbf{X} to \mathfrak{M} separate the points of $X = X_3 \dot{\cup} X_2$. Let $x, y \in X$ with $x \neq y$. If $x, y \in X_3$, then by Theorem 6.3(i), there is a continuous g -preserving map $\alpha_3 : X_3 \rightarrow 3$ with $\alpha_3(x) \neq \alpha_3(y)$. Thus the extension $\alpha = (\alpha_3, \alpha_2)$ of α_3 defined above satisfies $\alpha(x) \neq \alpha(y)$. If $x \in X_3$ and $y \in X_2$, then any morphism $\alpha = (\alpha_3, \alpha_2)$ from \mathbf{X} to \mathfrak{M} separates x and y : choose α_3 and α_2 to be constant maps onto $0 \in 3$ and $0 \in 2$, respectively. Now let $x, y \in X_2$ with $x \neq y$. Since $h \circ f = \text{id}_{X_2}$, we have $f(x) \neq f(y)$ in X_3 . By Theorem 6.3(i), there is a continuous g -preserving map $\alpha_3 : X_3 \rightarrow 3$ with $\alpha_3(f(x)) \neq \alpha_3(f(y))$. Let $\alpha = (\alpha_3, \alpha_2)$ be the extension of α_3 defined above. If $\alpha_2(x) = \alpha_2(y)$, then $f(\alpha_2(x)) = f(\alpha_2(y))$ and hence $\alpha_3(f(x)) = \alpha_3(f(y))$, a contradiction. Thus $\alpha_2(x) \neq \alpha_2(y)$, as required.

To prove \mathbf{X} is isomorphic to a substructure of \mathfrak{M}^S for some non-empty set S , let S be the set of all morphisms from \mathbf{X} into \mathfrak{M} . We require an \mathfrak{M} -sorted embedding $\varphi : \mathbf{X} \rightarrow \mathfrak{M}^S$. Since $X = X_3 \dot{\cup} X_2$ and since the underlying set of \mathfrak{M}^S is $3^S \dot{\cup} 2^S$, we may define $\varphi = (\varphi_3, \varphi_2) : \mathbf{X} \rightarrow \mathfrak{M}^S$ with $\varphi_3 : X_3 \rightarrow 3^S$ and $\varphi_2 : X_2 \rightarrow 2^S$ by

$$\varphi(x)(\alpha) = \alpha(x)$$

for all $x \in X$ and $\alpha \in S$. That is,

$$\varphi(x)(\alpha) = \begin{cases} \varphi_3(x)(\alpha) = \alpha_3(x) & \text{if } x \in X_3, \\ \varphi_2(x)(\alpha) = \alpha_2(x) & \text{if } x \in X_2, \end{cases}$$

where

$$\alpha(x) = \begin{cases} \alpha_3(x) & \text{if } x \in X_3, \\ \alpha_2(x) & \text{if } x \in X_2. \end{cases}$$

Since the topologies on $\mathbf{3}_\tau^S$ and $\mathbf{2}_\tau^S$ are the product topologies and since f^X, g^X and h^X are defined pointwise in \mathfrak{M}^S , it is straightforward to verify that φ is a morphism. Since the operations f, g and h are total, to prove that φ is an embedding it suffices to show that φ is one-to-one. Let $x, y \in X$ with $x \neq y$. Since the morphisms from \mathbf{X} into \mathfrak{M} separate the points of X , there exists $\alpha: \mathbf{X} \rightarrow \mathfrak{M}$ with $\alpha(x) \neq \alpha(y)$. Thus $\varphi(x)(\alpha) = \alpha(x) \neq \alpha(y) = \varphi(y)(\alpha)$ and hence $\varphi(x) \neq \varphi(y)$, as required.

(ii) By (i), the category $\mathcal{B}^{\mathfrak{M}}$ is isomorphic to \mathcal{X} and consequently $\mathcal{B}^{\mathfrak{M}}$ is dually equivalent to $\text{Var}(\mathbf{3})$. \square

6.3. Beyond the variety $\text{Var}(\mathbf{3})$

Let \mathcal{A} be a finitely generated variety of Heyting algebras, let \mathfrak{M} be a finite set of subdirectly irreducible Heyting algebras such that $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathfrak{M})$ and consider the structure

$$\mathfrak{M} := \langle \mathfrak{M}_0; \text{hom}(\mathfrak{M}), \mathcal{T} \rangle.$$

The category $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathfrak{M})$ of multisorted structures is isomorphic to a full subcategory of the functor category $\mathcal{B}^{\mathfrak{M}}$. Hence, if the duality between \mathcal{A} and $\mathcal{B}^{\mathfrak{M}}$ is a dual equivalence and $I_{\mathfrak{M}}$ is injective in $\mathcal{B}^{\mathfrak{M}}$, then \mathfrak{M} yields a full duality on \mathcal{A} and \mathfrak{M} is injective in \mathcal{X} . By Theorem 4.1, \mathfrak{M} yields a strong duality on \mathcal{A} and hence, by Theorem 5.2, \mathcal{A} is the variety generated by either $\mathbf{2}, \mathbf{3}$ or $\mathbf{2}^2 \oplus \mathbf{1}$. Thus, it remains to consider the variety $\text{Var}(\mathbf{2}^2 \oplus \mathbf{1})$ and the choices $\mathfrak{M} = \{\mathbf{2}^2 \oplus \mathbf{1}\}$ and $\mathfrak{M} = \{\mathbf{2}^2 \oplus \mathbf{1}, \mathbf{2}\}$.

Let $\mathbf{M} := \mathbf{2}^2 \oplus \mathbf{1}$ and assume that $M = \{0, a, b, c, 1\}$, where 0 and 1 are the bounds, a is the unique coatom and b and c are the atoms. The three non-identity endomorphisms of \mathbf{M} are the automorphism f , which interchanges the atoms and fixes the remaining elements, and the characteristic functions $g = \chi_{\{1, a, b\}}$ and $h = \chi_{\{1, a, c\}}$. Let $\underline{\mathbf{M}} := \langle M; f, g, h, \mathcal{T} \rangle$ and let $\mathfrak{M} := \{\mathbf{M}\}$. As per usual, the dual category $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underline{\mathbf{M}})$ is isomorphic to a full subcategory \mathcal{Y} of the functor category $\mathcal{B}^{\mathfrak{M}}$. We claim that there is an object in $\mathcal{B}^{\mathfrak{M}}$ which is not isomorphic to any object in the subcategory \mathcal{Y} and consequently the duality between \mathcal{A} and $\mathcal{B}^{\mathfrak{M}}$ is not a dual equivalence. The reason for this is quite simple. For each functor $\mathbf{X} \in \mathcal{B}^{\mathfrak{M}}$, the induced algebra $\langle X; f, g, h \rangle$, where X is the underlying set of $\mathbf{X}(\mathbf{M})$, belongs to the equational class \mathcal{K} generated by $\langle M; f, g, h \rangle$ while the underlying algebra $\langle X; f, g, h \rangle$ of a structure in \mathcal{X} belongs to the quasi-equational class \mathcal{Q} generated by $\langle M; f, g, h \rangle$ and \mathcal{Q} is a proper subclass of \mathcal{K} . Indeed, the algebra $\mathbf{M}^\circ := \langle M; f, g, h \rangle$ satisfies the quasi-equation

$$g(x) = g(y) \ \& \ f(x) = y \ \Rightarrow \ x = y, \tag{\sigma}$$

while \mathbf{M}°/θ fails (σ) , where θ is the congruence on \mathbf{M}° with corresponding partition $\{0, a, 1 \mid b \mid c\}$. By endowing \mathbf{M}°/θ with the discrete topology we obtain an object of $\mathcal{B}^{\mathfrak{M}}$ which has no isomorphic copy in the subcategory \mathcal{Y} , as required.

A simple modification of this argument, which we leave to the reader, shows that if we choose $\mathfrak{M} = \{\mathbf{2}^2 \oplus \mathbf{1}, \mathbf{2}\}$ or $\mathfrak{M} = \{\mathbf{2}^2 \oplus \mathbf{1}, \mathbf{3}, \mathbf{2}\}$ then the duality between $\text{Var}(\mathbf{2}^2 \oplus \mathbf{1})$ and $\mathcal{B}^{\mathfrak{M}}$, given by Theorem 3.6, is not a dual category equivalence.

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