Algebra univers. **53** (2005) 331–355
0002-5240/05/030331 – 25
DOI 10.1007/s00012-005-1920-6

Algebra Universalis

Dual categories for endodualisable Heyting algebras: optimization and axiomatization

B. A. Davey and M. R. Talukder

Abstract. This paper is both a contribution to *natural duality theory* as it applies to varieties of Heyting algebras and to the theory of *standard topological quasi-varieties* (see [3]) in general. We prove that the n-element Heyting chain $\mathbf{C}_n$ has an alter ego, consisting of $n-2$ endomorphisms, that yields an optimal duality on the variety generated by $\mathbf{C}_n$. In the case $n = 4$, we give a set of quasi-equations that describe the dual category. We also give a set of quasi-equations that describe the strong dual category that is obtained by adding a partial endomorphism of $\mathbf{C}_4$ to the type of the alter ego. A quasi-equational description of the optimal dual category for the variety of Heyting algebras generated by $\mathbf{2}^2 \oplus \mathbf{1}$ is also given. En route, we prove that if a finite topological unary algebra is injective amongst the finite algebras in the variety it generates, then it is standard. The results of [3] then allow us to give non-topological proofs of our topological descriptions of the dual categories.

1. Introduction

The varieties of Heyting algebras generated by finite chains have played a seminal role in the development of the theory of natural dualities. The fact that, for $n \geqslant 2$, the n-element Heyting chain $\mathbf{C}_n$ is endodualisable was first proved in Davey [5] in 1976. Seven years later, when Davey and Werner [15] set down the foundations of natural duality theory, they included a new proof of this result as an application of the NU Duality Theorem. In the mid-1980s, after a careful dissection of the proof in [5], Davey and Werner [16, 17] developed the piggyback approach to obtaining natural dualities and used it to give a simpler proof of the endodualisability of the chains $\mathbf{C}_n$ and proved that the (**N**on-chain) Heyting algebra $\mathbf{N} := \mathbf{2}^2 \oplus \mathbf{1}$ is also endodualisable. In 1996, Davey and Priestley [13] gave an extremely short proof that $\mathbf{C}_n$ is endodualisable and proved that the chains $\mathbf{C}_n$ and the non-chain $\mathbf{N}$ are the only subdirectly irreducible endodualisable Heyting algebras. The study of endodualisability and endoprimality (see [7, 8, 9, 10, 19]) grew out of these studies of endodualisable Heyting algebras.

Presented by N. Zaguia.
Received February 12, 2003; accepted in final form November 18, 2004.
2000 *Mathematics Subject Classification*: 06D20; 06D50, 08A60.
Key words and phrases: Heyting algebra, natural duality, optimal duality, standard topological quasi-varieties.

The fact that $\mathbf{C}_n$ is endodualisable was used in [5] to describe the free algebras and the injective and weakly injective algebras in the variety $\mathcal{C}_n := \mathrm{Var}(\mathbf{C}_n)$ generated by $\mathbf{C}_n$. Nevertheless, the usefulness of the duality was somewhat restricted by the fact that the dualising structure $\langle C_n; \mathrm{End}(\mathbf{C}_n), \mathcal{T}\rangle$ used all 2^{n-2} endomorphisms of $\mathbf{C}_n$ and by the fact that there was no axiomatic description of the objects in the dual category.

To reduce the complexity of the dual structure, we shall replace $\mathrm{End}(\mathbf{C}_n)$ by a natural subset G_n that is minimal with respect to generating $\mathrm{End}(\mathbf{C}_n)$ as a monoid. The fact that G_n generates the monoid $\mathrm{End}(\mathbf{C}_n)$ guarantees that the structure $\underset{\sim}{\mathbf{C}}_n = \langle C_n; G_n, \mathcal{T}\rangle$ yields a duality on $\mathcal{C}_n$. We prove in Section 2 that this duality is *optimal*, that is, for all $g \in G_n$, the simpler structure $\langle C_n; G_n\backslash\{g\}, \mathcal{T}\rangle$ does not yield a duality on $\mathcal{C}_n$. (The minimality of G_n does not automatically guarantee this.) Optimal natural dualities for the variety $\mathcal{N}$ generated by the remaining subdirectly irreducible endodualisable Heyting algebra, $\mathbf{N}$, were analysed in detail by Saramago (see [20, Chapter 5, Example 3]). She proved, in particular, that if G is a minimal generating set for the monoid $\mathrm{End}(\mathbf{N})$, then $\underset{\sim}{\mathbf{N}} := \langle N; G, \mathcal{T}\rangle$ yields an optimal duality on $\mathcal{N}$.

We would like to give sets of quasi-equations, and preferably small sets, that describe the objects in the dual categories $\mathfrak{X}_n := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_n)$ and $\mathfrak{Z} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$. To be more precise, we seek a set Σ_n of quasi-equations in the language of $\underset{\sim}{\mathbf{C}}_n$ such that a Boolean topological algebra $\mathbf{X}$ of the same type as $\underset{\sim}{\mathbf{C}}_n$ is in $\mathfrak{X}_n$ if and only if (the underlying algebra of) $\mathbf{X}$ is a model of Σ_n, and similarly for the topological quasi-variety $\mathfrak{Z}$. In general there is no reason why such sets of quasi-equations should exist. If such a set Σ exists, then the topological quasi-variety is said to be *standard*. The study of standardness was initiated in Clark, Davey, Haviar, Pitkethly and Talukder [3] and continued in Clark, Davey, Freese and Jackson [4].

In Section 3 we prove that a finite topological unary algebra $\underset{\sim}{\mathbf{M}}$ generates a standard topological quasi-variety provided the underlying algebra of $\underset{\sim}{\mathbf{M}}$ is injective amongst the finite algebras in the variety it generates. (This result is rather surprising as it gives purely algebraic and finitary conditions that guarantee the existence of a plethora of continuous homomorphisms.) By establishing this sufficient condition we then show, in Section 4, that both $\mathfrak{X}_4$ and $\mathfrak{Z}$ are standard. This guarantees the existence of the hoped-for sets of quasi-equations for these classes, but does not tell us what the sets are. In Section 5, we give a set of six quasi-equations that axiomatizes $\mathfrak{X}_4$ and a set of four quasi-equations that axiomatizes $\mathfrak{Z}$. (The latter is an unpublished result due to David Clark.) In both cases, since we already know that the class is standard, we can apply the results of [3] and restrict our attention to finite algebras, thereby avoiding any topological arguments. Axiomatizations are known for the classes $\mathfrak{X}_2$ and $\mathfrak{X}_3$: indeed, $\mathfrak{X}_2$ is simply the class of all Boolean spaces and so is axiomatized by the the empty set of quasi-equations while $\mathcal{X}_3$ is the category of Boolean spaces with a continuous self-map g satisfying $g(g(x) = g(x)$ (see Davey and Talukder [14]). Thus, we now have axiomatizations for the classes $\mathcal{Z}$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$. At this stage, we do not have axiomatizations of the topological quasi-varieties $\mathcal{X}_n$ for $n \geqslant 5$, indeed, we do not even know if these classes are standard.

By [2, Theorems 4.2.2, 4.2.3], the duality given by $\underset{\sim}{\mathbf{C}}_n$ is strong, and therefore full, if and only if $n = 2$ or $n = 3$. For $n \geqslant 4$, the duality can be upgraded to a strong duality by adding enough partial endomorphisms of $\mathbf{C}_n$ to the type of $\underset{\sim}{\mathbf{C}}_n$. We close the paper by proving, in Section 6, that, for $n = 4$, there is a six-element base for the quasi-equations of the resulting strong dual category which therefore is standard. Since the type now includes a proper partial operation, we cannot use the approach via injectivity from Section 3 and must handle arbitrary topological structures.

2. Optimal dualities for varieties generated by finite Heyting chains

We shall begin with a necessarily *very* brief refresher on Heyting algebras and natural dualities. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ is called a *Heyting algebra* if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with smallest element 0 and largest element 1 and for all $a, b \in A$, we have

$$\{x \in A \mid x \wedge a \leqslant b\} = \downarrow(a \to b).$$

For a detailed account of Heyting algebras, we refer the reader to [1]. Let $\mathbf{C}_n$ be an n-element Heyting chain, then for all $a, b \in \mathbf{C}_n$, we have

$$a \to b := \begin{cases} 1 \text{ if } a \leqslant b, \\ b \text{ if } a > b. \end{cases}$$

A finite Heyting algebra $\mathbf{M}$ is subdirectly irreducible if and only if it is of the form $\mathbf{M} = \mathbf{L} \oplus \mathbf{1}$ where $\mathbf{L}$ is a finite distributive lattice. Using this and Jónsson's Lemma it is easy to show that the $\mathcal{C}_n := \mathrm{Var}(\mathbf{C}_n) = \mathbb{ISP}(\mathbf{C}_n)$ and $\mathcal{N} := \mathrm{Var}(\mathbf{N}) = \mathbb{ISP}(\mathbf{N})$.

Let $\mathbf{M} := \langle M; F\rangle$ be an algebra. An n-ary relation r on M is *algebraic over* $\mathbf{M}$ if it is the underlying set of a subalgebra $\mathbf{r}$ of $\mathbf{M}^n$. An n-ary total or partial operation h is algebraic over $\mathbf{M}$ if its graph is algebraic over $\mathbf{M}$, or equivalently, if the domain of h is the underlying set of a subalgebra of $\mathbf{M}^n$ and h is a homomorphism. Given a finite algebra $\mathbf{M} = \langle M; F\rangle$, a topological structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ on the same underlying set is said to be an *alter ego* of $\mathbf{M}$ if each operation in G, each partial operation in H and each relation in R is algebraic over $\mathbf{M}$, and $\mathcal{T}$ is the discrete topology on M. The category $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ consisting of all isomorphic copies of topologically closed substructures of non-zero direct powers of $\underset{\sim}{\mathbf{M}}$ is called the *topological quasi-variety* generated by $\underset{\sim}{\mathbf{M}}$.

Let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M})$ be the quasi-variety generated by a finite algebra $\mathbf{M}$ and let $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ be the topological quasi-variety generated by $\underset{\sim}{\mathbf{M}}$, where $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathfrak{T}\rangle$ is some fixed alter ego of $\mathbf{M}$. For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the set $D(\mathbf{A})$ of all homomorphism from $\mathbf{A}$ into $\mathbf{M}$ forms a closed substructure of $\underset{\sim}{\mathbf{M}}^A$ and is in $\boldsymbol{\mathfrak{X}}$. The topological structure $D(\mathbf{A})$ is called the *dual* of $\mathbf{A}$. Similarly, for each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, the set $E(\mathbf{X})$ of all morphisms from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ forms a subalgebra of $\mathbf{M}^X$ and is in $\boldsymbol{\mathcal{A}}$. The algebra $E(\mathbf{X})$ is the *dual* of $\mathbf{X}$. For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, there are natural evaluation maps $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$. The quadruple $\langle D, E, e, \varepsilon\rangle$ forms a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$. We say that $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$) *yields a duality on* $\boldsymbol{\mathcal{A}}$ if, for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the evaluation map $e_{\mathbf{A}}$ is an isomorphism between $\mathbf{A}$ and its double dual $ED(\mathbf{A})$, in which case $\langle D, E, e, \varepsilon\rangle$ gives a dual equivalence between $\boldsymbol{\mathcal{A}}$ and a full subcategory of $\boldsymbol{\mathfrak{X}}$. (We refer to Clark and Davey [2] for the many missing details.)

An alter ego $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathfrak{T}\rangle$ yields an *optimal duality* on $\boldsymbol{\mathcal{A}}$ if $G \cup H \cup R$ yields a duality on $\boldsymbol{\mathcal{A}}$ and there is no proper subset of $G \cup H \cup R$ that yields a duality on $\boldsymbol{\mathcal{A}}$. The main tool in the study of optimal dualities is the theory of entailment. Let s be a fixed finitary algebraic relation or (partial) operation on $\mathbf{M}$. Given $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, we say that $G \cup H \cup R$ *entails* s on $D(\mathbf{A})$ if every continuous $(G \cup H \cup R)$-preserving map $\varphi \colon D(\mathbf{A}) \to M$ also preserves s. The set $G \cup H \cup R$ *entails* s if it entails s on $D(\mathbf{A})$ for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. If $G \cup H \cup R$ yields a duality on $\boldsymbol{\mathcal{A}}$ and $s \in G \cup H \cup R$, then $(G \cup H \cup R)\backslash\{s\}$ yields a duality on $\boldsymbol{\mathcal{A}}$ if and only if $(G \cup H \cup R)\backslash\{s\}$ entails s. Thus, to check whether $G \cup H \cup R$ entails s, we need to consider all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, which is unmanageable. The Test Algebra Lemma tells us that we need only consider the one algebra $\mathbf{s}$ corresponding to the algebraic relation s.

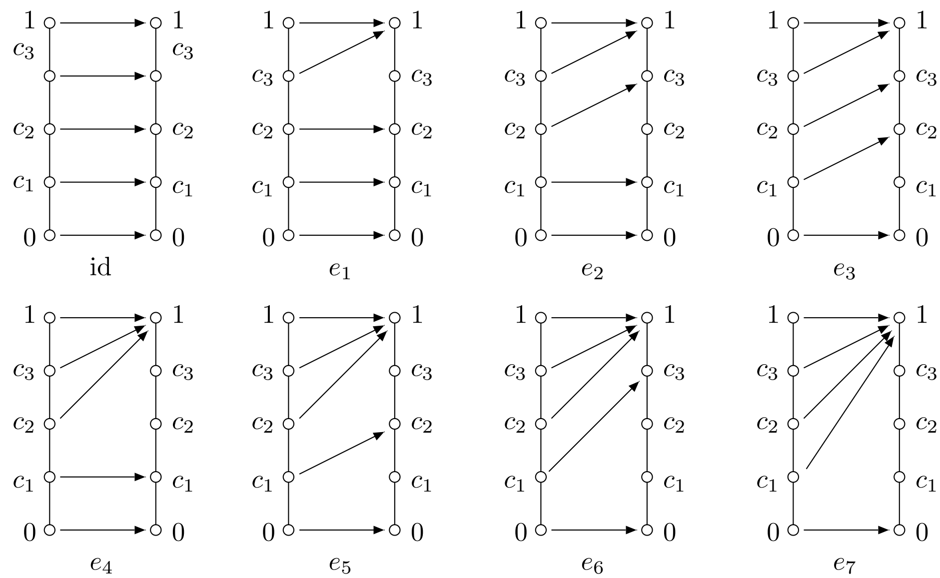
Lemma 2.1 (The Test Algebra Lemma [2] or [12]). *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathfrak{T}\rangle$ *be an alter ego of a finite algebra* $\mathbf{M}$ *and let* s *be a finitary algebraic relation or* (*partial*) *operation on* $\mathbf{M}$. *Then* $G \cup H \cup R$ *entails* s *if and only if* $G \cup H \cup R$ *entails* s *on* $D(\mathbf{s})$.

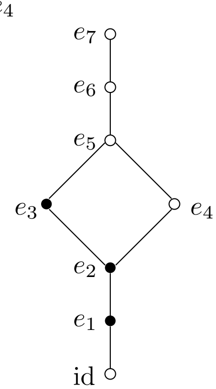
We now return to the n-element Heyting chain $\mathbf{C}_n$. We wish to use the Test Algebra Lemma to obtain an optimal duality on $\boldsymbol{\mathfrak{C}}_n := \mathbb{ISP}(\mathbf{C}_n)$. A crucial observation underpinning our proof is that, ordered pointwise, $\mathrm{End}(\mathbf{C}_n)$ has a particularly simple structure: see Lemma 2.2 and Figure 1. In fact, it is a lattice (see Davey and Priestley [11, Theorems 3.7 and 3.8]) but this will not be used explicitly.

2.1. The n-element Heyting chain. Let $\mathbf{C}_n = \langle C_n; \wedge, \vee, \to, 0, 1\rangle$ be the n-element Heyting chain, where

$$C_n = \{0, c_1, \dots c_{n-2}, 1\} \text{ with } 0 = c_0 < c_1 < c_2 < \cdots < c_{n-2} < c_{n-1} = 1.$$

As discussed earlier, we know that $\mathbf{C}_n$ is endodualisable, that is, the structure $\underset{\sim}{\mathbf{C}}_n = \langle C_n; \mathrm{End}(\mathbf{C}_n), \mathfrak{T}\rangle$ yields a duality on $\boldsymbol{\mathfrak{C}}_n := \mathbb{ISP}(\mathbf{C}_n)$. We shall show that





$\langle \mathrm{End}(\mathbf{C}_5); \leqslant \rangle$

$\circ$	id	e_1	e_2	e_3	e_4	e_5	e_6	e_7
id	id	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_1	e_4	e_5	e_4	e_5	e_7	e_7
e_2	e_2	e_2	e_4	e_6	e_4	e_6	e_7	e_7
e_3	e_3	e_3	e_5	e_6	e_5	e_6	e_7	e_7
e_4	e_4	e_4	e_4	e_7	e_4	e_7	e_7	e_7
e_5	e_5	e_5	e_5	e_7	e_5	e_7	e_7	e_7
e_6	e_6	e_6	e_6	e_7	e_6	e_7	e_7	e_7
e_7	e_7	e_7	e_7	e_7	e_7	e_7	e_7	e_7

Composition table of $\mathrm{End}(\mathbf{C}_5)$

FIGURE 1. The lattice and the composition table of $\mathrm{End}(\mathbf{C}_5)$

there is a natural generating set G_n of $\mathrm{End}(\mathbf{C}_n)$ that yields an optimal duality on $\mathfrak{C}_n$. For each $j \in \{1, 2, \ldots, n-2\}$, define a homomorphism e_j on $\mathbf{C}_n$ by

$$e_j(c_i) = \begin{cases} 1 & \text{if } i \in \{n-2, n-1\}, \\ c_{i+1} & \text{if } n-j-2 < i < n-2, \\ c_i & \text{if } 0 \leqslant i \leqslant n-j-2. \end{cases}$$

Let $G_n = \{e_1, \dots, e_{n-2}\}$. Note that, for all $e \in G_n$, we have $e^{-1}(1) = \{c_{n-2}, 1\}$ and that, in the pointwise order on $\mathrm{End}(\mathbf{C}_n)$, we have $\mathrm{id} < e_1 < \cdots < e_{n-2}$. Define

$$e_{n-1}(c) = \begin{cases} 1 & \text{if } c \in {\uparrow}c_{n-3}, \\ c & \text{otherwise.} \end{cases}$$

(See Figure 1 for $n = 5$: the elements of G_5 are shown with solid circles.)

Lemma 2.2. *In $\langle \mathrm{End}(\mathbf{C}_n); \circ, \leqslant \rangle$ we have*

(i) $e \circ e_1 = e$ *for all* $e \in \mathrm{End}(\mathbf{C}_n)\backslash\{\mathrm{id}\}$,
(ii) $e_2 \circ e_1 = e_2$ *and* $e_1 \circ e_2 = e_{n-1}$,
(iii) $e_i \circ e_j \geqslant e_{n-1}$ *for* $i, j \geqslant 2$,
(iv) $G_n \cup \{\mathrm{id}\} = {\downarrow}e_{n-2}$ *and is a chain of length* $n-2$,
(v) $\mathrm{End}(\mathbf{C}_n) = {\downarrow}e_{n-2} \cup {\uparrow}e_{n-1}$,
(vi) $e_i(c_{n-3}) = c_{n-2}$ *for* $i \in \{2, 3, \dots, n-3\}$.

Proof. This is very easy given that the endomorphisms of finite Heyting chain $\mathbf{C}$ are precisely the functions that map a proper non-empty filter F to 1, are one-to-one and order-preserving on $C\backslash F$ and map 0 to 0. □

Lemma 2.3. *G_n is a generating set for the monoid* $\mathrm{End}(\mathbf{C}_n)$.

Proof. Let $e \in \mathrm{End}(\mathbf{C}_n)\backslash(G_n \cup \{\mathrm{id}\})$. By Lemma 2.2(iv)(v), we have $e_{n-1} \leqslant e$. Then $e^{-1}(1) = {\uparrow}c_i$ for some $i \in \{1, 2, \dots, n-3\}$. Assume that $e(c_j) = c_k$ with $1 \leqslant j \leqslant i-1$ and $j \leqslant k \leqslant n-2$. Since, for each l with $1 \leqslant l \leqslant n-1$, there exists $g \in G_n$ such that $g(c_l) = c_{l+1}$, by composing the elements of G_n we are able to obtain e. □

We now show that the generating set G_n yields an optimal duality on $\mathcal{C}_n$. The proof requires the construction of a number of maps $\varphi_i \colon \mathrm{End}(\mathbf{C}_n) \to C_n$. Note that in each case we define $\varphi_i({\uparrow}e_{n-1}) = 1$ and vary the definition of φ_i on $\mathrm{End}(\mathbf{C}_n)\backslash{\uparrow}e_{n-1} = {\downarrow}e_{n-2}$ according to the value of i.

Theorem 2.4. *Let $\mathbf{C}_n$ be the n-element Heyting chain and let $\mathcal{C}_n := \mathbb{ISP}(\mathbf{C}_n)$. Then $\underset{\sim}{\mathrm{C}}_n = \langle C_n; G_n, \mathcal{T} \rangle$ yields an optimal duality on $\mathcal{C}_n$, where $G_n = \{e_1, \dots, e_{n-2}\}$ and $\mathcal{T}$ is the discrete topology on C_n.*

Proof. For each $e \in G_n$, we show that there is a map $\varphi \colon D(\mathbf{C}_n) \to C_n$ such that the map φ preserves the operations in $G_n\backslash\{e\}$ but does not preserve e, whence e cannot be deleted from G_n without destroying the duality.

Note again that $D(\mathbf{C}_n) = \mathrm{End}(\mathbf{C}_n)$ and define $\varphi_1 : D(\mathbf{C}_n) \to C_n$ by

$$\varphi_1(e) = \begin{cases} c_{n-2} & \text{if } e \in \{\mathrm{id}, e_1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then φ_1 preserves the maps in $G_n\backslash\{e_1\}$, since for each $i \in \{2, \dots, n-2\}$ and all $e \in \mathrm{End}(\mathbf{C}_n)$, we have

$$\varphi_1(e_i(e)) = \varphi_1(e_i \circ e) = 1 \text{ as } e_i \circ e \notin \{\mathrm{id}, e_1\}$$

and

$$e_i(\varphi_1(e)) = 1 \text{ as } \varphi_1(e) \in \{c_{n-2}, 1\}.$$

But φ_1 does not preserve e_1, since

$$\varphi_1(e_1(\mathrm{id})) = \varphi_1(e_1) = c_{n-2} \neq 1 = e_1(c_{n-2}) = e_1(\varphi_1(\mathrm{id})).$$

Now fix $e_i \in G_n\backslash\{e_1\}$. Define $\varphi_i : D(\mathbf{C}_n) \to C_n$ by

$$\varphi_i(e) = \begin{cases} 1 & \text{if } e = e_i \text{ or } e_{n-1} \leqslant e, \\ c_{n-2} & \text{if } e \in G\backslash\{\mathrm{id}, e_1, e_i\}, \\ c_{n-3} & \text{if } e \in \{\mathrm{id}, e_1\}. \end{cases}$$

First we show that φ_i preserves e_1. For all $e \in \mathrm{End}(\mathbf{C}_n)$ we have

$$\begin{aligned} \varphi_i(e_1(e)) &= \begin{cases} \varphi_i(e_1) & \text{if } e \in \{\mathrm{id}, e_1\}, \\ \varphi_i(f) \text{ where } e_{n-1} \leqslant f & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases} \\ &= \begin{cases} c_{n-3} & \text{if } e \in \{\mathrm{id}, e_1\}, \\ 1 & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} e_1(\varphi_i(e)) &= \begin{cases} e_1(c_{n-3}) & \text{if } e \in \{\mathrm{id}, e_1\}, \\ e_1(c) \text{ where } c \in \{c_{n-2}, 1\} & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases} \\ &= \begin{cases} c_{n-3} & \text{if } e \in \{\mathrm{id}, e_1\}, \\ 1 & \text{if } e \notin \{\mathrm{id}, e_1\}. \end{cases} \end{aligned}$$

Hence φ_i preserves e_1.

To prove φ_i preserves $e_k \in G_n\backslash\{e_1, e_i\}$, let $k \in \{2, 3, \dots, n-2\}\backslash\{i\}$. Then

$$\begin{aligned} \varphi_i(e_k(e)) &= \begin{cases} \varphi_i(e_k) & \text{if } e \in \{\mathrm{id}, e_1\}, \\ \varphi(f) \text{ where } e_{n-1} \leqslant f & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases} \\ &= \begin{cases} c_{n-2} & \text{if } e \in \{\mathrm{id}, e_1\}, \\ 1 & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases} \end{aligned}$$

and

$$e_k(\varphi_i(e)) = \begin{cases} e_k(c_{n-3}) & \text{if } e \in \{\mathrm{id}, e_1\}, \\ e_k(c) \text{ where } c \in \{c_{n-2}, 1\} & \text{if } e \notin \{\mathrm{id}, e_1\}, \end{cases}$$
$$= \begin{cases} c_{n-2} & \text{if } e \in \{\mathrm{id}, e_1\}, \\ 1 & \text{if } e \notin \{\mathrm{id}, e_1\}. \end{cases}$$

Hence φ_i preserves e_k.

Finally, we have

$$\varphi_i(e_i(\mathrm{id})) = \varphi_i(e_i) = 1 \neq c_{n-2} = e_i(c_{n-3}) = e_i(\varphi_i(\mathrm{id}))$$

Hence φ_i does not preserve e_i. □

3. Standard topological unary algebras and injectivity

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a finite discrete topological structure and let $\mathfrak{X}$ consist of all isomorphic copies of closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$, in symbols $\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$; then $\mathfrak{X}$ is called the *topological quasi-variety* generated by $\underset{\sim}{\mathbf{M}}$. The significance of allowing only non-zero powers is that $\mathfrak{X}$ will not necessarily contain a one-element structure. The importance of such topological quasi-varieties arises from the fact that they are precisely the dual categories of algebraic quasi-varieties under natural dualities: see Clark and Davey [2]. Every structure in $\mathfrak{X}$ is a Boolean topological structure of type $\langle G, H, R\rangle$ that satisfies every universal Horn sentence true in $\underset{\sim}{\mathbf{M}}$. We now make these notions precise.

A structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ is said to be a *Boolean* (*topological*) *structure* of type $\langle G, H, R\rangle$ if

(i) $\langle X; \mathcal{T}^{\mathbf{X}}\rangle$ is a Boolean space,
(ii) if $h \in G \cup H$ is n-ary, then $\mathrm{dom}(h^{\mathbf{X}})$ is a closed subset of $\langle X; \mathcal{T}^{\mathbf{X}}\rangle^n$ and the map $h^{\mathbf{X}} : \mathrm{dom}(h^{\mathbf{X}}) \to X$ is continuous, and
(iii) if $r \in R$ is n-ary, then $r^{\mathbf{X}}$ is a closed subset of $\langle X; \mathcal{T}\rangle^n$.

To simplify our notation we will omit the superscripts on $g^{\mathbf{X}}$, $h^{\mathbf{X}}$, $r^{\mathbf{X}}$ and $\mathcal{T}^{\mathbf{X}}$ except when to do so would cause ambiguity. In the first-order language of $\underset{\sim}{\mathbf{M}}$, a *universal Horn sentence* is a universally quantified expression of one of the forms

$$\varphi \quad \text{or} \quad \bigvee_{i=1}^{n} \neg\psi_i \quad \text{or} \quad \Big(\bigwedge_{i=1}^{n} \psi_i\Big) \Rightarrow \varphi \tag{$*$}$$

where φ and each ψ_i are atomic formulas. Let Σ be a set of universal Horn sentences in the first-order language of $\underset{\sim}{\mathbf{M}}$. The class of all Boolean structures of type

$\langle G, H, R\rangle$ that satisfy each universal Horn sentence in Σ is denoted by $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$: thus

$$\mathrm{Mod}_{\mathcal{T}}(\Sigma) = \{\mathbf{X} \mid \mathbf{X} \text{ is a Boolean structure of type } \langle G, H, R\rangle \text{ and } \mathbf{X} \models \Sigma\}.$$

The set of all universal Horn sentences that hold in $\underset{\sim}{\mathbf{M}}$ forms the *universal Horn theory* of $\underset{\sim}{\mathbf{M}}$ and is denoted by $\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$. In this paper, we shall be particularly interested in the case in which $R = \varnothing$.

Theorem 3.1 (Preservation Theorem [2]). *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a finite, discrete, topological structure and let $\mathbf{X} \in \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$. Then $\mathbf{X}$ is a Boolean topological structure that satisfies every universal Horn sentence satisfied by $\underset{\sim}{\mathbf{M}}$. Thus, $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}}) \subseteq \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$.*

If we have $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, then we have the possibility of a 'simple' description of the structures in $\mathcal{X} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ since the equality of these two classes says that the topological and model-theoretic aspects of structures in $\mathcal{X}$ are essentially independent and do not interact in a bad way. Following [3], if $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, then we say that $\mathcal{X} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ is a *standard topological quasi-variety*, or simply that $\underset{\sim}{\mathbf{M}}$ is *standard*. If $\mathcal{X}$ is axiomatized by some subset Σ of $\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$, that is if $\mathcal{X} = \mathrm{Mod}_{\mathcal{T}}(\Sigma)$, then $\mathcal{X}$ is certainly standard, and the axioms in Σ provide a description of its members. Many finite, discrete, topological structures are standard. For example, a finite topological cyclic group of order n is standard since the topological quasi-variety it generates is precisely the class of all Boolean topological abelian groups satisfying $x^n \approx 1$ [15]. Nevertheless, there are important topological quasi-varieties that are non-standard. The most well known is probably the class $\mathcal{P}$ of Priestley spaces. We have $\mathcal{P} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{2}})$, where $\underset{\sim}{\mathbf{2}} = \langle\{0, 1\}; \leqslant, \mathcal{T}\rangle$ is the two-element chain with the discrete topology. We have $\mathcal{P} \subsetneqq \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$ since $\mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$ is the class of all Boolean ordered spaces in which $\leqslant$ is topologically closed and Stralka [21] has given an example of such a space that is not a Priestley space.

The following result provides necessary and sufficient conditions for a structure to be in $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$.

Theorem 3.2 (Separation Theorem [2]). *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a finite, discrete, topological structure and let $\mathbf{X} = \langle X; G, H, R, \mathcal{T}\rangle$ be a compact topological structure of the same type as $\underset{\sim}{\mathbf{M}}$. Then $\mathbf{X} \in \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ if and only if there is at least one morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ and the following conditions hold:*

(i) *for each $x, y \in X$ with $x \neq y$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(x) \neq \alpha(y)$,*

(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus \mathrm{dom}(h^{\mathbf{X}})$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin \mathrm{dom}(h^{\underset{\sim}{\mathbf{M}}})$,*

(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \ldots, x_n) \in X^n \backslash r^{\mathbf{X}}$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin r^{\underset{\sim}{\mathbf{M}}}$.*

Based on this result, the "standard" method for proving that $\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is standard is to

(**I**) write down a set Σ of universal Horn sentences true in $\underset{\sim}{\mathbf{M}}$,

(**II**) prove that if $\mathbf{X}$ is a Boolean structure of type $\langle G, H, R\rangle$ such that $\mathbf{X} \models \Sigma$, then conditions (i), (ii) and (iii) of Theorem 3.2 hold.

While this method shows that Σ axiomatizes $\mathfrak{X}$, it has the distinct disadvantage that it requires us to define continuous morphisms on infinite Boolean topological structures. If we can prove *in advance* that $\mathfrak{X}$ is standard, then we may apply a result of [3] which allows us to establish that the set Σ axiomatizes $\mathfrak{X}$ without reference to topology.

Theorem 3.3. *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathfrak{T}\rangle$ be a finite, discrete, topological structure and assume that the topological quasi-variety $\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is standard. Then $\Sigma \subseteq \mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$ axiomatizes $\mathfrak{X}$ provided every model of Σ is locally finite and each finite model of Σ is in $\mathfrak{X}$.*

The following two results show that, in the case when $H = R = \varnothing$, the standardness or otherwise of the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is closely connected to the extent that $\underset{\sim}{\mathbf{M}}$ is injective in $\mathrm{Mod}_{\mathfrak{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$. In both proofs we use the fact that, for a finite algebra $\mathbf{A}$, we have $\mathrm{Mod}(\mathrm{Th}_{\mathrm{uH}}(\mathbf{A})) = \mathbb{ISP}^+(\mathbf{A})$ (see [18]). If $\mathbf{X} = \langle X; G, H, R, \mathfrak{T}\rangle$ is a topological structure, then we denote by $\mathbf{X}_{\backslash\mathfrak{T}} := \langle X; G, H, R\rangle$ the underlying structure of the same type as $\mathbf{X}$ without the topology.

Theorem 3.4. *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathfrak{T}\rangle$ be a finite, discrete topological algebra and assume that $\underset{\sim}{\mathbf{M}}$ is injective in $\mathrm{Mod}_{\mathfrak{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$. Then the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is standard.*

Proof. Let $\mathbf{X} \in \mathrm{Mod}_{\mathfrak{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, let $x, y \in X$ with $x \neq y$ and let $\mathbf{Y} := \mathrm{sg}_{\mathbf{X}}(\{x, y\})$. Since $\mathbf{X}_{\backslash\mathfrak{T}} \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}_{\backslash\mathfrak{T}})) = \mathbb{ISP}^+(\underset{\sim}{\mathbf{M}}_{\backslash\mathfrak{T}})$ and $\mathbb{ISP}^+(\underset{\sim}{\mathbf{M}}_{\backslash\mathfrak{T}})$ is locally finite, the structure $\mathbf{Y}$ is finite and therefore discrete. Since $\mathbf{Y}_{\backslash\mathfrak{T}} \in \mathbb{ISP}^+(\underset{\sim}{\mathbf{M}}_{\backslash\mathfrak{T}})$, there is a morphism $\beta\colon \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ with $\beta(x) \neq \beta(y)$. Finally, as $\underset{\sim}{\mathbf{M}}$ is injective in $\mathrm{Mod}_{\mathfrak{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ which extends β and hence satisfies $\alpha(x) \neq \alpha(y)$. Thus $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. □

This result has the disadvantage that it requires us to prove the injectivity of $\underset{\sim}{\mathbf{M}}$ in a class that includes infinite topological algebras. The following refinement shows that if we restrict our attention to unary algebras, then there is a purely algebraic and finitary sufficient condition for $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ to be standard.

For a class $\mathfrak{C}$ of structures, we denote the class of all finite members of $\mathfrak{C}$ by $\mathfrak{C}_{\mathrm{fin}}$.

Theorem 3.5. *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ *be a finite, discrete topological unary algebra. If* $\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}}$ *is injective in* $\mathrm{Var}(\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}})_{\mathrm{fin}}$*, then the topological quasi-variety* $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ *is standard.*

Proof. Without loss of generality we may assume that G is finite and forms a monoid. Let $\mathbf{X} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$. Suppose $x, y \in X$ with $x \neq y$. Then, as above, $\mathbf{Y} := \mathrm{sg}_{\mathbf{X}}(\{x, y\})$ is finite. Since $\mathbf{X}$ is totally disconnected, there is a family of pairwise disjoint clopen sets $\{U_z \mid z \in Y\}$ such that $z \in U_z$ and $U_z \cap (Y \setminus \{z\}) = \varnothing$ for all $z \in Y$. Define an equivalence relation θ on X by

$$(u, v) \in \theta \iff (\forall z \in Y)\, (u, v) \in U_z^2 \cup (X \setminus U_z)^2.$$

Then X/θ is finite, all the blocks of θ are clopen in X and $(x, y) \notin \theta$.

Now define an equivalence relation $\widehat{\theta}$ on X by

$$(u, v) \in \widehat{\theta} \iff (\forall g \in G)\, (g(u), g(v)) \in \theta.$$

We claim that

(a) $\widehat{\theta}$ is a congruence on X,
(b) all the blocks of $\widehat{\theta}$ are clopen,
(c) $(x, y) \notin \widehat{\theta}$,
(d) $X/\widehat{\theta}$ is finite.

Condition (a) is true, since G is a monoid. Condition (b) is true, since G is finite and since we have

$$\begin{aligned}
[x]\widehat{\theta} &= \{z \in X \mid (z, x) \in \widehat{\theta}\} \\
&= \{z \in X \mid (\forall g \in G)\, (g(z), g(x)) \in \theta\} \\
&= \{z \in X \mid (\forall g \in G)\, g(z) \in [g(x)]\theta\} \\
&= \bigcap_{g \in G} g^{-1}([g(x)]\theta).
\end{aligned}$$

Condition (c) is obvious as $\mathrm{id} \in G$ while (d) follows from (b) since $\langle X; \mathcal{T}\rangle$ is compact.

We have $\mathbf{Y}_{\setminus\mathcal{T}} \cong (\mathbf{Y}/\widehat{\theta})_{\setminus\mathcal{T}}$ and $(\mathbf{Y}/\widehat{\theta})_{\setminus\mathcal{T}}$, $(\mathbf{X}/\widehat{\theta})_{\setminus\mathcal{T}} \in \mathrm{Var}(\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}})_{\mathrm{fin}}$ with $(\mathbf{Y}/\widehat{\theta})_{\setminus\mathcal{T}}$ a subalgebra of $(\mathbf{X}/\widehat{\theta})_{\setminus\mathcal{T}}$. Since $x/\widehat{\theta} \neq y/\widehat{\theta}$, there is a homomorphism $\beta\colon (\mathbf{Y}/\widehat{\theta})_{\setminus\mathcal{T}} \to \underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}}$ with $\beta(x/\widehat{\theta}) \neq \beta(y/\widehat{\theta})$. Since $\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}}$ is injective in $\mathrm{Var}(\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}})_{\mathrm{fin}}$, the homomorphism β extends to a homomorphism $\beta'\colon (\mathbf{X}/\widehat{\theta})_{\setminus\mathcal{T}} \to \underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}}$. Let $\alpha\colon \mathbf{X} \to \mathbf{X}/\widehat{\theta}$ denote the natural quotient morphism. We now have a morphism $\beta' \circ \alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\beta' \circ \alpha(x) \neq \beta' \circ \alpha(y)$. Thus, $\mathbf{X} \in \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$. □

4. The optimal duals of $\mathrm{Var}(\mathbf{C}_4)$ and $\mathrm{Var}(\mathbf{N})$ are standard.

In this section we apply Theorem 3.5 to the optimal dualising structures of the four-element Heyting chain $\mathbf{C}_4$ and the Heyting algebra $\mathbf{N}$.

4.1. The four-element Heyting chain. Let $\mathbf{C}_4$ be the four-element Heyting chain based on $\{0, a, b, 1\}$ with $0 < a < b < 1$. By Theorem 2.4, the structure $\underset{\sim}{\mathbf{C}}_4 = \langle\{0, a, b, 1\}; e_1, e_2, \mathcal{T}\rangle$ yields an optimal duality on $\mathrm{Var}(\mathbf{C}_4)$, where e_1 is the endomorphism of $\mathbf{C}_4$ that moves b to 1 and fixes 0, a and 1, and e_2 is the endomorphism of $\mathbf{C}_4$ that moves a to b, b to 1 and fixes 0, and 1 (see Figure 2).

Theorem 4.1. *Let* $\underset{\sim}{\mathbf{C}}_4 = \langle\{0, a, b, 1\}; e_1, e_2, \mathcal{T}\rangle$ *be the optimal dualising structure for the four-element Heyting chain. Then* $\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}} = \langle\{0, a, b, 1\}; e_1, e_2\rangle$ *is injective in* $\mathrm{Var}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})_{\mathrm{fin}}$ *and hence the dual category* $\mathfrak{X}_4 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_4)$ *is standard.*

Proof. In order to apply Theorem 3.5, we must show that the underlying unary algebra $\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ is injective in $\mathrm{Var}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})_{\mathrm{fin}}$. We shall prove something slightly stronger, namely, that $\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ is injective in the larger class $\boldsymbol{\mathcal{V}}_{\mathrm{fin}}$ where $\boldsymbol{\mathcal{V}}$ is the variety of all unary algebras $\langle X; e_1, e_2\rangle$ satisfying

(C_1) $e_1e_1(x) = e_1(x)$,
(C_2) $e_2e_2e_2(x) = e_2e_2(x)$,
(C_3) $e_2e_1(x) = e_2(x)$.

Let $\mathbf{X} \in \boldsymbol{\mathcal{V}}_{\mathrm{fin}}$, let $\mathbf{Y}$ be a subalgebra of $\mathbf{X}$ and let $\beta\colon \mathbf{Y} \to \underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ be a homomorphism. Define

$$O := \beta^{-1}(0), \quad I := \beta^{-1}(1), \quad A := \beta^{-1}(a), \quad B := \beta^{-1}(b).$$

(See Figure 2.) Then $Y = O \mathbin{\dot\cup} I \mathbin{\dot\cup} A \mathbin{\dot\cup} B$. Define

$$A' := e_2^{-1}(B), \quad I' := (e_1^{-1}(I) \cup e_2^{-1}(I))\backslash(B \cup A') \text{ and } O' := X\backslash(B \cup A' \cup I').$$

Then A' is disjoint from B and I as I and B are disjoint.

We claim that

(a) $e_1(A') \subseteq A'$ and $e_2(A') \subseteq B$,
(b) $e_i(B) \subseteq I'$ for all $i \in \{1, 2\}$,
(c) $e_i(I') \subseteq I'$ for all $i \in \{1, 2\}$,
(d) $e_i(O') \subseteq O'$ for all $i \in \{1, 2\}$.

Using (C_3), we have $e_1(A') \subseteq A'$, and $e_2(A') \subseteq B$ is obvious. Thus (a) holds. Condition (b) follows from the fact that for all $i \in \{1, 2\}$ we have

$$x \in B \Longrightarrow \beta(x) = b \Longrightarrow e_i(\beta(x)) = 1 \Longrightarrow \beta(e_i(x)) = 1 \Longrightarrow e_i(x) \in I \subseteq I'.$$

To prove (c), let $x \in I'$. Then for any $i \in \{1, 2\}$, we have $e_i(x) \in I$ which implies $e_ie_i(x) \in I$ and $e_i(x) \notin B \cup A'$. Hence $e_i(x) \in I'$. Thus (c) holds. To prove (d), it is enough to show that $e_i(x) \in B \cup A' \cup I'$ implies that $x \in B \cup A' \cup I'$, for all $i \in \{1, 2\}$.

If $e_1(x) \in B$, then

$$\begin{aligned}\beta(e_1(x)) = b &\Longrightarrow e_1(\beta(x)) = b\\ &\Longrightarrow e_1e_1(\beta(x)) = 1\end{aligned}$$

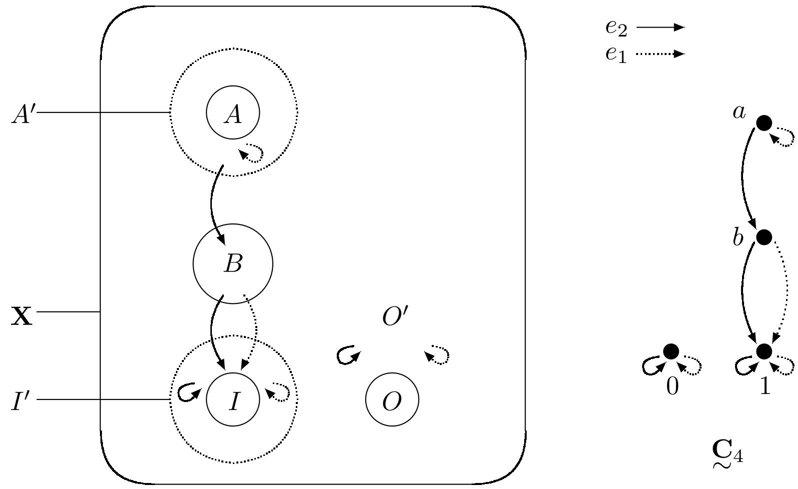


FIGURE 2

$$\implies e_1(\beta(x)) = 1 \text{ by } (\mathrm{C}_1)$$
$$\implies \beta(e_1(x)) = 1$$

which implies that $e_1(x) \in I$, contradicting the fact that B and I are disjoint. Moreover, by the definition, $e_2(x) \in B$ if and only if $x \in A'$.

If $e_1(x) \in A'$, then, by the definition, $e_2e_1(x) \in B$ if and only if $e_2(x) \in B$, by (C_3) and hence $x \in A'$. If $e_2(x) \in A'$, then $e_2e_2(x) \in B$. This implies that $e_2e_2e_2(x) \in I'$, by (b), and hence, by (C_2), we have $e_2e_2(x) \in I'$, which contradicts the fact that B and I' are disjoint.

If $e_1(x) \in I'$, then either $e_1e_1(x) \in I$ or $e_2e_1(x) \in I$. This implies that either $e_1(x) \in I$, by (C_1), or $e_2(x) \in I$, by (C_3). Thus either $\beta(e_1(x)) = 1$ or $\beta(e_2(x)) = 1$, that is, either $e_1(\beta(x)) = 1$ or $e_2(\beta(x)) = 1$. This implies that $\beta(x) \in \{b, 1\}$ and hence $x \in I \cup B$. Finally, if $e_2(x) \in I'$, then, by the above process, we have $\beta(e_2(x)) \in \{b, 1\}$. Hence $\beta(x) \in \{a, b, 1\}$. Consequently, $x \in I \cup A \cup B$.

Hence, we conclude that $e_i(x) \in B \cup A' \cup I'$ implies that $x \in B \cup A' \cup I'$, for all $i \in \{1, 2\}$.

Now define $\alpha\colon X \to C_4$ by $O' \mapsto 0$, $A' \mapsto a$, $B \mapsto b$ and $I' \mapsto 1$. Then clearly α is a homomorphism that extends β. Hence $\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ is injective in $\boldsymbol{\mathcal{V}}_{\mathrm{fin}}$. It follows that $\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ is injective in $\mathrm{Var}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})_{\mathrm{fin}}$ and so, by Theorem 3.5, $\boldsymbol{\mathcal{X}}_4$ is standard. $\square$

In Clark, Davey, Freese and Jackson [4] it is proved that the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by a finite topological unary algebra $\underset{\sim}{\mathbf{M}}$ is standard provided the (algebraic) quasi-variety $\mathbb{ISP}(\underset{\sim}{\mathbf{M}}_{\setminus\mathcal{T}})$ is a variety. We cannot apply this result here as $\mathbb{ISP}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})$ is not closed under homomorphic images. In Figure 3,

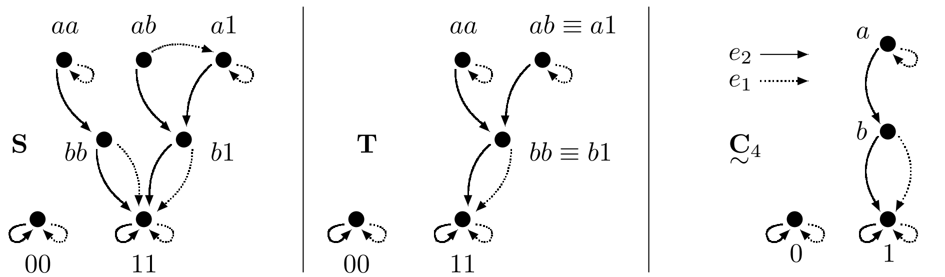


FIGURE 3. The quasi-variety is not a variety

$\mathbf{S}$ is a substructure of $\underset{\sim}{\mathbf{C}}_4^2$ and $\mathbf{T}$ is a homomorphic image of $\mathbf{S}$. Observe that in $\mathbf{T}$ we have $e_2(aa) = e_2(ab)$ but $e_1(aa) \neq e_1(ab)$. This shows that $\mathbf{T}$ does not satisfy quasi-equation $e_2(x) = e_2(y) \Rightarrow e_1(x) = e_1(y)$ which holds in $\underset{\sim}{\mathbf{C}}_4$. Hence $\mathbf{T}$ belongs to $\mathrm{Var}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})$ but not to $\mathbb{ISP}(\underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}})$.

We do not know if $\mathfrak{X}_n$ is standard for $n \geqslant 5$, but we do know that we cannot apply Theorem 3.5 in order to prove it. Let $\mathbf{C}_5$ be the five-element Heyting chain based on $\{0, a, b, c, 1\}$ with $0 < a < b < c < 1$. By Theorem 2.4, $\underset{\sim}{\mathbf{C}}_5 = \langle\{0, a, b, c, 1\}; e_1, e_2, e_3, \mathcal{T}\rangle$ yields an optimal duality on $\mathrm{Var}(\mathbf{C}_5)$, where e_1, e_2 and e_3 are as given in Figures 1 and 4. On the left of Figure 4 is a substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{C}}^2_{5\setminus\mathcal{T}}$. The map β is defined on the substructure $\mathbf{Y}$ of $\mathbf{X}$, where $Y = X \setminus \{ba\}$. It is easily seen that β is a homomorphism into $\underset{\sim}{\mathbf{C}}_{5\setminus\mathcal{T}}$ and has no extension to $\mathbf{X}$. Consequently, $\underset{\sim}{\mathbf{C}}_{5\setminus\mathcal{T}}$ is not injective in $\mathrm{Var}(\underset{\sim}{\mathbf{C}}_{5\setminus\mathcal{T}})$.

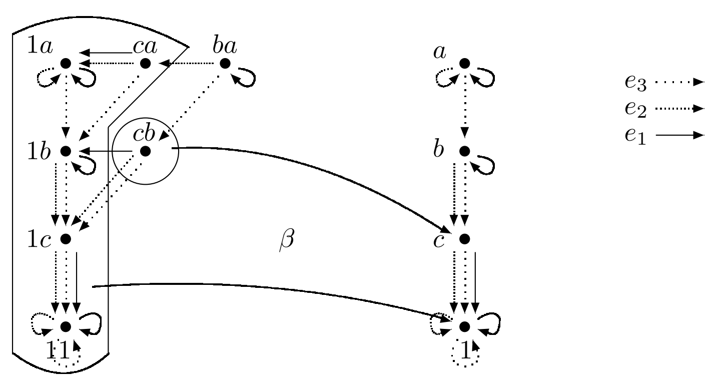
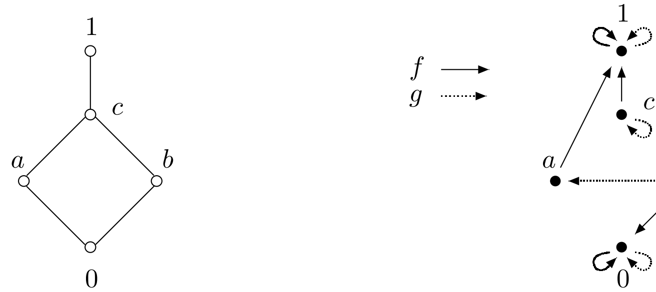


FIGURE 4. The dual of $\mathbf{C}_5$ is not injective

4.2. The Heyting algebra N. Consider the Heyting algebra $\mathbf{N} = \mathbf{2^2} \oplus \mathbf{1}$ as given in Figure 5. Saramago [20] proved that the structure $\underset{\sim}{\mathbf{N}} = \langle \{0, a, b, c, 1\}; f, g, \mathcal{T}\rangle$, given in Figure 5, yields an optimal duality on $\mathrm{Var}(\mathbf{N})$.



The algebra $\mathbf{N} = \mathbf{2^2} \oplus \mathbf{1}$ The structure $\underset{\sim}{\mathbf{N}}$

FIGURE 5

Theorem 4.2. *Let* $\underset{\sim}{\mathbf{N}} = \langle \{0, a, b, c, 1\}; f, g, \mathcal{T}\rangle$ *be the optimal dualising structure for the Heyting algebra* $\mathbf{N} = \mathbf{2^2} \oplus \mathbf{1}$*. Then the algebra* $\underset{\sim}{\mathbf{N}}_{\backslash\mathcal{T}} = \langle \{0, a, b, c, 1\}; f, g\rangle$ *is injective in* $\mathrm{Var}(\underset{\sim}{\mathbf{N}}_{\backslash\mathcal{T}})_{\mathrm{fin}}$ *and hence the dual category* $\mathfrak{Z} := \mathbb{I}\mathbb{S}_c\mathbb{P}^{+}(\underset{\sim}{\mathbf{N}})$ *is standard.*

Proof. Once again, we shall prove something stronger, namely, that $\underset{\sim}{\mathbf{N}}_{\backslash\mathcal{T}}$ is injective in the class $\mathcal{W}_{\mathrm{fin}}$, where $\mathcal{W}$ is the variety of all unary algebras $\langle X; f, g\rangle$ satisfying

(N_1) $ff(x) = f(x)$,

(N_2) $gg(x) = x$,

(N_3) $gf(x) = f(x)$.

Let $\mathbf{X} \in \mathcal{W}_{\mathrm{fin}}$, let $\mathbf{Y}$ be a subalgebra of $\mathbf{X}$ and let $\beta\colon \mathbf{Y} \to \underset{\sim}{\mathbf{N}}_{\backslash\mathcal{T}}$ be a homomorphism. Define

$$O := \beta^{-1}(0), \quad A := \beta^{-1}(a), \quad B := \beta^{-1}(b), \quad C := \beta^{-1}(c), \quad I := \beta^{-1}(1).$$

Then $Y = O \,\dot\cup\, A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, I$. Define

$$\begin{aligned} O' &:= \{x \in X \mid f(x) \in O,\ fg(x) \in O\},\\ B' &:= \{x \in X \mid f(x) \in O,\ fg(x) \notin O\},\\ A' &:= \{x \in X \mid g(x) \in B'\},\\ C' &:= C,\\ I' &:= X \backslash (O' \cup A' \cup B' \cup C'). \end{aligned}$$

Then clearly, O', A', B', C' and I' are pairwise disjoint and contain O, A, B, C and I respectively. We show that

(a) $f(O') \subseteq O'$ and $g(O') \subseteq O'$,

(b) $f(B') \subseteq O'$ and $g(B') \subseteq A'$,
(c) $f(C') \subseteq I'$ and $g(C') \subseteq C'$,
(d) $f(A') \subseteq I'$ and $g(A') \subseteq B'$,
(e) $f(I') \subseteq I'$ and $g(I') \subseteq I'$.

(a) Let $x \in O'$. Then $f(x) \in O$ and $fg(x) \in O$. Now $f(x) \in O$ implies $ff(x) \in O$, by (N_1), and $gf(x) \in O$, by (N_3). This implies $ff(x) \in O$ and $fgf(x) \in O$. Hence $f(x) \in O'$. Also $f(x) \in O$ implies $fgg(x) \in O$, by (N_2). Hence $g(x) \in O'$.

(b) Let $x \in B'$. Then $f(x) \in O \subseteq O'$. Also, by (N_2), we have $gg(x) \in B'$. This implies $g(x) \in A'$, by definition.

(c) Since $I \subseteq I'$ and $C = C'$, this is trivial.

(d) First we show that for all $x \in X$, we have $f(x) \notin A' \cup B' \cup C'$. For any $x \in X$, if $f(x) \in A'$, then $gf(x) = f(x) \in B'$, a contradiction since A' and B' are disjoint. If $f(x) \in B'$, then $ff(x) = f(x) \in O'$, a contradiction since O' and B' are disjoint. If $f(x) \in C' = C$, then $f(x) = ff(x) \in I$, a contradiction since C and I are disjoint.

Now let $x \in A'$. Then $g(x) \in B'$. If $f(x) \in O'$, then $ff(x) = f(x) = fgg(x) \in O$. But $g(x) \in B'$ implies $fgg(x) \notin O$, which is a contradiction. Hence $f(x) \in I'$.

(e) If $f(x) \notin I'$, then, by (d), we have $f(x) \in O'$. Hence $f(x) = ff(x) \in O$. This implies either $x \in O'$ or $x \in B'$ and so $x \notin I'$.

Now $g(x) \in O'$ implies $fg(x) \in O$ and $fgg(x) \in O$. Hence, by (N_2), we have $f(x) \in O$ and $fg(x) \in O$. Therefore, $x \in O'$. If $g(x) \in B'$, then, by definition, $x \in A'$. If $g(x) \in A'$, then $gg(x) = x \in B'$. If $g(x) \in C' = C$, then $x = gg(x) \in C'$, as C is closed under g. Hence $g(x) \notin I'$ implies $x \notin I'$.

Now define $\alpha\colon X \to N$ by $O' \mapsto 0$, $A' \mapsto a$, $B' \mapsto b$, $C' \mapsto c$ and $I' \mapsto 1$. Then clearly α is a homomorphism that extends β. Hence $\underset{\sim}{\mathbf{N}}_{\setminus\mathcal{T}}$ is injective in $\boldsymbol{\mathcal{W}}_{\mathrm{fin}}$. Therefore $\underset{\sim}{\mathbf{N}}_{\setminus\mathcal{T}}$ is injective in $\mathrm{Var}(\underset{\sim}{\mathbf{N}}_{\setminus\mathcal{T}})_{\mathrm{fin}}$ and so, by Theorem 3.5, $\boldsymbol{\mathfrak{Z}}$ is standard. $\square$

5. Axiomatization of the optimal duals of $\mathrm{Var}(\mathbf{C}_4)$ and $\mathrm{Var}(\mathbf{N})$

We now wish to give axiomatic descriptions of the dual categories $\boldsymbol{\mathfrak{X}}_4$ and $\boldsymbol{\mathfrak{Z}}$. We know in advance, by Theorem 4.1, that $\boldsymbol{\mathfrak{X}}_4$ is standard. Hence, by Theorems 3.3 and 3.2, to show that a set $\Sigma \subseteq \mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{C}}_4)$ axiomatizes $\boldsymbol{\mathfrak{X}}_4$ it suffices to

(Ax_1) show that every model of Σ is locally finite, and
(Ax_2) show that if $\mathbf{X}$ is a finite model of Σ, then for all $x, y \in X$ with $x \neq y$ there exists a homomorphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{C}}_{4\setminus\mathcal{T}}$ with $\alpha(x) \neq \alpha(y)$.

Since, by Theorem 4.2, $\boldsymbol{\mathfrak{Z}}$ is also standard, a similar statement holds for $\boldsymbol{\mathfrak{Z}}$.

5.1. Axiomatization of $\mathcal{X}_4$. Let $\Sigma_{\underset{\sim}{\mathbf{C}}_4}$ consist of the following six equations and quasi-equations:

(C_1) $e_1e_1(x) = e_1(x)$,
(C_2) $e_2e_2e_2(x) = e_2e_2(x)$,
(C_3) $e_2e_1(x) = e_2(x)$,
(C_4) $e_2e_2(x) = e_1e_2(x)$,
(C_5) $e_2(x) = e_2(y) \Longrightarrow e_1(x) = e_1(y)$,
(C_6) $e_2(x) = x \Longrightarrow e_1(x) = x$.

The next lemma will be used to construct some of the morphisms required in (Ax_2).

Lemma 5.1. *Let* $\mathbf{X} = \langle X; e_1, e_2\rangle$ *be a unary algebra satisfying equations* (C_1)–(C_4) *above. Let* U *be a subset of* X *and define*

$$V := U \cap e_1^{-1}(U) \cap e_2^{-1}(U) \cap e_2^{-1}e_1^{-1}(U) \quad \textit{and} \quad W := e_2^{-1}(V)\backslash e_1^{-1}(V).$$

Then the map $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{C}}_{4\backslash\mathcal{T}}$ *given by*

$$\alpha(x) = \begin{cases} 1 & \textit{if} \;\; x \in U_1 := V \cup W, \\ b & \textit{if} \;\; x \in U_b := e_1^{-1}(V)\backslash V, \\ a & \textit{if} \;\; u \in U_a := e_2^{-1}e_1^{-1}(V)\backslash e_2^{-1}(V), \\ 0 & \textit{if} \;\; u \in U_0 := X\backslash e_2^{-1}e_1^{-1}(V), \end{cases}$$

is a well-defined homomorphism.

Proof. The fact that $\{U_0, U_a, U_b, U_1\}$ is a partition of X and that α is a homomorphism will be almost immediate once we establish the following facts (see Figure 6):

(a) $e_i(V) \subseteq V$,
(b) $e_1^{-1}(V) \subseteq e_2^{-1}(V) \subseteq e_2^{-1}e_1^{-1}(V)$,
(c) $e_i(U_b) \subseteq V$, for all $i \in \{1, 2\}$,
(d) $e_1(W) \subseteq W$ and $e_2(W) \subseteq V$,
(e) $e_1(U_a) \subseteq U_a$ and $e_2(U_a) \subseteq U_b$,
(f) $e_i(U_0) \subseteq U_0$, for all $i \in \{1, 2\}$.

(a) Equations (C_1) to (C_4) imply that the monoid generated by e_1 and e_2 consists of the four maps id, e_1, e_2 and e_1e_2. It follows easily that V is closed under e_1 and e_2. (More generally, if G is a monoid on maps on X and $U \subseteq X$, then $\bigcap\{\, e^{-1}(U) \mid e \in G\,\}$ is closed under every map in G.)

(b) This is clear since, using equation (C_3) and claim (a), we have

$$e_1(x) \in V \implies e_2(x) = e_2e_1(x) \in V \implies e_1e_2(x) \in V.$$

(c) Trivially $e_1(U_b) \subseteq V$, and $e_2(U_b) \subseteq V$ follows from the fact that, by (b), we have $U_b \subseteq e_1^{-1}(V) \subseteq e_2^{-1}(V)$.

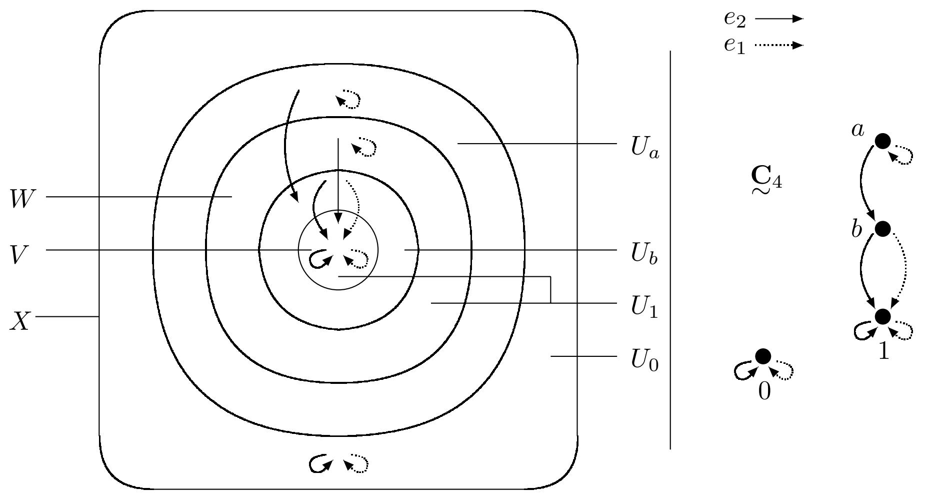


FIGURE 6

(d) Clearly $e_2(W) \subseteq V$. Now let $x \in W$; then, by (C_3), $e_2e_1(x) = e_2(x) \in V$ and, by (C_1), $e_1e_1(x) = e_1(x) \notin V$. Hence $e_1(x) \in W := e_2^{-1}(V) \backslash e_1^{-1}(V)$.

(e) Let $x \in U_a$. Then $e_1e_2(x) \in V$ and $e_2(x) \notin V$. This implies $e_2(x) \in U_b$. Also $e_1e_2(x) \in V$ and $e_2(x) \notin V$ implies $e_1e_2e_1(x) \in V$ and $e_2e_1(x) \notin V$, by (C_3). Hence $e_1(x) \in U_a$.

(f) Let $x \in U_0$. Then $e_1e_2(x) \notin V$. This implies $e_1e_2e_1(x) \notin V$, by (C_3). That is, $e_1(x) \in U_0$. Also $e_1e_2(x) \notin V$ implies $e_2e_2(x) \notin V$, by (C_4). Hence, by (C_2), $e_2e_2e_2(x) \notin V$. Thus, by (C_4), $e_1e_2e_2(x) \notin V$. Hence $e_2(x) \in U_0$. □

Theorem 5.2. *The dual category* $\mathfrak{X}_4 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_4)$ *of* $\mathfrak{C}_4$ *is exactly the category of Boolean topological unary algebras* $\mathbf{X} = \langle X; e_1, e_2, \mathcal{T}\rangle$ *satisfying* $\Sigma_{\underset{\sim}{\mathbf{C}}_4}$.

Proof. We shall establish (Ax_1) and (Ax_2) for the set $\Sigma_{\underset{\sim}{\mathbf{C}}_4}$. Let $\mathbf{X} = \langle X; e_1, e_2, \mathcal{T}\rangle$ be a Boolean topological unary algebra satisfying $\Sigma_{\underset{\sim}{\mathbf{C}}_4}$. The equations ($C_1$) to ($C_4$) imply that the substructure of $\mathbf{X}$ generated by an element $x \in X$ is equal to $\{x, e_1(x), e_2(x), e_1e_2(x)\}$. Since the type is unary it follows that an n-generated substructure has at most $4n$ elements. Thus (Ax_1) holds.

In order to prove (Ax_2), let $\mathbf{X} = \langle X; e_1, e_2\rangle$ be a finite model of $\Sigma_{\underset{\sim}{\mathbf{C}}_4}$ and let $x, y \in X$ with $x \neq y$. We shall define a homomorphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{C}}_{4\backslash\mathcal{T}}$ with $\alpha(x) \neq \alpha(y)$.

Case 1: $e_2(x) \neq e_2(y)$. First assume that $e_2e_2(x) \neq e_2e_2(y)$ and define

$$U := e_2^{-1}e_2^{-1}(\{e_2e_2(x)\}).$$

Then $x \in U$ and $y \notin U$. By (C_2) and (C_3), we have

$$e_2e_2(e_2(z)) = e_2e_2(z) \ \text{ and } \ e_2e_2(e_1(z)) = e_2e_2(z),$$

for all $z \in X$. It follows from these equations that $e_i(U) \subseteq U$ and $e_i(X \backslash U) \subseteq X \backslash U$, for $i = 1, 2$. Hence the map $\alpha\colon X \to C_4$ given by $X\backslash U \mapsto 0$ and $U \mapsto 1$ is a homomorphism with $\alpha(x) \neq \alpha(y)$.

Assume that $e_2e_2(x) = e_2e_2(y)$. Since $e_2(x) \neq e_2(y)$, we cannot have both $e_2(y) = e_2e_2(x)$ and $e_2(x) = e_2e_2(y)$. Thus, without loss of generality, we may assume that $e_2(y) \notin U$, where $U := \{e_2(x), e_2e_2(x)\}$. Let $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{C}}_{4\backslash\mathcal{T}}$ be the homomorphism defined in Lemma 5.1 using this choice of U. We shall show that $\alpha(e_2(x)) \neq \alpha(e_2(y))$, and hence $\alpha(x) \neq \alpha(y)$. Indeed, in the notation of Lemma 5.1, we claim that $e_2(x) \in U_1$ and $e_2(y) \in U_b$.

To prove that $e_2(x) \in U_1$ it suffices to show that $e_2(x) \in V$, that is,

$$e_2(x) \in U, \ e_1e_2(x) \in U, \ e_2e_2(x) \in U, \ e_1e_2e_2(x) \in U.$$

The first statement holds by definition while the axioms quickly yield that the last three elements listed above are all equal to $e_2e_2(x)$ and hence are in U.

We now prove that $e_2(y) \in U_b$, that is, $e_2(y) \in e_1^{-1}(V)\backslash V$. Since $V \subseteq U$ and $e_2(y) \notin U$, it is trivial that $e_2(y) \notin V$. To prove that $e_2(y) \in e_1^{-1}(V)$ we must show that $e_1e_2(y) \in V$, that is,

$$e_1e_2(y) \in U, \ e_1e_1e_2(y) \in U, \ e_2e_1e_2(y) \in U, \ e_1e_2e_1e_2(y) \in U.$$

Using the axioms we easily see that each of these four elements equals $e_2e_2(y)$ and so is in U, as required.

Case 2: $e_2(x) = e_2(y)$. It follows that $e_2e_2(x) = e_2e_2(y)$ and, by (C_5), that $e_1(x) = e_1(y)$. Define $U^x := \{x, e_1(x), e_2(x), e_2e_2(x)\}$ and define U^y similarly. We claim that either $y \notin U^x$ or $x \notin U^y$.

Assume that $x = e_2e_2(y)$. Then $x = e_2e_2(x)$ and so, by (C_2), $e_2(x) = e_2e_2(x)$. Thus $x = e_2(x)$ and this implies that $x = e_1(x)$, by (C_6). Hence $x = e_1(x) = e_2(x) = e_2e_2(x)$. So, in this case, we have $y \notin U^x$.

Assume $x = e_2(y)$. Then $x = e_2(x)$ and so $x = e_2e_2(x)$. Thus, by the argument in the previous paragraph, we again have $y \notin U^x$.

Hence, by symmetry, we may assume that $x \neq e_2e_2(y)$, $x \neq e_2(y)$, $y \neq e_2e_2(x)$ and $y \neq e_2(x)$. Suppose that $x \in U^y$ and $y \in U^x$. Since $x \neq y$ we must have $x = e_1(y)$ and, similarly, $y = e_1(x)$. Thus $x \neq y$ gives $e_1(x) \neq e_1(y)$ which, by (C_5), yields the contradiction $e_2(x) \neq e_2(y)$.

By symmetry, we may now assume that $y \notin U^x$. Let $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{C}}_{4\backslash\mathcal{T}}$ be the homomorphism defined in Lemma 5.1 with $U = U^x$. It is immediate from the definition of U^x that $x \in V \subseteq U_1$, whence $\alpha(x) = 1$. By (C_5) we have $e_1(y) = e_1(x)$

and from this and the axioms it is easy to check that $y \in U_b$, whence $\alpha(y) = b$. Hence $\alpha(x) \neq \alpha(y)$, as required.

This establishes condition (Ax_2) and completes the proof. $\square$

5.2. Axiomatization of $\mathfrak{Z}$. Let $\Sigma_{\underset{\sim}{\mathbf{N}}}$ consist of the following four equations and quasi-equations:

(N_1) $ff(x) = f(x)$,
(N_2) $gg(x) = x$,
(N_3) $gf(x) = f(x)$,
(N_4) $f(x) = f(y)\ \&\ g(x) = y \implies x = y$.

The following is an unpublished result due to David Clark. His proof was direct and required topological arguments. Since we now know that $\mathfrak{Z}$ is standard, we can give a simpler proof that requires no topology.

Theorem 5.3. *The dual category $\mathfrak{Z} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$ of $\mathbf{N}$ is exactly the category of Boolean topological unary algebras $\mathbf{X} = \langle X; f, g, \mathcal{T}\rangle$ satisfying $\Sigma_{\underset{\sim}{\mathbf{N}}}$.*

Proof. We shall prove (Ax_1) and (Ax_2) for the set $\Sigma_{\underset{\sim}{\mathbf{N}}}$. Let $\mathbf{X} = \langle X; f, g, \mathcal{T}\rangle$ be a Boolean topological unary algebra satisfying $\Sigma_{\underset{\sim}{\mathbf{N}}}$. By equations ($\mathrm{N}_1$) to ($\mathrm{N}_3$), the substructure of $\mathbf{X}$ generated by an element $x \in X$ equals $\{x, f(x), g(x), fg(x)\}$. Hence an n-generated substructure has at most $4n$ elements and so (Ax_1) holds.

Now let $\mathbf{X} = \langle X; f, g\rangle$ be a finite model of $\Sigma_{\underset{\sim}{\mathbf{N}}}$ and $x, y \in X$ with $x \neq y$. We shall show that there exists a homomorphism $\alpha\colon \widetilde{\mathbf{X}} \to \underset{\sim}{\mathbf{N}}_{\setminus\mathcal{T}}$ with $\alpha(x) \neq \alpha(y)$. This will verify (Ax_2).

Case 1: $f(x) \neq f(y)$. Define $U := f^{-1}\{f(x)\}$ and $V := X\setminus U$. By (N_1), we have $f(x) \in U$, $f(y) \in V$, $f(U) \subseteq U$ and $f(V) \subseteq V$. Define

$$O := V \cap g(V),\ A := U\setminus g(U),\ B := V\setminus g(V),\ I := U \cap g(U).$$

Since, by (N_2), g is a bijection, $\{O|A|B|I\}$ is a partition of X.

Let $u \in I$. Then $f(u) \in I$, by (N_3), and $g(u) \in I$, by (N_2). Hence $f(I) \subseteq I$ and $g(I) \subseteq I$.

Let $u \in A$. Then, by (N_3), $f(u) \in U \cap g(U) = I$, whence $f(A) \subseteq I$. Since $u \notin g(U)$ we have $gg(u) \notin g(U)$, by (N_2), and hence $g(u) \notin U$, that is $g(u) \in V$. Also, $u \in U$ implies $u \notin V$ and so $g(u) \notin g(V)$, by (N_2). Thus, $g(u) \in V\setminus g(V) = B$, whence $g(A) \subseteq B$.

Analogous arguments show that $f(O) \subseteq O$, $g(O) \subseteq O$, $f(B) \subseteq O$ and $g(B) \subseteq A$. Thus the map $\alpha\colon X \to N$, given by $O \mapsto 0$, $A \mapsto a$, $B \mapsto b$ and $I \mapsto 1$, is a homomorphism that separates $f(x)$ and $f(y)$ and so separates x and y.

Case 2: $f(x) = f(y)$. We may assume that $y \neq f(y)$, since otherwise we have $x \neq f(x)$. By (N_1) we have $y \notin f(X)$ and, by (N_4) we have $y \neq g(x)$. Define $U := f(X) \cup \{x, g(x)\}$. Thus, $x \in U$ and $y \notin U$. By construction, we have $f(X) \subseteq U$

and, by (N_1), we have $f(U) \subseteq U$. From (N_3) and (N_2) we obtain $g(U) = U$ and so, by (N_2), we also have $g(X \backslash U) = X \backslash U$. Thus the map $\alpha\colon X \to N$, given by $U \mapsto 1$ and $X \backslash U \mapsto c$, is a homomorphism that separates x and y. □

6. Axiomatization of an optimal strong dual category for $\mathrm{Var}(\mathbf{C}_4)$

If $\underset{\sim}{\mathbf{M}}$ is an alter ego of a finite algebra $\mathbf{M}$ that yields a duality on $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M})$, then the category $\boldsymbol{\mathcal{A}}$ is dually equivalent to some subcategory of $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. An even tighter connection exists between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ when $\underset{\sim}{\mathbf{M}}$ yields a *strong duality* on $\boldsymbol{\mathcal{A}}$. In that case, $\boldsymbol{\mathcal{A}}$ is dually equivalent to the category $\boldsymbol{\mathfrak{X}}$ rather than to some unknown subcategory of $\boldsymbol{\mathfrak{X}}$. Thus it is particularly important to have an axiomatization of $\boldsymbol{\mathfrak{X}}$ in this case. (See [2, Chapter 3] for a full discussion of strong dualities.)

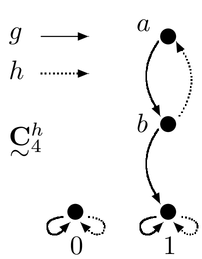
By [2, Theorem 4.2.3 (iii)], we know that

$$\underset{\sim}{\mathbf{C}}'_4 = \langle \{0, a, b, 1\}; \mathrm{End}(\mathbf{C}_4), h, \mathcal{T} \rangle$$

yields a strong duality on $\mathcal{C}_4$, where $h\colon 0 \mapsto 0, b \mapsto a, 1 \mapsto 1$ is the non-extendable partial endomorphism of $\mathbf{C}_4$. We want to give an axiomatic description of the topological quasi-variety generated by $\underset{\sim}{\mathbf{C}}'_4$. In order to do so, we need a manageable generating set for $\mathrm{End}(\mathbf{C}_4) \cup \{h\}$. The composition table of $\mathrm{End}(\mathbf{C}_4) \cup \{h\}$ is given in Figure 7, where $\times$ stands for *is undefined*. Clearly, $\{e_2, h\}$ is a generating set for $\mathrm{End}(\mathbf{C}_4) \cup \{h\}$. Let $g := e_2$. Then

$$\underset{\sim}{\mathbf{C}}^h_4 = \langle \{0, a, b, 1\}; g, h, \mathcal{T} \rangle$$

yields a strong duality on $\mathcal{C}_4$. The structure $\underset{\sim}{\mathbf{C}}^h_4$ is shown in Figure 7.



$\circ$	id	e_1	e_2	e_3	h
id	id	e_1	e_2	e_3	h
e_1	e_1	e_1	e_3	e_3	h
e_2	e_2	e_3	e_3	e_3	$\mathrm{id}_{\mathrm{dom}(h)}$
e_3	e_3	e_3	e_3	e_3	$\times$
h	$\times$	$\times$	e_1	e_3	$\times$

Composition table of $\mathrm{End}(\mathbf{C}_4) \cup \{h\}$

FIGURE 7

In the theorem below we give quasi-equations that describe the topological quasi-variety $\boldsymbol{\mathfrak{X}}^h_4 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}^h_4)$. It then follows that $\boldsymbol{\mathfrak{X}}^h_4$ is standard. (The statements (S_i) below are equivalent to equations or quasi-equations since, for example, $x \in \mathrm{dom}(h)$ is equivalent to the equation $h(x) = h(x)$.) This proof illustrates just how difficult it can be to axiomatize a topological quasi-variety generated by a finite topological structure when we do not know in advance that it is standard. It also illustrates the subtle additional complications caused by partial operations. The authors would like to thank Jane Pitkethly for her generous advice on this proof.

Theorem 6.1. *The strong dual category* $\mathfrak{X}_4^h := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_4^h)$ *of* $\mathfrak{C}_4$ *is exactly the category of Boolean structures* $\mathbf{X} = \langle X; g, h, \mathcal{T}\rangle$ *satisfying the following axioms:*

(S$_1$) $ggg(x) = gg(x)$,
(S$_2$) $x \in \mathrm{dom}(h) \Longleftrightarrow gg(x) = g(x)$,
(S$_3$) $g(x) = x \Longleftrightarrow (x \in \mathrm{dom}(h)\ \&\ h(x) = x)$,
(S$_4$) $x \in \mathrm{dom}(h) \Longrightarrow gh(x) = x$,
(S$_5$) $g(x) \in \mathrm{dom}(h)$.

Proof. It is easy to check that $\underset{\sim}{\mathbf{C}}_4^h$ satisfies (S$_1$)–(S$_5$). Hence, by Theorems 3.1 and 3.2, it suffices to show that if $\mathbf{X} = \langle X; g, h, \mathcal{T}\rangle$ is a Boolean structure that satisfies (S$_1$)–(S$_5$), then conditions (i) and (ii) of Theorem 3.2 hold.

Assume that $\mathbf{X} = \langle X; g, h, \mathcal{T}\rangle$ is a Boolean structure that satisfies (S$_1$)–(S$_5$) and let $x, y \in X$ with $x \neq y$.

Case 1: $gg(x) \neq gg(y)$. Since $\langle X; \mathcal{T}\rangle$ is a Boolean space, there exists a clopen subset U' of X such that $gg(x) \in U', gg(y) \notin U'$. Define

$$U := g^{-1}g^{-1}(U').$$

By (S$_1$), U is closed under g. To prove U is closed under h, let $u \in U \cap \mathrm{dom}(h)$. Then $gg(u) \in U'$ and hence, by (S$_4$), $gggh(u) \in U'$. This implies, by (S$_1$), that $ggh(u) \in U'$. Hence $h(u) \in U$. Define

$$V := X\backslash U.$$

Then clearly, by (S$_1$), V is closed under g. Now if $v \in V \cap \mathrm{dom}(h)$, then, using (S$_4$) and (S$_2$), we can show that $h(v) \in V$. Hence V is closed under h. Define a map $\alpha : X \to C_4$ by $U \mapsto 1$ and $V \mapsto 0$. Then clearly α is a separating morphism for x and y.

Case 2: $gg(x) = gg(y)$. Define

$$\begin{aligned}
\mathrm{Level}\,0 &:= \{z \in X \mid g(z) = z\},\\
\mathrm{Level}\,1 &:= \{z \in X \mid gg(z) = g(z) \neq z\},\\
\mathrm{Level}\,2 &:= \{z \in X \mid ggg(z) = gg(z) \neq g(z)\}.
\end{aligned}$$

(See Figure 8.) Then, by (S$_1$), $X = \mathrm{Level}\,0 \cup \mathrm{Level}\,1 \cup \mathrm{Level}\,2$.

We claim that $\mathrm{Level}\,1 \cap \mathrm{Im}(h) = \varnothing$. Let $z \in \mathrm{Level}\,1$. If $z \in \mathrm{Im}(h)$, then $z = h(u)$ for some $u \in \mathrm{dom}(h)$. This implies $g(z) = gh(u)$ and hence, by (S$_4$), we have $g(z) = u$. Hence $hg(z) = h(u) = z$. Now $z \in \mathrm{Level}\,1$ implies $gg(z) = g(z)$

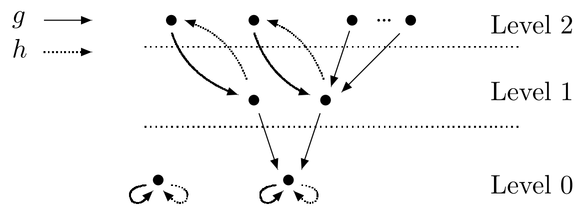


FIGURE 8

and hence, by (S_3), $g(z) \in \mathrm{dom}(h)$ and $hg(z) = g(z)$. Therefore $g(z) = z$ which contradicts the fact that $z \in$ Level 1. Hence $z \notin \mathrm{Im}(h)$.

First assume that $x \in$ Level 1. Then there is a clopen subset U of X such that $x \in U$ and $g(x), y \notin U$. We may assume that $U \cap g^{-1}(U) = \varnothing$ (otherwise, replace U by $U \backslash g^{-1}(U)$). Then, for each $u \in U$, we have $g(u) \notin U$. In particular, U does not contain any fixed point. Let $h(z) = t \in U$ for some $z \in \mathrm{dom}(h)$. Then $t \neq x$ as $x \in$ Level 1 and hence there is a subset U_t of U, which is clopen in X, such that $t \in U_t$ and $x \notin U_t$. The subset $h^{-1}(U)$ of X is closed. Since h is continuous, for each $t \in \mathrm{Im}(h) \cap U$, the set $h^{-1}(U_t)$ is clopen in $\mathrm{dom}(h)$, so $h^{-1}(U_t)$ is clopen in $h^{-1}(U)$. We can easily check that $h^{-1}(U) = \bigcup\{h^{-1}(U_t) \mid t \in \mathrm{Im}(h) \cap U\}$. Since $h^{-1}(U)$ is compact, there is a finite subset F of $\mathrm{Im}(h) \cap U$ such that

$$h^{-1}(U) = \bigcup\{h^{-1}(U_t) \mid t \in F\}.$$

Now define

$$W := U \backslash \bigcup\{U_t \mid t \in F\}.$$

Then W is clopen and $x \in W$, $y \notin W$. Define a map $\alpha\colon X \to C_4$ by

$$\alpha(u) = \begin{cases} a & \text{if } u \in g^{-1}(W), \\ b & \text{if } u \in W, \\ 1 & \text{if } u \in X \backslash (W \cup g^{-1}(W)). \end{cases}$$

Let $u \in W$. Since $U \cap g^{-1}(U) = \varnothing$, we have $W \cap g^{-1}(W) = \varnothing$. This implies $g(u) \notin W$. If $gg(z) \in W$ for some $z \in X$, then, by (S_1), we have $ggg(z) \in W$ which contradicts the fact that $W \cap g^{-1}(W) = \varnothing$. Hence $gg(z) \notin W$ for all $z \in X$. Therefore, for all $u \in W$, we have $g(u) \in X \backslash (W \cup g^{-1}(W))$. If $u \notin W \cup g^{-1}(W)$, then clearly, $g(u) \notin W \cup g^{-1}(W)$. Now let $u \in g^{-1}(W) \cap \mathrm{dom}(h)$. Then, by ($S_2$), $g(u) = gg(u)$ and $g(u) \in W$ which contradicts the fact that $W \cap g^{-1}(W) = \varnothing$. If $u \in W \cap \mathrm{dom}(h)$, then, by ($S_4$), $gh(u) \in W$ and hence $h(u) \in g^{-1}(W)$. Finally, let $u \in \mathrm{dom}(h)$ and $u \notin W \cup g^{-1}(W)$. Then, by ($S_4$), $h(u) \notin g^{-1}(W)$. Furthermore, for all $z \in \mathrm{dom}(h)$, we have $h(z) \notin W$. For, if $h(z) \in U$, then $z \in h^{-1}(U)$ and hence $z \in h^{-1}(U_s)$ for some $s \in F$. This implies $h(z) \in U_s$ for some $s \in F$. Hence $h(z) \notin W$. Thus, α is the required separating morphism for x and y.

Next assume that $x \in \text{Level}\,2$ and $x \notin \text{Im}(h)$. Define $Y = \text{dom}(h) \cup h(\text{dom}(h))$. Then, by ($S_2$), we have $x \notin \text{dom}(h)$. Since Y is closed and $x \notin Y$, there exists a clopen subset U of X such that $x \in U$, $y \notin U$ and $U \cap Y = \varnothing$. Clearly $X \backslash U$ is closed under h. By (S_5), $X \backslash U$ is also closed under g and $g(U) \subseteq \text{dom}(h) \subseteq Y$. The map $\alpha\colon X \to C_4$ defined by $U \mapsto b$ and $X \backslash U \mapsto 1$ is the required separating morphism for x and y.

Now assume that $x \in \text{Level}\,2$ and $x \in \text{Im}(h)$. Since $x \in \text{Level}\,2$, it follows that $g(x) \in \text{Level}\,1$. By symmetry, the argument in the previous paragraph allows us to assume that $y \in \text{Im}(h)$ also. Then $g(y) \neq g(x)$ (for otherwise $y = h(g(y)) = h(g(x)) = x$) and hence, by the Level 1 subcase, we have a separating morphism α for $g(x)$ and $g(y)$ which therefore separates x and y.

Finally, assume $x \in \text{Level}\,0$. Then $y \notin \text{Level}\,0$ (for otherwise $y = gg(y) = gg(x) = x$, a contradiction). Hence, from the cases above (applied to y), there is a separating morphism for x and y.

We have now proved that $\mathbf{X}$ satisfies (i) of Theorem 3.2. It remains to prove that $\mathbf{X}$ satisfies (ii) of Theorem 3.2. Let $x \notin \text{dom}(h)$. Then, by (S_2), $gg(x) \neq g(x)$ and hence, by (i), there is a morphism $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{C}}_4^h$ such that $\alpha(gg(x)) \neq \alpha(g(x))$. Now if $\alpha(x) \in \text{dom}(h)$, then, by ($S_2$), $gg(\alpha(x)) = g(\alpha(x))$ and hence $\alpha(gg(x)) = \alpha(g(x))$, which is a contradiction. Therefore $\alpha(x) \notin \text{dom}(h)$, as required. □

References

[1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, (1974).

[2] D. M. Clark and B. A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, (1998).

[3] D. M. Clark, B. A. Davey, M. Haviar, J. G. Pitkethly and M. R. Talukder, *Standard topological quasi-varieties*, Houston Math J. **29** (2003), 859–887.

[4] D. M. Clark, B. A. Davey, R. Freese and M. Jackson, *Standard topological algebras: syntactic and principal congruences and profiniteness*, Algebra Universalis **52** (2004), 343–376.

[5] B. A. Davey, *Dualities for equational classes of Brouwerian algebras and Heyting algebras*, Trans. Amer. Math. Soc. **221** (1976), 119–146.

[6] B. A. Davey, *Dualities for Stone algebras, double Stone algebras, and relative Stone algebras*, Coll. Math. **46** (1982), 1–14.

[7] B. A. Davey, *Dualisability in general and endodualisability in particular*, in: A. Ursini and P. Aglianò, eds, Logic and Algebra (Pontignano, 1994), Lecture Notes in Pure and Applied Mathematics **180**, Marcel Dekker, New York, 1996, pp. 437–455.

[8] B. A. Davey and J. G. Pitkethly, *Endoprimal algebras*, Algebra Universalis **38** (1997), 266–288.

[9] B. A. Davey, M. Haviar and H. A. Priestley, *Endoprimal distributive lattices are endodualisable*, Algebra Universalis **34** (1995), 444–453.

[10] B. A. Davey, M. Haviar and H. A. Priestley, *Kleene algebras: a case study of clones and dualities from endomorphisms*, Acta Sci. Math. (Szeged) **67** (2001), 325–351.
[11] B. A. Davey and H. A. Priestley, *Lattices of homomorphisms*, J. Austral. Math. Soc. (Series A) **40** (1986), 364–406.
[12] B. A. Davey and H. A. Priestley, *Optimal natural dualities*, Trans. Amer. Math. Soc. **338** (1993), 655–677.
[13] B. A. Davey and H. A. Priestley, *Optimal dualities for varieties of Heyting algebras*, Studia Logica **56** (1996), 67–96.
[14] B. A. Davey and M. R. Talukder, *Functor category dualities for varieties of Heyting algebras*, J. Pure Appl. Algebra **178** (2003), 49–71.
[15] B. A. Davey and H. Werner, *Dualities and equivalences for varieties of algebras*, in A. P. Huhn and E. T. Schmidt, eds, Contributions to Lattice Theory (Szeged, 1980), Coll. Math. Soc. János Bolyai **33**, North-Holland, Amsterdam, 1983, pp. 101–275.
[16] B. A. Davey and H. Werner, *Piggyback-dualitäten*, Bull. Austral. Math. Soc. **32** (1985), 1–32.
[17] B. A. Davey and H. Werner, *Piggyback dualities*, in L. Szabó and A. Szendrei, eds, Lectures in Universal Algebra, Coll. Math. Soc. János Bolyai, Vol. **43**, North-Holland, Amsterdam, 1986, pp. 61–83.
[18] I. A. Mal'cev, *On the general theory of algebraic systems* (*Russian*), Mat. Sb. **35** (1954), 3–20.
[19] J. G. Pitkethly, *Endoprimal implication algebras*, Algebra Universalis **41** (1999), 201–211.
[20] M. J. A. D. G. Saramago, *A study of natural dualities, including an analysis of the structure of failsets*, D.Phil. Thesis, Universidade de Lisboa (1998).
[21] A. Stralka, *A partially ordered space which is not a Priestley space*, Semigroup Forum **20** (1980), 293–297.

B. A. Davey
School of Mathematical and Statistical Science, La Trobe University, Victoria 3086, Australia
e-mail: B.Davey@latrobe.edu.au

M. R. Talukder
Department of Mathematics, Shahjalal University of Science and Technology, Sylhet 3114, Bangladesh
e-mail: r.talukder-math@sust.edu