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From the subalgebras of the square to the discriminator

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How do we prove that a finite, non-trivial algebra A is quasi-primal? In general we will not use the definition which requires that for all $n \geq 1$ every map $f : A^n \to A$ which preserves internal isomorphisms of A is a term function. (By an *internal isomorphism* we mean an isomorphism between subalgebras of A.) We could apply Pixley's characterization [2] which says that A will be quasi-primal if and only if there is a ternary term function $t : A^3 \to A$ satisfying

$$t(abc) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } a \neq b. \end{cases}$$

Such a function is called the *discriminator* on A. On small examples a search for such a term is possible but on larger algebras with more complicated operations this can be rather daunting. The alternative is to use the characterization which states that A is quasi-primal if and only if A has a ternary term function $m : A^3 \to A$ which is a *majority function*, i.e.

$$m(aab) = m(aba) = m(baa) = a \qquad \text{for all } a, b \in A,$$

and every subalgebra of A^2 is either a product of subalgebras of A or the graph of an internal isomorphism. (This characterization follows from Baker and Pixley [1] since the discriminator preserves precisely those subsets of $A \times A$ which are either products of subsets of A or graphs of bijections between them.) Note that if A has an underlying lattice structure then we get the majority term for free via $m(xyz) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. If we use this approach and thereby show that A is quasi-primal, the problem of finding an explicit term to represent the discriminator remains. Our aim in this paper is to show that for a large and natural class of algebras it is possible to read off a discriminator term from the proof that

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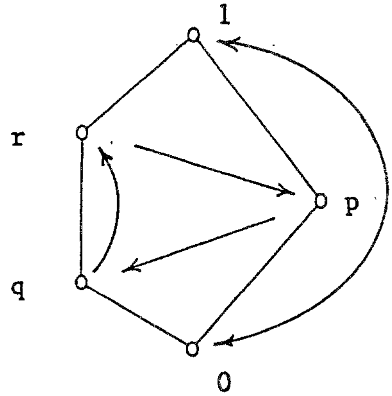


Figure 1

every subalgebra of A^2 is either a product of two subalgebras of A or the graph of an internal isomorphism. Consider for example the algebra $A = \langle N_5; \vee, \wedge, ' \rangle$ based on the five-element, non-modular lattice with unary operation $'$ as indicated in Figure 1. The proof that A is quasi-primal has two parts. First we characterize those subalgebras R of A^2 which are products of subalgebras of A. The characterization is based on the simple observation that if R contains either $(1, 0)$ or $(0, 1)$ then R is a product of subalgebras: if $(a, b), (c, d), (1, 0) \in R$, then

$$(a, d) = ((a, b) \wedge (1, 0)) \vee ((c, d) \wedge (1, 0)') \in R.$$

A power $x^{(n)}$ of x is defined from $'$ in the obvious way:

$$x^{(1)} = x' \quad \text{and} \quad (\forall k \in \mathbf{N}) x^{(k+1)} = x^{(k)'}.$$

Note that each of the elements p, q, r in the middle of A has an even power which is a (meet) semi-complement of it. Indeed, if we define

$$e_r(x) := x \wedge x^{(4)}, \qquad e_p(x) = e_q(x) := x \wedge x^{(2)},$$

then for each $u \in \{p, q, r\}$ and all $a \in A$ we have

$$e_u(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a = u, \\ * & \text{otherwise.} \end{cases}$$

(Here and later the star $(*)$ is a form of wild card indicating that the exact value is of no particular importance.)

It is much easier to work with $\{0, 1\}$-valued term functions – since they take values in the two-element Boolean subalgebra $\{0, 1\}$ of A we can apply the propositional calculus. We refer to such term functions as *predicates* and where possible give them meaningful titles. For example, by taking the meet of the terms $e_u(x)$ we obtain the term

$$one(x) := \bigwedge(e_u(x) \mid u \in \{p, q, r\}) = x \wedge x^{(2)} \wedge x^{(4)}.$$

On A this yields the predicate $one : A \to A$ with

$$one(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a \neq 1, \end{cases}$$

which is the characteristic function of $\{1\}$. Since $u' \neq 1$ for all $u \in \{p, q, r\}$, it follows easily that the term

$$zero(x) := one(x')$$

yields the predicate $zero : A \to A$ which is the characteristic function of $\{0\}$:

$$zero(a) = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \neq 0. \end{cases}$$

Note now that if a subalgebra R of A^2 contains $(1, u)$ with $u \neq 1$ then

$$(1, 0) = one((1, u)) \in R,$$

and if $(0, v) \in R$ with $v \neq 0$, then

$$(1, 0) = zero((0, v)) \in R.$$

Hence R is a product of subalgebras of A if and only if it contains an element of the form $(1, u)$ or $(u, 1)$ with $u \neq 1$, or $(v, 0)$ or $(0, v)$ with $v \neq 0$.

In the second part of the proof we assume that R is a subalgebra of A^2 which is not the graph of an internal isomorphism and show that R must be a product of subalgebras. Since R is not the graph of an isomorphism, there exist elements $u, v, w \in A$, $v \neq w$, with $(u, v), (u, w) \in R$ or $(v, u), (w, u) \in R$, say the former. Without loss of generality we may assume that $v < w$. By our characterization of products of subalgebras, we may assume that $u, v, w \notin \{0, 1\}$. Hence the only possibilities for (uvw) are (rqr), (pqr) and (qqr). It suffices to show that in each of

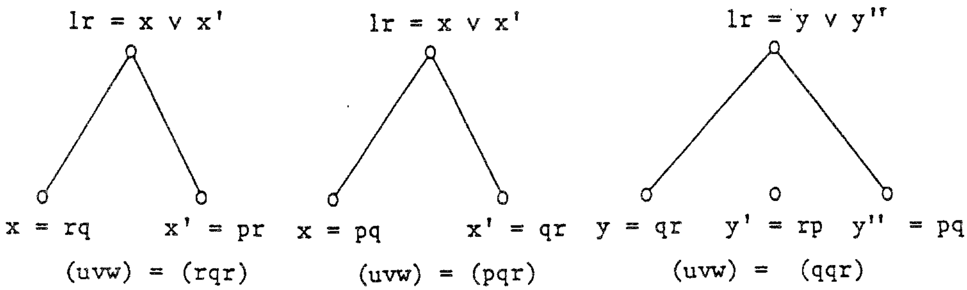


Figure 2

these cases the subalgebra of A^2 generated by $\{x = (u, v), y = (u, w)\}$ contains an element of the form $(1, c)$ with $c \neq 1$: this is easily done as shown in Figure 2. The case where $(v, u), (w, u) \in R$ is symmetric. Hence A is quasi-primal. Our calculations now provide us with a term for the discriminator as follows. From Figure 2 we read off terms

$$t_{rqr}(xy) = t_{pqr}(xy) := x \vee x' \quad \text{and} \quad t_{qqr}(xy) := y \vee y''$$

such that for all $a, b \in A$:

$$t_{uvw}(ab) = \begin{cases} 1 & \text{if } a = u = b, \\ \neq 1 & \text{if } a = v < w = b, \\ * & \text{otherwise.} \end{cases} \qquad \text{(TREN)}$$

In order to build the discriminator, for each triple (uvw) with $v < w$ we require a term $t_{uvw}(xy)$ such that (TREN) holds. All of the remaining cases are trivial and follow directly from our proof of the characterization of subalgebras of products. The required terms are given by

$$t_{1vw}(xy) := x, \qquad t_{0vw}(xy) := y',$$

$$t_{u0w}(xy) := zero(x)', \qquad \text{when } u \notin \{0, 1\},$$

$$t_{uv1}(xy) := one(x)', \qquad \text{when } u \notin \{0, 1\}.$$

We now paste these terms together to obtain a binary predicate sep_{vw} which separates the pair (vw) from the diagonal. For $v < w$ define

$$sep_{vw}(xy) := \bigvee \{one(t_{uvw}(xy)) \mid u \in A\}$$

then for all $a, b \in A$,

$$sep_{vw}(ab) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a = v < w = b, \\ 0, 1 & \text{otherwise.} \end{cases}$$

The meet of these terms provides us with the characteristic function of the diagonal:

$$equal(xy) := \bigwedge \{sep_{v \wedge w, v \vee w}(xy) \mid v \neq w\}.$$

With the predicate *equal* in hand the discriminator is given by

$$t(xyz) := (z \wedge equal(xy)) \vee (x \wedge equal(xy)').$$

The resulting term would be something of a mess and we refrain from writing it out.

Having worked through this example, the reader will find the proof of the main result of the next section quite straightforward. In the last section we analyse a large number of examples and show how to use other natural predicates to produce a more compact term for the discriminator.

1. The main result

We say that a nontrivial algebra A is a *helau* (with respect to 0, 1) if there exist binary terms $\vee$ and $\wedge$, a unary term $'$ and distinct elements 0, 1, each of which is the value of a constant unary term, satisfying the following equations for all $x \in A$:

$$x \wedge 1 = x, \quad 0 \wedge x = 0 = x \wedge 0, \quad x \vee 0 = x = 0 \vee x, \quad 0' = 1, \quad 1' = 0.$$

As a simple exercise the reader should show that the assumption that 0 and 1 are the values of constant unary terms can be replaced by the (apparently weaker) assumption that 0 and 1 belong to every subalgebra of A.

Note that $\langle \{0, 1\}; \vee, \wedge, ', 0, 1 \rangle$ is a two-element Boolean algebra. Any term function with values in $\{0, 1\}$ will be called a *predicate*.

A nontrivial algebra A is a *discriminator algebra* if there is a term which yields the discriminator on A. Thus quasi-primal algebras are precisely the finite discriminator algebras. Our first result, which says that every discriminator algebra is almost a helau comes from Werner [4]. Of course it is easy to find a discriminator algebra which has a one-element subalgebra and therefore is not a helau.

PROPOSITION 1.1. *Let A be a discriminator algebra which has distinct elements* 0, 1 *each of which is the value of a constant unary term on A. Then A is a helau with respect to* 0, 1 *and moreover A has a binary predicate equal and unary predicates one and zero such that for all $a, b \in A$*

$$equal(ab) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

$$one(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a \neq 1, \end{cases} \qquad zero(a) = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \neq 0. \end{cases}$$

Proof. The reader should check that the terms defined below have the desired properties (to make the notation more compact we denote $t(xyz)$ by (xyz)): $x \vee y := (x0y)$, $x \wedge y := (0(0yx)x)$, $x' := (0x1)$, $equal(xy) := ((xy1)(xy0)0)$, $one(x) := (1(1x0)0)$, $zero(x) := (0x1)$. □

The remainder of this section is devoted to proving a characterization of those helaus A which are discriminator algebras; en route we obtain an algorithm for obtaining the discriminator term from an analysis of the proof that the only subalgebras of A^2 are products of subalgebras of A or graphs of internal isomorphisms. It will become apparent that all that is required is a careful dissection of the proof from the previous section that $A = \langle N_5; \vee, \wedge, ' \rangle$ is quasi-primal. Our initial observation on subalgebras of A^2 carries over directly.

LEMMA 1.2. *Let A be a helau with respect to* 0, 1 *and let R be a subalgebra of A^2. Then R is a product of subalgebras of A if and only if it contains either* $(1, 0)$ *or* $(0, 1)$ *(and therefore both).* □

It follows from this lemma that the products of subalgebras form a principal filter in the lattice of subalgebras of A^2.

We denote $A \backslash \{0, 1\}$ by Mid(A) and think of its as the *middle* of A. The following two results expose the true role of the predicates *one* and *zero*.

PROPOSITION 1.3. *Let A be a helau with respect to* 0, 1. *Then the following are equivalent when A is finite and are related by (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftrightarrow$ (v) in general:*

(i) *if R is a subalgebra of A^2 and there exists $u \neq 1$ with $(1, u) \in R$ (or $(u, 1) \in R$) then R is a product of subalgebras of A;*
(ii) *for all $u \in$ Mid(A), the subalgebra of A^2 generated by $(1, u)$ is a product of subalgebras of A;*

(iii) *for all* $u \in \mathrm{Mid}(A)$ *there is a term* $e_u(x)$ *such that for all* $a \in A$,

$$e_u(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a = u, \\ * & \text{otherwise}; \end{cases} \qquad \text{(EIN)}$$

(iv) *there is a finite set* E *of unary terms such that for all* $u \in \mathrm{Mid}(A)$ *there is a term* $e_u(x) \in E$ *such that* (EIN) *holds*;

(v) *the function one* : $A \to A$ *is a unary predicate on* A.

Proof. (iv) $\Rightarrow$ (iii) is trivial and in the light of the previous lemma the equivalence of (i), (ii) and (iii) is clear. Assume that (iv) holds and let $E = \{e_1(x), \ldots, e_k(x)\}$, then the term

$$one(x) := x \wedge e_1(x) \wedge \cdots \wedge e_k(x)$$

induces the function *one* on A, whence (v) holds. That (v) implies (iv) is trivial since $one(x)$ will serve for $e_u(x)$ for all $u \in \mathrm{Mid}(A)$. □

For future reference we state the corresponding result for *zero*.

PROPOSITION 1.4. *Let* A *be a helau with respect to* 0, 1. *Then the following are equivalent for finite* A *and are related by (i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Leftarrow$ *(iv)* $\Leftrightarrow$ *(v) in general*:

(i) *if* R *is a subalgebra of* A^2 *and there exists* $v \neq 0$ *with* $(0, v) \in R$ *(or* $(v, 0) \in R$*) then* R *is a product of subalgebras of* A;

(ii) *for all* $v \in \mathrm{Mid}(A)$ *the subalgebra of* A^2 *generated by* $(0, v)$ *is a product of subalgebras of* A;

(iii) *for all* $v \in \mathrm{Mid}(A)$ *there is a term* $n_v(x)$ *such that for all* $a \in A$

$$n_v(a) = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a = v, \\ * & \text{otherwise}; \end{cases} \qquad \text{(NULL)}$$

(iv) *there is a finite set* N *of unary terms such that for all* $v \in \mathrm{Mid}(A)$ *there is a term* $n_v(x) \in N$ *such that* (NULL) *holds*;

(v) *the function zero* : $A \to A$ *is a unary predicate on* A.

Note that given the set $N = \{n_1(x), \ldots, n_k(x)\}$ of terms from (iv), the function *zero* is induced by the term

$$zero(x) := x' \wedge n_1(x) \wedge \cdots \wedge n_k(x).$$

As was the case for N_5, the following observation is often useful.

LEMMA 1.5. *Let A be a helau with respect to* 0, 1 *and assume that the function one is a unary predicate on A. Then the term one(x') yields the predicate zero if $u' \neq 1$ for all $u \in \mathrm{Mid}(A)$.* □

In view of Proposition 1.1, the following result amounts to a characterization of discriminator algebras with at least two constant unary terms.

THEOREM 1.6. *Let A be a helau with respect to* 0, 1. *The following are equivalent for finite A and related by (i) ⇔ (ii) ⇔ (iii) ⇐ (iv) ⇔ (v) ⇔ (vi) in general:*

(i) *one and zero are unary predicates on A and every subalgebras of A^2 is either a product of subalgebras of A or the graph of an internal isomorphism;*

(ii) *one and zero are unary predicates on A and for all $u, v, w \in \mathrm{Mid}(A)$ with $v \neq w$ the subalgebra of A^2 generated by $\{(u, v), (u, w)\}$ is a product of subalgebras of A;*

(iii) *one and zero are unary predicates on A and for all $u, v, w \in \mathrm{Mid}(A)$ with $v \neq w$ there exists binary terms $t_{uvw}(xy)$ such that for all $a, b \in A$*

$$t_{uvw}(ab) = \begin{cases} 1 & \text{if } a = u = b, \\ \neq 1 & \text{if } a = v \neq w = b, \\ * & \text{otherwise;} \end{cases} \qquad \text{(TREN)}$$

(iv) *one and zero are unary predicates on A and there exists a finite set T of binary terms such that for all u, v, $w \in \mathrm{Mid}(A)$ with $v \neq w$ there exists $t_{uvw}(xy) \in T$ such that* (TREN) *holds;*

(v) *equal is a binary predicate on A;*

(vi) *A is a discriminator algebra.*

Proof. That (i) implies (iii) is trivial. Assume that (ii) holds and let $u, v, w \in \mathrm{Mid}(A)$ with $v \neq w$. Thus the subalgebra generated by $\{(u, v), (u, w)\}$ is a product of subalgebras and so contains $(1, z)$ for some $z \neq 1$. Thus there is a binary term $t_{uvw}(xy)$ with $t_{uvw}((u, v), (u, w)) = (1, z)$, which says precisely that (TREN) holds. Hence (iii) holds.

Let (iii) hold and let R be a subalgebra of A^2 which is not the graph of an isomorphism. By Proposition 1.3, in order to establish (i) it suffices to show that R contains an element of the form $(1, z)$ or $(z, 1)$ with $z \neq 1$. Note that we can replace R by its inverse $\check{R} := \{(a, b) \mid (b, a) \in R\}$. Hence, since R is not the graph of an internal isomorphism, without loss of generality there exist $(u, v), (u, w) \in R$ with $v \neq w$. If u, v, $w \in \mathrm{Mid}(A)$, then the binary term $t_{uvw}(xy)$ (given by (iii)) when applied to $x = (u, v)$ and $y = (u, w)$ yields an element of the form $(1, z)$ with $z \neq 1$. If $\{u, v, w\} \cap \{0, 1\} \neq \emptyset$, then the required term $t_{uvw}(xy)$ is obtained from the predicates *one* and *zero* as indicated below:

$$t_{1vw}(xy) := \begin{cases} x & \text{if } v \neq 1, \\ y & \text{if } v = 1, \end{cases} \qquad t_{0vw}(xy) := \begin{cases} zero(x) & \text{if } v \neq 0, \\ zero(y) & \text{if } v = 0, \end{cases}$$

and for $u \notin \{0, 1\}$, $t_{u0w}(xy) := zero(x)'$ $t_{u1w}(xy) := one(x)'$, $t_{uv0}(xy) := zero(y)'$ and $t_{uv1}(xy) := one(y')$. Thus (iii) implies (i).

Assume now that (iv) holds; we shall build the predicate *equal* from the terms $t_{uvw}(xy)$. First let T^* be obtained by adjoining to T the eight terms listed above corresponding to the case where $\{u, v, w\} \cap \{0, 1\} \neq \emptyset$. Thus T^* is a finite set of binary terms such that for all $u, v, w \in A$ with $v \neq w$ there exists $t_{uvw}(xy) \in T^*$ such that (TREN) holds. Fix $v \neq w$ and let

$$T^*(vw) := \{t(xy) \in T^* \mid (\exists u \in A) t(xy) \text{ satisfies (TREN)}\}.$$

Now define

$$sep_{vw}(xy) := \bigvee (one(t(xy)) \mid t(xy) \in T^*(vw)).$$

The corresponding predicate $sep_{vw} : A^2 \to A$ separates (v, w) from the diagonal: for all $a, b \in A$

$$sep_{vw}(ab) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a = v \neq w = b, \\ * & \text{otherwise.} \end{cases}$$

Since $T^*(uw) \subseteq T^*$ and T^* is finite, the set of terms

$$S := \{sep_{vw}(xy) \mid (v, w) \in A^2 \;\&\; v \neq w\}$$

is finite. Thus we can define *equal*(xy) by

$$equal(xy) := \bigwedge (s(xy) \mid s(xy) \in S).$$

Thus (iv) implies (v), and that (v) implies (iv) follows at once since we can define the discriminator from *equal*(xy) by

$$t(xyz) := (z \wedge equal(xy)) \vee (x \wedge equal(xy)').$$

As noted in Proposition 1.1, *equal*(xy) is definable from the discriminator and the constants 0 and 1. Since *equal*(xy) satisfies (TREN) for all $u, v, w \in A$ with $v \neq w$, it follows that (vi) implies (iv). Finally, it is trivial that (iv) implies (iii). □

Our algorithm for deciding whether a finite helau is a discriminator algebra, and for producing a discriminator term en route, is now clear. First show that for all $u \in \text{Mid}(A)$ the subalgebra of A^2 generated by $(1, u)$ contains $(1, 0)$ thereby producing the terms $e_u(x)$ from which the unary predicate *one* is defined as in the proof of Proposition 1.3. If $u' \neq 1$ for all $u \in \text{Mid}(A)$, then by Lemma 1.5, *one*(x') yields the predicate *zero*. Otherwise show that for all $v \in \text{Mid}(A)$ the subalgebra of A^2 generated by $(0, v)$ contains $(1, 0)$ and so produce the terms $n_v(x)$ needed to define the predicate *zero* as indicated after the statement of Proposition 1.4. Next show that for all $u, v, w \in \text{Mid}(A)$ with $v \neq w$ the subalgebra of A^2 generated by $\{(u, v), (u, w)\}$ contains an element of the form $(1, c)$ for some $c \neq 1$, thereby obtaining the terms $t_{uvw}(xy)$ of Theorem 1.6. We then know that A is a discriminator algebra and a discriminator term is obtained from the $t_{uvw}(xy)$ and the terms *one*(x) and *zero*(x) as in the proof of (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) in Theorem 1.6. This is precisely what was done in the introduction for the helau based on N_5. The same procedure can be applied when A is infinite provided we can find the finite set of terms required by part (iv) in each of Propositions 1.3 and 1.4 and Theorem 1.6.

It is of interest to note that whereas the Baker–Pixley characterization of finite discriminator (equals quasi-primal) algebras in terms of subalgebras of the square requires the existence of a tenary majority term, the algroithm given above produces a discriminator term without specific use of a majority term.

2. Examples

In this section we give a list of examples showing how the complexity of the discriminator term can be reduced considerably by an intelligent choice of the predicates from which it is built. This task is particularly simple if one has an underlying ring or lattice structure.

In Werner [4, 1.20] the third author gave a long list of algebras having a discriminator term. The examples having an underlying ring structure are special cases of the following general situation: the algebra A has a constant 0 and three binary terms $+, -$, and $\cdot$ such that the following conditions are satisfied

$$x - x = 0$$

$$(x - y) + y = x$$

$$x \cdot 0 = 0$$

$$x \cdot 1 = x$$

for some element $1 \in A$ independent of x. (In the ring case these examples are almost helaus, with $x' := 1 - x$, except for the fact that 1 need not be a constant. We cannot apply our theorem in this situation, but fortunately we do not have to.) These conditions imply that $+$ has a one-sided unit, 0, and on the same side a one-sided (unary) inverse operation, $0 - x$, and the third operation $\cdot$ has 0 as a one-sided absorbing element and 1 as a one-sided unity. Note that the conditions also imply that $0 + x = x$ and importantly

$$x - y = 0 \Leftrightarrow x = y.$$

If we can create a predicate

$$\mathit{notzero}(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then it is easy to define the ternary discriminator by

$$t(xyz) := ((x - z) \cdot \mathit{notzero}(x - y)) + z.$$

For a field of order q we can choose $\mathit{notzero}(x) := x^{q-1}$. A biregular ring has a unary operation $^+$ which assigns to each element a the central idempotent a^+ such that a and a^+ generate the same ideal. A simple biregular ring has a unit element, 1, and $^+$ is the predicate *notzero*. For a Baer *-ring the unary operation * assigns to each element a the central idempotent a^* generating the annihilator of a. In a simple Baer *-ring there is always a unit element, 1, and * is the predicate *zero*; hence $\mathit{notzero}(x)$ is given by x^{**}.

Note that all such examples satisfy the equations $0 + x = x, x \cdot 0 = 0$ and $x \cdot 1 = x$ and, in the special cases considered above, $x + 0 = x$ and $0 \cdot x = 0$ also. In the general case the two extra conditions on 0 are not necessarily satisfied; however the operation $+$ must be almost a quasigroup operation. Each example based on a ring with unit (meaning that 1 is the value of a constant unary term or that we include the constant 1 in the type) is a helau with $x' := 1 - x$. Note that if we do not include 1 in the type of a simple biregular ring A then $\{0\}$ is a subalgebra and hence A is not a helau since 1 is not the value of a constant unary term. (Nevertheless, it comes as close as it possibly could since $x^+ = 1$ for all x other than 0.)

We now turn to examples of discriminator algebras which are based on some bounded lattice structure $\langle A; \vee, \wedge, 0, 1\rangle$. Clearly the equations $x \wedge 1 = x, 0 \wedge x = 0 = x \wedge 0, x \vee 0 = x = 0 \vee x$ hold in such an algebra and hence we have a helau provided we can define $' : A \to A$ which is a complementation on the bounds: $0' = 1$ and $1' = 0$. Along the way we must create the predicate *one* which (by Proposition 1.3) can be obtained as the meet of "enough" terms $e_u(x)$ satisfying $e_u(1) = 1$ & $e_u(u) = 0$. If we already have *one* then applying it to a unary term $c(x)$ satisfying $c(0) = 1$ & $c(1) \neq 1$ will yield a unary term function $'$ which is a complementation on the bounds.

Instead of creating the predicate *equal* we can settle, in the lattice case, for the predicate *equal* under the further proviso that the two entries are already in the "less than or equal" relation. Thus we need the restricted predicate *r-equal* satisfying:

$$r\text{-}equal(xy) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x < y, \\ * & \text{otherwise.} \end{cases}$$

The predicate equal is then given by

$$equal(xy) = r\text{-}equal(x \wedge y, x \vee y),$$

and, of course, the discriminator is then

$$t(xyz) = (z \wedge equal(xy)) \vee (x \wedge equal(xy)').$$

(Although the function *r-equal* need not map into $\{0, 1\}$, and so need not be a predicate, composing it with the predicate *one* converts it into a predicate.) Following the proof of Theorem 1.6, the predicate *r-equal* can be obtained as join and meet of "enough" terms $t_{uvw}(xy)$ satisfying $t_{uvw}(uu) = 1$ & $t_{uvw}(vw) \neq 1$ for $v < w$. The aim must clearly be to use as few as possible of the terms $t_{uvw}(xy)$ in order to keep the formula for the discriminator short.

If the underlying lattice has a relative pseudocomplementation, $\rightarrow$, then *r-equal*(xy) is given by *one*$(y \rightarrow x)$ since, given that $x \leq y$, we have $y \rightarrow x = 1$ if and only if $x = y$. Complementation on the bounds is given by $x \rightarrow 0$. Thus it is sufficient to find the predicate *one*.

In the two-element *Boolean algebra* there is an even more compact term for the discriminator,

$$t(xyz) = ((x \wedge z) \vee y') \wedge (x \vee z),$$

which is the relative complement of y in the interval $[x \wedge z, x \vee z]$. This shows that the two-element *relatively complemented lattice* is also a discriminator algebra.

On the n-element chain $0 < 1 < \cdots < n - 1$ there are several ways to define an n-valued logic. We describe here three of the best known ways each of which leads to a discriminator algebra. Define $\mathbf{1} := n - 1$.

Łukasiewicz algebras have an involutary negation $\bar{x} := \mathbf{1} - x$ and unary operations

$$D_i(x) = \begin{cases} \mathbf{1} & \text{if } x \geq i, \\ 0 & \text{if } x < i, \end{cases}$$

for $i = 1, \ldots, n-1$. Since

$$D_i(x) \vee \overline{D_i(y)} = \begin{cases} 0 & \text{if } x < i \leq y, \\ \mathbf{1} & \text{otherwise}, \end{cases}$$

we can construct *r-equal* as the meet of these $n-1$ terms.

Post algebras have the same unary operations D_i as Łukasiewicz algebras; however the negation is given by

$$\bar{x} := \begin{cases} \mathbf{1} & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Nevertheless *r-equal* is given by the same term as for Łukasiewicz algebras.

Complemented semigroups have two binary operations given on $\{0, 1, \ldots, n-1\}$ by $x \cdot y := \min\{n-1, x+y\}$ and $x * y := y - \min\{x, y\}$. Clearly

$$x \vee y := x \cdot (x * y) = \max\{x, y\} \quad \text{and} \quad x \wedge y := (x * y) * y = \min\{x, y\}$$

are the lattice operations $x * \mathbf{1} = \mathbf{1} - x$ is the Łukasiewicz negation. The predicate *notzero* is given by the term

$$x^{n-1} = x \cdot x \cdot \cdots \cdot x = \begin{cases} \mathbf{1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easily checked that the term $x \nabla y := (x * y) \cdot (y * x)$ satisfies $x \nabla y = |y - x|$ for all x and y. Thus the predicate *notequal* is given by

$$\textit{notequal}(xy) := \textit{notzero}(x \nabla y).$$

As an algebra of type $\langle 2, 2\rangle$, this algebra is not a helau as $\{0\}$ is a subalgebra. If $\mathbf{1}$ is included as a nullary then we have a helau and the predicate equal is given simply by

$$\textit{equal}(xy) := \textit{notequal}(xy) * \mathbf{1}.$$

Without **1** as a constant we can build the discriminator directly from the predicate *notequal* and the operations $\vee$, $\wedge$ and $*$; using the fact that $0 * z = z$ and $\mathbf{1} * z = 0$ it is straightforward to check that

$$t(xyz) := (notequal(xy) * z) \vee (notequal(xy) \wedge x)$$

defines the discriminator.

There are further examples coming from logic such as *cylindric algebras*, *monadic algebras* and *relation algebras* which have an underlying Boolean algebra structure with further operations and the simple members are characterized by the fact that some particular term yields the predicate *one*. Thus these are examples of discriminator algebras too.

We now turn to new examples which are similar in construction to the example in the introduction based on the lattice N_5. An easy way to convert a bounded lattice L into a helau which has both *zero* and *one* as predicates is to define a map $g : \mathrm{Mid}(A) \to \mathrm{Mid}(A)$ where $\mathrm{Mid}(A) := L \backslash \{0, 1\}$ and then define the negation $' : L \to L$ by

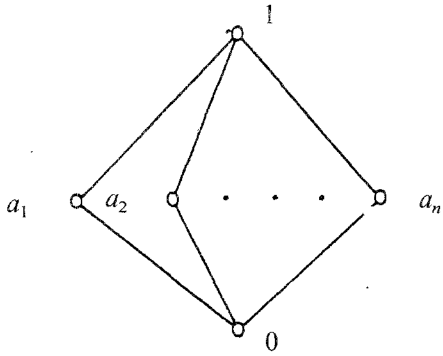
$$x' := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1, \\ g(x) & \text{otherwise.} \end{cases}$$

If we now assume that there is a finite set S of positive, even integers such that for all $u \in \mathrm{Mid}(A)$ there exists $s \in S$ with $u \wedge u^{(s)} = 0$, then the predicate *one* is given by the term

$$x \wedge x^{(s_1)} \wedge \cdots \wedge x^{(s_m)},$$

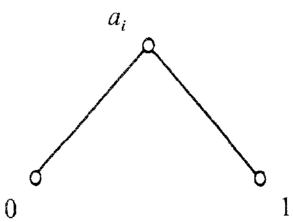
where $S = \{s_1, \ldots, s_m\}$, and by Lemma 1.5, *one*(x') yields the predicate *zero*.

If we start from the lattice M_n of Figure 3, then $A = \langle M_n; \vee, \wedge, ', 0, 1\rangle$ will be discriminator algebra if and only if for all i we have $g(g(a_i)) \neq a_i$; that is, g has no fixed points and no 2-cycles. If $g(a_i) \neq a_i = g(g(a_i))$, then $\{0, a_i, g(a_i), 1\}$ is a subalgebra isomorphic to a four-element Boolean algebra which is not simple, and hence the algebra A does not have a discriminator term. If $g(a_i) = a_i$, then $K := \{0, a_i, 1\}$ is a subalgebra isomorphic to the three-element Kleene algebra, which is not a discriminator algebra since the partial order of Figure 4 is a subalgebra of K^2 which is neither a product of subalgebras nor the graph of an internal isomorphism. Since every subalgebra of a discriminator algebra is also a discriminator algebra, it follows that A is not a discriminator algebra.



$g(g(a_i)) \neq a_i$

Figure 3



$\leqslant \; = \{(00), (a_i a_i), (11), (0a_i), (1a_i)\}$

Figure 4

Finally assume that g has no fixed points and no 2-cycles. Then we have

$$one(x) := x \wedge x'', \qquad zero(x) = x' \wedge x''',$$

$$less(xy) := (zero(x) \wedge zero(y)') \vee (one(x)' \wedge one(y)),$$

$$r\text{-}equal(xy) := less(x \wedge y, x \vee y)',$$

whence A is a discriminator algebra.

Let L be the lattice of Figure 5 with g as indicated by the arrows. Again we have

$$one(x) := x \wedge x'' \quad \text{and} \quad zero(x) := x' \wedge x''$$

and we have a new predicate

$$atom(x) := zero(x \wedge x')'$$

from which we construct the restricted-less predicate:

$$r\text{-}less(xy) := (zero(x) \wedge zero(y)') \vee (atom(x) \wedge atom(y)') \vee (one(x)' \wedge one(y)).$$

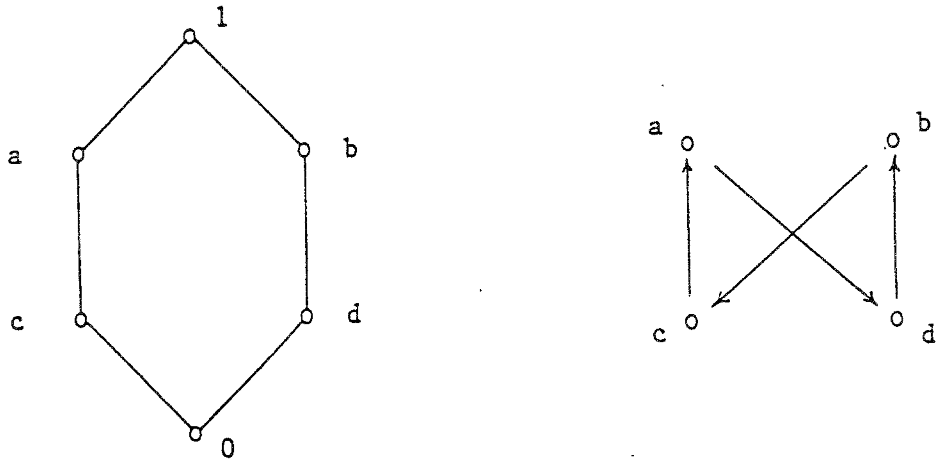


Figure 5

As before we can now go on to define the restricted-equal predicate and the discriminator. (In fact, the unrestricted predicate *less* is given by

$$less(xy) := (zero(x) \wedge zero(y)') \vee atom(x \vee y''') \vee (one(x)' \wedge one(y)),$$

but this is a little harder to see.)

If on the same six-element lattice we choose g to be the 4-cycle on the left of Figure 6, then we can choose the same terms as above for the predicates *one* and *zero*; however instead of using the predicate *atom* we colour a and d blue and b and c yellow. The corresponding predicates

$$blue(x) := zero(x' \wedge x'')' \quad \text{and} \quad yellow(x) := zero(x \wedge x')'$$

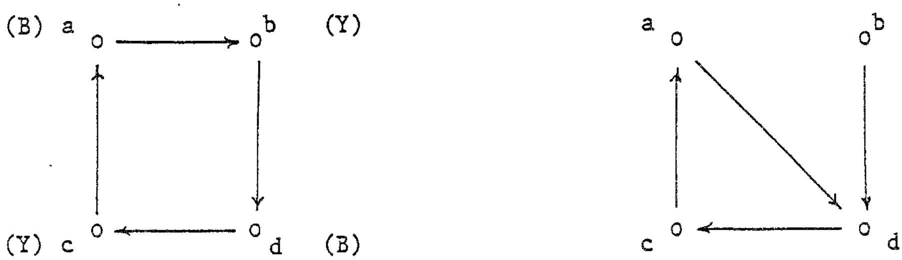


Figure 6

then yield the restricted-less predicate:

$$r\text{-}less(xy) := (zero(x) \wedge zero(y)') \vee (blue(x) \wedge blue(y)') \vee (yellow(x) \wedge yellow(y)') \vee (one(x)' \wedge one(y)).$$

For g we can also take a non-permutation. If g is as given on the right of Figure 6 we obtain a discriminator algebra. The terms must be altered slightly as shown below:

$$one(x) := x \wedge x'' \wedge x^{(4)}, \qquad zero(x) := one(x'),$$

$$atom\text{-}d(x) := zero(x' \wedge x'')', \qquad coatom\text{-}a(x) := zero(x \wedge x^{(5)})',$$

$$blue(x) := atom\text{-}d(x) \vee coatom\text{-}a(x), \qquad yellow(x) := zero(x \wedge x')',$$

and then the term for *r-less*, given just above, still works.

If we choose g to be the permutation on the left of Figure 7 we do not obtain a discriminator algebra since collapsing a and c yields a nontrivial congruence on the algebra. The same is true if we allow g to zig-zag between the sides as on the right of Figure 7 because collapsing each side yields a nontrivial congruence on the algebra.

Although our introductory example, N_5, is a subalgebra of one of the examples above, we investigate it once more to find a shorter version of the discriminator in this special case: see Figure 1. In this case we have:

$$one(x) := x \wedge x'' \wedge x^{(4)}, \qquad zero(x) := x' \wedge x''' \wedge x^{(5)},$$

$$atom(x) := zero(x)' \wedge zero(x \wedge x''),$$

$$r\text{-}less(xy) := (zero(x) \wedge zero(y)') \vee (atom(x) \wedge atom(y)') \vee (one(x)' \wedge one(y)).$$

Figure 7

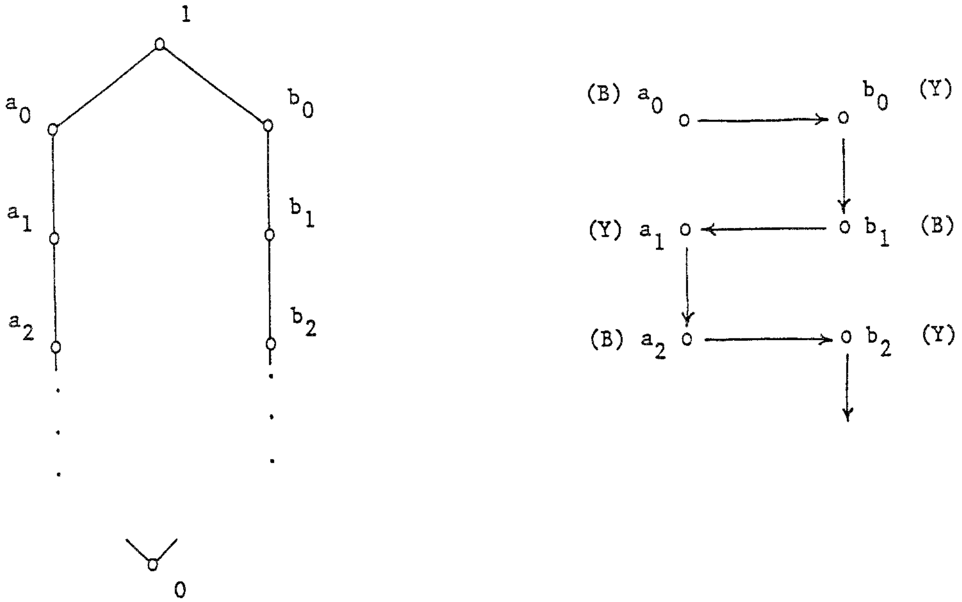


Figure 8

Finally consider the infinite example given in Figure 8. Again we have $one(x) := x \wedge x''$ and $zero(x) := one(x')$. A nonbound a is coloured blue if $a \wedge a' = 0$ and yellow if $a' < a$. We define corresponding predicates by

$$yellow(x) := zero(x \wedge x')',$$

$$blue(x) := yellow(x') = zero(x' \wedge x'')',$$

$$bound(x) := zero((x \wedge x') \vee (x' \wedge x'')).$$

Now for each triple (uvw) with $v < w$ we define a term $t_{uvw}(xy)$ satisfying:

$$t_{uvw}(ab) = \begin{cases} 1 & \text{if } a = u = b, \\ \neq 1 & \text{if } a = v < w = b, \\ * & \text{otherwise.} \end{cases} \tag{TREN}$$

By Theorem 1.6 it is sufficient to consider the cases where $0, 1 \notin \{u, v, w\}$ since the other cases are essentially trivial. Indeed, from the proof of Theorem 1.6 we have (since $v < w$ and therefore $v \neq 1$ and $w \neq 0$):

$$t_{1vw}(xy) := x, \qquad t_{0vw}(xy) := \begin{cases} zero(x) & \text{if } v \neq 0, \\ zero(y) & \text{if } v = 0, \end{cases}$$

and for $u \notin \{0, 1\}$, $t_{u0w}(xy) := zero(x)'$ and $t_{uv1}(xy) := one(y)'$. Now assume that $u, v, w \in \mathrm{Mid}(A)$ with $v < w$. If u is blue we have to distinguish cases according to the colours of v and w:

$$t_{uvw}(xy) := \begin{cases} x' \vee y & \text{if } v \text{ is yellow,} \\ x \vee y' & \text{if } w \text{ is yellow,} \\ x \vee (x \vee y''')' & \text{if } v \text{ and } w \text{ are blue.} \end{cases}$$

If u is yellow we have a similar distinction:

$$t_{uvw}(xy) := \begin{cases} x''' \vee y & \text{if } v \text{ is blue,} \\ x \vee y''' & \text{if } w \text{ is blue,} \\ y \vee (x \vee y')''' & \text{if } v \text{ and } w \text{ are yellow.} \end{cases}$$

These terms are easy to find except for the case where u, v and w are the same colour. To find the term in the case where u, v and w are blue we apply the algorithm and look in the subalgebra of A^2 generated by $x = (a_0, a_2)$ and $y = (a_0, a_0)$ searching for an element of the form $(1, c)$ with $c \neq 1$. First we find $y''' = (a_1, a_1)$ in the subalgebra and then

$$x \vee y''' = (a_0, a_2) \vee (a_1, a_1) = (a_0, a_1),$$

and finally we get

$$x \vee (x \vee y''')' = (a_0, a_2) \vee (b_0, a_2) = (1, a_2).$$

Now an easy check shows that this term works equally well for all blue u, v, w such that $v < w$. If u, v and w are yellow we consider the subalgebra generated by $x = (b_0, b_2)$ and $y = (b_0, b_0)$ as a concrete example. The same kind of calculation yields the second term which works for all yellow u, v and w with $v < w$. Joins of the $t_{uvw}(xy)$ terms yield the $sep_{vw}(xy)$ terms for $v < w$ as follows:

for $0 < w$,

$$sep_{0w}(xy) := zero(x)' \vee zero(y),$$

for $0 < v < 1$,

$$sep_{v1}(xy) := one(x) \vee one(y)'$$

for v and w both blue,

$$sep_{vw}(xy) := one(x''' \vee y) \vee one(x \vee (x \vee y''')'),$$

for v and w both yellow,

$$sep_{vw}(xy) := one(x' \vee y) \vee one(y \vee (x \vee y')'''),$$

for v yellow and w blue,

$$sep_{vw}(xy) := one(x' \vee y) \vee one(x \vee y'''),$$

and for v blue and w yellow,

$$sep_{vw}(xy) := one(x \vee y') \vee one(x''' \vee y).$$

Each of these terms was obtained by applying the algorithm given in the proof of Theorem 1.6 and then deleting superfluous factors from the join. The meet of these six terms gives *r-equal* from which we again obtain the discriminator.

For an infinite example like this one we need a finite number of terms $t_{uvw}(xy)$ working for all u and $v < w$. If for example we changed the definition of g so that for all $n \in \mathbf{N}$ there is some index k such that $g(a_{k+i}) = a_{k+i+1}$ for $i = 0, 1, \ldots, n$, then no discriminator term can exist on this algebra. Indeed, if n is the number of operation symbols in some term $t(xyz)$ and k is chosen as above, then this term cannot be the discriminator on the subset $\{1, a_k, a_{k+1}, \ldots, a_{k+n+1}, 0\}$ since redefining $g(a_{k+n+1}) := 0$ does not change $t(xyz)$ on this subset yet turns the subset into a non-simple algebra.

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