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Partition-induced natural dualities for varieties of pseudo-complemented distributive lattices

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Abstract

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A natural duality is obtained for each finitely generated variety $\mathbf{B}_n$ $(n < \omega)$ of distributive p-algebras. The duality for $\mathbf{B}_n$ is based on a schizophrenic object: $\underline{P}_n$ in $\mathbf{B}_n$ is the algebra $\mathbf{2}^n \oplus \mathbf{1}$ which generates the variety and $\underset{\sim}{P}_n$ is a topological relational structure carrying the discrete topology and a set of algebraic relations. The relations are (i) the graphs of a (3-element) generating set for the endomorphism monoid of $\underline{P}_n$ and (ii) a set of subalgebras of $\underline{P}_n^2$ in one-to-one correspondence with partitions of the integer n. Each of the latter class of relations, regarded as a digraph, is 'nearly' the union of two isomorphic trees. The duality is obtained by the piggyback method of Davey and Werner (which has previously yielded a duality in case $n \leqslant 2$), combined with use of the restriction to finite p-algebras of the duality for bounded distributive lattices, which enables the relations suggested by the general theory to be concretely described.

1. Introduction

A distributive p-algebra is an algebra $(A; \vee, \wedge, {}^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A; \vee, \wedge, 0, 1)$ is a distributive lattice with zero, 0, and identity, 1, and * is an operation of pseudocomplementation, that is,

$$a^* = \max\{b \in A \mid a \wedge b = 0\}.$$

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The distributive p-algebras form a variety $\boldsymbol{B}_\omega$. The lattice of subvarieties of $\boldsymbol{B}_\omega$ was determined by Lee in [14]. It is an $\omega+1$ chain

$$\boldsymbol{B}_{-1} \subset \boldsymbol{B}_0 \subset \boldsymbol{B}_1 \subset \cdots \subset \boldsymbol{B}_n \subset \cdots \subset \boldsymbol{B}_\omega.$$

Here $\boldsymbol{B}_{-1}$ is the trivial variety, $\boldsymbol{B}_0$ is the variety of Boolean algebras and $\boldsymbol{B}_1$ the variety of Stone algebras. The equational characterisation of $\boldsymbol{B}_n$ is given in [13, p. 167]. For our purposes the alternative characterisation based on Birkhoff's subdirect product theorem is more useful. This gives $\boldsymbol{B}_n = \mathbb{ISP}(\underset{\sim}{P}_n)$, where the p-algebra $\underset{\sim}{P}_n = (P_n; \vee, \wedge, {}^*, 0, 1)$ is obtained by adjoining a new unit to the n-atom Boolean lattice. We shall represent P_n as $\mathscr{P}(\{1, \ldots, n\}) \,\dot\cup\, \{\top\}$ and denote the empty set by $\bot$ and the set $\{1, \ldots, n\}$ by d. Then, for $a \in P_n$,

$$a^* = \begin{cases} d \backslash a & \text{if } a \subseteq d, \\ \bot & \text{if } a = \top. \end{cases}$$

One tool used to study the varieties $\boldsymbol{B}_n$ has been the duality for $\{0,1\}$-distributive lattices. Surveys of this duality can be found in [5] and [16], and an introductory account in [8]. The duality yields a category $\mathscr{Y}_n$ dually equivalent to $\boldsymbol{B}_n$ and allows the algebras in $\boldsymbol{B}_n$ to be concretely represented as families of sets. This representation (whose details we recall in Section 3) is appealingly pictorial, but has one major drawback. Although the duality between $\boldsymbol{B}_n$ and $\mathscr{Y}_n$ was successfully used by Davey and Goldberg in [6] to identify coproducts, and in particular free algebras in $\boldsymbol{B}_n$, it is not the natural tool to employ. For $n \geq 2$, products in $\mathscr{Y}_n$ are not cartesian. A good duality for handling free algebras should have the property that products in the dual category are cartesian. A duality with this property, and many other categorically desirable features, does exist for $\boldsymbol{B}_n$ $(n < \omega)$, as the results of [10] show. In that paper Davey and Werner undertook a major study of natural dualities. They showed that, under suitable conditions, a prevariety of the form $\mathscr{A} = \mathbb{ISP}(\underset{\sim}{P})$ has a duality defined by hom-functors and based on a schizophrenic object. This object is required to live, as $\underset{\sim}{P}$, in $\mathscr{A}$. It also lives, as a topological relational structure $\underset{\sim}{P}$, on the same underlying set P as $\underset{\sim}{P}$, in the dual category formed by taking isomorphic copies of closed substructures of powers of $\underset{\sim}{P}$. The structure $\underset{\sim}{P}$ acts as the dual of the free algebra $\boldsymbol{F}\mathscr{A}(1)$, and $\underset{\sim}{P}^\kappa$ (with product topology and pointwise defined relational strueture) as the dual of $\boldsymbol{F}\mathscr{A}(\kappa)$. When, as is the case for $\boldsymbol{B}_n$, the generating algebra $\underset{\sim}{P}$ is finite, the topology on $\underset{\sim}{P}$ is discrete and plays no role. The study of free algebras is then purely combinatorial in nature.

It is relatively easy to cast appropriate known dualities in the canonical form of the Davey–Werner theory (see the many examples given in [10]). However, as we recall in more detail in the next section, it is much harder to find the right candidate for $\underset{\sim}{P}$ when a natural duality is not already available.

The search for a description of a natural duality for each $\boldsymbol{B}_n$, $n < \omega$, has been the subject of a long-running serial. This paper and its companion, [9], unveil the last two episodes. The first instalment was written by M.H. Stone when he obtained his famous duality for $\boldsymbol{B}_0$ (Boolean algebras). The second was contributed 40 years later by Davey: in [4] he described a natural duality for $\boldsymbol{B}_1$. In [10] the dualities for $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ were used to illustrate the general theory (indeed they provided some of the motivation for it). The NU-Duality Theorem (Theorem 1.18 of [10]) uses results of Baker and Pixley [3] and applies to varieties with a near-unanimity term. The theorem shows that for a variety of distributive-lattice-ordered algebras (such as $\boldsymbol{B}_n$) it is possible to obtain a duality by taking the relational structure of $\underset{\sim}{P}$ to consist of all subalgebras of $\underset{\sim}{P}^2$. This led to new dualities for certain varieties, including de Morgan and Kleene algebras, but left $\boldsymbol{B}_n$ $(n \geqslant 2)$ out of reach because the number of subalgebras of $\underset{\sim}{P}_n^2$ appeared uninvitingly large.

The next episode in the story concerns piggyback dualities ([11, 12]). The piggyback technique applies to prevarieties whose members have reducts in a variety (such as $\{0, 1\}$-distributive lattices) for which a duality is already available. Applied to a prevariety $\mathbb{ISP}(\underset{\sim}{P})$ of distributive-lattice-ordered algebras the method identifies a restricted set of subalgebras of $\underset{\sim}{P}^2$ which serve to define $\underset{\sim}{P}$. It was successfully used in [11, 12] to produce a workable duality for $\boldsymbol{B}_2$. The necessary subalgebras of $\underset{\sim}{P}_2^2$ were found by algebraic means. The corresponding subalgebras required for a piggyback duality for $\boldsymbol{B}_n$ $(n \geqslant 3)$ were not exhibited. The story to date ends with a comment in [11] asserting that these subalgebras are 'many and ugly'. This paper refutes the claim that they are ugly; [9] addresses the question of the number of relations needed.

The key that unlocks the piggyback subalgebras is the 'old' duality—that between $\boldsymbol{B}_n$ and $\mathcal{Y}_n$. This allows us to derive a natural duality for $\boldsymbol{B}_n$ for $n \geqslant 3$ (Theorem 3.6). This duality has $\underset{\sim}{P}n = (P_n; \mathcal{T}, R)$ with $\mathcal{T}$ the discrete topology and R a set of $p(n) + 3$ relations, where $p(n)$ denotes the number of partitions of the integer n. The relations in R are:

(i) the graphs of a set of three endomorphisms which serve to generate the endomorphism monoid of $\underset{\sim}{P}_n$;

(ii) a set of subalgebras of $\underset{\sim}{P}_n^2$ in one-to-one correspondence with the partitions of n.

In (ii) the partitions into a fixed number of parts all give isomorphic subalgebras. Each relation in (ii) has a representations as a digraph. These digraphs are described in Section 4. The elementary theory of partitions can be found in Andrews, [2, Chapter 1].

Until now, every natural duality that has been explicitly described has involved only a very small number of relations. In such cases there is no incentive to discover whether the set of relations is redundant. Given the rapid growth of $p(n)$ with n we are obviously led to enquire whether a duality exists for $\boldsymbol{B}_n$ with fewer than $p(n) + 3$ relations, in our companion paper [9] we show, using a new range of techniques, that $n + 3$ relations in fact suffice. This problem and its solution open up a new branch of duality theory—the study of 'optimal' dualities.

2. Natural dualities, piggyback fashion

In this section we outline the duality theory presented in [10–12] as it applies to a (pre-)variety of distributive-lattice-ordered algebras generated by a single finite algebra.

Assume that $\mathscr{A} = \mathbb{ISP}(\underline{P})$ is a prevariety such that $\underline{P}$ has a reduct $\underline{\underline{P}}$ in $\boldsymbol{D}$, the variety of $\{0, 1\}$-distributive lattices. Assume that the underlying set, P, of $\underline{P}$ is finite. Consider

$$\underset{\sim}{P} = (P, \mathscr{T}, R),$$

where $\mathscr{T}$ is the discrete topology on P and R is a set of algebraic relations. (By an algebraic relation we mean one which is a subalgebra of some finite power $\underline{P}^m$). We shall, where convenient, identify a homomorphism from $\underline{P}^m$ to $\underline{P}$ with its graph, and so regard operations as (algebraic) relations.

We equip an arbitrary power $\underset{\sim}{P}^S$ with the product topology and pointwise-defined relations. A closed substructure of $\underset{\sim}{P}^S$ is a topologically closed subset of $\underset{\sim}{P}^S$ which is also a substructure with respect to the relations in R. Now define $\mathscr{X}$ to be the following category. A topological relational structure X belongs to $\mathscr{X}$ if it embeds as a closed structure in some $\underset{\sim}{P}^S$; the morphisms in $\mathscr{X}$ are the continuous relation-preserving maps. For $A \in \mathscr{A}$, define the dual of A to be

$$D(A) := \mathscr{A}(A, \underline{P}),$$

the set of $\mathscr{A}$-homomorphisms from A into $\underline{P}$. This can be regarded as a subset of $\underset{\sim}{P}^A$ and in fact belongs to $\mathscr{X}$ ([10, Lemma 1.3]). Then define the dual of $X \in \mathscr{X}$ to be

$$E(X) := \mathscr{X}(X, \underset{\sim}{P}),$$

the set of $\mathscr{X}$-morphisms of X into $\underset{\sim}{P}$. Because the relations are algebraic, this subset of $\underline{P}^X$ forms a subalgebra ([10, Lemma 1.3]). The maps D and E extend to morphisms to give hom-functors $D: \mathscr{A} \to \mathscr{X}$ and $E: \mathscr{X} \to \mathscr{A}$. Lemmas 1.3–1.5 of [10] show that these functors form an adjoint pair such that the evaluation maps, which are the units of the adjunction, are embeddings.

If $\underset{\sim}{P}$ is chosen in such a way that the evaluation map $a \to e_a$ from A to $ED(A)$ is an isomorphism for each $A \in \mathscr{A}$ we say that we have a *duality* for $\mathscr{A}$. (We shall not need to address the question of whether the duality is *full*, that is whether $X \cong DE(X)$ for all $X \in \mathscr{X}$).

The duality for the variety $\boldsymbol{D}$ of $\{0, 1\}$-distributive lattices as given in [16] (or see [8]) fits into this framework ([10, 2.8]). We have $\boldsymbol{D} = \mathbb{ISP}(\mathbf{2})$, where $\mathbf{2}$ denotes the 2-element chain as a lattice. The 2-element chain, as an ordered set, with the discrete topology is denoted by $\mathbf{2}$. The schizophrenic object is then the 2-element chain, living as $\mathbf{2}$ in $\mathbf{D}$ and as $\mathbf{2}$ in the dual category $\boldsymbol{P}$, which may be shown to consist of all compact totally order-disconnected spaces and continuous order-preserving maps. In this context we shall use the letters H, K in place of D, E to denote the hom-functors $\boldsymbol{D}(-, \mathbf{2})$, $\boldsymbol{P}(-, \mathbf{2})$ (and also their restrictions to subcategories—not necessarily full—of $\boldsymbol{D}$ and $\boldsymbol{P}$). Then, for each $A \in \boldsymbol{D}$, the evaluation map from A to $KH(A)$ (which we denote by k_A) is an isomorphism.

Assuming that the hom-functors D and E do establish a duality for the prevariety $\mathscr{A} = \mathbb{ISP}(\underset{\sim}{P})$, the following properties hold (see [10, p. 106]):

(i) $\underset{\sim}{P}$ is the dual of the free algebra $F\mathscr{A}(1)$, so that $F\mathscr{A}(1) = \mathscr{X}(\underset{\sim}{P}, \underset{\sim}{P})$;

(ii) products in $\mathscr{X}$ are cartesian, so that $F\mathscr{A}(\kappa) = \mathscr{X}(\underset{\sim}{P}^{\kappa}, \underset{\sim}{P})$.

Therefore if we choose the structure $\underset{\sim}{P}$ so that we indeed have a duality, we have access to the free algebras in $\mathscr{A}$.

As we mentioned in Section 1, we do obtain a duality for $\mathscr{A}$ by taking the set of R of relations on $\underset{\sim}{P}$ to consist of all subalgebras of $\underset{\sim}{P}^2$. This result can be refined by observing that it is sufficient to take a restricted set of subalgebras which 'generates' the entire set of subalgebras, in the sense defined in [10, p. 140]. Even this refinement may not yield a workable duality: the subalgebras may be hard to describe and a 'small' generating set may not be apparent. The piggyback method developed in [11] and [12] identifies a restricted set of subalgebras which suffices to define a duality. Assume that some topological relational structure $\underset{\sim}{P}$ has been put forward. We seek conditions under which the embedding $e_A : A \to ED(A)$ is surjective, for each $A \in \mathscr{A}$. The idea is to exploit the fact that, for any A, the evaluation map $k_A : A \to KH(A)$ is onto. If it is possible to construct an injective map $\Delta : ED(A) \to KH(A)$ such that $\Delta \circ e_A = k_A$, then e_A is forced to be surjective.

To define Δ we need to associate with each continuous morphism φ from $D(A)$ $(= \mathscr{A}(A, \underset{\sim}{P}))$ to $\underset{\sim}{P}$ a continuous order-preserving map $\Delta(\varphi)$ from $H(A)$ (= $\boldsymbol{D}(A, \underset{\approx}{P})$) to $\mathbf{2}$ such that $(\Delta \circ e_A)(a) = k_A(a)$ for all $a \in A$. A natural way to try to construct $\Delta(\varphi)$ is to seek a map $\alpha : \underset{\sim}{P} \to \mathbf{2}$ and a surjective map Φ_α from $D(A)$ to $H(A)$ such the diagram below commutes. In the diagram, α is some member of $H(\underset{\sim}{P})$ and $\Phi_\alpha := \alpha \circ -$.

$$\begin{array}{ccc} D(A) = \mathscr{A}(A, \underset{\sim}{P}) & \xrightarrow{\varphi} & \underset{\sim}{P} \\ {\scriptstyle \Phi_\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H(A) = \boldsymbol{D}(A, \underset{\approx}{P}) & \xrightarrow{\Delta(\varphi)} & \mathbf{2} \end{array}$$

The commutativity of the diagram means that, for $y \in H(A)$ and $x \in D(A)$,

$$(\Delta(\varphi))(y) = \alpha(\varphi(x)) \quad \text{where } y(a) = \alpha(x(a)) \text{ for all } a \in A,$$

and it follows from this that $\Delta \circ e_A = k_A$.

To carry out this construction we need:

(i) $\alpha : \underset{\sim}{P} \to \mathbf{2}$ such that Φ_α is surjective;

(ii) $\Delta(\varphi)$, given by $(\Delta(\varphi))(y) = \alpha(\varphi(x))$ where $y = \Phi_\alpha(x)$, is well defined on $H(A) = \operatorname{Im} \Phi_\alpha$, and is order-preserving and continuous, for each φ;

(iii) Δ is one-to-one.

As noted in [11, p. 68], (i) will be automatic when the underlying duality is that for $\boldsymbol{D}$. Now consider (ii). Assume that $y_1 = \Phi_\alpha(x_1)$ and $y_2 = \Phi_\alpha(x_2)$. If we can show that $y_1 \leqslant y_2$ implies

$$(\Delta(\varphi))(y_1) = \alpha(\varphi(x_1)) \leqslant \alpha(\varphi(x_2)) = (\Delta(\varphi))(y_2),$$

then $\Delta(\varphi)$ must be well defined (because $=$ is $\leqslant \cap \geqslant$). In the Piggyback Duality Theorem of [10] the structure of $\underset{\sim}{P}$ includes a family of relations which serves to ensure that $\Delta(\varphi)$ is well defined. The observation above shows that these relations can be omitted in the special case we are considering. The following theorem, which specialises the Piggyback Theorem to a prevariety whose algebras have a reduct in $\boldsymbol{D}$, takes account of this. We say that a subset R_1 of a set of relations R on $\underset{\sim}{P}$ *generates* R if, for every $A \in \mathscr{A}$, whenever a morphism $\varphi: D(A) \to \underset{\sim}{P}$ preserves each $r \in R_1$, it also preserves each $r \in R$. In the theorem, the inclusion of a generating set for the subalgebras of $\underset{\sim}{P}^2$ maximal in

$$\alpha^{-1}(\leqslant) := \{(b, c) \in \underset{\sim}{P}^2 \mid \alpha(b) \leqslant \alpha(c)\}$$

ensures that $\Delta(\varphi)$ is order-preserving; its continuity is easily verified. The separating set of endomorphisms makes Δ one-to-one. See [12] for the details of the proof of the theorem.

Theorem 2.1. *Suppose that $\mathscr{A} = \mathbb{ISP}(\underset{\sim}{P})$ is a prevariety generated by a finite algebra $\underset{\sim}{P}$ such that $A \in \mathscr{A}$ has a reduct in $\boldsymbol{D}$. Fix $\alpha \in H(\underset{\sim}{P})$.*

Let $\underset{\sim}{P} = (P; \mathscr{T}, R)$ be a topological relational structure on the underlying set P of $\underset{\sim}{P}$ in which $R = S \cup G$, where:

(i) *$\mathscr{T}$ is the discrete topology,*

(ii) *S is a generating set for the collection of subalgebras of $\underset{\sim}{P}^2$ maximal in $\alpha^{-1}(\leqslant)$,*

(iii) *G is the set of graphs of a family E of endomorphisms satisfying the separation condition:*

$$\text{for all } a, b \in P \text{ with } a \neq b, \text{ there exists } u \text{ in the submonoid of the endomorphism monoid generated by } E \text{ such that } \alpha(u(a)) \neq \alpha(u(b)). \quad \text{(C)}$$

Then the hom-functors $D: A \mapsto \mathscr{A}(A, \underset{\sim}{P})$ and $E: X \mapsto \mathscr{X}(X, \underset{\sim}{P})$ set up a duality for $\mathscr{A}$.

3. The duality for $\boldsymbol{B}_n$

We seek to apply the Piggyback Duality Theorem to $\boldsymbol{B}_n$. To do so, we use the duality for $\boldsymbol{D}$ to identify the relations defining $\underset{\sim}{P}_n$. We begin by stating this duality in the form in which it gives the most pictorial representations. To relate the representation below to that stated in the previous section simply note that the lattice of clopen upper sets of an ordered topological space Y is isomorphic to the lattice of continuous order-preserving maps from Y into $\mathbf{2}$, via the map assigning to a set U its characteristic function χ_U. A set U is an *upper set* if $y \in U$ and $z \geqslant y$ imply $z \in U$. For a detailed account of this representation see [5, 16], or [8, Chapters 8–10].

Theorem 3.1. *Let $L \in \boldsymbol{D}$. Then L is isomorphic to the lattice of clopen upper sets ($=$order filters) of its dual space $H(L) := \boldsymbol{D}(L, \mathbf{2})$, which is topologised as a* *subspace of $\mathbf{2}^L$ and ordered pointwise. As an ordered set, $H(L)$ is isomorphic to the set of prime filters of L ordered by inclusion, and in case L is finite, $H(L)$ has the discrete topology and is order anti-isomorphic to the set of join-irreducible elements of L.*

Henceforth we shall, where expedient, identify $L \in \boldsymbol{D}$ with the lattice of clopen upper sets of $H(L)$.

For morphisms we have the following result.

Theorem 3.2. *Given $L, M \in \boldsymbol{D}$, there exists a bijective correspondence between $\boldsymbol{D}(L, M)$ (the set of $\{0, 1\}$-lattice homomorphisms from L to M) and $\boldsymbol{P}(H(M), H(L))$ (the set of continuous order-preserving maps from $H(M)$ to $H(L)$). This associates to $f \in \boldsymbol{D}(L, M)$ the map $H(f) := - \circ f$*; specifically,

$$((H(f))(y))(a) = y(f(a)) \quad \text{for } a \in L,\ y \in H(M). \tag{†}$$

Further, f is surjective if and only if $H(f)$ is an order-embedding and f is a lattice-embedding if and only if $H(f)$ is surjective.

It follows immediately from this last result that, up to isomorphism, L is a $\{0, 1\}$-sublattice of $M \in \boldsymbol{D}$ if and only if there is a continuous order-preserving surjection from $H(M)$ onto $H(L)$.

The final fact we need about the duality for $\boldsymbol{D}$ concerns products of finite lattices. Suppose L_1 and L_2 are finite members of $\boldsymbol{D}$. Then

$$H(L_1) \mathbin{\dot\cup} H(L_2) \cong H(L_1 \times L_2).$$

It was shown in [15] (following Adams [1]) that $L \in \mathbf{D}$ is pseudocomplemented if and only if $Y = H(L)$ is a p-space, that is, it is a compact totally order-disconnected space with the property that for eaeh clopen upper set U in Y, the set

$$\downarrow U := \{z \in Y \mid z \leq y \text{ for some } y \in U\}$$

is clopen. This condition on upper sets is satisfied automatically if Y (or equivalently L) is finite. The pseudocomplement of a clopen upper set U in a p-space Y is given by

$$U^* = Y \setminus \downarrow U.$$

Given $A, B \in \boldsymbol{B}_\omega$, a map $f \in \mathbf{D}(A, B)$ preserves the operation * of pseudocomplementation if and only if $\varphi = H(f)$ is a p-morphism, that is, it is a continuous order-preserving map with the property that

$$\varphi(\max y) = \max \varphi(y) \quad \text{for all } y \in H(B).$$

Here $\max z$ denotes the set of maximal points above z.

The subvarieties $\boldsymbol{B}_n$ of $\boldsymbol{B}_\omega$ can be characterised in terms of prime filters: an algebra $A \in \boldsymbol{B}_\omega$ belongs to $\boldsymbol{B}_n$ if and only if each prime filter in A is contained in at

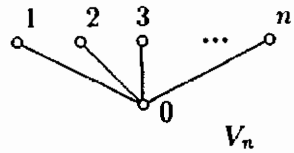


Fig. 1.

most n maximal filters ([14]). Since $H(A)$ is order-isomorphic to the set of prime filters of A ordered by inclusion, we have the following theorem ([1, 15]).

Theorem 3.3. *The restrictions of the functors H and K establish a contravariant category equivalence between* $\boldsymbol{B}_n$ and $\mathscr{Y}_n$, *where* $\mathscr{Y}_n$ *is the category whose objects are p-spaces in which each point is majorised by at most n maximal points and whose morphisms are the p-morphisms.*

We now have all the machinery for analysing subalgebras of $\underline{P}_n^2$. We use the notation introduced in Section 1. The join-irreducible elements of $\underline{P}_n$ are the atoms $\{i\}$ $(i = 1, \ldots, n)$ together with $\top$. Hence $H(\underline{P}_n)$ is the ordered set V_n shown in Fig. 1. We henceforth identify $\underline{P}_n$ with the lattice of npper sets of V_n.

We require a map $\alpha : \underline{P}_n \to \mathbf{2} = \{0, 1\}$ to support the piggyback construction. We take α to be the element 0 of V_n. This is the map which sends $\top$ in $\underline{P}_n$ to 1 and all other elements of $\underline{P}_n$ to 0.

Endomorphisms of $\underline{P}_n$ can be identified with p-morphisms from V_n to V_n. A map $\varphi : V_n \to V_n$ is a p-morphism if and only if:

(i) $i \geq 1$ implies $\varphi(i) \geq 1$, and

(ii) if $\varphi(0) \neq 0$, the map φ is constant, and if $\varphi(0) = 0$, the restriction of φ to $\{1, \ldots, n\}$ is a permutation of $\{1, \ldots, n\}$.

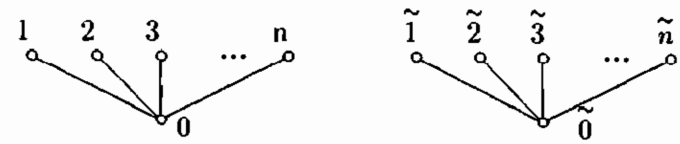
[It can in fact be seen that (i) is implied by (ii).] Since the symmetric group S_n is generated by the cycle $\sigma = (12 \cdots n)$ and the transposition $\tau = (12)$, we deduce that the endomorphism monoid of $\underline{P}_n$ is generated by three maps, f_σ, f_τ and e. These have the following p-morphisms as their duals:

$$(H(f_\sigma))(0) = 0, \qquad H(f_\sigma) \restriction \{1, 2, \ldots, n\} = \sigma,$$

$$(H(f_\tau))(0) = 0, \qquad H(f_\tau) \restriction \{1, 2, \ldots, n\} = \tau,$$

$$(H(e))(i) = 1 \quad \text{for all } i = 0, 1, \ldots, n.$$

We claim that the separation condition (C) in the Piggyback Duality Theorem is satisfied. Take $a \neq b$ in $\underline{P}_n$. We must find an endomorphism u such that exactly one of $u(a)$, $u(b)$ equals $\top$. Certainly $0 \notin a \cap b$. If 0 belongs to just one of a and b, then we take $u = \mathrm{id}_{\underline{P}_n}$. If $0 \notin a \cup b$, then there exists $i \geq 1$ such that, without loss of generality, $i \in a$ and $i \notin b$. There exists $\theta \in S_n$ such that $\theta(1) = i$. Then θ induces an automorphism f_θ of $\underline{P}_n$ which is a composite of powers of f_σ and f_τ such that $(e \circ f_\theta)(a) = \top$ and $(e \circ f_\theta)(b) \neq \top$.



$V_n \dot{\cup} V_n$

Fig. 2.

Our next task is to identify the subalgebras of $\underline{P}_n^2$ maximal in $\alpha^{-1}(\leq)$. Dually, each subalgebra, M, of $\underline{P}_n^2$ is determined by a p-morphism φ_M defined on $V_n \dot{\cup} V_n$; M is recaptured as $M = K(\operatorname{Im} \varphi_M)$. We label the elements of $V_n \dot{\cup} V_n$ as in Fig. 2.

The sublattice $\alpha^{-1}(\leq)$ is dual to W, the ordered set obtained by strengthening the order on $V_n \dot{\cup} V_n$ by putting $0 < \tilde{0}$. Lemma 3.4 establishes a little more than this.

Lemma 3.4. *A subalgebra M of $\underline{P}_n^2$ is contained in $\alpha^{-1}(\leq)$ if and only if $\varphi_M(0) \leq \varphi_M(\tilde{0})$.*

Proof. Let f_M be the embedding dual to φ_M. We have, by (†) in 3.2,

$$\begin{aligned}\varphi_M(0) \leq \varphi_M(\tilde{0}) &\Leftrightarrow ((\forall a \in M)(\varphi_M(0))(a) \leq (\varphi_M(\tilde{0}))(a)) \\ &\Leftrightarrow ((\forall a \in M)(\varphi_M(0))(a) = 1 \Rightarrow (\varphi_M(\tilde{0}))(a) = 1) \\ &\Leftrightarrow ((\forall a \in M)0(f_M(a)) = 1 \Rightarrow \tilde{0}(f_M(a))) \\ &\Leftrightarrow ((\forall (b, c) \in M)0((b, c)) = 1 \Rightarrow \tilde{0}((b, c) = 1) \\ &\Leftrightarrow ((\forall (b, c) \in M)\alpha(b) = 1 \Rightarrow \alpha(c)) = 1) \\ &\Leftrightarrow M \subseteq \alpha^{-1}(\leq). \qquad \square\end{aligned}$$

We denote the set of subalgebras of $\underline{P}^2$ maximal in $\alpha^{-1}(\leq)$ by $\mathcal{M}$.

Lemma 3.5. *Let M be a subalgebra of $\underline{P}^2$. Then $M \in \mathcal{M}$ if and only if φ_M satisfies:*

(i) $\varphi_M(0) < \varphi_M(\tilde{0})$,

(ii) $\varphi_M(\tilde{0})$ *is nonmaximal*,

(iii) $|\varphi_M(\{1, 2, \ldots, n\})| = n$.

Proof. In order that M be nonmaximal it is necessary that there should exist a subalgebra M' such that $M \subset M' \subseteq \alpha^{-1}(\leq)$. By Theorem 3.2 there would then exist a surjective order-preserving map $\eta : \operatorname{Im} \varphi_{M'} \to \operatorname{Im} \varphi_M$ such that $\eta \circ \varphi_{M'} = \varphi_M$ and η is not an order-isomorphism. The conditions in the statement of the lemma are exactly those needed to ensure that $\operatorname{Im} \varphi_M$ cannot be 'expanded' in this way. For example, consider condition (i). Assume that $\varphi_M(0) = \varphi_M(\tilde{0})$. Let Z denote $\operatorname{Im} \varphi_M$ with a new bottom element, c, adjoined; certainly $Z \in \mathcal{Y}_n$. Define

$\psi: V_n \,\dot\cup\, \tilde V_n \to Z$ and $\eta: Z \to \operatorname{Im} \varphi_M$ as follows:

$$\psi(y) = \begin{cases} \varphi(y) & \text{if } y \neq 0, \\ c & \text{if } y = 0, \end{cases}$$

$$\eta(y) = \begin{cases} y & \text{if } y \neq c, \\ \varphi(0) & \text{if } y = c. \end{cases}$$

Then ψ is a p-morphism, η is order-preserving and φ_M has the nontrivial factorisation $\varphi_M = \eta \circ \psi$. Hence M is not maximal. Conditions (ii) and (iii) are handled similarly. □

We deduce that, as an ordered set, the dual $H(M)$ of a subalgebra M of $\underline{P}^2$ maximal in $\alpha^{-1}(\leq)$ is as in Fig. 3, where r lies between 1 and n. On this ordered set we superimpose the labels $0, 1, 2, \ldots, n, \tilde 0, \tilde 1, \tilde 2, \ldots, \tilde n$ to indicate the point of $\operatorname{Im} \varphi_M$ to which these points of $V_n \,\dot\cup\, \tilde V_n$ are mapped. Lemma 3.3 implies that the n maximal points of $H(M)$ are labelled bijectively with $1, 2, \ldots, n$ and that the labels $\tilde 1, \tilde 2, \ldots, \tilde n$ are distributed surjectively among the maximal points above the point labelled $\tilde 0$. Up to a permutation $1, 2, \ldots, n$ and a permutation of $\tilde 1, \tilde 2, \ldots, \tilde n$, the labelling is as shown in Fig. 3.

Each $\tilde X_i$ is a string of labels each of the form $\tilde x$, where $1 \leq x \leq n$. Define $X_i = \{x \mid \tilde x \text{ occurs in the string } \tilde X_i\}$. The sets $X_1, \ldots, X_r$ have the following properties:

(i) $i \neq j$ implies $X_i \cap X_j = \emptyset$,

(ii) $X_1 \cup \cdots \cup X_r = \{1, 2, \ldots, n\}$,

(iii) $k_1 \geq k_2 \geq \cdots \geq k_r$, where $k_i := |X_i|$,

(iv) $i < j$ implies $(\forall s \in X_i)(\forall t \in X_j) s < t$.

Given such a labelling (which we call *left-packed*), there is an associated partition $p = (k_1, k_2, \ldots, k_r)$ of the integer n. Conversely every partition $p = (k_1, k_2, \ldots, k_r)$ of n gives rise to a unique labelling satisfying (i)–(iv). Given p

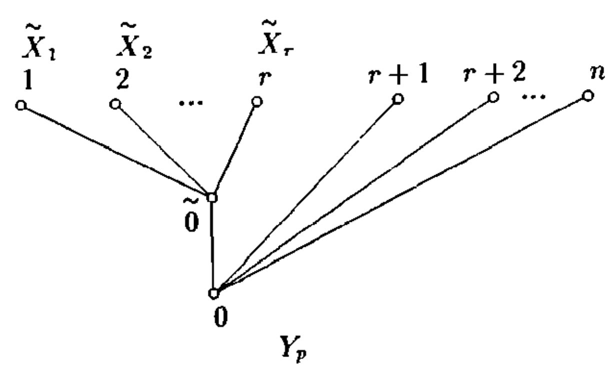


Fig. 3.

we define

$$X_1 = \{1, \ldots, k_1\}, \qquad X_2 = \{k_1 + 1, \ldots, k_1 + k_2\},$$
$$X_3 = \{k_1 + k_2 + 1, \ldots, k_1 + k_2 + k_3\}, \ldots, \qquad X_r = \{n - k_r + 1, \ldots, n\}.$$

We denote the maximal subalgebra of $\alpha^{-1}(\leqslant)$ associated with the partition $p = (k_1, k_2, \ldots, k_r)$ by $M(k_1, k_2, \ldots, k_r)$ and denote the associated labelled ordered set by Y_p.

Our final task in this section is to show that we have isolated a sufficient set of subalgebras to define a duality. Given $M \in \mathcal{M}$, there exists a partition $(k_1, k_2, \ldots, k_r)$ of n and automorphisms f and g of $\underset{\sim}{P}_n$ such that $(b, c) \in M$ if and only if $(f(b), g(c)) \in L := M(k_1, k_2, \ldots, k_r)$. This is simply the formal statement of our earlier observation that each M is, to within permutations of the labels, of the form $M(k_1, k_2, \ldots, k_r)$. Regarding M and L as relations, we may write

$$M = ((L \circ f)^{-1} \circ g)^{-1}.$$

Here $\circ$ denotes the relational product given by

$$r \circ h = \{(a, b) \in P^2 \mid (h(a), b) \in r\},$$

for a binary relation r and a map h.

Each of f and g is a composite of powers of f_σ and f_τ. It follows from [10, pp. 140–142, (b) and (j)], that a map $\varphi : D(A) \to \underset{\sim}{P}$ preserving f_σ, f_τ and L also preserves M. We therefore obtain the promised duality, needing at most $p(n) + 3$ relations, as a corollary of the Piggyback Duality Theorem, 2.1.

Theorem 3.6. *Let* $\underset{\sim}{P}_n := (P_n; \mathcal{T}, S \cup G)$, *with* $n \geqslant 1$, *where*:

(i) $\mathcal{T}$ *is the discrete topology*,

(ii) $S = \{M(k_1, k_2, \ldots, k_r) \mid (k_1, k_2, \ldots, k_r)$ *is a partition of* $n\}$,

(iii) G *is (the set of graphs of) a generating set for the endomorphism monoid of* $\underset{\sim}{P}_n$;

$$G = \begin{cases} \{e\} & \text{if } n = 1, \\ \{f_\sigma, e\} & \text{if } n = 2, \\ \{f_\sigma, f_\tau, e\} & \text{if } n \geqslant 3, \end{cases}$$

suffices, where f_σ, f_τ *and* e *are defined as above.*

4. The partition-induced relations

Let $p = (k_1, k_2, \ldots, k_r)$ be a partition of n and consider the associated subalgebra $L := M(k_1, k_2, \ldots, k_r)$ of $\alpha^{-1}(\leqslant)$. The algebra L is recaptured as the lattice of upper sets of $H(L)$, the ordered set in Fig. 3. Denote the lattice of upper sets of a finite ordered set Q by $\mathcal{U}(Q)$. We have

(i) $\mathcal{U}(1 \oplus Q) \cong \mathcal{U}(Q) \oplus \mathbf{1}$ (where $\mathbf{1}$ denotes the 1-element chain and $\oplus$ is linear sum), and

(ii) $\mathcal{U}(Q_1 \cup Q_2) \cong \mathcal{U}(Q_1) \times \mathcal{U}(Q_2)$.

Hence, as a lattice,

$$M(k_1, k_2, \ldots, k_r) \cong (\mathbf{2}^{n-r} \times (\mathbf{2}^r \oplus \mathbf{1})) \oplus \mathbf{1}$$

(where $\mathbf{2}$ is the 2-element chain). Hence the subalgebras $M(k_1, k_2, \ldots, k_r)$ and $M(l_1, l_2, \ldots, l_s)$ are isomorphic as algebras if and only if $r = s$, that is, if and only if the associated partitions have the same number of parts.

We have identified L as a lattice, but have not yet described how the labelling in Fig. 3 encodes the way L sits in $\underset{\sim}{P}_n^2$. Let $Y = \{\tilde{0}, 0, 1, 2, \ldots, n\}$ be the ordered set obtained from Y_p by deleting the tilda-ed labels, except $\tilde{0}$. For $Z \subseteq Y$, define $Z^{\uparrow}$ and $Z^{\downarrow}$ by

$$Z^{\uparrow} := \begin{cases} \bigcup \{X_x \mid x \in Z\} & \text{if } \tilde{0} \notin Z, \\ \{0, 1, 2, \ldots, n\} & \text{if } \tilde{0} \in Z, \end{cases}$$

$$Z^{\downarrow} := \begin{cases} Z \setminus \{\tilde{0}\} & \text{if } 0 \notin Z, \\ \{0, 1, 2, \ldots, n\} & \text{if } 0 \in Z. \end{cases}$$

For each upper set Z in Y, the sets $Z^{\uparrow}$ and $Z^{\downarrow}$ belong to $\underset{\sim}{P}_n$ (concretely realised in the way described in Section 1). Then

$$M(k_1, k_2, \ldots, k_r) = \{(Z^{\downarrow}, Z^{\uparrow}) \mid Z \in \mathcal{U}(Y)\}.$$

We can alternatively specify $M(k_1, k_2, \ldots, k_r)$ as the algebra $(2^{n-r} \times (2^r \oplus 1)) \oplus 1$ with atoms $\{(i, X_i) \mid i = 1, \ldots, n\}$, where $X_i = \emptyset$ for $i > r$.

Before we analyse in detail the subalgebras $M(k_1, k_2, \ldots, k_r)$ *qua* relations, we consider the $\boldsymbol{B}_n$ duality for small values of n.

When $n = 1$ there is only the single partition, (1), so that, as in [11] and [12], we obtain a duality with just one relation. When $n = 2$ we have two partitions, (1, 1) and (2). The corresponding subalgebras are those given for $\boldsymbol{B}_2$ in [11, 12].

Figs. 4 and 5 show the duals of the necessary maximal subalgebras in case $n = 3$ and $n = 4$. For $n = 3$ the associated subalgebras are shown alongside.

For n equal to 1 or 2 the relational structore of $\underset{\sim}{P}_n$ given in [11] and [12] included a partial order, different from the partial order of $\underset{\sim}{P}_n$. This happens for every value of n. Consider the partition $(1, 1, \ldots, 1)$. The associated subalgebra is

$$M(1, 1, \ldots, 1) = \{(a, a) \mid a \in \underset{\sim}{P}_n\} \cup \{(d, \top)\}.$$

This corresponds to the partial order on P_n in which the only non-trivial comparability is $d < \top$.

In [11] the relation for $\boldsymbol{B}_2$ corresponding to the partition (2) was described as an 'almost order'—an antisymmetric, transitive relation which satisfies the reflexivity condition only on certain elements. The case $n = 2$ is too special to

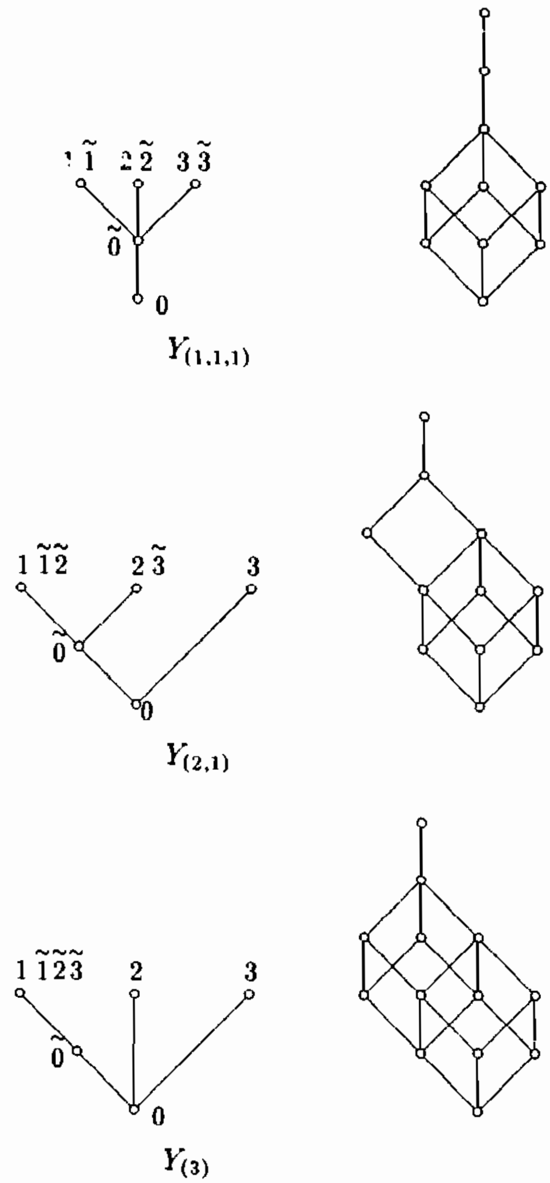


Fig. 4.

reveal the general pattern. It turns out that the subalgebras $M(k_1, k_2, \ldots, k_r)$, as relations, are best described as digraphs. In what follows we exclude the special partition $(1, 1, \ldots, 1)$, which we have already considered.

Fix $(k_1, k_2, \ldots, k_r) \neq (1, 1, \ldots, 1)$ and regard $\rho = M(k_1, k_2, \ldots, k_r)$ as a digraph $G = (P_n, \rho)$. We shall show that $P_n \backslash \{\top\}$ splits into the union of two disjoint connected subgraphs, $G_\perp$ and $G_\top$, each of which is almost a tree, in a

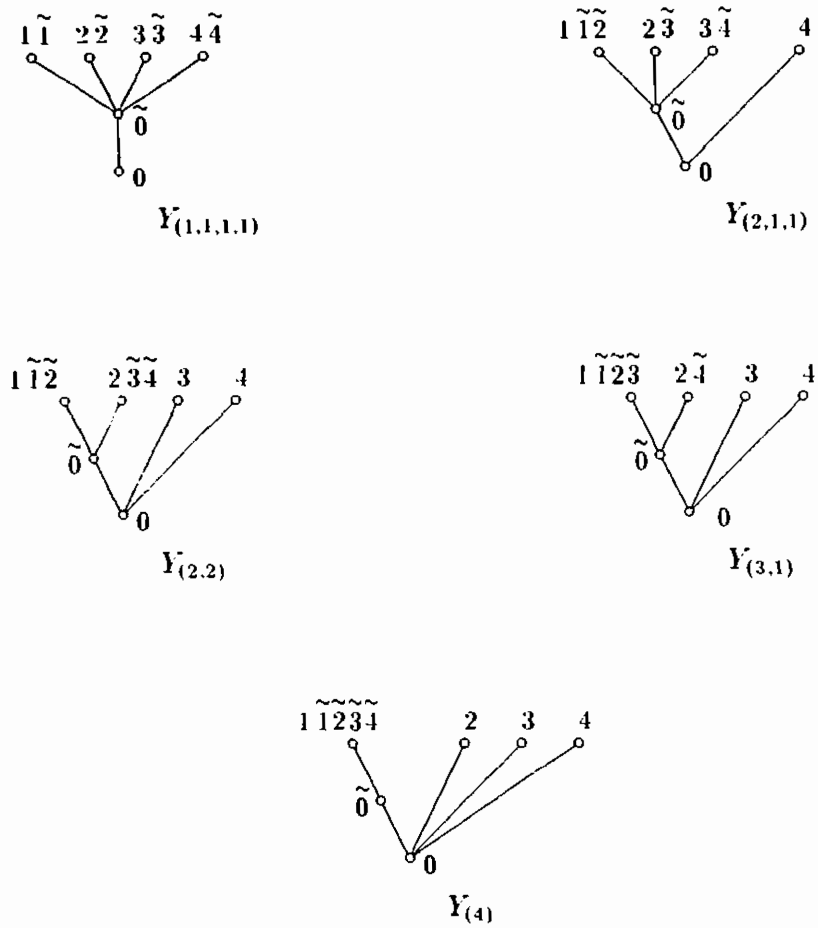


Fig. 5.

sense to be made precise. Moreover the components $G_\perp$ and $G_\top$ turn out to be isomorphic, so that each is of size 2^{n-1}. We define

$$G_\perp = \{A \in \mathscr{P}(\{1, 2, \ldots, n\}) \mid 1 \notin A\},$$

$$G_\top = \{A \in \mathscr{P}(\{1, 2, \ldots, n\}) \mid 1 \in A\}.$$

Certainly $P_n = G_\perp \cup G_\top \cup \{\top\}$ with $|G_\perp| = |G_\top| = 2^{n-1}$. The following proposition describes G.

Proposition 4.1. *Assume that $\rho = M(k_1, k_2, \ldots, k_r)$ is the relation determined by a partition p of n other than $(1, 1, \ldots, 1)$ and let $G = (P_n, \rho)$ be the associated digraph. Let $A \in P_n$. Then:*

(i) $(d, \top) \in \rho$.

(ii) *there is a loop at A if and only if $A \in \{\perp, d, \top\}$.*

(iii) *If* $(A, d) \in \rho$ *then* $(A, \top) \in \rho$.

(iv) *The digraph* G *is the disjoint union of the subgraphs* $G_\perp$ *and* $G_\top \cup \{\top\}$.

(v) *If the loops at* $\perp$, d *and* $\top$ *are deleted and, for all* A *with* $(A, d) \in \rho$, *the edge* $(A, \top) \in \rho$ *is deleted, then* $G_\perp$ *and* $G_\top$ *become trees. Moreover the map* $A \mapsto \{1, 2, \ldots, n\} \backslash A$ *is an isomorphism between the tree* $G_\perp$ *and the tree* $G_\top$.

Proof. Since $\bar{0}$ is not maximal in Y_p, we may take $A = \max Y_p$ to obtain $(d, d) = (A^\downarrow, A^\uparrow) \in \rho$. Since $0 < \bar{0}$, taking $A = \max Y_p \cup \{\bar{0}\}$ gives $(d, \top) = (A^\downarrow, A^\uparrow) \in \rho$. It is trivial that $(\perp, \perp)$ and $(\top, \top)$ are in ρ.

Since $p \neq (1, 1, \ldots, 1)$, we have $\{1, 2\} \subseteq X_1$. It follows that if A is a proper non-empty upper set in Y_p, then $A^\downarrow \neq A^\uparrow$. Thus the only loops occur at $\perp$, d and $\top$.

Assume that $(A, d) \in \rho$. Then there exists $U \subseteq Y_p$ with $U^\downarrow = A$ and $U^\uparrow = d$. Thus $\bar{0} \notin U$ while $\max \bar{0} = \{1, 2, \ldots, r\} \subseteq U$. Hence $V := U \cup \{\bar{0}\}$ is an upper set and yields $(A, \top) = (V^\downarrow, V^\uparrow) \in \rho$.

Now let $A \in P_n \backslash \{\top\}$ with $1 \in A$. Then we claim that there is a path $A = A_1, A_2, \ldots, A_l = \top$ from A to $\top$ in $G_\top$. Since there is an edge from d to 1 in $\top$, we may assume that $A \subsetneqq \{1, 2, \ldots\} = d$. We construct a path from A to d. Let $A_1 = A$ and, for all $i \geqslant 1$, define $A_{i+1} = A_i^\uparrow$. (This is well defined since $A_i \subseteq \max Y$ implies $A_i^\uparrow \subseteq \max Y$, whence $A_i^\uparrow$ is an upper set.) As $1 \in A = A_i$ and $1 \in X_1$, it follows that $1 \in A_i$ for all i. By construction we have $(A_i, A_{i+1}) \in \rho$ (because $0 \notin A_i$, so that $A_i^\downarrow = A_i$). Since $\{1, 2\} \subseteq X_1$, it follows that if $\{1, 2, \ldots, s\} \subseteq A_i$, then $\{1, 2, \ldots, s, s+1\} \subseteq A_i^\uparrow = A_{i+1}$. Consequently for some l we have $A_l = d$. Note that the path from A to d is unique, from which it follows that $G_\top$ is a tree.

We now take $A \in P_n$ with $1 \notin A$. We claim that there is a path $A = A_1, A_2, \ldots, A_l = \perp$ $(=\emptyset)$ in $G_\perp$. Again the path is uniquely given by $A_{i+1} = A_i^\uparrow$. Since $1 \notin A$, the least element of $A = A_1$ is greater than 1. We know that $\{1, 2\} \subseteq X_1$. Thus if the least element of A_i is k, then the least element of $A_{i+1} = A_i^\uparrow$ is at least $k+1$. Eventually the least element of A_i will be at least $r+1$, in which case $A_{i+1} = \emptyset$. Since $G_\perp$ is connected and the path from $A \in G_\perp$ to $\perp$ is unique, we conclude that $G_\perp$ is a tree.

It is clear that complementation is a graph-isomorphism between the trees $G_\top$ and $G_\perp$. Finally, since each element of $G_\perp \cup G_\top$ other than $\perp$ and d has a unique upper cover, it follows that G is the disjoint union of $G_\perp$ and $G_\top \cup \{\top\}$. □

We can now see the extent to which the relation induced by a partition (other than $(1, 1, \ldots, 1)$) fails to be a partial order. We have $(A, A) \in \rho$ in $G_\perp \cup G_\top$ if and only if $A \in \{\perp, d, \top\}$. Given $A, B, C \in P_n$ such that $(A, B) \in \rho$, $(B, C) \in \rho$, $A \neq B$ and $B \neq C$, we have $B = d$ and $C = \top$. Consider the associated trees drawn with their roots uppermost. In $G_\perp$ we have reflexivity only at the top level, and no nontrivial transitivity. In $G_\top$ we have reflexivity at the top two levels, and nontrivial transitivity restricted to the top three levels. Fig. 6 shows the tree structures obtained in the case $n = 4$.

Table 1

(5, 1)	$q_1 = 2$.
(4, 2)	$q_1 = 2$.
(3, 3)	$q_1 = 2$.
(4, 1, 1)	$q_1 = 2$.
(3, 2, 1)	$q_1 = 2$.
(2, 2, 2)	$q_1 = 3$, $q_2 = 2$.
(3, 1, 1, 1)	$q_1 = 3$, $q_2 = 2$.
(2, 2, 1, 1)	$q_1 = 3$, $q_2 = 2$.
(2, 1, 1, 1, 1)	$q_1 = 5$, $q_2 = 4$, $q_3 = 3$, $q_4 = 2$.

We conclude with an explicit description of the tree G_ι which uniquely determines $M(k_1, k_2, \ldots, k_r)$, *qua* digraph. As usual we shall assume that $(k_1, k_2, \ldots, k_r)$ is a partition of n with $k_1 > 1$ (or, equivalently, $r < n$). Note that since each $k_i \geq 1$ and $k_1 > 1$, we have $k_1 + \cdots + k_{r-1} + 1 \geq r + 1$, whence

$$X_r \subseteq \{r+1, \ldots, n\}.$$

In case $r > 1$, we now define a sequence $r + 1 = q_0 > q_1 > \cdots > q_{k-1} > q_k = 2$ of integers in the following way:

$$q_0 = r + 1, \; q_k = 2,$$
$$q_j = \min\{i \mid X_i \subseteq \{q_{j-1}, \ldots, n\}\} \quad \text{for } 1 \leq j \leq k.$$

We interpose some simple illustrations before giving our characterisation of the trees associated with partitions. For both of the partitions (3, 1) and (2, 2) of the integer 4 we have $k = 1$ and $q_1 = 2$. Now consider partitions of 6. We obtain the sequences shown in Table 1.

We adopt the following additional notation. Given a set A we write $\mathcal{P}'(A)$ for $\mathcal{P}(A)\setminus\{\emptyset\}$ and let $\mathcal{P}^B(A)$ be the family of sets of the form $B \cup C$, where $C \subseteq A$. Finally, for $\emptyset \neq J \subseteq \{1, \ldots, n\}$, let $X_J = \bigcup \{X_i \mid i \in J\}$.

The statements in Proposition 4.2 supplement those in Proposition 4.1. All follow from Proposition 4.1 and the formula

$$M(k_1, k_2, \ldots, k_r) = \{(Z^\downarrow, Z^\uparrow) \mid Z \in \mathcal{U}(Y)\}.$$

Proposition 4.2 implies that the sequence of integers $q_0, q_1, \ldots, q_k$ derived from the partition $(k_1, k_2, \ldots, k_r)$ uniquely determines up to isomorphism the trees associated with $M(k_1, k_2, \ldots, k_r)$.

Proposition 4.2. *Let G_ι be as in Proposition 4.1. Assume that $1 < r < n$ and let $I_0 = \{r+1, \ldots, n\}$. For $1 \leq j \leq k$, let*

$$I_j = \{q_j, \ldots, q_{j-1} - 1\}.$$

(i) *The vertices of G_ι are the elements of $\mathcal{P}(\{2, \ldots, n\})$, with $\emptyset$ as the root.*
(ii) *The set of vertices at depth 1 is $\mathcal{P}'(I_0)$.*

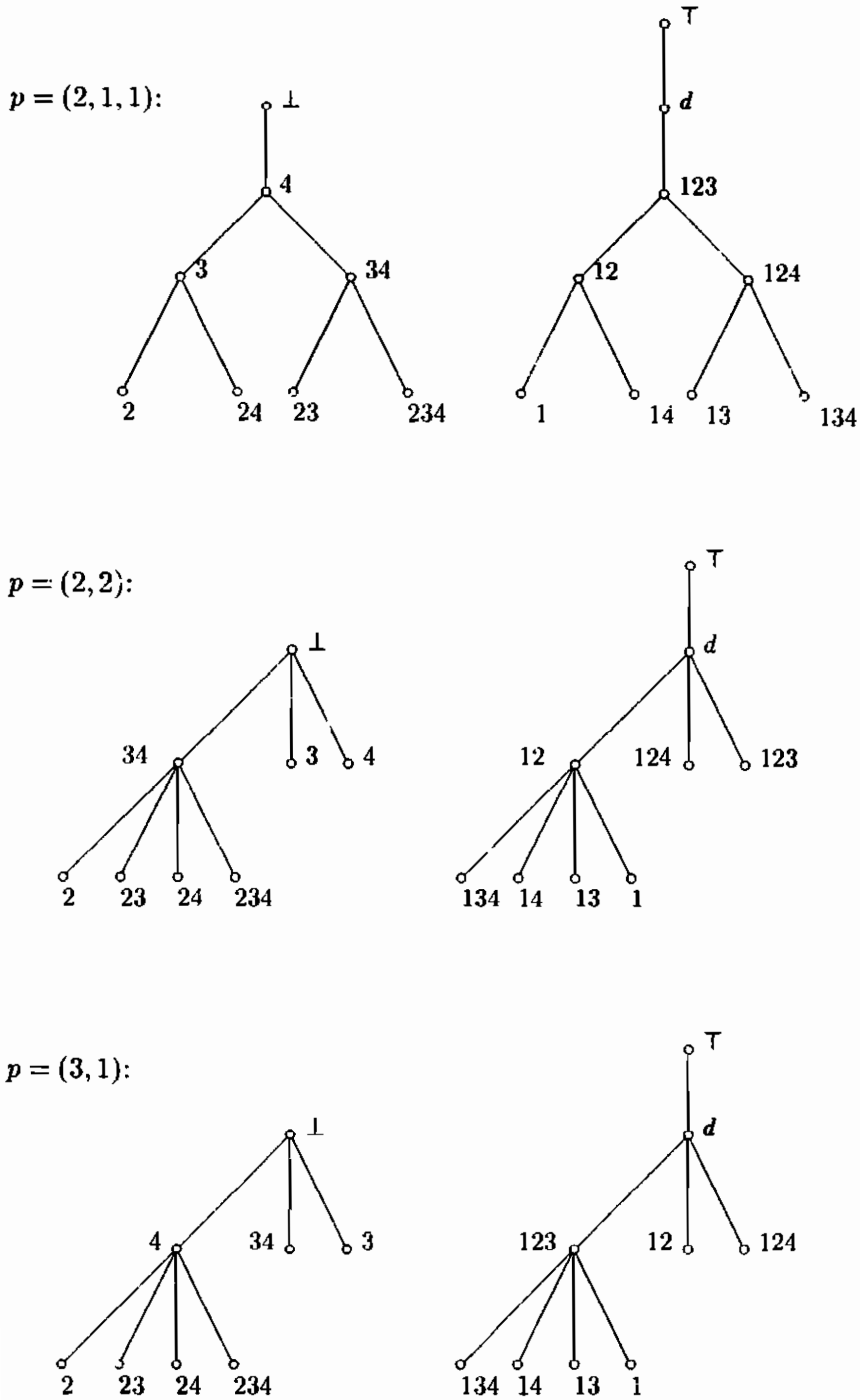

$p = (2,1,1)$:
⊥
4
3
34
2
24
23
234
⊤
d
123
12
124
1
14
13
134
$p = (2,2)$:
⊥
34
3
4
2
23
24
234
⊤
d
12
124
123
134
14
13
1
$p = (3,1)$:
⊥
4
34
3
2
23
24
234
⊤
d
123
12
124
134
14
13
1

Fig. 6.

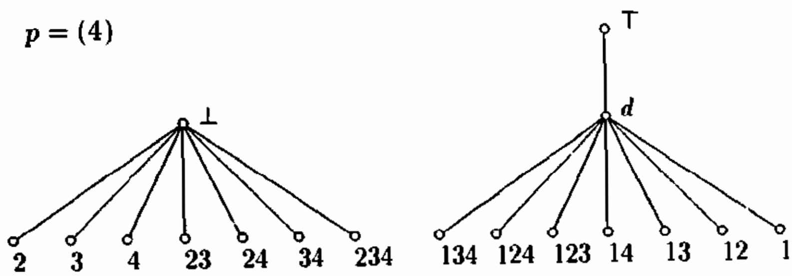


Fig. 6. (Cont.).

(iii) *The height of the tree is* $k+1$; *the depth of* $S \neq \emptyset$ *is*

$$\max\{i \mid S \cap I_i \neq \emptyset\} + 1.$$

(iv) *The nonleaves, other than the root, are precisely the sets of the form* X_J *for some non-empty subset* J *of* $\{1, \ldots, r\}$. *If the smallest integer in* J *is* i *and* $i \in I_j$, *then the depth of* X_J *is* j. *The descendants of* X_J *are the members of* $\mathcal{P}^J(I_0)$.

If $r = n$, *the tree* $G_\perp$ *has height* 1; *the empty set is the root and all non-empty subsets of* $\{2, \ldots, n\}$ *are leaves.*

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