

# OPTIMAL NATURAL DUALITIES FOR VARIETIES OF HEYTING ALGEBRAS

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ABSTRACT. The techniques of natural duality theory are applied to certain finitely generated varieties of Heyting algebras to obtain optimal dualities for these varieties, and thereby to address algebraic questions about them. In particular, a complete characterisation is given of the endodualisable finite subdirectly irreducible Heyting algebras. The procedures involved rely heavily on Priestley duality for Heyting algebras.

## 1. Introduction.

We may sum up the aims and methods of this paper as follows. We are interested in the symbiotic relationship between the solution of problems in universal algebra and the understanding of these problems interpreted in specific classes of algebras. We contend that, where it can be applied, duality theory has an important contribution to make in this area. We illustrate below how optimal dualities can be set up for varieties of distributive-lattice-ordered algebras, taking as examples some important finitely generated varieties of Heyting algebras. These dualities may be regarded as parallels to Priestley duality for distributive lattices. In order to derive them, we make extensive use of distributive lattice duality itself, as it restricts to Heyting algebras. Dualities in hand, we can answer purely algebraic questions. In this instance, we consider endoprimal algebras, a natural generalisation of primal algebras, and algebras close to these.

In a series of earlier papers, published over the past ten years and conveniently summarised in [6], or [4], a theory of natural dualities has been developed for suitable classes of algebras  $\mathbb{ISP}(\underline{M})$ . This theory allows algebras in such classes to be concretely represented as algebras of functions, in a way that enables free algebras to be described very directly. The general theory encompasses Stone duality for Boolean algebras and Priestley duality for distributive lattices. Many other varieties of interest in algebraic logic come within the scope of the theory: for example, de Morgan

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algebras, Kleene algebras,  $n$ -valued Łukasiewicz-Moisil algebras, and the varieties  $\mathbf{L}_n$  of relative-Stone Heyting algebras generated by the finite chains (which helped to motivate the general theory; see [5]). Alternatively, all these distributive-lattice-ordered algebras can be studied by setting up categorical equivalences based on Priestley duality for the lattice reducts. This latter approach has proved very profitable in practice, but such ‘borrowed’ dualities may have disadvantages. For example, they do not in general give easy access to free algebras (because products in the dual category fail to be concrete). Therefore it is beneficial to have a natural duality available when dealing with free algebras and consequently also with term functions.

Assume that  $\underline{M}$  is a finite non-trivial algebra, and that  $\tilde{M} = (M; R, \tau)$  is a structure in which  $R$  is a set of algebraic relations on  $\underline{M}$  (that is,  $R \subseteq \bigcup_{n \geq 1} \mathbb{S}(\underline{M}^n)$ ) and  $\tau$  is the discrete topology. Assume that  $R$  yields a duality on  $\mathcal{A}$  (we recall in the next section exactly what this means). Then we have an immediate description of the clone of term functions on  $\underline{M}$ : the  $n$ -ary term functions are given by the  $R$ -preserving maps from  $\tilde{M}^n$  to  $\tilde{M}$ . When  $R$  can be taken to be the empty set then every function from  $\tilde{M}^n$  to  $\tilde{M}$  ( $n \geq 1$ ) is a term function, that is,  $\underline{M}$  is primal. Various generalisations of primality have been studied, and lead to interesting structural properties of the associated varieties. For a discussion, for example, of quasiprimality, see [19] and [12], pp. 161–168. The class of endoprimal algebras is rather different, and less well understood. An algebra  $\underline{M}$  (not necessarily finite) is said to be *endoprimal* if every function commuting with the endomorphisms of  $\underline{M}$  is a term function. A finite algebra  $\underline{M}$  is *endodualisable* if the set of (graphs of) endomorphisms of  $\underline{M}$  yields a duality on  $\mathbb{ISP}(\underline{M})$ . The remarks above indicate that every endodualisable algebra  $\underline{M}$  is endoprimal.

The lattice of subvarieties of the variety,  $\mathbf{H}$ , of Heyting algebras is very rich. Recall that every finite distributive lattice with a new unit element is subdirectly irreducible as a Heyting algebra, and that every finite subdirectly irreducible algebra in  $\mathbf{H}$  is of this form. Such algebras provide a wealth of test case examples for general duality theory, of which we shall consider here some important instances. *Inter alia* we shall correct the list of endodualisable finite subdirectly irreducible Heyting algebras given in [14], and further prove that these algebras are exactly the finite chains and the algebra  $\mathbf{2}^2 \oplus \mathbf{1}$ . Also, our study of optimal dualities for finitely generated quasivarieties  $\mathbb{ISP}(\underline{M})$  of Heyting algebras allows us to show that many finite Heyting algebras  $\underline{M}$  are *almost endodualisable*, meaning that there is a set  $R$  yielding a duality which contains only one relation which is not the graph of an endomorphism.

For the classes of Heyting algebras we wish to consider it is the case that the variety  $\mathbb{HSP}(\underline{M})$  coincides with the quasivariety  $\mathbb{ISP}(\underline{M})$  for which our theorems are stated. In [14], a discussion is given of circumstances under which the variety and quasivariety do coincide: it is shown that this happens if and only if

- (S) every subdirectly irreducible subalgebra of a homomorphic image of  $\underline{M}$  has a Heyting embedding into  $\underline{M} \oplus \mathbf{1}$ .

This condition is satisfied whenever  $\underline{M}$  is built from the 2-element chain  $\mathbf{2}$  by taking cartesian products, linear sums and reduced linear sums (the **reduced linear sum** of

finite lattices  $L_1$  and  $L_2$  is derived from the linear sum  $L_1 \oplus L_2$  by identifying the top element of  $L_1$  with the bottom element of  $L_2$ ). The restrictions imposed by condition (S) can be circumvented by working with the generalised duality theory developed in [6]. We shall only to a very limited extent consider such dualities in this paper.

## 2. Preliminaries: natural dualities for distributive-lattice-ordered algebras

The variety  $\mathbf{D}$  of  $\{0, 1\}$ -distributive lattices has the two element chain  $\underline{\mathbf{2}} = \{0, 1\}$  as its only subdirectly irreducible algebra, and so by Birkhoff's Subdirect Product Theorem we have  $\mathbf{D} = \mathbb{ISP}(\underline{\mathbf{2}})$ . Recall that an ordered space  $(X; \leq, \tau)$  is *totally order-disconnected* if and only if given  $x \not\leq y$  in  $X$ , there exists a clopen up-set  $U$  such that  $x \in U$  and  $y \notin U$ . Let  $\mathbf{P}$  be the category whose objects are the compact totally order-disconnected spaces (Priestley spaces) and morphisms the continuous order-preserving maps. Let  $\underline{\mathbf{2}}$  be the 2-element chain, with the discrete topology. Then, for any set  $S$ , the space  $\underline{\mathbf{2}}^S$ , with product topology and pointwise order, belongs to  $\mathbf{P}$ . In fact the objects in  $\mathbf{P}$  are precisely those ordered spaces which can be identified with some closed subspace of a space  $\underline{\mathbf{2}}^S$  (with induced order and topology). We may therefore write, in shorthand,  $\mathbf{P} = \mathbb{IS}_c\mathbb{P}(\underline{\mathbf{2}})$ , the class of isomorphic copies of substructures of powers of  $\underline{\mathbf{2}}$  (the subscript  $c$  indicating that substructures are required to be topologically closed).

We set up functors as follows:

On objects:

$$\begin{aligned} H: L &\mapsto \mathbf{D}(L, \underline{\mathbf{2}}) \leq \underline{\mathbf{2}}^L, \\ K: Y &\mapsto \mathbf{P}(Y, \underline{\mathbf{2}}) \leq \underline{\mathbf{2}}^Y. \end{aligned}$$

On morphisms:

$$\begin{aligned} H: f &\mapsto - \circ f, \\ K: \varphi &\mapsto - \circ \varphi. \end{aligned}$$

(Here  $\leq$  means 'is a substructure of'.) Then  $H$  and  $K$  define an adjunction between  $\mathbf{D}$  and  $\mathbf{P}$  with unit and co-unit the evaluation maps  $\eta_L$  and  $\varepsilon_Y$  ( $L \in \mathbf{D}$  and  $Y \in \mathbf{P}$ ). Further,

$$\begin{aligned} (\forall L \in \mathbf{D}) \eta_L: L &\rightarrow KH(L) \text{ is an isomorphism,} \\ (\forall Y \in \mathbf{P}) \varepsilon_Y: Y &\rightarrow HK(Y) \text{ is an isomorphism,} \end{aligned}$$

and we have a categorical equivalence between  $\mathbf{D}$  and  $\mathbf{P}$ . The first of the above isomorphism assertions is the statement that we have a *duality*, the second that this duality is *full*. We also note that  $\underline{\mathbf{2}} = D(\mathbf{FD}(1))$  and, because products in  $\mathbf{P}$  are concrete, that  $\underline{\mathbf{2}}^\kappa = D(\mathbf{FD}(\kappa))$ . Since the distinction will be clear from the context we shall from now on suppress the symbols indicating to which category the 2-element chain belongs.

We parallel this procedure for a quasivariety  $\mathcal{A} := \mathbb{ISP}(\underline{M})$  which is generated by a given finite algebra  $\underline{M} = (M; \vee, \wedge, 0, 1, F)$  having a  $\mathbf{D}$ -reduct  $(M; \vee, \wedge, 0, 1)$ . (From now on we shall not distinguish notationally between an algebra  $A$  in  $\mathcal{A}$  and

its  $\mathbf{D}$ -reduct.) Let  $\tau$  be the discrete topology on  $M$ . We always assume subsets of powers of  $M$  to carry the induced topology derived from the product topology. We shall be concerned almost exclusively with finite objects, and on these the topology can be suppressed. We seek to form a structure  $\underline{M} = (M; R, \tau)$  so that

$$(\forall A \in \mathcal{A}) A \cong ED(A),$$

where  $D$  and  $E$  are the hom functors given on objects by

$$D: A \mapsto \mathcal{A}(A, \underline{M}) \leq \underline{M}^A \in \mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underline{M}),$$

$$E: X \mapsto \mathcal{X}(X, \underline{M}) \leq \underline{M}^X \in \mathcal{A} = \mathbb{ISP}(\underline{M}),$$

and on morphisms by composition. Here operations (in  $\underline{M}$ ) and relations (in  $\underline{M}$ ) are extended pointwise to subsets of powers. When  $A \cong ED(A)$  (under the evaluation map  $\eta_A$ ) for all  $A \in \mathcal{A}$  we say  $R$  yields a duality on  $\mathcal{A}$  or that  $R$  is a dualising set for  $\mathcal{A}$ . In this situation, the free algebras are given as follows:  $F\mathcal{A}(\kappa) = \mathcal{X}(\underline{M}^\kappa, \underline{M})$ . It turns out (see [12], [6] or [16] for details) that the functors are well-defined whenever  $R$  consists of algebraic relations. Further, because  $\underline{M}$  has a lattice reduct, so that the median provides a 3-ary near-unanimity term, the NU-Duality Theorem applies (see [12], Theorem 1.19, [6], Theorem 2.8 or [16], Theorem 2.4). This implies that  $R = \mathbb{S}(\underline{M}^2)$  yields a duality on  $\mathcal{A}$ . Here we have assumed a purely relational structure, and shall encompass endomorphisms and partial endomorphisms by regarding them as identified with their graphs. In the present context this is legitimate and convenient. Were we concerned with full dualities we should need explicitly to allow full and partial operations in the type of  $\underline{M}$ ; see [6].

The set of subalgebras  $\mathbb{S}(\underline{M}^2)$  is in general far too large and unwieldy to give a workable duality. Much effort has been expended over recent years in devising methods for identifying small dualising sets and, latterly, optimal dualising sets. The first major step in this direction was the piggyback method, [13], [14], [6]. We require only the special case given by the following theorem.

**Theorem 2.1. (The Piggyback Duality Theorem)** *Let  $\mathcal{A}$  and  $\underline{M}$  be as above. Let  $\Omega$  be a family of  $\mathbf{D}$ -homomorphisms from  $\underline{M}$  into  $\mathbf{2}$  and  $G$  a set of  $\mathcal{A}$ -endomorphisms of  $\underline{M}$  such that, for each  $A \in \mathcal{A}$ ,*

$$\{ \alpha \circ g : M \rightarrow \mathbf{2} \mid \alpha \in \Omega, g \in G \}$$

*separates the points of  $M$ . Define  $S$  to be the set of all  $\mathcal{A}$ -subalgebras of  $\underline{M}^2$  which are maximal in*

$$(\alpha, \beta)^{-1}(\leq) := \{ (a, b) \in M^2 \mid \alpha(a) \leq \beta(b) \},$$

*where  $\alpha, \beta$  range over  $\Omega$ . Then  $\underline{M} := (M; G \cup S, \tau)$  is such that  $G \cup S$  yields a duality on  $\mathcal{A} = \mathbb{ISP}(\underline{M})$ .*

When  $\alpha = \beta$  we write  $\alpha^{-1}(\leq)$  rather than  $(\alpha, \alpha)^{-1}(\leq)$ . Hereafter we shall, for brevity, adopt an abuse of terminology and refer to ‘a maximal subalgebra of  $\alpha^{-1}(\leq)$ ’ rather than to ‘a subalgebra of  $\underline{M}^2$  which is maximal in  $\alpha^{-1}(\leq)$ ’. When  $\underline{M}$  is a heyting algebra satisfying (S) it turns out that we can take the set  $\Omega$  in the preceding theorem to have just one element. The following result is proved in [14] (Section 3.5, Theorem D).

**Theorem 2.2.** *Let  $\underline{M}$  be a finite subdirectly irreducible Heyting algebra satisfying (S) and let  $\alpha: \underline{M} \rightarrow \mathbf{2}$  be the  $\mathbf{D}$ -homomorphism given by  $\alpha^{-1}(1) = \{1\}$ . Let  $S$  be the set of subalgebras of  $\underline{M}^2$  which are maximal in  $\alpha^{-1}(\leq)$ . Then  $\text{End } \underline{M} \cup S$  yields a duality on  $\mathbb{ISP}(\underline{M})$ .*

Subalgebras of  $\alpha^{-1}(\leq)$  were described algebraically in [14] and recognised as being the graphs of partial endomorphisms, that is, homomorphisms from  $\underline{N}$  to  $\underline{M}$ , where  $\underline{N}$  is a subalgebra of  $\underline{M}$ ; see [14].

While piggyback dualities are generally much more economical than those supplied by the NU-Duality Theorem, they may still contain many redundant relations. This possibility was first explored in [10], in which the finitely generated varieties  $\mathbf{B}_n = \mathbb{ISP}(\mathbf{2}^n \oplus \mathbf{1})$  of pseudocomplemented distributive lattices were investigated. The Piggyback Theorem supplied a dualising set for  $\mathbf{B}_n$  with  $O(2^n)$  members (see [6]). The test algebra technique developed in [10] allowed a dualising set with  $n + 3$  elements to be found. This technique relies on the observation that any algebraic relation,  $s$ , on an algebra  $\underline{M}$  exists as an algebra,  $\underline{s}$ , in  $\mathcal{A} = \mathbb{ISP}(\underline{M})$ . We say that  $R$  yields a duality on an individual algebra  $A \in \mathcal{A}$  if  $A \cong ED(A)$ . Further, a set  $R$  of finitary algebraic relations on  $\underline{M}$  is said to *entail* (or to *generate*) the relation  $s$  if for each  $A \in \mathcal{A}$  any map  $u: D(A) \rightarrow M$  which preserves  $R$  also preserves  $s$ .

**Lemma 2.3. (The Test Algebra Lemma, [10], Propositions 2.2, 2.3)** *Let  $R$  be a family of finitary algebraic relations on  $\underline{M}$ . If  $R$  yields a duality on the test algebra  $\underline{s} \leq \underline{M}^n$ , then  $R$  entails  $s$ . In particular, if  $R$  yields a duality on  $\mathcal{A}$  and if  $R \setminus \{s\}$  yields a duality on the test algebra  $\underline{s} \leq \underline{M}^n$ , then  $R \setminus \{s\}$  yields a duality on  $\mathcal{A}$ .*

It is of interest to see how far the behaviour of the varieties  $\mathbf{B}_n = \mathbb{ISP}(\mathbf{2}^n \oplus \mathbf{1})$  of pseudocomplemented distributive lattices is typical, in the number and type of relations in their dualising sets. We accordingly consider the Heyting varieties  $\mathbf{H}_n := \mathbb{ISP}(\mathbf{2}^n \oplus \mathbf{1})$ . The number of relations in the piggyback duality for  $\mathbf{H}_n$  grows even more rapidly with  $n$  than that in the piggyback duality for  $\mathbf{B}_n$ . Nevertheless we are able to show that, for every  $n$ , there is a 4-element subset yielding a duality. For each  $n \geq 3$ , we take three endomorphisms and are forced also to include just one relation which is not an endomorphism. In this case it happens to be extremely easy to show that the endomorphism monoid of  $\mathbf{2}^n \oplus \mathbf{1}$  is generated by 3 elements. In general we shall be concerned principally with dualities which are optimal ‘modulo the endomorphism monoid’. In other words, we shall seek dualising sets of the form  $R = \text{End } \underline{M} \cup S$ , where  $R \setminus \{s\}$  does not yield a duality for any  $s \in S$ .

### 3. Priestley duality for Heyting algebras

In order to apply Theorem 2.2 and the Test Algebra Lemma we re-cast the necessary notions in terms of Priestley duality. In the preceding section, this duality was set up in terms of  $\{0, 1\}$ -valued functions, in order to stress the parallels with the natural dualities we seek. When using Priestley duality as a practical tool, to describe piggybacking subalgebras for example, we shall work either with sets or with functions, as seems most convenient. We identify a lattice  $L$  in  $\mathbf{D}$  with the lattice of clopen up-sets of its dual space  $Y = H(L)$ ; this lattice of sets is isomorphic to the lattice of functions  $K(Y)$ . In fact, in later sections we shall switch backwards and forwards without comment between algebraic concepts and their dual interpretations. We shall henceforth write  $\mathbf{2}$  to denote both  $\underline{\mathbf{2}} \in \mathbf{D}$  and  $\underline{\mathbf{2}} \in \mathbf{P}$ , since it will be clear from the context which is intended.

We begin by recalling some folklore (see for example [1], [3], [18]). For undefined notation, see [7].

**Proposition 3.1.** *Let  $Y \in \mathbf{P}$ . Then the following are equivalent:*

- (1)  $K(Y)$  is (the reduct of) a Heyting algebra;
- (2)  $Y$  is such that  $\downarrow(a \wedge b) \in K(Y)$  for all  $a, b \in K(Y)$ ;
- (3)  $\downarrow c$  is open for every open subset  $c$  of  $Y$ .

If the conditions of the proposition are satisfied we say  $Y$  is a *Heyting space*. If  $Y$  is such a space then implication in  $K(Y)$  is given by

$$a \rightarrow b = Y \setminus \downarrow(a \wedge b).$$

Of course, any finite ordered set equipped with the discrete topology is a Heyting space.

**Proposition 3.2.** *Let  $A$  and  $B$  be Heyting algebras and  $f: A \rightarrow B$  be a  $\mathbf{D}$ -morphism. Then  $f$  is an  $\mathbf{H}$ -morphism if and only if  $H(f) = - \circ f: H(B) \rightarrow H(A)$  is such that*

$$\varphi(\uparrow y) = \uparrow \varphi(y) \text{ for all } y \in H(B),$$

where  $\varphi = H(f)$ .

Maps between Heyting spaces commuting with  $\uparrow$  are called *Heyting morphisms*. Summing up we have Theorem 3.3.

**Theorem 3.3.** *The functors  $H$  and  $K$  between  $\mathbf{D}$  and  $\mathbf{P}$  restrict to yield a dual equivalence between the category  $\mathbf{H}$  of Heyting algebras and the category  $\mathcal{Y}$  of Heyting spaces and Heyting morphisms.*

In order to identify the relations in the Piggyback Duality Theorem we shall need to combine the preceding result with some elementary facts about duality for finite distributive lattices, as set out in Proposition 3.4.

**Proposition 3.4.** *Let  $L_1$  and  $L_2$  be finite distributive lattices. Then*

- (i)  $L_1 \times L_2 \cong K(H(L_1) \dot{\cup} H(L_2))$ ;
- (ii)  $L_1 \oplus \mathbf{1} \cong K(\mathbf{1} \oplus H(L_1))$ ;
- (iii)  $f \in \mathbf{D}(L_1, L_2)$  is injective if and only if  $H(f) \in \mathbf{P}(H(L_2), H(L_1))$  is surjective.

Assume that we have a quasivariety  $\mathcal{A} = \mathbb{ISP}(\underline{M})$  where  $\underline{M}$  is a finite subdirectly irreducible Heyting algebra. The lattice  $\underline{M}$  is of the form  $L \oplus \mathbf{1}$  for some finite distributive lattice  $L$ . We denote the top element of  $L$  by  $d$ . Let  $V := H(\underline{M})$ . Let  $\alpha: \underline{M} \rightarrow \mathbf{2}$  be such that  $\alpha(a) = 1$  if and only if  $a = 1$  in  $\underline{M}$  (as in Theorem 2.2). Note that  $\alpha \in \mathbf{D}(\underline{M}, \mathbf{2}) = V$ . It is the bottom element of  $V$ . We shall denote the bottom element of a finite ordered set by  $\perp$ . We remark that  $\{\alpha \circ g \mid g \in \text{End } \underline{M}\}$  separates the points of  $\underline{M}$  if and only if  $V = \{\alpha \circ H(g) \mid g \in \text{End } \underline{M}\}$ ; cf. Theorem 2.1.

We wish to identify Heyting subalgebras of  $\underline{M}^2$  contained in  $\alpha^{-1}(\leq)$ . By Proposition 3.4, we shall need to consider  $\mathcal{Y}$ -morphic images of the disjoint union of two copies of  $V$ . For this purpose it will be convenient, where appropriate, to distinguish the copies of  $V$  by referring to the second copy as  $V_*$  and tagging its elements, and maps out of it, with  $*$ . Let  $W := V \dot{\cup} V_*$ . Subalgebras of  $\alpha^{-1}(\leq)$  are given dually by surjective  $\mathcal{Y}$ -morphisms  $\varphi$  with domain  $W$  such that  $\varphi(\perp) \leq \varphi(\perp_*)$  (see [16], Section 4). Let  $\varphi$  be such a map, and denote its image by  $Y$ . Let  $\sigma := \varphi|_V$  and  $\sigma_* := \varphi|_{V_*}$ . Since  $W = \uparrow\perp \dot{\cup} \uparrow\perp_*$  and  $\varphi$  is a Heyting morphism with  $\varphi(\perp) \leq \varphi(\perp_*)$ , we see that  $\sigma$  maps  $V$  onto  $Y$ , and belongs to  $\mathcal{Y}(V, Y)$ . Also  $\sigma_* \in \mathcal{Y}(V_*, Y)$ . Conversely any such pair  $\sigma, \sigma_*$  gives rise to a well-defined Heyting morphism on  $V \dot{\cup} V_*$ . We thus have the following characterisation.

**Lemma 3.5.** *There is a bijective correspondence between subalgebras of  $\alpha^{-1}(\leq)$  and pairs  $\sigma, \sigma_*$  of Heyting morphisms with domain  $V$  and such that  $\sigma$  is surjective and  $\text{im } \sigma_* \subseteq \text{im } \sigma$ . Under this correspondence the subalgebra associated with  $\sigma, \sigma_*$  is*

$$Q[\sigma, \sigma_*] := \{ (a \circ \sigma, a \circ \sigma_*) \in \underline{M}^2 \mid a \in K(\text{im } \sigma) \},$$

and  $H(Q[\sigma, \sigma_*]) = \text{im } \sigma$ .

Note that different pairs  $\sigma, \sigma_*$  can give rise to the same subalgebra of  $\underline{M}^2$ . We have  $Q[\sigma, \sigma_*] = Q[\sigma', \sigma'_*]$  if and only if there exists an automorphism  $g$  of  $K(\text{im } \sigma)$  such that  $\sigma' = H(g) \circ \sigma$  and  $\sigma'_* = H(g) \circ \sigma_*$ .

We recalled earlier that the subalgebras of  $\alpha^{-1}(\leq)$  are the graphs of partial endomorphisms of  $\underline{M}$ . We note that  $Q[\sigma, \sigma_*]$  is the graph of a partial isomorphism if and only if  $\text{im } \sigma = \text{im } \sigma_*$  (equivalently,  $\sigma(\perp) = \sigma_*(\perp_*)$ ). It is the graph of an endomorphism if and only if we can choose  $\sigma'_*$  such that  $Q[\sigma, \sigma_*] = Q[\text{id}_V, \sigma'_*]$ .

In order to apply the test algebra technique we want to know the action of the relations on  $D(\underline{s})$  for each relation  $s$  in a duality. Lemma 3.6 is a special case of a much more general result presented in [11] (Lemma 2.1). Before stating it, two items of notation are needed. Given a subalgebra  $\underline{r}$  of  $\underline{M}^2$ , we denote by  $\rho_1, \rho_2$  the restrictions to  $\underline{r}$  of the natural projections  $\pi_1, \pi_2: \underline{M}^2 \rightarrow \underline{M}$ . Also, given homomorphisms

$x_1, x_2 \in \mathbf{H}(\underline{s}, \underline{M})$ , we define  $x_1 \sqcap x_2: \underline{s} \rightarrow \underline{M}^2$  by  $(\forall a \in \underline{s}) (x_1 \sqcap x_2)(a) = (x_1(a), x_2(a))$ . Recall that  $(x_1, x_2) \in r$  on  $D(\underline{s})$  means  $(\forall a \in \underline{s}) (x_1(a), x_2(a)) \in r$ .

**Lemma 3.6.** *Let  $\underline{M}$  be a finite Heyting algebra, let  $\underline{r}$  and  $\underline{s}$  be subalgebras of  $\underline{M}^2$  and let  $x_1, x_2 \in D(\underline{s})$ . Then the following are equivalent:*

- (a)  $(x_1, x_2) \in r$  on  $D(\underline{s})$ ;
- (b) *there is a (necessarily unique) homomorphism, namely  $\gamma = x_1 \sqcap x_2$ , from  $\underline{s}$  to  $\underline{r}$  such that  $x_i = \rho_i \circ \gamma$  for  $i = 1, 2$ .*

**Proof.** Assume that  $(x_1, x_2) \in r$  on  $D(\underline{s})$ . Then  $(x_1, x_2)(\underline{s}) \subseteq r$  and hence the map  $\gamma := x_1 \sqcap x_2: \underline{s} \rightarrow \underline{r}$  is well defined and satisfies  $x_i = \rho_i \circ \gamma$  for  $i = 1, 2$ . The categorical definition of product asserts that  $\gamma$  is the unique homomorphism satisfying  $x_i = \rho_i \circ \gamma$  for all  $i$ . Thus (a) implies (b).

For the converse, first observe that  $(\rho_1, \rho_2) \in r$  on  $D(\underline{r})$ . It then follows immediately that  $(\rho_1 \circ \gamma, \rho_2 \circ \gamma) \in r$  on  $D(\underline{s})$ .  $\square$

We shall want subsequently to identify members  $g$  of  $D(\underline{s}) := \mathbf{H}(\underline{s}, \underline{M})$  with the corresponding members  $H(g)$  of  $\mathcal{Y}(V, H(\underline{s}))$ . We henceforth write  $Y_s$  for  $H(\underline{s})$ . We have the following corollary to Lemma 3.6.

**Corollary 3.7.** *Let  $\underline{r} = Q[\sigma, \sigma_*]$  be a subalgebra of  $\alpha^{-1}(\leq)$  and let  $\underline{s}$  be a subalgebra of  $\underline{M}^2$ . Let  $g_1, g_2 \in D(\underline{s})$ . Then  $(g_1, g_2) \in r$  on  $D(\underline{s})$  if and only if there exists  $\mu \in \mathcal{Y}(Y_r, Y_s)$  such that the diagram below commutes, that is, such that*

$$H(g_1) = \mu \circ \sigma \text{ and } H(g_2) = \mu \circ \sigma_*.$$

$$\begin{array}{ccc}
 V & \xrightarrow{H(g_1)} & Y_s \\
 \sigma \downarrow & \nearrow \mu & \uparrow H(g_2) \\
 Y_r & \xleftarrow{\sigma_*} & V
 \end{array}$$

In case the relation  $r$  is the graph of an endomorphism  $f$ , we may assume that  $\sigma = \text{id}_V$  and  $\sigma_* = H(f)$ . Then, in the above corollary, we have  $(g_1, g_2) \in r$  on  $D(\underline{s})$  if and only if  $H(g_1) \circ \sigma_* = H(g_2)$ . This is just the anticipated action by composition of the endomorphism monoid. We shall henceforth identify  $D(\underline{s})$  with  $\mathcal{Y}(V, Y_s)$  via the functor  $H$ . We write  $\text{End } V$  ( $\text{Aut } V$ ) for the image under the functor  $H$  of the endomorphism monoid (the automorphism group) of  $\underline{M}$ .

We now turn to *maximal* subalgebras of  $\alpha^{-1}(\leq)$ .

**Lemma 3.8.** *Let  $\varphi \in \mathcal{Y}(V \dot{\cup} V_*, Y)$  with  $\varphi$  surjective and  $\varphi(\perp) \leq \varphi(\perp_*)$ . Then  $Q[\varphi|_V, \varphi|_{V_*}]$  is a subalgebra of  $\alpha^{-1}(\leq)$  which is maximal in  $\alpha^{-1}(\leq)$  if and only if it is not possible to find  $Y' \in \mathcal{Y}$  such that there exist*

- (i) *a surjective map  $\varphi' \in \mathcal{Y}(V \dot{\cup} V_*, Y')$ , and*
- (ii)  *$\psi \in \mathcal{Y}(Y', Y)$  which is not an order-isomorphism*

*such that the diagram below commutes.*

$$\begin{array}{ccc}
 V & \xrightarrow{\varphi'} & Y' \\
 & \searrow \varphi & \downarrow \psi \\
 & & Y
 \end{array}$$

**Proof.** We observe that it is elementary to check that, if  $A$  and  $B$  are Heyting subalgebras of  $C$  and  $A \subseteq B$ , then  $A$  is a Heyting subalgebra of  $B$ . Alternatively, we may argue as follows. Were the assertion in (ii) that  $\psi$  be a  $\mathcal{Y}$ -morphism rather than a  $\mathbf{P}$ -morphism, then we would just have the definition of  $Q[\varphi|_V, \varphi|_{V_*}]$  being maximal as a subalgebra of  $\underline{M}^2$  contained in the sublattice  $\alpha^{-1}(\leq)$ . Hence if we assume only that  $\psi$  is a  $\mathbf{P}$ -morphism, we need to show that  $\psi$  is forced to be a Heyting morphism. Let  $z \in Y$  and  $\varphi'(u) \in Y'$  with  $z \geq \psi(\varphi'(u))$ . But  $\psi \circ \varphi' = \varphi$ , which is a Heyting morphism, so that there exists  $x \in V$  for which  $z = \varphi(x)$  and  $x \geq u$ . Since  $\varphi'$  is order-preserving,  $\varphi'(x) \geq \varphi'(u)$ . Thus for any  $y \in Y'$  we have  $\uparrow\psi(y) \subseteq \psi(\uparrow y)$ . The reverse inclusion holds because  $\psi$  is order-preserving, so  $\psi$  is indeed a Heyting morphism.  $\square$

We wish to be able to recognise that  $Q[\sigma, \sigma_*]$  is maximal in  $\alpha^{-1}(\leq)$  directly from the maps  $\sigma$  and  $\sigma_*$ . Theorem 3.10 achieves this. To obtain it we need the following dual characterisation of the maximal subalgebras of a Heyting algebra.

**Lemma 3.9.** (M.E. Adams [1], Lemma 4) *Let  $X$  be the dual space of a finite Heyting algebra  $A$  and let  $Y$  be a finite Heyting space. Then  $Y$  is (order-isomorphic to) the dual space of a maximal subalgebra of  $A$  if and only if there exist distinct points  $p$  and  $q$  of  $X$  satisfying*

$$\{p\} \cup \uparrow q = \{q\} \cup \uparrow p$$

*such that  $Y = \tau_{pq}(X)$ , where  $\tau_{pq}$  is the map on  $X$  which identifies  $p$  and  $q$ .*

We note that the finiteness restriction is not essential. However by imposing it we avoid any discussion of topology. It follows immediately from Lemma 3.9 that any surjective Heyting map  $\varphi: Y' \rightarrow Y$ , where  $Y$  and  $Y'$  are finite Heyting spaces with  $Y'$  not order-isomorphic to  $Y$ , can be expressed as the composition of a finite sequence of maps of the form  $\tau_{pq}$ .

Given  $Y := H(Q[\sigma, \sigma_*])$  we shall write, for each  $p \in Y$ ,

$$X^p := \sigma^{-1}(p) \text{ and } X_*^p := \sigma_*^{-1}(p).$$

Note that each  $X^p \neq \emptyset$  and that  $X_*^p \neq \emptyset$  if and only if  $p \geq \sigma_*(\perp_*)$ .

The following theorem is easier to understand in practice than in theory. The reader may like to read the proof in parallel with the illustrative example which follows it.

**Theorem 3.10.** *Let  $Q[\sigma, \sigma_*]$  be a subalgebra of  $\alpha^{-1}(\leq)$  with dual space  $Y$ . Then  $Q[\sigma, \sigma_*]$  is not maximal in  $\alpha^{-1}(\leq)$  if and only if, for some  $p \in Y$ , one of the following occurs:*

- (a)  $p$  is such that  $X^p$  can be decomposed as a linear sum  $F \oplus G$  where  $F \neq \emptyset$ ,  $G \neq \emptyset$  and, in case  $p \in \text{im } \sigma_* \setminus \{\sigma_*(\perp_*)\}$ ,  $X_*^p$  can be decomposed as a linear sum  $F_* \oplus G_*$  where  $F_* \neq \emptyset$ ,  $G_* \neq \emptyset$ ;
- (b)  $p \neq \perp$ ,  $X^p$  can be written as the disjoint union of non-empty sets  $F$  and  $G$  such that

$$\downarrow X^p \setminus X^p = \downarrow F \setminus F = \downarrow G \setminus G,$$

and, in case  $p \in \text{im } \sigma_* \setminus \{\sigma_*(\perp_*)\}$ ,  $X_*^p$  can be written as the disjoint union of non-empty sets  $F_*$  and  $G_*$  such that

$$\downarrow X_*^p \setminus X_*^p = \downarrow F_* \setminus F_* = \downarrow G_* \setminus G_*.$$

**Proof.** We begin with some general observations. Suppose we know that  $Y$  is the image of a Heyting space  $Y'$  under some Heyting map  $\tau_{uv}$ . Two cases arise: either  $u$  and  $v$  are comparable (without loss of generality  $u < v$ ) or they are incomparable. In the former case,  $\uparrow u = \{u\} \cup \uparrow v$  in  $Y'$  and in the latter  $\uparrow u \cup \{v\} = \uparrow v \cup \{u\}$ . Now consider reversing the process. Suppose  $Y$  is given and take any point  $p$  in  $Y$ . Now replace  $p$  by two points,  $u$  and  $v$  and let  $Y' := (Y \setminus \{p\}) \cup \{u, v\}$ . We can define a strict order relation on  $Y'$  in the following way:

- (i) if  $x, y \notin \{u, v\}$  then  $x < y$  in  $Y'$  if and only if  $x < y$  in  $Y$ ,
- (ii) if  $x = u$  or  $x = v$  and  $y \notin \{u, v\}$  then  $x < y$  in  $Y'$  if and only if  $p < y$  in  $Y$  and  $x > y$  in  $Y'$  if and only if  $p > y$  in  $Y$ , and
- (iii) if  $\{x, y\} = \{u, v\}$ , then  $x$  and  $y$  are incomparable in  $Y'$ .

Alternatively we may replace (iii) by the requirement that  $u < v$  in  $Y'$ . In both cases we get a well-defined order relation, and the map  $\tau_{uv}: Y' \rightarrow Y$  is a Heyting morphism. We say  $Y'$  is obtained by a *horizontal split* if  $u \parallel v$  and by a *vertical split* if  $u < v$ .

We conclude that any obstacle to ‘splitting’ points of  $Y = H(Q[\sigma, \sigma_*])$  in the manner described above comes from the restrictions to be imposed upon  $Y'$ —namely the requirement that it be of the form  $H(Q[\tau, \tau_*])$  for some  $\tau, \tau_*$ .

Suppose  $Y'$  is obtained from  $Y$  by splitting  $p$  into  $u$  and  $v$ . In order to recognise  $Y'$  as  $H(Q[\tau, \tau_*])$  we must specify  $\tau$  and  $\tau_*$  by defining  $Y^u := \tau^{-1}(u)$ ,  $Y^v := \tau^{-1}(v)$ ,

$Y_*^u := \tau_*^{-1}(u)$ , and  $Y_*^v := \tau_*^{-1}(v)$ ; for  $y \notin \{u, v\}$  we must make  $\tau^{-1}(y) = \sigma^{-1}(y)$  and  $\tau_*^{-1}(y) = \sigma_*^{-1}(y)$ . Thus the problem reduces the question of when it is possible to decompose  $X^p$  into disjoint sets  $Y^u$  and  $Y^v$  and  $X_*^p$  into disjoint sets  $Y_*^u$  and  $Y_*^v$  so that  $\tau$  and  $\tau_*$  really do define a legitimate subalgebra  $Q[\tau, \tau_*]$  either when  $u < v$  in  $Y'$  or when  $u \parallel v$ .

Notice that because  $\tau$  and  $\tau_*$  are defined on different components of a disjoint union of ordered sets, the tagged and untagged sets can be handled separately, with the proviso that either  $u < v$  or  $u \parallel v$  must be permissible for both simultaneously.

Since  $\tau$  is surjective, each of  $Y^u$  and  $Y^v$  must be non-empty. Thus it is impossible to split  $p$  if  $X^p$  has just one element. Similarly, since  $\tau_*$  maps onto  $\uparrow\tau_*(\perp_*)$ , it is impossible to split  $p \in \uparrow\tau_*(\perp_*) \setminus \{\tau_*(\perp_*)\}$  if  $X_*^p$  has just one element. We assume henceforth that these restrictions are not met.

Suppose that we try to split  $p$ . Note that  $X^p$  is order-convex, and that the Heyting morphism condition  $\tau(\uparrow x) = \uparrow\tau(x)$  automatically holds for  $x$  strictly above a maximal element of  $X^p$  because the corresponding condition holds with  $\sigma$  in place of  $\tau$ . Thus we only need to ensure that  $\tau(\uparrow x) = \uparrow\tau(x)$  holds for  $x \in \downarrow X^p$ .

If  $p = \perp$ , then we can split  $p$  into  $u$  and  $v$  with  $u < v$  and define  $Y^u = \{\perp\}$ ,  $Y^v = X^p \setminus \{\perp\}$ ,  $Y_*^u = \emptyset$ , and  $Y_*^v = X_*^p$ . Assume henceforth that  $p \neq \perp$ .

We now seek to split  $p$  in case  $p \neq \perp$  and  $X^p$  has more than one element. There exists by hypothesis  $q \in Y$  with  $q < p$ . Since we require  $\tau(\uparrow x) = \uparrow\tau(x)$  for each  $x \in X^q$  we are forced to have both  $Y^u$  and  $Y^v$  non-empty for a split to be possible. Additionally we must define  $Y^u$  and  $Y^v$  to ensure that, for  $x \in Y^u$  or  $x \in Y^v$ ,  $\tau(\uparrow x) = \uparrow\tau(x)$ . If we are to make a split with  $u < v$  then it is necessary that  $X^p$  be the linear sum  $F \oplus G$  of two non-empty subsets  $F$  and  $G$ , and, conversely, when this occurs we can define a vertical split by taking  $Y^u := F$  and  $Y^v := G$ .

Finally assume that  $X^p$  has more than one element and is not a non-trivial linear sum. In order for a horizontal split to be possible in which at least  $\tau(\uparrow x) = \uparrow\tau(x)$  for those  $x$  which lie in  $\uparrow X^p$  we must have  $X^p$  as the disjoint union of non-empty subsets  $F$  and  $G$ , in which case we can take  $Y^u = F$  and  $Y^v = G$ . It is not guaranteed that such a decomposition will make  $\tau$  a Heyting morphism. In order that it should be it is necessary and sufficient that any element strictly below a minimal element of  $X^p$  should be both below an element of  $F$  and an element of  $G$ . In symbols we require

$$\downarrow X^p \setminus X^p = \downarrow F \setminus F = \downarrow G \setminus G.$$

We now sum up. The cases enumerated in the statement of the theorem cover the various ways non-maximality can occur:

- (a) a vertical split of a point  $p$ ,
- (b) a horizontal split of a point  $p$ .

For case (a) the appropriate non-trivial vertical decomposition of  $X^p$  must be possible, and for case (b) the appropriate horizontal decomposition. If  $p = \perp$  then only (a) is a possibility. Turning to the tagged sets we note that unless  $\tau_*(\perp_*)$  lies strictly below  $p$  there is no need for  $Y_*^u$  to be non-empty, and we may take  $Y_*^u = \emptyset$  and  $Y_*^v = X_*^p$

both in case (a) or case (b). If however  $\tau_*(\perp_*)$  lies strictly below  $p$  we must impose the same decomposability requirement on  $X_*^p$  as on  $X^p$ .  $\square$

As promised, we now indicate how Theorem 3.10 works in a particular case. We take  $\underline{M}$  to be  $\mathbf{2}^2 \oplus \mathbf{2}^2 \oplus \mathbf{1}$ , so that  $H(\underline{M})$  is  $X$ , as shown in Figure 1. Suppose that  $Q[\sigma, \sigma_*]$  is a subalgebra  $\underline{s}$  of  $\alpha^{-1}(\leq)$  with  $H(\underline{s}) = Y$ , say. Then  $Y$  may be  $X$  itself, a chain with at most 3 elements, a 2-element antichain with a bottom element adjoined, or one of the spaces  $Y_1, Y_2$  or  $Y_3$  shown in Figure 1. The fact that  $\sigma$  and  $\sigma_*$  are Heyting morphisms with  $\text{Im } \sigma = Y$  allows us to make the following observations. The images of the maximal points in  $X$  under  $\sigma$  and  $\sigma_*$  are maximal in  $Y$ , and  $\sigma(5) = \perp$  in  $Y$ . Under  $\sigma$ , either  $\sigma(3) = \sigma(1) = \sigma(2)$  or 3 is mapped by  $\sigma$  to a common lower cover of  $\sigma(1)$  and  $\sigma(2)$ . The same applies to  $\sigma(4), \sigma_*(3)$  and  $\sigma_*(4)$ . If  $Y$  contains a 3-element chain then at least one of  $\sigma(3)$  and  $\sigma(4)$  is non-maximal.

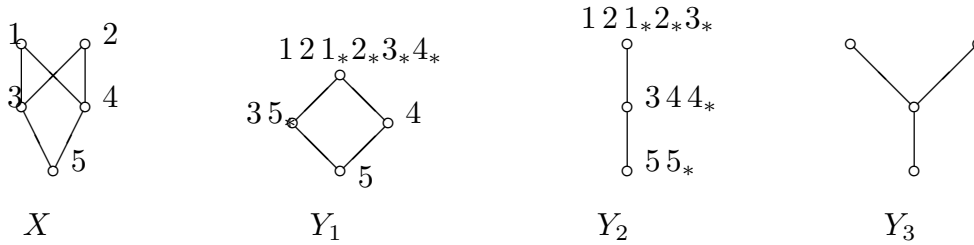


Figure 1

The space  $X$  itself (with  $\sigma = \sigma_* = \text{id}$ ) is the dual of the diagonal subalgebra of  $\underline{M}^2$ , and of subalgebras obtained from this by the action of automorphisms. Now suppose that  $Y$  is a 2-element chain  $\perp < \top$ . If  $X^\perp$  contains a point other than  $\perp$  then  $\perp$  splits vertically. If  $X^\top = X \setminus \{\perp\}$  then  $\top$  splits horizontally. One of these cases must occur. Hence there is no maximal subalgebra  $\underline{s}$  of  $\alpha^{-1}(\leq)$  with  $H(\underline{s})$  a 2-element chain, and, even more simply, none with  $|H(\underline{s})| = 1$ . In case  $Y$  is a 2-element antichain with a bottom element adjoined we have  $X^\perp = \{3, 4, 5\}$ , so that  $\perp$  splits vertically. Again  $Y$  is not the dual of any maximal subalgebra of  $\alpha^{-1}(\leq)$ . If  $Y = Y_3$  then the upper cover of  $\perp$  in  $Y$  splits horizontally, and no maximal subalgebra arises.

It is possible to find maps  $\sigma$  and  $\sigma_*$  onto each of  $Y_1$  and  $Y_2$  so that  $\underline{s} = Q[\sigma, \sigma_*]$  is maximal in  $\alpha^{-1}(\leq)$ . Suitable maps are indicated in Figure 1, with starred and unstarred figures marking the images of the corresponding elements of  $X$  under  $\sigma$  and  $\sigma_*$ . It is easily seen that no splitting is possible for either of these configurations. Other maximal subalgebras are obtained by permuting the labels the labels 3 and 4 (such a permutation corresponding to action by an automorphism on the associated subalgebras). It is readily checked that no other subalgebra  $\underline{s}$  with  $H(\underline{s}) = Y_1$  or  $Y_2$  is maximal.

In the last section we shall derive piggyback dualities in a number of simple cases. In practice it is easier to resort to computer assistance to find subalgebras of  $\alpha^{-1}(\leq)$  and to test them for maximality rather than to argue in the manner we did above. Theorem 3.10 principally serves to reveal the diversity of piggybacking subalgebras that can arise for arbitrary  $\underline{M}$ .

#### 4. Endodualisable Heyting algebras

This section uses the machinery we have set up to characterise those finite subdirectly irreducible Heyting algebras  $\underline{M}$  such that  $\text{End } \underline{M}$  yields a duality on the quasivariety generated by  $\underline{M}$ . It was proved in [5] that the chain  $\mathbf{n}$  is endodualisable for each  $n \geq 2$  and in [13] that  $\mathbf{2}^2 \oplus \mathbf{1}$  is endodualisable. In [14], Section 3.6, Theorem C, the claim was made that  $\mathbf{2}^2 \oplus \mathbf{n}$  is endodualisable for any  $n \geq 2$ . This claim was not fully substantiated, and is wrong: see Lemma 4.5 below. We prove in Theorem 4.7 that the finite subdirectly irreducible Heyting algebras which are endodualisable are just  $\mathbf{n}$  ( $n \geq 2$ ) and  $\mathbf{2}^2 \oplus \mathbf{1}$ . En route to our characterisation we first establish two general lemmas which allow us to show very easily that the algebras above are indeed endodualisable.

**Lemma 4.1.** *Let  $\underline{M}$  be any finite algebra. Let  $\underline{r}$  be a subalgebra of  $\underline{M}$  and let  $R$  be the set of partial endomorphisms with domain  $\underline{r}$ . Then  $R$  yields a duality on  $\underline{r}$ . In particular,  $\text{End } \underline{M}$  yields a duality on  $\underline{M}$ .*

**Proof.** Let  $u: D(\underline{r}) \rightarrow \underline{M}$  preserve  $R$ . Let  $\iota: \underline{r} \rightarrow \underline{M}$  be the inclusion map, and let  $a := u(\iota)$ . We claim that  $u = \eta_{\underline{r}}(a)$  (where  $\eta_{\underline{r}}: \underline{r} \rightarrow ED(\underline{r})$  is the evaluation map). First note that  $a \in \underline{r}$  (because  $u$  preserves  $\iota$ ). Take  $f \in R$ . Then  $f: \underline{r} \rightarrow \underline{M}$ . Since  $\iota \in \text{dom}_{D(\underline{r})} f$  we have  $u(f(\iota)) = f(u(\iota))$ . Hence

$$\eta_{\underline{r}}(a)(f) = f(a) = f(u(\iota)) = u(f(\iota)) = u(\iota \circ f) = u(f).$$

This proves our claim.  $\square$

**Lemma 4.2.** *Let  $\underline{M}$  be any finite algebra and  $\mathcal{A} = \text{ISP}(\underline{M})$ . If a set  $R$  of finitary algebraic relations on  $\underline{M}$  yields a duality on the algebra  $A \in \mathcal{A}$  then  $R$  yields a duality on (every isomorphic copy of) a retract of  $A$ .*

**Proof.** Let  $\underline{M} = (M; R, \tau)$ . Assume  $B$  is a retract of  $A$ . Let  $f: B \rightarrow A$  and  $g: A \rightarrow B$  be homomorphisms with  $g \circ f = \text{id}_B$ . Let  $\varphi: D(B) \rightarrow \underline{M}$  be in the second dual  $ED(B)$  of  $B$ . Then  $\varphi \circ D(f): D(A) \rightarrow \underline{M}$  is in the second dual of  $A$ . Hence  $\varphi \circ D(f) = \eta_A(a)$  for some  $a \in A$ . We claim that  $\varphi = \eta_B(b)$ , where  $b := g(a)$ . Let  $\theta \in D(B)$ . Then

$$\begin{aligned} \eta_B(b)(\theta) &= \theta(b) = \theta(g(a)) = (\theta \circ g)(a) = \eta_A(a)(\theta \circ g) \\ &= (\varphi \circ D(f))(\theta \circ g) = \varphi(\theta \circ g \circ f) = \varphi(\theta), \end{aligned}$$

as required.  $\square$

**Proposition 4.3.** *Let  $\underline{M}$  be a finite subdirectly irreducible Heyting algebra satisfying condition (S). If the domain of every non-extendable partial endomorphism is isomorphic to a retract of  $\underline{M}$ , then  $\text{End } \underline{M}$  yields a duality on  $\mathcal{A} = \text{ISP}(\underline{M})$ .*

**Proof.** Theorem 2.2 applies. Proposition 4.1 and Lemma 4.2 guarantee that  $\text{End } \underline{M}$  yields a duality on every test algebra arising from a maximal partial endomorphism. Hence every such relation may be deleted, by the Test Algebra Lemma.  $\square$

We can deduce the following corollary.

**Corollary 4.4.** *The chain  $\mathbf{n}$  ( $n \geq 2$ ) and  $\mathbf{2}^2 \oplus \mathbf{1}$  are endodualisable.*

**Proof.** For  $\mathbf{2}^2 \oplus \mathbf{1}$  it is easily seen that every partial endomorphism extends to an endomorphism (that is,  $\mathbf{2}^2 \oplus \mathbf{1}$  is self-injective). Now consider  $\mathbf{n}$ . Every subalgebra of  $\mathbf{n}$  is isomorphic to  $N_a := \downarrow a \cup \{1\}$  for some  $a \in \mathbf{n}$  and  $N_a$  is a retract of  $\mathbf{n}$ .  $\square$

We shall now prove the converse. Of course, if a set  $R$  of relations fails to yield a duality on some fixed  $A \in \mathcal{A}$ , then  $R$  fails to yield a duality on  $\mathcal{A}$ . Thus in order to show that a finite subdirectly irreducible algebra  $\underline{M}$  is not endodualisable it certainly suffices to find a subalgebra  $\underline{r}$  of  $\underline{M}^2$  on which  $\text{End } \underline{M}$  does not yield a duality. We shall always choose  $\underline{r}$  to be a maximal subalgebra of  $\alpha^{-1}(\leq)$ . Note however that we do not need to assume that (S) is satisfied.

Our first lemma refutes the claim in [14], Section 3.6, Theorem C.

**Lemma 4.5.** *The algebra  $\mathbf{2}^2 \oplus \mathbf{n}$  is not endodualisable for any  $n \geq 2$ .*

**Proof.** The dual space,  $V$ , of  $\underline{M} := \mathbf{2}^2 \oplus \mathbf{n}$  is of the form  $\mathbf{n} \oplus \bar{\mathbf{2}}$ , where  $\bar{\mathbf{2}}$  denotes a 2-element antichain. We label the elements of  $V$  as  $1, 2, \dots, n+2$ , with  $\max V = \{1, 2\}$  and  $3 > 4 > \dots > n+2$ . Let  $T = V \setminus \{1\}$ . Define  $\sigma, \sigma_* \in \mathcal{Y}(V, T)$  by

$$\begin{aligned} \sigma(1) &= 2, & \sigma(i) &= i \text{ for } i \geq 2, \\ \sigma_*(1) &= \sigma_*(2) = 2, & \sigma_*(i) &= i - 1 \text{ for } i \geq 3. \end{aligned}$$

Note that  $(1, d) \notin Q[\sigma, \sigma_*]$  (by Lemma 3.5). Define  $u(\varphi) = d$  if  $\varphi = \sigma_*$  and  $u(\varphi) = 1$  otherwise. We claim that there exists  $u: \mathcal{Y}(V, T) \rightarrow M$  such that  $u$  preserves each  $g \in \text{End } \underline{M}$  but does not preserve the relation  $r := Q[\sigma, \sigma_*]$ . Certainly  $(\sigma, \sigma_*) \in r$  on  $D(\underline{r})$  and  $(u(\sigma), u(\sigma_*)) \notin r$ .

Let  $\xi := H(g) \in \text{End } V$  and  $(x_1, x_2) \in g$  on  $D(\underline{r})$ . Then  $x_2 = x_1 \circ \xi$  (by the remarks following Corollary 3.7). Note that every element of  $\mathcal{Y}(V, T)$  is invariant under the action of  $\text{Aut } V$ , since  $\mathcal{Y}$ -morphisms map maximal points to maximal points and the only non-identity automorphism simply interchanges 1, 2 in  $V$  and fixes all other points. Thus  $u$  preserves each automorphism. Assume  $\xi \in \text{End } V \setminus \text{Aut } V$ . Then  $\perp < \xi(\perp)$ , and this, interpreted algebraically, says  $g(d) = 1$ . We have  $x_1(\xi(\perp)) = x_2(\perp)$ . But  $x_1(v) \geq v$  for every  $v \in V$ , so  $x_2(\perp) = x_1(\xi(\perp)) \geq \xi(\perp) > \perp$ . Suppose  $x_2 = \sigma_*$ . Then  $x_1(n+1) = n+1$ , so  $x_1 \neq \sigma_*$ . Consequently,  $g(u(x_1)) = g(1) = 1 = g(d) = g(u(x_2))$ . If  $x_2 \neq \sigma_*$  then  $g(u(x_1)) \geq g(d) = 1 = u(x_2)$ , so again  $g(u(x_1)) = u(x_2)$ . We conclude that  $u$  does indeed preserve  $\text{End } V$ .  $\square$

The remaining non-endodualisable algebras will all be recognised by applying the following lemma. It gives conditions which ensure that  $\underline{M}$  is not endualisable, by exhibiting a test algebra  $\underline{r}$  such that the endomorphism monoid does not yield a duality on  $\underline{r}$ .

**Lemma 4.6.** *Let  $\underline{M}$  be a finite subdirectly irreducible Heyting algebra and  $V = H(\underline{M})$ . Let  $T$  be a finite ordered set and assume that  $\sigma, \sigma_* \in \mathcal{Y}(V, T)$  are such that*

there exists  $w \in T$  such that  $\sigma^{-1}(w) = \sigma_*^{-1}(w) = \{\perp\}$ . Assume further that for any  $\xi \in \text{End } V$  and any  $\tau \in \mathcal{Y}(V, T)$ ,

$$\begin{aligned}\sigma \circ \xi &\neq \sigma_*, \\ \tau \circ \xi &\neq \sigma \text{ if } \xi \notin \text{Aut } V.\end{aligned}$$

Then there exists  $u: \mathcal{Y}(V, T) \rightarrow \mathbf{P}(V, \mathbf{2})$  such that  $u$  preserves each  $\xi \in \text{End } V$  but does not preserve the relation  $Q[\sigma, \sigma_*]$ .

**Proof.** By assumption, the relation  $r := Q[\sigma, \sigma_*]$  is the graph of a partial isomorphism  $\theta$  having  $d$  in its domain and such that  $\theta(d) = d$ . The diagram below commutes.

$$\begin{array}{ccc} V & \xrightarrow{d} & \mathbf{2} \\ \sigma \downarrow & \nearrow \alpha & \uparrow \theta(d) \\ T & \xleftarrow{\sigma_*} & V \end{array}$$

We let  $O_\sigma$  denote the orbit of  $\sigma$  under the natural action of  $\text{Aut } V$ , so that  $\tau \in O_\sigma$  if and only if  $\sigma \circ \xi = \tau$  for some  $\xi \in \text{Aut } V$ .

We need to construct  $u: \mathcal{Y}(V, T) \rightarrow M = \mathbf{P}(V, \mathbf{2})$  which preserves all  $\xi \in \text{End } V$  but such that  $u$  does not preserve  $r$ . Define  $u$  by

$$u(\varphi) = \begin{cases} d & \text{if } \varphi \in O_\sigma, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that  $u$  is certainly well defined. Consider  $\xi \in \text{End } V$  and  $\varphi \in \mathcal{Y}(V, T)$ . We distinguish the following cases.

(i) If  $u(\varphi) = d$  and  $\xi \in \text{Aut } V$ , then

$$u(\varphi \circ \xi) = d = d \circ \xi = u(\varphi) \circ \xi.$$

(ii) If  $u(\varphi) = d$  and  $\xi \notin \text{Aut } V$ , then

$$u(\varphi \circ \xi) = 1 = d \circ \xi = u(\varphi) \circ \xi.$$

(iii) If  $u(\varphi) = 1$  then  $\xi$  is such that  $\varphi \circ \xi \notin O_\sigma$ , so

$$u(\varphi \circ \xi) = 1 = u(\varphi) \circ \xi.$$

We have proved that  $u$  preserves all elements  $\xi \in \text{End } V$ . However, by construction,  $(\sigma, \sigma_*) \in r$ , but  $(u(\sigma), u(\sigma_*)) = (d, 1) \notin r$ , since  $\sigma_* \notin O_\sigma$ .  $\square$

**Theorem 4.7.** *Let  $\underline{M}$  be a finite subdirectly irreducible Heyting algebra. Then the following are equivalent:*

- (1)  $\text{End } \underline{M}$  yields a duality on  $\text{HSP}(\underline{M})$ ;
- (2)  $\text{End } \underline{M}$  yields a duality on  $\text{ISP}(\underline{M})$ ;
- (3)  $\underline{M}$  is either a finite chain or is  $\mathbf{2}^2 \oplus \mathbf{1}$ .

**Proof.** Obviously (1) implies (2), and we have already proved that (3) implies (2), and hence also (1), since (S) holds for each of the algebras in (3). To complete the proof we shall show that (2) implies (3).

For the converse, let  $\underline{M}$  be a finite subdirectly irreducible Heyting algebra and let  $V := H(\underline{M})$ . Let  $n = 1 + \text{length}(V)$ . Label the elements of the  $n$ -element chain  $\mathbf{n}$  as  $1 > 2 > \cdots > n$ .

For  $a \in V$  define

$$\text{depth}(a) := \max\{\text{length}(C) \mid C \text{ is a chain from } a \text{ to } 1 \text{ in } V\},$$

and define subsets  $L_t$  of  $V$  (the *levels* of  $V$ ) by

$$L_t = \{x \in V \mid \text{depth}(x) = t - 1\}.$$

Define  $\sigma: V \rightarrow \mathbf{n}$  by

$$\sigma(a) = \text{depth}(a) + 1.$$

By construction  $\sigma \in \mathcal{Y}(V, \mathbf{n})$  and  $L_t = \sigma^{-1}(t)$ . Also, for any  $\tau \in \mathcal{Y}(V, \mathbf{n})$ , we have

$$(\forall a \in V) \sigma(a) \leq \tau(a).$$

This is easily established by induction on  $\text{depth}(a)$ .

We now distinguish two cases. First assume that there exists  $\ell \geq 2$  for which  $|L_\ell| \geq 2$ . Then there exists  $z$  such that  $z$  is a node in  $V$  and  $z$  is the largest node such  $\downarrow z$  is a chain. Define

$$k := \max\{t \mid |L_t| \geq 1\}.$$

Then  $n > k \geq 2$ . Pick  $w \in L_k$  and define  $\sigma_*: V \rightarrow \mathbf{n}$  by

$$\sigma_* \upharpoonright_{(V \setminus \{w\})} = \sigma \upharpoonright_{(V \setminus \{w\})}, \text{ and } \sigma_*(w) = k - 1.$$

Then  $\sigma_* \in \mathcal{Y}(V, \mathbf{n})$  and  $\sigma_*^{-1}(n) = \sigma^{-1}(n) = \{\perp\}$ .

We claim that, for all  $\xi \in \text{End } V$ , we have  $\sigma \circ \xi \neq \sigma_*$ . Suppose otherwise. Then in particular  $\sigma(\xi(\perp)) = \sigma_*(\perp) = n$ . Since  $\sigma^{-1}(n) = \{\perp\}$ , we deduce that  $\xi(\perp) = \perp$ . Then  $\xi(V) = \xi(\uparrow \perp) = \uparrow \xi(\perp)$ . Thus  $\xi$  is a surjective Heyting morphism. It follows that  $\xi$  is an order-isomorphism and so preserves the levels. Hence  $\xi(L_k) = L_k$ . But then  $\sigma(L_k) = \sigma_*(L_k)$ , contrary to hypothesis.

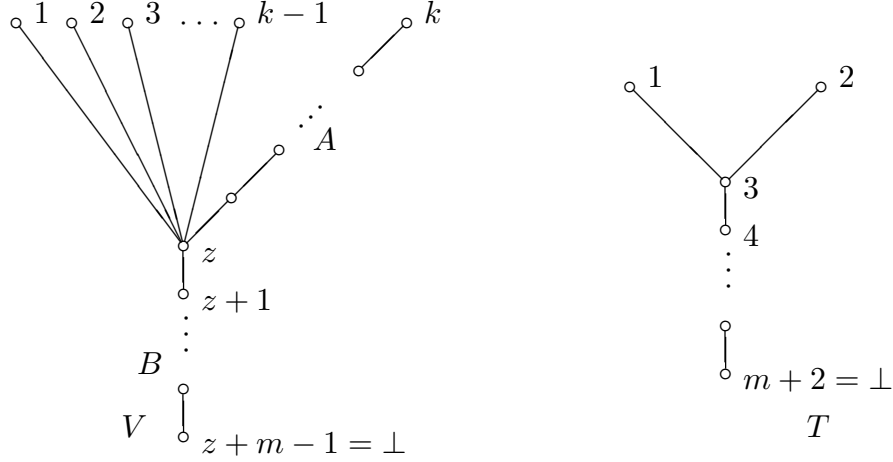
Note that if it were true that  $\tau \circ \xi = \sigma$  for some  $\tau \in \mathcal{Y}(V, \mathbf{n})$  and  $\xi \in \text{End } V$  then  $\tau(\xi(\perp)) = \sigma(\perp) = n$ . But, as observed above,  $\tau(\xi(\perp)) \geq \sigma(\perp)$ . Thus if  $\xi(\perp) \neq \perp$ , then  $\tau(\xi(\perp)) < n$ , which is a contradiction. So  $\xi(\perp) = \perp$  which again forces  $\xi$  to

be an automorphism. We conclude from Lemma 4.6, with  $T = \mathbf{n}$ , that  $\underline{M}$  is not endodualisable in case  $|L_\ell| \geq 2$  for some  $\ell \geq 2$ .

Now assume that  $|L_\ell| = 1$  for all  $\ell \geq 2$  and that  $V$  is neither a chain nor  $H(\mathbf{2}^2 \oplus \mathbf{2}) = \mathbf{1} \oplus \overline{\mathbf{2}}$ , where  $\overline{\mathbf{2}}$  denotes a 2-element antichain. This means that  $V$  is of the form

$$B \oplus (\overline{\mathbf{k-1}} \dot{\cup} A),$$

where  $A$  is a non-empty chain, whose largest element we denote by  $k$ ,  $B$  is an  $m$ -element chain,  $z > z+1 > \dots > z+m-1 = \perp$ , and  $\overline{\mathbf{k-1}}$  is an antichain with elements  $1, \dots, k-1$  (see Figure 2).



**Figure 2**

First consider the case in which  $A \setminus \{k\} \neq \emptyset$ . Define  $T$  to be  $C \oplus \overline{\mathbf{2}}$ , where  $C$  is the  $m$ -element chain  $3 > 4 > \dots > m+2 = \perp$  (see Figure 2). Let  $\sigma, \sigma_*: V \rightarrow T$  be as follows:

$$\begin{aligned} \sigma(1) &= 1, \quad \sigma(a) = 2 \text{ for all } a \in \max(V \setminus \{1\}), \\ \sigma_*(1) &= 2, \quad \sigma_*(a) = 1 \text{ for all } a \in \max(V \setminus \{1\}), \\ \sigma(z+j) &= \sigma_*(z+j) = j+3 \text{ for } j = 0, \dots, m-1, \\ \sigma(a) &= \sigma_*(a) = 3 \text{ for all } a \in A \setminus \{k\}. \end{aligned}$$

We claim that  $\sigma \circ \xi \neq \sigma_*$  for any  $\xi \in \text{End } V$ . Suppose not. As previously we obtain that  $\sigma(\xi(\perp)) = \sigma_*(\perp) = \perp$ , from which we deduce that  $\xi(\perp) = \perp$  and thence that  $\xi$  is an automorphism. Then  $\xi(z) = z$  and  $\xi(\downarrow k) = \downarrow k$ , which would force  $A = \{k\}$ , contrary to hypothesis.

We now aim to show that

$$(\forall \xi \in \text{End } V \setminus \text{Aut } V)(\forall \tau \in \mathcal{Y}(V, T)) \tau \circ \xi \neq \sigma.$$

Suppose we did have  $\tau \circ \xi = \sigma$  for some  $\tau$  and  $\xi$ . Then  $\tau(\xi(\perp)) = \sigma(\perp) = \perp$ . Thence, because  $\tau$  is a Heyting morphism,  $\xi(\perp) = \perp$ , which would force  $\xi$  to be an automorphism. Thus Lemma 4.6 again applies.

Now suppose  $A = \{k\}$ , so that  $V = \mathbf{m} \oplus \overline{\mathbf{k}}$  for some  $m \geq 1$  and some  $k \geq 2$ . First consider  $k \geq 3$ . Take  $T = \mathbf{m} \oplus \overline{\mathbf{k}-1}$ . Label the antichains of maximal points in  $V$  and  $T$  respectively by  $1, 2, \dots, k$  and  $1, 2, \dots, k-1$ , and the remaining elements as  $z > z+1 > \dots > z+m-1 = \perp$ . Define  $\sigma, \sigma_* \in \mathcal{Y}(V, T)$  by

$$\begin{aligned} \sigma(i) &= \sigma_*(i) = i \text{ if } 1 \leq i \leq k-3, \\ \sigma(k-2) &= \sigma(k-1) = k-1, \quad \sigma(k) = k-1, \\ \sigma_*(k-2) &= k-1, \quad \sigma_*(k-1) = \sigma(k) = k-1, \\ \sigma(z+j) &= \sigma_*(z+j) = z+j \text{ for } 0 \leq j \leq m-1. \end{aligned}$$

As before, we check that  $\sigma \circ \xi \neq \sigma_*$  for any  $\xi \in \text{End } V$ . If we had  $\sigma \circ \xi = \sigma_*$  we would have  $\xi \in \text{Aut } V$ . Choose  $p, q$  such that  $\xi(p) = k-2$ ,  $\xi(q) = k-1$ . Then  $\sigma(\xi(p)) = \sigma(\xi(q)) = k-1$ . So  $\{\xi(p), \xi(q)\} \subseteq \sigma_*^{-1}(k-1) = \{k-2\}$ , which contradicts the injectivity of  $\xi$ . For any  $\tau \in \mathcal{Y}(V, T)$  we now require  $\tau \circ \xi \neq \sigma$  if  $\xi \notin \text{Aut } V$ . For any non-maximal point  $s \in V$ , we must have  $\tau(s) \geq s$ . Thus  $\sigma(\perp) = \perp < \xi(\perp) \leq \tau(\xi(\perp))$ . So  $\tau \circ \xi \neq \sigma$ . Hence Lemma 4.6 applies once again.

Finally, if  $k = 2$  then Lemma 4.5 implies that  $\underline{M}$  is not endodualisable if  $m > 1$ .  $\square$

**Corollary 4.8.** *The following finite Heyting algebras are endoprimal: the chain  $\mathbf{n}$  ( $n \geq 1$ ) and the algebra  $\mathbf{2}^2 \oplus \mathbf{1}$ .*

It remains an open question whether there are any finite endoprimal Heyting algebras which are not endodualisable.

## 5. Almost endodualisable Heyting algebras.

We showed in the preceding section that  $\underline{M} := \mathbf{2}^2 \oplus \mathbf{n}$  is not endodualisable, by exhibiting a subalgebra  $\underline{r}$  of  $\underline{M}^2$  not entailed by  $\text{End } \underline{M}$ . We now reveal that  $\underline{M}$  is almost endodualisable, for every  $n \geq 2$ , with  $\text{End } \underline{M} \cup \{r\}$  yielding the required duality.

**Theorem 5.1.** *Let  $n \geq 2$  and let  $\underline{M} := \mathbf{2}^2 \oplus \mathbf{n}$ . Let  $V = H(\underline{M})$  labelled so that  $\max V = \{1, 2\}$  and  $3 > 4 > \dots > n+2$  and  $T = V \setminus \{1\}$ . Let  $\underline{r}$  be the  $n+2$ -element chain in  $\underline{M}^2$  for which  $\underline{r}$  is  $Q[\sigma, \sigma_*]$  where  $\sigma, \sigma_* \in \mathcal{Y}(V, T)$  are given by*

$$\begin{aligned} \sigma(1) &= 2, \quad \sigma(i) = i \text{ for } i \geq 2, \\ \sigma_*(1) &= \sigma_*(2) = 2, \quad \sigma_*(i) = i-1 \text{ for } i \geq 3. \end{aligned}$$

*Then  $\text{End } \underline{M} \cup \{r\}$  yields a duality on  $\underline{M}$ .*

**Proof.** Every maximal subalgebra of  $\alpha^{-1}(\leq)$  is of the form  $Q[\sigma, \sigma_*]$  where  $\text{im } \sigma$  is a  $\mathcal{Y}$ -morphic image  $W$  of  $V$ . The possibilities for  $W$  are (i) a chain with  $k$  elements, where  $k \leq n+1$  and (ii) the linear sum of a  $k$ -element chain ( $1 \leq k \leq n$ ) and a two element antichain. Each of these is isomorphic to a  $\mathcal{Y}$ -retract either of  $V$  or of  $T$ . It

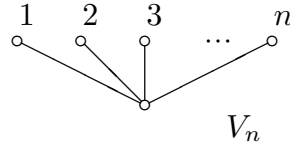
follows from Proposition 4.1 and Theorem 2.2 that  $\text{End } \underline{M} \cup S$  yield a duality on  $\underline{M}$ , where  $S$  is the set of maximal subalgebras of  $\alpha^{-1}(\leq)$  isomorphic to  $(n+2)$ . The set  $S$  includes  $r$ . To complete the proof it will be sufficient to prove that every  $s$  in  $S$  can be expressed in the form  $g \circ r$ , where  $g \in \text{End } \underline{M}$ . The relation  $g \circ r$  is  $Q[\sigma, \sigma'_*]$ , where  $\sigma$  is as above and  $\sigma'_* = \sigma_* \circ H(g)$ . Take  $\sigma'_* \in \mathcal{Y}(V, T)$  with  $\underline{s} := Q[\sigma, \sigma'_*] \in S$ . If  $\sigma'_*(3) = 3$  then  $\sigma^{-1}(2) = \sigma'^{-1}(2) = \{1, 2\}$  so  $\underline{s}$  is not maximal, by Theorem 3.10 (the point  $2 \in T$  can be split horizontally). Therefore  $\sigma'_*(3) = 2$  and consequently the  $\mathcal{Y}$ -morphism  $\sigma'_*$  maps into  $T \setminus \{n+2\}$ . Define  $\xi: V \rightarrow V$  by  $\xi(1) = 1$ ,  $\xi(2) = 2$  and  $\xi(i) = \sigma'_*(i) + 1$  for  $i \geq 3$ . Then  $\xi$  is a well-defined element of  $\mathcal{Y}(V, T)$  and  $\sigma'_* = \sigma_* \circ \xi$ . Take  $g = K(\xi) \in \text{End } \underline{M}$ .  $\square$

We next consider a special, but very natural, class of varieties. We take  $\mathbf{H}_n := \mathbb{ISP}(\underline{B}_n)$  ( $n \geq 1$ ), where  $\underline{B}_n$  is the Boolean lattice  $\mathbf{2}^n$  with a new top element, regarded as a Heyting algebra. As noted already, the class  $\mathbf{H}_n$  is in fact a variety.

F. Bellissima presents in [2] a method for deriving identities for finitely generated subvarieties of  $\mathbf{H}$ . J. Pretorius, in his thesis [15], pursues these ideas further and in particular interprets Bellissima's description of free algebras in terms of duality. The identities obtained by Bellissima's techniques are in general very complicated, and, as Pretorius has worked out, this is the case for the varieties  $\mathbf{H}_n$  despite the simple description of their generating algebras. For  $n = 2$  the identities do simplify nicely. The class  $\mathbf{H}_2 = \mathbb{ISP}(\mathbf{2}^2 \oplus \mathbf{1})$  is given within  $\mathbf{H}$  by the law

$$\neg\neg p \vee (\neg\neg p \rightarrow p) \approx 1.$$

We already know that  $\underline{B}_n$  is not endodualisable for  $n \geq 3$ . We shall show that it is almost endodualisable. Assume henceforth that  $n \geq 3$ . We denote  $H(\underline{B}_n)$  by  $V_n$ , as shown in Figure 3.



**Figure 3**

Observe that the  $\mathcal{Y}$ -morphic images of  $V_n$  with more than one element are exactly the Heyting spaces  $V_m$  for  $m \leq n$ . The Heyting morphisms from  $V_n$  onto  $V_m$  map  $\perp$  to  $\perp$  and are in one-to-one correspondence with the set of all maps from  $\{1, \dots, n\}$  onto  $\{1, \dots, m\}$ . As before we take  $\alpha = \perp$  in  $V_n$ . The following characterisation of maximal subalgebras of  $\alpha^{-1}(\leq)$  is a special case of Theorem 3.10. As it is so simple, we give a self-contained proof.

**Proposition 5.2.** *Let  $Q[\sigma, \sigma_*] \leq \underline{B}_n^2$  be contained in  $\alpha^{-1}(\leq)$ . Then  $Q[\sigma, \sigma_*]$  is maximal in  $\alpha^{-1}(\leq)$  if and only if*

- (i)  $\text{im } \sigma = \text{im } \sigma_*$ , and

(ii) if  $i \in \text{im } \sigma = \text{im } \sigma_*$  then

$$(\forall i \in \text{im } \sigma_*) \quad |\sigma^{-1}(i)| > 1 \implies |\sigma_*^{-1}(i)| = 1.$$

**Proof.** It is easily seen that if  $\sigma, \sigma_*: V_n \rightarrow V_m = \text{im } \sigma$  are Heyting morphisms then  $Q[\sigma, \sigma_*]$  is not a maximal subalgebra of  $\alpha^{-1}(\leq)$  if  $\text{im } \sigma \subsetneq \text{im } \sigma_*$ : if the inclusion were proper we could replace  $\sigma_*$  by a Heyting morphism with larger image. If  $\text{im } \sigma = \text{im } \sigma_*$  but (ii) fails for some  $i \in \text{im } \sigma_*$ , then we can ‘split’ the image of  $i$ . Clearly (i) and (ii) are necessary for maximality.  $\square$

The automorphism group  $\text{Aut } V_n$  consists of the maps  $\xi: V_n \rightarrow V_n$  which fix  $\perp$  and permute the maximal elements  $\max V_n$  of  $V_n$ . Consequently the maximal subalgebras required for piggybacking correspond, up to the action of the automorphism group, to double partitions of  $\{1, \dots, n\}$  into  $m$  parts for  $2 \leq m \leq n - 1$ . By an  $m$ -part double partition of  $n$  we mean a pair of  $m$ -tuples of natural numbers  $(\lambda_1, \dots, \lambda_m)$  and  $(\lambda_1^*, \dots, \lambda_m^*)$  of  $n$  so that

- (i)  $\sum_{i=1}^m \lambda_i = \sum_{i=1}^m \lambda_i^* = n$ ,
- (ii)  $\lambda_1 \geq \dots \geq \lambda_m$ , and
- (iii) for each  $i = 1, \dots, m$ , either  $\lambda_i = 1$  or  $\lambda_i^* = 1$  (and possibly both).

For such a double partition the associated algebra  $Q[\sigma, \sigma_*]$  has  $\sigma$  given by  $\sigma(\perp) = \perp$  and

$$\sigma(i) = j \text{ if } \lambda_0 + \dots + \lambda_{j-1} < i \leq \lambda_0 + \dots + \lambda_j \text{ for } i = 1, \dots, n$$

(where  $\lambda_0 = 0$ ), and  $\sigma_*$  is obtained likewise from  $(\lambda_1^*, \dots, \lambda_m^*)$ . Our test algebras are accordingly the algebras  $\underline{B}_m = H(V_m)$  for  $2 \leq m \leq n - 1$ . We shall follow the terminology of [10] and refer to any test algebra isomorphic to  $\underline{B}_m$  as having *shape*  $m$ .

**Proposition 5.3.** *Let  $n \geq 3$  and let  $\underline{r}$  be the subalgebra of  $\underline{B}_n^2$  of shape  $n - 1$  associated with the double partition  $(2, 1, 1, \dots, 1), (1, 2, 1, \dots, 1)$ . Let  $R = \text{End } \underline{B}_n \cup \{\underline{r}\}$ . Then  $R$  yields a duality on  $\underline{B}_m$  for any  $m \leq n - 1$ .*

**Proof.** The algebra  $\underline{B}_m$  may be identified with  $\mathbf{D}(V_m, \mathbf{2})$ . Its elements may therefore be labelled by  $m$ -tuples  $\mathbf{x} = (x_0, x_1, \dots, x_m)$  where  $x_i = 0$  or  $1$  ( $0 \leq i < m$ ) and where  $x_0 = 1$  implies  $x_i = 1$  for all  $i \geq 1$ . The dual  $D(\underline{B}_m)$  is identified with  $\mathcal{Y}(V_n, V_m)$ , and so consists of

- (a)  $m$  constant maps, and
- (b) maps  $\psi$  sending  $\perp$  to  $\perp$  and mapping  $\{1, \dots, n\}$  onto  $\{1, \dots, m\}$ .

The constant map onto  $c$  will be denoted by  $\vec{c}$ . Each map  $\psi$  in (ii) is determined by its restriction to  $\{1, \dots, n\}$ ; we write  $[y_1 y_2 \dots y_n]$  for the map sending  $i$  to  $y_i$  ( $1 \leq y_i \leq m, 1 \leq i \leq n$ ).

It will be sufficient to show there exist exactly  $2^m + 1$  maps  $u: D(\underline{r}) \rightarrow M$  which preserve the relations in  $R$ . We proceed as follows. We let  $\chi \in D(\underline{r})$  be the map determined by  $[1 \dots 1 2 \dots m]$  (where there are  $n - m + 1$  1s). We first show by diagram-chasing that the only possible images for  $u(\chi)$  are those  $\mathbf{x} =$

$(x_0, x_1, x_2, \dots, x_m)$  for which  $x_1 = x_2 = \dots = x_{n-m+1}$  (Claim 1), and then that the image under  $u$  of every element of  $D(\underline{r})$  is uniquely determined by  $u(\chi)$  (Claim 2). Once Claims 1 and 2 are established we can conclude that  $|ED(\underline{r})| \leq |\underline{r}|$ . Since we know that a set containing  $R$  yields a duality on  $\underline{r}$ , this is sufficient to ensure that  $R$  itself yields a duality on  $\underline{r}$ .

**Proof of Claim 1:** Fix  $k$  with  $1 \leq k \leq n - m + 1$  and let  $\gamma$  be the automorphism of  $\underline{B}_n$  which acts as the cycle  $(1 \ 2 \ \dots \ n)^k$  on the atoms of  $\underline{B}_n$ , and let  $g$  be the endomorphism for which  $H(g): V_n \rightarrow V_n$  is the constant map onto 1. Suppose  $u(\chi) = (x_0, x_1, x_2, \dots, x_n)$ . Recalling that endomorphisms act by composition, we see that the diagram below commutes.

$$\begin{array}{ccc}
 [1 \ 1 \ \dots \ 1 \ 2 \ \dots \ (n-1)] & \xrightarrow{u} & (x_0, x_1, \dots, x_n) \\
 \downarrow \gamma & \searrow \xi & \downarrow \xi \\
 [1 \ 1 \ \dots \ 1] & \xrightarrow{u} & \vec{c} \\
 \uparrow \xi & \swarrow \xi & \uparrow \xi \\
 [1 \ \dots \ 1 \ 2 \ \dots \ (n-1)] & \xrightarrow{u} & (x_0, x_k, \dots, x_n, x_1, \dots, x_{k-1})
 \end{array}$$

From the upper parallelogram we see that

$$\vec{c} = \begin{cases} \vec{0} & \text{if } x_1 = 0, \\ \vec{1} & \text{if } x_1 = 1, \end{cases}$$

while from the lower parallelogram we have

$$\vec{c} = \begin{cases} \vec{0} & \text{if } x_k = 0, \\ \vec{1} & \text{if } x_k = 1. \end{cases}$$

We deduce that  $x_1 = x_k$  whenever  $1 \leq k \leq n - m + 1$ , as claimed.

**Proof of Claim 2:** Each of the constant maps in  $D(\underline{r})$  is the image of  $\chi$  under  $\xi \circ \zeta$ , where  $\zeta$  is some automorphism of  $V_n$ . Thus the images of the constant maps are uniquely determined by  $u(\chi)$ . Write  $D_m$  for the set of non-constant maps in  $D(\underline{r})$ . Now suppose that  $\psi, \psi' \in D_m$ , and that  $(\psi, \psi') \in r$  on  $D(\underline{r})$ . Suppose that  $u(\psi) = (x_0, x_1, x_2, \dots, x_m)$  and that  $u(\psi') = (x'_0, x'_1, x'_2, \dots, x'_m)$ . Then  $(u(\psi), u(\psi')) \in r$ . Lemma 3.5 implies that there exists an order-preserving map  $\mu: V_m \rightarrow \mathbf{2}$  such that

$$\begin{aligned}
 (\mu(\perp), \mu(1), \mu(1), \mu(2), \mu(3), \dots, \mu(m)) &= (x_0, x_1, x_2, x_3, \dots, x_m), \\
 (\mu(\perp), \mu(1), \mu(2), \mu(2), \mu(3), \dots, \mu(m)) &= (x'_0, x'_1, x'_2, x'_3, \dots, x'_m).
 \end{aligned}$$

Thus  $x'_i = x_i$  for  $i \neq 2$  and  $x'_2 = x_1$ , so that  $u(\psi')$  is uniquely determined by  $u(\psi)$ . Thus on  $D_m$  the relation  $r$  is the graph of a function,  $f_r$  say. It will now be enough

to show that every element of  $D_m$  lies in the orbit of  $\chi$  under the action of  $f_r$  and the automorphism group of  $\underline{B}_n$ .

Let  $\Lambda$  be the set of  $m$ -part partitions of  $m$ . Associate with each  $\psi \in D_m$  the partition  $P_\psi \in \Lambda$  which has parts of sizes  $\psi^{-1}(i)$  for  $i = 1, \dots, m$ . Note that if  $P_\psi = P_{\psi'}$  then  $\psi'$  can be obtained from  $\psi$  by permuting  $\{1, 2, \dots, m\}$ , that is, by the action of an automorphism. Therefore it will be enough to show that for each  $\underline{\lambda} \in \lambda$  we can find  $\psi \in D_m$  such that  $P_\psi = \underline{\lambda}$  and  $\psi$  lies in the orbit of  $\chi$ . Order the elements of  $\lambda$  lexicographically. Under this ordering the partition  $(n - m + 1, 1, 1, \dots, 1)$  is the greatest element, and successive elements

$$\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m) \succ (\lambda'_1, \lambda'_2, \dots, \lambda'_m) = \underline{\lambda}'$$

differ only in that there exist  $p, q$  with  $p < q$  so that

$$\lambda'_p = \lambda_p + 1 \quad \text{and} \quad \lambda'_q = \lambda_q - 1.$$

Suppose  $\psi$  is such that

$$\psi^{-1}(j) = \{\lambda_0 + \dots + \lambda_{j-1} + 1, \dots, \lambda_0 + \dots + \lambda_j\}.$$

Define  $\psi'$  by  $\psi'(i) = \psi(i)$  except when  $i = u := \lambda_1 + \dots + \lambda_p$ ,  $\psi'(u) = q$ . Then  $P_{\psi'} = \underline{\lambda}'$ . We seek to show that  $(\psi, \psi') \in r$  on  $\underline{B}_m$ . Let  $v := \lambda_1 + \dots + \lambda_{q-1} + 1$ . Then

$$\begin{aligned} \psi(u-1) &= \psi(u) = p, & \psi(v) &= q, \\ \psi'(u-1) &= p, & \psi'(u) &= \psi'(v) = q, \end{aligned}$$

and on  $\{1, \dots, n\} \setminus \{u, v\}$  we have  $\psi = \psi'$ . Now choose a function  $\mu: V_{n-1} \rightarrow V_m$  such that  $\mu(1) = p$ ,  $\mu(2) = q$ , and  $\mu(i) = \psi(i)$  for  $i \neq 1, 2$ . There exists a permutation  $\pi$  of  $\{1, 2, \dots, n\}$  such that

$$\psi \circ \pi = \mu \circ \sigma \quad \text{and} \quad \psi' \circ \pi = \mu \circ \sigma_* \quad \text{on } \max V_n.$$

Extend  $\pi$  and  $\mu$  by setting  $\pi(\perp) = \perp$  and  $\mu(\perp) = \perp$ . Then  $\pi \in \mathcal{Y}(V_n, V_n)$  and  $\mu \in \mathcal{Y}(V_{n-1}, V_m)$  and

$$\psi \circ \pi = \mu \circ \sigma \quad \text{and} \quad \psi' \circ \pi = \mu \circ \sigma_* \quad \text{on } V_n.$$

By Lemma 3.6, this means that  $(\psi \circ \pi, \psi' \circ \pi) \in r$  on  $\underline{B}_m$ . Equivalently,  $(\psi, \psi') \in (h \circ (h \circ r)^\smile)^\smile$  on  $\underline{B}_m$ , where  $H(h) = \pi \in \text{Aut } \underline{B}_n$  and  $^\smile$  denotes converse. Hence starting from  $\chi$  and relating successive elements in the order we obtain that every element of  $D_m$  is in the orbit of  $\chi$ .  $\square$

**Theorem 5.4.** *Let  $r$  be any shape  $n - 1$  relation and let  $R = \text{End } \underline{B}_n \cup \{r\}$ . Then  $R$  yields a duality on  $\mathbf{H}_n := \mathbb{ISP}(\underline{B}_n)$ .*

**Proof.** Theorem 2.2 tells us that we obtain a duality for  $\mathbf{H}_n$  by taking the relations  $\text{End } \underline{B}_n$  and a set of maximal subalgebras of  $\alpha^{-1}(\leq)$  sufficient to entail, together with  $\text{Aut } \underline{B}_n$ , all shape  $m$  relations for  $m \leq n - 1$ . The Test Algebra Lemma and Proposition 5.2 tell us that every shape  $m$  relation for  $m < n - 1$  can be deleted.

$\underline{M}$	$V$	pairs $\sigma, \sigma_*$
$1 \oplus 2^2 \oplus 1$		
$2^2 \oplus 1 \oplus 2^2 \oplus 1$		
$(2^2 \oplus 1)^2 \oplus 1$		
$2^2 \bar{\oplus} 2^2 \oplus 1$		
$2^3 \oplus 2$		

Table 1

Any two shape  $n - 1$  relations are obtainable from each other via automorphisms, so any one of them may be included in the duality.  $\square$

The endomorphism monoid  $\text{End } \underline{B}_n$  consists of  $n$  maps with codomain  $\{0, 1\}$

(whose images under  $H$  are the  $n$  constant maps onto the points of  $\max V_n$ ), together with the automorphisms of  $\underline{B}_n$ . The automorphism group of  $\underline{B}_n$  is isomorphic to the symmetric group  $S_n$ , each element being uniquely determined by its action on the atoms of  $\mathbf{2}^n \oplus \mathbf{1}$ . We therefore get a generating set for  $\text{End } \underline{B}_n$  by taking  $g, g_\sigma, g_\tau$ , where  $H(g)$  is the constant map onto 1,  $H(g_\sigma)$  fixes  $\perp$  and is the cycle  $(1\ 2\ \dots\ n)$  on  $\max V_n$  and  $H(g_\tau)$  fixes  $\perp$  and is the cycle  $(1\ 2)$  on  $\max V_n$ . We deduce the following corollary to Theorem 5.4.

**Corollary 5.5.** *Let  $n \geq 3$ . Then there exists an optimal duality for  $\mathbf{H}_n$  in which the dualising set contains 4 binary relations.*

**Proof.** We already know that  $\{r, g, g_\sigma, g_\tau\}$ , is a dualising set, where  $r$  has shape  $n-1$  and the endomorphisms  $g, g_\sigma, g_\tau$  are as above. Further, we know that  $r$  cannot be deleted, by Theorem 4.7. It is easily seen, using  $\underline{B}_n$  as the test algebra, that none of  $g, g_\sigma$  or  $g_\tau$  can be deleted without destroying the duality. It is therefore optimal.  $\square$

We have now characterised those finite subdirectly irreducible Heyting algebras which are endodualisable and have exhibited two infinite families of such algebras which are almost endodualisable. For each of these almost endodualisable algebras, the one non-endomorphism included in the duality is a (proper non-extendable) partial endomorphism. We conclude with some examples to indicate that other almost endodualisable algebras exist, and also finite subdirectly irreducible algebras  $\underline{M}$  satisfying (S) for which any dualising set of partial endomorphisms must contain two proper non-extendable partial endomorphisms. All the examples we present were computer-generated, using the programming techniques developed in [17]. These allow us completely to automate the process of setting up a duality for  $\mathbb{ISP}(\underline{M})$  when  $\underline{M}$  is small, and reducing it to an optimal duality with the aid of the Test Algebra Lemma. In this process it is in practice easier to identify the piggybacking subalgebras by using the maximality criterion given in Lemma 3.8 rather than by appealing to Theorem 3.10. Table 1 indicates for each of the listed generating algebras  $\underline{M}$ , pairs  $\sigma, \sigma_*$  such that the set  $S$  of the associated algebras  $Q[\sigma, \sigma_*]$  gives, together with  $\text{End } \underline{M}$ , a duality which is optimal in the sense that  $\text{End } \underline{M} \cup (S \setminus \{s\})$  does not yield a duality for any  $s \in S$ . For each pair  $\sigma, \sigma_*$  we depict the dual space of the relation  $Q[\sigma, \sigma_*]$ , with the figures (unstarred for  $\sigma$  and starred for  $\sigma_*$ ) giving the images of the elements of  $V$  under the maps  $\sigma$  and  $\sigma_*$ . This varied sample of results suggests that we cannot expect any universal result characterising which piggybacking relations arise in optimal dualities for finitely generated varieties of Heyting algebras.

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