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GENERALISED PIGGYBACK DUALITIES AND APPLICATIONS TO OCKHAM ALGEBRAS

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B. A. Davey and H. Werner developed in [11] a general procedure for creating a natural duality between a prevariety $\mathcal{A} = \mathrm{ISP}(\underset{\sim}{\mathrm{P}})$ generated by a single algebra $\underset{\sim}{\mathrm{P}}$ and a category $\mathcal{X}$ of topological structures. In [12,13] they showed how certain natural dualities could be obtained, piggyback-fashion, from existing dualities. (In the examples presented, either the Priestley duality for bounded distributive lattices or the Hofmann-Mislove-Stralka duality for semilattices was used.) This new approach had the great merit of indicating, in a way that the theorems of [11] did not, how best to construct $\mathcal{X}$. Among the varieties for which dualities were constructed, in either [11], [12] or [13], were Stone algebras, Ockham algebras and the subvarieties $\mathcal{P}_{m,n}$, and certain varieties of pseudocomplemented distributive lattices and Heyting algebras. With the exception of Kleene algebras, all these varieties come within the scope of the Piggyback Duality Theorem in [12,13].

This paper deals with a generalisation of the duality theory of [11,12,13] applicable to a wide range of varieties, including the rogue example of Kleene algebras mentioned above. To motivate our strategy we begin with an informal discussion, without proofs, of duality for Kleene algebras. Sections 1 and 2 contain the general theory inspired by this example. In the final section we apply our results to Ockham algebras and obtain a natural duality for every variety generated by a finite subdirectly irreducible algebra. These dualities are such that products in the dual categories are cartesian, but are in some ways less easy to work with than the previously employed restricted Priestley duality. We indicate the relationship between the two available dualities on a given variety, so that these can be used in conjunction as seems

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expedient. We discuss applications of our theory to other varieties of distributive-lattice-ordered algebras in a later paper, in which we also consider the problem of constructing full dualities.

We recall that the variety, $\mathcal{K}$, of Kleene algebras is $\mathbf{ISP}(\underline{K})$, with $\underline{K} = (K; \wedge, \vee, 0, 1, \sim)$ where $K = \{0, c, 1\}$ and $0 = {\sim}1 < c = {\sim}c < 1 = {\sim}0$. It is characterised algebraically by the equations for bounded distributive lattices together with

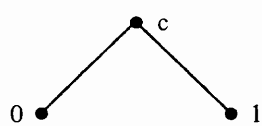
$$\sim\sim a = a, \sim(a \wedge b) = \sim a \vee \sim b, \sim 0 = 1 \text{ and } a \wedge \sim a \leqslant b \vee \sim b.$$

The duality for $\mathcal{K}$ in [11] (pages 175-178) is arrived at as follows. A topological structure is imposed on the underlying set of K, *viz.*

$$\underset{\sim}{K} = (K; \tau, \leqslant, K_0, -)$$

where τ is the discrete topology, K_0 is the subspace containing the points 0 and 1 (and may be regarded as the relation $\{(0,0),(1,1)\}$), $-$ is the relation $K^2 \backslash \{(0,1),(1,0)\}$ and $\leqslant$ is the partial order shown below:

$\leqslant$:=



Each of the relations introduced is a subalgebra of $\underline{K}^2$. The category $\mathbf{IS_cP}(\underset{\sim}{K})$ of isomorphic copies of closed substructures of powers of $\underset{\sim}{K}$ is denoted by $\mathcal{X}$. There exist natural hom-functors D: $\mathcal{K} \to \mathcal{X}$ and E: $\mathcal{X} \to \mathcal{K}$ given by

$$D\colon A \mapsto \mathcal{K}(A, \underline{K}) \leqslant \underset{\sim}{K}^A$$

and

$$E\colon X \mapsto \mathcal{X}(X, \underset{\sim}{K}) \leqslant \underline{K}^X.$$

The NU-Duality Theorem ([11], Theorem 1.19) implies that, for each $A \in \mathcal{A}$, the evaluation $e_A\colon A \to ED(A)$ is an isomorphism. The justification at this stage for the choice of relations defining $\underset{\sim}{K}$ seems to be merely that it makes the NU-Duality Theorem applicable: the relations are algebraic and sufficient to generate (in the sense of [11], page 140) all subalgebras of $\underline{K}^2$.

To provide a rationale for the choice we need to review briefly the piggyback approach to duality, insofar as it applies to a prevariety $\mathcal{A} = \mathbf{ISP}(\underline{P})$ of bounded-distributive-lattice-ordered algebras. We assume familiarity with Priestley duality (for which we take the recent survey articles [9] and [16] as our reference). We denote the category of bounded distributive lattices by $\boldsymbol{D}$ and the dual category of compact totally order-disconnected spaces by $\boldsymbol{P}$. The functors establishing the duality are defined, for $A \in \boldsymbol{D}$ and $X \in \boldsymbol{P}$ by

$$\mathrm{H}\colon A \mapsto \boldsymbol{D}(A,\underset{=}{2}) \leqslant \underset{\approx}{2}^{A}$$

and

$$\mathrm{K}\colon X \mapsto \boldsymbol{P}(X,\underset{\approx}{2}) \leqslant \underset{=}{2}^{A}$$

where $\underset{=}{2}$ is the 2-element chain in $\boldsymbol{D}$ and $\underset{\approx}{2}$ is the 2-element chain with the discrete topology, regarded as lying in $\boldsymbol{P}$. The evaluation map from A to KH(A) is denoted by k_A; it is an isomorphism for every A in $\boldsymbol{D}$. Given the prevariety $\boldsymbol{A} = \mathbf{ISP}(\underline{P})$ we may seek a topological structure $\underset{\sim}{P}$ on the underlying set P of $\underline{P}$ which is such that the evaluation maps $e_A\colon A \to ED(A)$ are isomorphisms, where $D(A) := \boldsymbol{A}(A,\underline{P})$, $E(X) := \boldsymbol{X}(X,\underset{\sim}{P})$ and $\boldsymbol{X} := \mathbf{IS}_c\mathbf{P}(\underset{\sim}{P})$. The idea in [12,13] is to choose $\underset{\sim}{P}$ so that, for each $A \in \boldsymbol{A}$, a one-to-one map $\Delta\colon ED(A) \to KH(A)$ exists such that $\Delta \circ e_A = k_A$, and thereby to show that each map k_A is an isomorphism. (Here $A \in \boldsymbol{A}$ and its reduct in $\boldsymbol{D}$ are denoted by the same symbol and the functors H and K of the $\boldsymbol{D}$-$\boldsymbol{P}$ duality are suitably restricted.) We now need to construct Δ. A natural way to proceed is to take a $\boldsymbol{D}$-homomorphism $\alpha\colon P \to 2$ and to let $\Phi_\alpha := \alpha \circ -$. Assuming Φ_α is onto we may try to define Δ by means of the diagram

$$\begin{array}{ccc}
\boldsymbol{A}(A,\underline{P}) & \xrightarrow{\ \phi\ } & \underset{\sim}{P} \\
{\scriptstyle \Phi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\
\boldsymbol{D}(A,\underset{=}{2}) & \xrightarrow[\ \Delta(\phi)\ ]{} & \underset{\approx}{2}
\end{array}$$

in which $\phi \in ED(A) = \boldsymbol{X}(\boldsymbol{A}(A,\underline{P}),\underset{\sim}{P})$. To carry out this programme we require

(i) Φ_α onto,

(ii) $\underset{\sim}{P}$ such that $\Delta(\phi)$ is well-defined, continuous and relation-preserving, and

(iii) $\underset{\sim}{P}$ such that Δ is one-to-one.

It turns out that Condition (ii) can be met by including among the relations of $\underset{\sim}{P}$ every maximal $\boldsymbol{A}$-subalgebra of $\ker(\alpha)$ ($\subseteq \underline{P} \times \underline{P}$) not contained in the diagonal and every maximal $\boldsymbol{A}$-subalgebra of

$$\alpha^{-1}(\leqslant) := \{(\phi,\psi) \in \underline{P} \times \underline{P} | \alpha(\phi) \leqslant \alpha(\psi)\}.$$

Conditions (i) and (iii) can be met by including among the relations of $\underset{\sim}{P}$ the graphs of a collection of endomorphisms of $\underline{P}$ which together with α separate the points of $\underline{P}$.

The reason that this programme fails for $\boldsymbol{A} = \boldsymbol{K}$ is that it is not possible to choose α: K → 2 such that Φ_α is always onto. There are two $\boldsymbol{D}$-homomorphisms from K to 2, α and β, defined by

$$\alpha(0) = 0, \alpha(c) = \alpha(1) = 1 \text{ and } \beta(0) = \beta(c) = 0, \beta(1) = 1.$$

Take $A = \underline{K}$. The hom-set $\boldsymbol{K}(A,\underline{K})$ consists of the identity, id, alone. Neither Φ_α: id $\mapsto \alpha$ nor Φ_β: id $\mapsto \beta$ is onto $\boldsymbol{D}(A,\underline{\underline{2}})$. To see what happens for an arbitrary $A \in \boldsymbol{K}$ it is convenient to make use of the restricted $\boldsymbol{D}$-$\boldsymbol{P}$ duality for $\boldsymbol{K}$. Let $\boldsymbol{Y}$ be the category whose objects are pairs (Y;g), where $Y \in \boldsymbol{P}$ and $g: Y \to Y$ is a continuous order-reversing map, with g^2 equal to the identity map, and such that, for every $y \in Y$, y and g(y) are comparable, and whose morphisms are continuous order-preserving maps commuting with g. Then H and K establish a (full) duality between $\boldsymbol{K}$ and $\boldsymbol{Y}$. Negation on $A \in \boldsymbol{K}$ and the g-map on $H(A) \in \boldsymbol{Y}$ are linked by

$$(\forall a \in A)(\forall y \in H(A)) \;\; g(y)(a) = 1 \Leftrightarrow y(\sim a) = 0.$$

Fix $A \in \boldsymbol{K}$ and let $x \in \boldsymbol{K}(A,\underline{K})$. Then $\alpha \circ x \in \boldsymbol{D}(A,\underline{\underline{2}})$ and, for all $a \in A$,

$$\begin{aligned} g(\alpha \circ x)(a) = 1 &\Leftrightarrow (\alpha \circ x)(\sim a) = 0 \\ &\Leftrightarrow \alpha(x(\sim a)) = 0 \\ &\Leftrightarrow \alpha(\sim x(a)) = 0 \text{ (since x preserves} \sim) \\ &\Leftrightarrow \sim x(a) = 0 \qquad \text{(by definition of } \alpha) \\ &\Leftrightarrow x(a) = 1. \end{aligned}$$

It follows that $g(\alpha \circ x)(a) = 1$ implies $(\alpha \circ x)(a) = 1$, whence $\alpha \circ x \in \{y \in \boldsymbol{D}(A,\underline{\underline{2}}) | g(y) \leqslant y\}$. Conversely, every $y \in \boldsymbol{D}(A,\underline{\underline{2}})$ with $g(y) \leqslant y$ is of the form $\alpha \circ x$ for some $x \in \boldsymbol{K}(A,\underline{K})$. The required x is defined by

$$(\forall a \in A) \quad x(a) = \begin{cases} 0 \text{ if } y(a) = 0, y(\sim a) = 1, \\ c \text{ if } y(a) = y(\sim a) = 1, \\ 1 \text{ if } y(a) = 1, y(\sim a) = 0. \end{cases}$$

Thus $\operatorname{Im} \Phi_\alpha = \{y \in H(A) | g(y) \leqslant y\}$. A similar argument shows that $\operatorname{Im} \Phi_\beta =$

$\{ y \in H(A) | g(y) \geqslant y\}$. Thus although neither Φ_α nor Φ_β is onto in general, the maps Φ_α and Φ_β are always *jointly onto*: $\mathrm{Im}\,\Phi_\alpha \cup \mathrm{Im}\,\Phi_\beta = H(A)$. This suggests that we should use both α and β to build a piggyback duality for K. Let us review the relations of $\underset{\sim}{K} = (K;\tau,\leqslant,K_0,-)$ in this light. The partial order $\leqslant := \{(0,0),(c,c),(1,1),(0,c),(1,c)\}$ is the unique maximal K-subalgebra of $\alpha^{-1}(\leqslant)$. The distinguished subspace K_0 (*qua* relation) is just $\ker(\alpha,\beta) := \{(\phi,\psi) \in \underline{K} \times \underline{K} | \alpha(\phi) = \beta(\psi)\}$, which is already a K-subalgebra of $\underline{K}^2$. Finally $- = K^2 \backslash \{(0,1),(1,0)\}$ coincides with the unique maximal K-subalgebra of

$$(\beta,\alpha)^{-1}(\leqslant) := \{(\phi,\psi) \in \underline{K} \times \underline{K} | \beta(\phi) \leqslant \alpha(\psi)\}.$$

This provides confirmation that for a prevariety $A = \mathbf{ISP}(\underline{P})$ we should in general consider a family $\Omega_{\underline{P}}$ of D-homomorphism onto $\underline{\underline{2}}$ for which the induced maps Φ_α: $A(\mathrm{A},\underline{P}) \to D(\mathrm{A},\underline{\underline{2}})$ $(\alpha \in \Omega_{\underline{P}})$ are jointly onto and define the topological structure $\underset{\sim}{P}$ using maximal A-subalgebras of $\ker(\alpha_i,\alpha_j)$ $(\alpha_i,\alpha_j \in \Omega_{\underline{P}})$, maximal A-subalgebras of $(\alpha_i,\alpha_j)^{-1}(\leqslant)$ $(\alpha_i,\alpha_j \in \Omega_{\underline{P}})$ and a suitable set of endomorphisms of $\underline{P}$.

The theory in [11,12,13] concerns prevarieties. It handles varieties only where these happen to be of the form $\mathbf{ISP}(\underline{P})$ for a single (subdirectly irreducible) algebra $\underline{P}$. Very many familiar varieties are of this type, Kleene algebras included, but, for varieties of Ockham algebras at least, it is the exception rather than the rule that a variety $\mathbf{HSP}(\underline{P})$ coincides with the prevariety $\mathbf{ISP}(\underline{P})$. However, Birkhoff's Theorem implies that we do always have $\mathbf{HSP}(\underline{P}) = \mathbf{ISP}(\underline{\Pi})$, where $\underline{\Pi}$ is the class of subdirectly irreducible algebras in the variety. Section 1 is devoted to a discussion of what is entailed in extending the duality theory in [11] to a (pre)variety, A, generated by a family of algebras, $\underline{\Pi}$. The dual of an algebra A in A is taken to be a disjoint union, $\dot{\cup}(A(\mathrm{A},\underline{P}) | \underline{P} \in \underline{\Pi}) \leqslant \underset{\sim}{\Pi}^{\mathrm{A}}$, where $\underset{\sim}{\Pi} = \dot{\cup}(\mathrm{P} | \underline{P} \in \underline{\Pi})$, endowed with relations which are subalgebras of products $\underline{P}_1 \times \cdots \times \underline{P}_n$ $(\underline{P}_i \in \underline{\Pi})$. This leads on to a very general piggyback duality theorem applicable to prevarieties $\mathbf{ISP}(\underline{\Pi})$ and allowing for a family of homomorphisms $\Omega_{\underline{P}}$ for each $\underline{P} \in \underline{\Pi}$.

For Kleene algebras it is not necessary to replace $\underline{K}$ by a family of algebras $\underline{\Pi}$, so long as we allow a 2-element set $\Omega_{\underline{K}} = \{\alpha,\beta\}$. An alternative, yielding an isomorphic dual category, is to take $\underline{\Pi} = \{\underline{P}_0,\underline{P}_1\}$ where each of $\underline{P}_0,\underline{P}_1$ is $\underline{K}$ and to let each $\Omega_{\underline{P}_i}$ contain a single homomorphism α_i, where $\alpha_0 = \alpha$, $\alpha_1 = \beta$. We then have, for each

$A \in K$, a dual D(A) which is the disjoint union $K(A,\underline{P}_0) \dot\cup K(A,\underline{P}_1)$. The generalised Piggyback Duality Theorem indicates that we should structure $P_0 \dot\cup P_1$ (and thence, pointwise, D(A)) by taking the following relations and homomorphisms:

(a) the maximum K-subalgebras of $\alpha_0^{-1}(\leqslant)$, $\alpha_1^{-1}(\leqslant)$, $(\alpha_0,\alpha_1)^{-1}(\leqslant)$ and $(\alpha_1,\alpha_0)^{-1}(\leqslant)$, which we denote, respectively, by $\leqslant_0$, $\leqslant_1$, $\leftarrow$ and $\succ\!\!-$;

(b) $g_0: \underline{P}_0 \to \underline{P}_1$ and $g_1: \underline{P}_1 \to \underline{P}_0$ defined, for $\phi \in \underline{P}_i$, by $g_i(\phi) = \phi$.

The components $K(A,\underline{P}_0)$ and $K(A,\underline{P}_1)$ of D(A) correspond, under the bijections Φ_{α_0} and Φ_{α_1} to the subspaces $H(A)^0 := \{y \in H(A) | g(y) \leqslant y\}$ and $H(A)^1 := \{y \in H(A) | g(y) \geqslant y\}$. When the relations and homomorphisms on D(A) are transferred to $H(A)^0$ and $H(A)^1$ via these bijections we find that they relate to the structure on H(A) (an object of Y) in the following way. The pointwise extensions of $\leqslant_0$ and $\leqslant_1$ give the partial orders induced on $H(A)^0$ and $H(A)^1$ from H(A), while $\leftarrow$ and $\succ\!\!-$ give "interconnecting orders," specifying which elements of $H(A)^1$ majorise which elements of $H(A)^0$ and vice versa. The homomorphisms g_0 and g_1, extended pointwise, give g on $H(A)^0$ and $H(A)^1$ respectively. The spaces $K(A,\underline{P}_i)$ and $H(A)^i$ are homeomorphic. Thus the duals, D(A) and H(A), of $A \in K$ for the two dualities are related in a very natural way. The chosen structure on $P_0 \dot\cup P_1$ is just what is required to reconstruct H(A) from $K(A,\underline{P}_0)$ and $K(A,\underline{P}_1)$. Set theoretically H(A) is obtained from $K(A,\underline{P}_0) \dot\cup K(A,\underline{P}_1)$ by identifying $x_0 \in K(A,\underline{P}_0)$ with $x_1 \in K(A,\underline{P}_1)$ if and only if for all $a \in A$, $x_1(a) \succ\!\!- x_0(a)$ and $x_0(a) \leftarrow x_1(a)$. On the resulting set, the topologies patch together correctly, $\leqslant_0$, $\leqslant_1$, $\succ\!\!-$ and $\leftarrow$ combine to give a partial order, and g_0 and g_1 combine to give a continuous order-reversing map. The resulting space is isomorphic to H(A). The major advantage of D(A) over H(A) is that the former has algebraic relations, and lives in a category which has "cartesian" products and "free" objects (neither of which holds in the restricted ***D-P*** dual category Y).

1. **Natural dualities for varieties.** In B. A. Davey and H. Werner [11, 12, 13], a general framework is described for creating a (full) duality between the prevariety $A := \mathrm{ISP}(\underline{P})$ generated by an algebra $\underline{P}$ and a category X of topological structures. We refer to such dualities as "natural" since

(i) the algebra $\underline{P} \in A$ plays a schizophrenic role typical of duality theorems and lives as an A-algebra $\underset{\sim}{P}$ in the category X;

(ii) the structure on $\underset{\sim}{P} \in \mathcal{X}$ consists of operations, partial operations and relations which are algebraic over $\underline{P}$; that is, the relations and the graphs of the (partial) operations are subalgebras of the appropriate powers of $\underline{P}$;

(iii) the duality between $\mathcal{A}$ and $\mathcal{X}$ is given by naturally defined hom-functors

$$\mathcal{A}(-,\underline{P})\colon \mathcal{A} \to \mathcal{X} \text{ and } \mathcal{X}(-,\underset{\sim}{P})\colon \mathcal{X} \to \mathcal{A};$$

(iv) the category $\mathcal{X}$ has free objects, and products in $\mathcal{X}$ are simply cartesian products with the relations and (partial) operations extended pointwise.

We shall now show that some straightforward modifications allow us to set up a similar framework for the prevariety generated by a family $\underline{\Pi}$ of algebras. This allows us, for example, to extend the duality from [11] for the prevariety **ISP**($\underline{P}$) generated by a finite lattice-ordered algebra $\underline{P}$ up to a duality for the whole variety **HSP**($\underline{P}$) generated by $\underline{P}$.

Let $\underline{\Pi}$ be a family of algebras and assume that the underlying set P of each algebra $\underline{P} = (P;F)$ in $\underline{\Pi}$ is equipped with a compact Hausdorff topology with respect to which each operation in F is continuous. Consider an object

$$\underset{\sim}{\Pi} = (\dot{\bigcup}(P \mid \underline{P} \in \underline{\Pi});G,H,R,\tau)$$

where

(a) G is a set of maps each of which is a homomorphism $g\colon \underline{P}_1 \times \cdots \times \underline{P}_n \to \underline{P}_{n+1}$ for some $\underline{P}_1,\ldots,\underline{P}_{n+1} \in \underline{\Pi}$,

(b) H is a set of maps each of which is a homomorphism $h\colon \underline{D} \to \underline{P}_{n+1}$, with $\underline{D}$ a proper subalgebra of $\underline{P}_1 \times \cdots \times \underline{P}_n$ for some $\underline{P}_1,\ldots,\underline{P}_{n+1} \in \underline{\Pi}$,

(c) R is a set of relations each of which is a subalgebra of $\underline{P}_1 \times \cdots \times \underline{P}_n$ for some $\underline{P}_1,\ldots,\underline{P}_n \in \underline{\Pi}$,

(d) τ is the disjoint union topology.

By analogy with the case where $\underline{\Pi}$ consists of a single algebra, we refer to the maps in G as *operations* and those in H as *partial operations* and we summarize (a), (b) and (c) by saying that *the structure on* $\underset{\sim}{\Pi}$ *is algebraic over* $\underline{\Pi}$.

Assume that $(X;\tau)$ is a topological space and that X is written as a disjoint union indexed by $\underline{\Pi}$:

$$X = \dot{\bigcup}(X_{\underline{P}} \mid \underline{P} \in \underline{\Pi}).$$

If for every operation $g: \underline{P}_1 \times \cdots \times \underline{P}_n \to \underline{P}_{n+1}$ in G there is a corresponding map $g: X_{\underline{P}_1} \times \cdots \times X_{\underline{P}_n} \to X_{\underline{P}_{n+1}}$, and similarly for the partial operations in H and the relations in R, then we refer to

$$\underset{\sim}{X} = (X;G,H,R,\tau)$$

as a $\underline{\Pi}$-*indexed structure* (*of the same type as* $\underset{\sim}{\Pi}$), and $X_{\underline{P}}$ is called the $\underline{P}$-*component* of $\underset{\sim}{X}$. If $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are both $\underline{\Pi}$-indexed structures, then a map $\phi: X \to Y$ is a *morphism* provided it maps $X_{\underline{P}}$ into $Y_{\underline{P}}$ for all $\underline{P} \in \underline{\Pi}$ and preserves the relations and (partial) operations in the obvious sense. On any subset Y of a $\underline{\Pi}$-indexed structure $\underset{\sim}{X}$, the relations in R may be interpreted by restriction. If Y is closed under the operations and partial operations, then with the induced structure it becomes a $\underline{\Pi}$-indexed structure $\underset{\sim}{Y}$. Such $\underline{\Pi}$-indexed structures will be referred to as *subalgebras* of $\underset{\sim}{X}$. A morphism which is an isomorphism onto a subalgebra will be called an *embedding*. For any set S, we obtain a $\underline{\Pi}$-indexed structure $\underset{\sim}{\Pi}^S$ whose underlying set is

$$\Pi^S := \dot{\cup}(P^S | \underline{P} \in \underline{\Pi})$$

with the obvious topology, and operations, partial operations, and relations extended pointwise in the natural manner.

We now define $\mathcal{X}$ to be the category of all $\underline{\Pi}$-indexed structures which are isomorphic to a closed subalgebra of some power $\underset{\sim}{\Pi}^S$ of $\underset{\sim}{\Pi}$ (in symbols, $\mathcal{X} := \mathbf{IS}_c\mathbf{P}(\underset{\sim}{\Pi})$). The category $\mathcal{A} := \mathbf{ISP}(\underline{\Pi})$ is simply the prevariety generated by the family $\underline{\Pi}$ of algebras.

It is an easy matter to mimic the early development in B. A. Davey and H. Werner [11]. For every $\underline{A} \in \mathcal{A}$,

$$D(\underline{A}) := \dot{\cup}(\mathcal{A}(\underline{A},\underline{P}) | \underline{P} \in \underline{\Pi})$$

is an $\mathcal{X}$-subalgebra of $\underset{\sim}{\Pi}^A$, and for every $\underset{\sim}{X} \in \mathcal{X}$,

$$E(\underset{\sim}{X}) := \mathcal{X}(\underset{\sim}{X},\underset{\sim}{\Pi})$$

is an $\mathcal{A}$-subalgebra of $\prod(\underline{Q}_{\underline{P}} | \underline{P} \in \underline{\Pi})$ where $\underline{Q}_{\underline{P}}$ is $\underline{P}$ raised to the power $X_{\underline{P}}$. Thus we obtain contravariant functors

$$D: \mathcal{A} \to \mathcal{X} \text{ and } E: \mathcal{X} \to \mathcal{A}$$

which are adjoint on the right. Moreover the evaluation maps (the units of the adjunction)

$$e_{\underline{A}}: \underline{A} \to ED(\underline{A}),\ e_{\underline{A}}(a): x \mapsto x(a)$$

and

$$\epsilon_{\underset{\sim}{X}}: \underset{\sim}{X} \to DE(\underset{\sim}{X}),\ \epsilon_{\underset{\sim}{X}}(x): a \mapsto a(x)$$

are embeddings. This situation we call a *natural proto-duality* and if $e_{\underline{A}}$ is onto (and therefore an isomorphism) for all $\underline{A} \in \mathcal{A}$ we say that we have a *natural duality* between $\mathcal{A}$ and $\mathcal{X}$. If $\epsilon_{\underset{\sim}{X}}$ is also onto (and therefore an isomorphism) for all $\underset{\sim}{X} \in \mathcal{X}$, then we say that the duality between $\mathcal{A}$ and $\mathcal{X}$ is *full*.

The category $\mathcal{X}$ is particularly well behaved.

1.1 LEMMA. *Products in $\mathcal{X}$ are cartesian on components. If $\{\underset{\sim}{X}^i | i \in I\}$ is a family of $\underline{\Pi}$-indexed structures from $\mathcal{X}$, then the product $\prod(\underset{\sim}{X}^i | i \in I)$ in $\mathcal{X}$ has its $\underline{P}$-component given by*

$$\prod(\underset{\sim}{X}^i | i \in I)_{\underline{P}} = \prod(X^i_{\underline{P}} | i \in I)$$

(where the product on the right is cartesian): the topology is the obvious one and the relations and (partial) operations are the pointwise extensions of those on the coordinates. The i-*th projection*

$$\pi_i: \prod(\underset{\sim}{X}^i | i \in I) \to \underset{\sim}{X}^i$$

is the union of the natural projections on the $\underline{P}$-components.

PROOF. That $\prod(\underset{\sim}{X}^i | i \in I)$ as defined belongs to $\mathcal{X}$ follows from the fact that a product of powers of $\underset{\sim}{\Pi}$ is a power of $\underset{\sim}{\Pi}$ and a product of embeddings is an embedding. The rest of the proof is straightforward.■

The first part of the following lemma is an immediate consequence of Lemma 1.1 and the second part follows, as in Lemma 1.6 on page 120 of [11], from the fact that the structure on $\underset{\sim}{\Pi}$ is algebraic over $\underline{\Pi}$. Denote by $F\mathcal{A}(S)$ the free algebra in $\mathcal{A}$ generated by the set S.

1.2 LEMMA. *The object $\underset{\sim}{\Pi}^S$ is the* S-*fold power of $\underset{\sim}{\Pi}$ in $\mathcal{X}$ and is isomorphic to* $D(F\mathcal{A}(S))$. *The map ρ: $D(F\mathcal{A}(S)) \to \underset{\sim}{\Pi}^S$, which restricts each $x \in D(F\mathcal{A}(S))$ to* S, *is an $\mathcal{X}$-isomorphism.*

For any set S, each S-ary term $\underset{\sim}{t}$ in the language of $\mathcal{A}$ induces an $\mathcal{X}$-morphism $t: \underset{\sim}{\Pi}^S \to \underset{\sim}{\Pi}$ which is the disjoint union of the induced S-ary term functions $t: P^S \to P$ for $\underline{P} \in \underline{\Pi}$. We refer to such an $\mathcal{X}$-morphism as an (S-ary) *term function on* $\underline{\Pi}$.

Before we can state the Duality Theorem we need the following result whose proof is a simple extension of the proof of Lemma 1.7 on page 120 of [11] and so is omitted.

1.3 LEMMA. *If* $\underline{A},\underline{B} \in \mathcal{A}$ *and* $k: \underline{A} \to \underline{B}$ *is an onto homomorphism then* $D(k): D(\underline{B}) \to D(\underline{A})$ *is an embedding in* $\mathcal{X}$.

The Duality Theorem is also presented without proof. It is a straightforward extension of the corresponding result in [11]; Theorems 1.8 and F.1 (on pages 121 and 251).

1.4 THE DUALITY THEOREM. *The natural proto-duality* D,E *between* $\mathcal{A}$ *and* $\mathcal{X}$ *is a duality if and only if*

(D1) $\underset{\sim}{\Pi}$ *is injective in* $\mathcal{X}$ *with respect to the embeddings* $D(k): D(\underline{B}) \to D(\underline{A})$ *where* $k: \underline{A} \to \underline{B}$ *is an onto homomorphism in* $\mathcal{A}$, *that is, for each* $\phi \in \mathcal{X}(D(\underline{B}),\underset{\sim}{\Pi})$ *there exists* $\phi' \in \mathcal{X}(D(\underline{A}),\underset{\sim}{\Pi})$ *with* $\phi' \circ D(k) = \phi$;

(D2) *for each positive integer* n, *every* $\phi \in \mathcal{X}(\underset{\sim}{\Pi}^n,\underset{\sim}{\Pi})$ *is an* n*-ary term function on* $\underline{\Pi}$;

(D3) *for each set* S, *every* $\phi \in \mathcal{X}(\underset{\sim}{\Pi}^S,\underset{\sim}{\Pi})$ *has a finite support, that is, there exist* $s_1,\ldots,s_n \in S$ *such that for all* $\underline{P} \in \underline{\Pi}$ *and all* $x,y \in P^S$, *if* $x(s_i) = y(s_i)$ *for* $1 \leqslant i \leqslant n$, *then* $\phi(x) = \phi(y)$.

Furthermore, if $\underline{\Pi}$ *is a finite set of finite subalgebras, then* D,E *is a duality if and only if* (D1) *and* (D2) *hold.*

If D,E is a duality between $\mathcal{A}$ and $\mathcal{X}$, then by restricting the codomain of D and the domain of E we obtain a full duality between $\mathcal{A}$ and $D(\mathcal{A}) \subseteq \mathcal{X}$. Of course in practice we want the full dual category to be closed under isomorphisms so

$$\mathcal{X}_f = ID(\mathcal{A}) = \{X \in \mathcal{X} \mid \epsilon_X \text{ is an isomorphism}\}$$

is a more natural choice. In many cases we find that $\mathcal{X}_f = \mathcal{X}$, but $\mathcal{X}_f$ can be a proper subcategory of $\mathcal{X}$. Some conditions under which $\mathcal{X}$ itself is the full dual are given in D. M. Clark and P. H. Krauss [6]. Unfortunately there is no general theory which produces a workable axiomatisation of $\mathcal{X}_f$. In practice we choose some isomorphism-closed category $\mathcal{X}'$ with $D(\mathcal{A}) \subseteq \mathcal{X}' \subseteq \mathcal{X}$ and we try to ascertain whether ϵ_X is an isomorphism for all $X \in \mathcal{X}'$.

Both of the full duality theorems of [11] carry over to the present setting, but a little more work is required to set them up.

The category $\underline{\Pi}$-*Set* is defined in the obvious way: an object is a $\underline{\Pi}$-indexed disjoint union of sets and a morphism is a set map which maps the $\underline{P}$-component of its domain into the $\underline{P}$-component of its codomain. Note that if $\underline{\Pi}$ consists of a single algebra, then the categories *Set* and $\underline{\Pi}$-*Set* are isomorphic. There is an obvious forgetful functor from X into $\underline{\Pi}$-*Set*.

Given a $\underline{\Pi}$-indexed set S, say $S = \dot{\cup}(S_{\underline{P}} | \underline{P} \in \underline{\Pi})$, define the A-algebra $\underline{\Pi}^S$ to be $\Pi(\underline{Q}_{\underline{P}} | \underline{P} \in \underline{\Pi})$ where $\underline{Q}_{\underline{P}}$ is $\underline{P}$ raised to the power $S_{\underline{P}}$.

Note that if $|\underline{\Pi}| \geqslant 2$, then free objects will not exist in X_f: clearly $FX_f(1)$ does not exist since there is no canonical choice for the component which contains the free generator.

1.5 LEMMA. *Assume that* D,E *is a duality between* A *and* X. *Then the forgetful functor,* $|\text{-}|$, *from* X_f *into* $\underline{\Pi}$-*Set has a left adjoint, the free* X_f-*object generated by a* $\underline{\Pi}$-*indexed set* $S = \dot{\cup}(S_{\underline{P}} | \underline{P} \in \underline{\Pi})$ *being* $FX_f(S) = D(\underline{\Pi}^S)$. *The canonical* $\underline{\Pi}$-*indexed-set map from* S *into* $FX_f(S)$ *maps* $s \in S_{\underline{P}}$ *to the projection* $\pi_s \colon \underline{Q}_{\underline{P}} \to \underline{P}$ *for each* $\underline{P} \in \underline{\Pi}$. *When* $|\underline{\Pi}| = 1$ *this yields the free* X_f-*object generated by the set* S *as* $FX_f(S) = D(\underline{P}^S)$, *where* $\underline{\Pi} = \{\underline{P}\}$, *the free generators being the projections.*

PROOF. Denote the natural contravariant $\underline{\Pi}$-indexed hom-set functor from A to $\underline{\Pi}$-*Set* by U, that is,

$$U(\underline{A}) := \dot{\cup}(A(\underline{A},\underline{P}) | \underline{P} \in \underline{\Pi}).$$

Define a contravariant functor V: $\underline{\Pi}$-*Set* $\to A$ by $V(S) := \underline{\Pi}^S$. It is easily seen that U and V are adjoint on the right. The chain of natural bijections below now establishes the result. For all $S \in \underline{\Pi}$-*Set* and all $\underset{\sim}{X} \in X_0$,

$$\underline{\Pi}\text{-}\mathit{Set}(S,|\underset{\sim}{X}|) \cong \underline{\Pi}\text{-}\mathit{Set}(S,|DE(\underset{\sim}{X})|) = \underline{\Pi}\text{-}\mathit{Set}(S,UE(\underset{\sim}{X})) \cong A(E(\underset{\sim}{X}),V(S))$$
$$\cong X(DV(S),DE(\underset{\sim}{X})) \cong X(DV(S),\underset{\sim}{X}). \blacksquare$$

1.6 LEMMA. *Assume that* D,E *is a duality between* A *and* X. *Then each algebra* $\underline{P} \in \underline{\Pi}$ *is injective in* A *with respect to the embeddings* $E(\phi) \colon E(\underset{\sim}{Y}) \to E(\underset{\sim}{X})$ *where* $\phi \colon \underset{\sim}{X} \to \underset{\sim}{Y}$ *is an onto morphism in* X_f, *that is, for each* $k \in A(E(\underset{\sim}{Y}),\underline{P})$ *there exists* $k' \in A(E(\underset{\sim}{X}),\underline{P})$ *with* $k' \circ E(\phi) = k$.

PROOF. Let $\underline{P} \in \underline{\Pi}$, let $\underset{\sim}{X},\underset{\sim}{Y} \in X_f$, assume that $\phi \colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is an onto X-morphism and that k: $E(\underset{\sim}{Y}) \to ED(\underline{P}) \cong \underline{P}$ is an A-homomorphism. Since the duality between A and X_f is full, there exists ψ: $D(\underline{P}) \to \underset{\sim}{Y}$ with $k = E(\psi)$. Let S be the $\underline{\Pi}$-indexed set all of whose components are empty except for the $\underline{P}$-component which is a singleton. Then by Lemma 1.5, $FX_f(S) \cong D(\underline{P})$ and so, since ϕ is onto, there exists an X-morphism ψ': $D(\underline{P}) \to \underset{\sim}{X}$ with $\phi \circ \psi' = \psi$ and hence $k = E(\psi) = E(\psi') \circ E(\phi)$. Thus we choose $k' = E(\psi')$.■

We have established the necessity of the conditions (E1) and (E2) which appear in the result below; their sufficiency is proved as in [11]: Theorems 1.9 and F.3 on pages 125 and 256.

1.7 THE FIRST FULL-DUALITY THEOREM. *Assume that the natural proto-duality between A and X is a duality and that* $D(A) \subseteq X' \subseteq X$. *Then* D,E *is a full duality between A and X' if and only if*

(E1) *each algebra* $\underline{P} \in \underline{\Pi}$ *is injective in A with respect to the embeddings* $E(\phi)$: $E(\underset{\sim}{Y}) \to E(\underset{\sim}{X})$ *where* ϕ: $\underset{\sim}{X} \to \underset{\sim}{Y}$ *is an onto morphism in* X';

(E2) *for every* $\underline{\Pi}$-*indexed set* S, $D(\underline{\Pi}^S)$ *is freely generated in X' by the* $\underline{\Pi}$-*indexed set of projections.*

The Second Full-Duality Theorem and its proof transfer to the more general setting with only a few notational changes: see Theorems 1.10 and F.4 on pages 127 and 258 of [11].

1.8 THE SECOND FULL-DUALITY THEOREM. *Assume that the natural proto-duality between A and X is a duality and that* $D(A) \subseteq X' \subseteq X$. *Then* D,E *is a full duality between A and X' provided*

(E3) *if* μ: $\underset{\sim}{X} \to \underset{\sim}{Y}$ *is a proper embedding in* X', *then* $E(\mu)$ *is not one-to-one, that is, there exist X-morphisms* ϕ, ψ: $\underset{\sim}{Y} \to \underset{\sim}{\Pi}$ *with* $\phi \neq \psi$ *but* $\phi \circ \mu = \psi \circ \mu$.

For a discussion of the necessity of (E3) if D,E is to be a full duality between A and X' we refer to Propositions 1.11 and F.7 on pages 128 and 262 of [11]: injectivity conditions on $\underline{P}$ must now be replaced by the same conditions on all $\underline{P} \in \underline{\Pi}$. For our purposes it suffices to note that (a) although some or all of the $\underline{P} \in \underline{\Pi}$ may fail to be injective in A, in every example of a full duality known so far $\underset{\sim}{\Pi}$ is injective in X_f, and (b) if $\underset{\sim}{\Pi}$ is injective in X', then (E3) is necessary for D,E to be a full duality between A and X'.

In the remainder of the first section of [11], the assumption that $\underline{P}$ is finite is used to eliminate the topology completely from the conditions for D,E to be a (full) duality between A and X. All the results carry over to the case where $\underline{\Pi}$ is a finite set of finite algebras.

We shall say that the structure on $\underset{\sim}{\Pi}$ *generates* some relation $r \leqslant \underline{P}_1 \times \cdots \times \underline{P}_k$ if for every closed subalgebra X of a power of $\underset{\sim}{\Pi}$ each X-morphism preserves the relation r. Recall that a (k+1)-ary term, t, in the language of A is a *near unanimity term* for $\underline{\Pi}$ and therefore for A if each $\underline{P} \in \underline{\Pi}$ satisfies the identities:

$$t(x,y,\ldots,y) = t(y,x,y,\ldots,y) = \cdots = t(y,y,\ldots,y,x) = y.$$

1.9 THE NU-DUALITY THEOREM. *Let* $\underline{\Pi}$ *be a finite set of finite algebras having a* (k+1)-*ary near unanimity term. If the structure on* $\underset{\sim}{\Pi}$ *generates every subalgebra of* $\underline{P}_1 \times \cdots \times \underline{P}_k$ *for all choices of* $\underline{P}_1,\ldots,\underline{P}_k \in \underline{\Pi}$, *then the natural proto-duality* D,E *between* A *and* X *is a duality and* $\underset{\sim}{\Pi}$ *is injective in* X.

PROOF. In [11], the NU-Duality Theorem is the culmination of the results 1.13 to 1.17. Each of these is easily restated and proved in the present setting: subalgebras of finite powers of $\underline{P}$ must be replaced by subalgebras of finite products of algebras from $\underline{\Pi}$. ■

In [11] on pages 140-142 a list of constructs is given which can be used to show that the structure on $\underset{\sim}{P}$ generates all subalgebras of $\underline{P}^k$. As expected, the corresponding constructs also work in the more general setting.

The following result allows us to eliminate the topology from Condition (E3). As its proof is a direct translation of the proofs of 1.12 and 1.20 from [11] it is omitted.

1.10 THEOREM. *Suppose that* $\underline{\Pi}$ *is a finite set of finite algebras and that* $\underset{\sim}{\Pi}$ *satisfies either*

(a) *the set* H *of partial operations on* $\underset{\sim}{\Pi}$ *is empty, or*

(b) *the operations and partial operations in* $G \cup H$ *are at most unary.*

Assume that the natural proto-duality D,E *between* A *and* X *is a duality. Then* D,E *is a full duality between* A *and* X *provided* $(E3)_F$ *holds, that is,* (E3) *holds for all finite members* $\underset{\sim}{X},\underset{\sim}{Y}$ *of* X.

Whenever we use the NU-Duality Theorem we have k = 2 and hence Case (b) of this theorem is applicable.

We shall require the following result in Section 2, and in the particular case $\mathcal{A} = \boldsymbol{D}$ and $\mathcal{X} = \boldsymbol{P}$, in Section 3. We say that a subset X of $D(\underline{A})$ *separates* $\underline{A}$ if for $a \neq b$ in A there exists $x \in X$ with $x(a) \neq x(b)$.

1.11 PROPOSITION. *Assume that* D,E *is a full duality between $\mathcal{A}$ and $\mathcal{X}$ and that $\underset{\sim}{\Pi}$ is injective in $\mathcal{X}$. If $\underset{\sim}{X}$ is a closed subalgebra of* $D(\underline{A})$ *which separates* $\underline{A}$ *then* $\underset{\sim}{X} = D(\underline{A})$.

PROOF. Let $\underset{\sim}{X}$ be a closed subalgebra of $D(\underline{A})$ which separates $\underline{A}$. Let $\mu\colon \underset{\sim}{X} \to D(\underline{A})$ be the inclusion map and consider $E(\mu)\colon ED(\underline{A}) \to E(\underset{\sim}{X})$. If $\phi \neq \psi$ in $ED(\underline{A})$, then we find $a \neq b$ in A with $\phi = k_{A}(a)$ and $\psi = k_{A}(b)$. Since $\underset{\sim}{X}$ separates $\underline{A}$ there exists $x \in X$ with $x(a) \neq x(b)$, whence

$$\phi(x) = k_{A}(a)(x) = x(a) \neq x(b) = k_{A}(b)(x) = \psi(x).$$

Hence $\phi\restriction_{\underset{\sim}{X}} \neq \psi\restriction_{\underset{\sim}{X}}$ and consequently $E(\mu)$ is one-to-one. Since $\underset{\sim}{\Pi}$ is injective in $\mathcal{X}$, $E(\mu)$ is onto and so is an isomorphism. As D,E is a full duality it follows that μ is an isomorphism and thus $\underset{\sim}{X} = D(\underline{A})$.■

Although the following result can be obtained via a general category-theoretic argument, we give a direct proof which involves less hand waving.

Note that if θ is a congruence on $\underline{A} \in \mathcal{A}$ and $\phi\colon \underline{A} \to \underline{A}/\theta$ is the induced homomorphism, then

$$\theta^* := \{x \in D(\underline{A}) \mid x \text{ factors through } \phi\}$$

is a closed subalgebra of $D(\underline{A})$ since it is the image of the embedding $D(\phi)$.

1.12 PROPOSITION. *Assume that* D,E *is a full duality between $\mathcal{A}$ and $\mathcal{X}$ and that $\underset{\sim}{\Pi}$ is injective in $\mathcal{X}$. Then $\theta \mapsto \theta^*$ is a dual lattice-isomorphism between the lattice of congruences on $\underline{A} \in \mathcal{A}$ and the lattice of closed subalgebras of* $D(\underline{A})$. *If $\underset{\sim}{X} \in \mathcal{X}$ and $\underset{\sim}{Y}$ is a closed subalgebra of $\underset{\sim}{X}$, then the corresponding congruence $\theta_{\underset{\sim}{Y}}$ on* $E(\underset{\sim}{X})$ *is given by*

$$(\forall\, \phi,\psi \in E(\underset{\sim}{X}) = \mathcal{X}(\underset{\sim}{X},\underset{\sim}{\Pi}))\,\phi \equiv \psi(\theta_{\underset{\sim}{Y}}) \Leftrightarrow \phi\restriction_{\underset{\sim}{Y}} = \psi\restriction_{\underset{\sim}{Y}}.$$

Moreover, $E(\underset{\sim}{X})/\theta_{\underset{\sim}{Y}} \cong E(\underset{\sim}{Y})$.

PROOF. Let θ_1,θ_2 be congruences on $\underline{A}$ with induced maps $\phi_1\colon \underline{A} \to \underline{A}/\theta_1$ and $\phi_2\colon \underline{A} \to \underline{A}/\theta_2$. If $\theta_1 \leqslant \theta_2$ then ϕ_2 factors through ϕ_1 and hence $\theta_2^* \leqslant \theta_1^*$. Assume that $\theta_2^* \leqslant \theta_1^*$. If $a \not\equiv b(\theta_2)$, then $\phi_2(a) \neq \phi_2(b)$ and hence (since $\underline{A} \in \mathcal{A} = \mathbf{ISP}(\underline{\Pi})$)

there exists $\underline{P} \in \underline{\Pi}$ and x: $\underline{A}/\theta_2 \to \underline{P}$ with $x(\phi_2(a)) \neq x(\phi_2(b))$. Since $x \circ \phi_2 \in \theta_2^* \leqslant \theta_1^*$ there exists x′: $\underline{A}/\theta_1 \to \underline{P}$ with $x \circ \phi_2 = x' \circ \phi_1$. It follows that $\phi_1(a) \neq \phi_1(b)$ whence $a \not\equiv b(\theta_1)$. Thus $\theta_1 \leqslant \theta_2$.

Let $\underset{\sim}{Y}$ be a subalgebra of $D(\underline{A})$ and let j: $\underset{\sim}{Y} \to D(\underline{A})$ be the inclusion. Since $\underset{\sim}{\Pi}$ is injective in *X*, E(j): $ED(\underline{A}) \to E(\underset{\sim}{Y})$ is onto. We claim that $\underset{\sim}{Y} = \theta^*$ where θ is the kernel of $E(j) \circ e_{\underline{A}}$: $\underline{A} \to E(\underset{\sim}{Y})$. Clearly

$$\theta^* = \{ x \in D(\underline{A}) \mid x \text{ factors through } E(j) \circ e_{\underline{A}} \}.$$

Since D and E are adjoint on the right we have

(*) $j = D(E(j) \circ e_{\underline{A}}) \circ \epsilon_{\underset{\sim}{Y}}$, and so $(\forall\, y \in Y)$, $y = \epsilon_{\underset{\sim}{Y}}(y) \circ E(j) \circ e_{\underline{A}}$;

see Lemma 1.5(4) on page 118 of [11]. It follows at once that $\underset{\sim}{Y} \leqslant \theta^*$. If $x \in \theta^*$, then $x = z \circ E(j) \circ e_{\underline{A}}$ for some $z \in DE(\underset{\sim}{Y})$. Since the duality between *A* and *X* is full, $\epsilon_{\underset{\sim}{Y}}$ is onto and so there exists $y \in Y$ with $z = \epsilon_{\underset{\sim}{Y}}(y)$. It now follows from (*) that $x = y \in Y$ and so $\theta^* \leqslant \underset{\sim}{Y}$. The final claim in the proposition follows easily. ■

Since we wish the dual of $\underline{A} \in A$ to be as simple as possible, we minimize the size of $\underline{\Pi}$. If $\underline{P}, \underline{Q} \in \underline{\Pi}$ and $\underline{P}$ can be embedded into $\underline{Q}$, we may delete $\underline{P}$ from $\underline{\Pi}$ without destroying the duality. Reducing $\underline{\Pi}$ to a single algebra has the added advantage that the full dual category will then have free objects (see Lemma 1.5).

Assume that A = **ISP**($\underline{\Pi}$) and $A^{\bullet}$ = **ISP**($\underline{\Pi}^{\bullet}$) where $\underline{\Pi}^{\bullet} \subseteq \underline{\Pi}$, and suppose we have a $\underline{\Pi}^{\bullet}$-indexed structure $\underset{\sim}{\Pi}^{\bullet}$ such that the natural proto-duality $D^{\bullet}, E^{\bullet}$ between $A^{\bullet}$ and $X^{\bullet}$:= $\mathbf{IS_cP}(\underset{\sim}{\Pi}^{\bullet})$ is a duality. Since the definition of $D^{\bullet}(\underline{A})$ makes sense for any algebra $\underline{A}$ of the given type, we may extend $D^{\bullet}: A^{\bullet} \to X^{\bullet}$ to a functor $D^{\bullet}: A \to X^{\bullet}$.

1.13 PROPOSITION. *Let A, $A^{\bullet}$ and $X^{\bullet}$ be as above and assume that $\underset{\sim}{\Pi}^{\bullet}$ is injective in $X^{\bullet}$. Then for all* $A \in A$, *the algebra* $E^{\bullet}D^{\bullet}(\underline{A}) \in A^{\bullet}$ *is the maximum homomorphic image of* $\underline{A}$ *in* $A^{\bullet}$: *the homomorphism* $e_{\underline{A}}: \underline{A} \to E^{\bullet}D^{\bullet}(\underline{A})$ *is onto and if* $\underline{B} \in A$ *and* k: $\underline{A} \to \underline{B}$ *is a homomorphism, then there exists a homomorphism* k′: $E^{\bullet}D^{\bullet}(\underline{A}) \to \underline{B}$ *with* $k' \circ e_{\underline{A}} = k$ (*that is*, $\ker e_{\underline{A}} \subseteq \ker k$).

PROOF. Let $\underline{A} \in A$. Suppose that k: $\underline{A} \to \underline{B}$ is a homomorphism with $\underline{B} \in A^{\bullet}$. Since $e_{\underline{B}} \circ k = E^{\bullet}D^{\bullet}(k) \circ e_{\underline{A}}$ and since $e_{\underline{B}}$ is an isomorphism, it follows that k factors through $e_{\underline{A}}$. It remains to show that $e_{\underline{A}}$ is onto.

Clearly $D^{\bullet}(FA(S)) \cong \underset{\sim}{\Pi}^{S} \cong D^{\bullet}(FA^{\bullet}(S))$ and it follows easily that

$$e_{FA(S)}: FA(S) \to E^{\bullet}D^{\bullet}(FA(S)) \cong FA^{\bullet}(S)$$

is onto. Let $\underline{F} = FA(A)$ and let k: $FA(A) \to \underline{A}$ be the natural homomorphism. Since k is onto $D^{\bullet}(k)$ is an embedding, and hence $E^{\bullet}D^{\bullet}(k)$ is onto as $\underset{\sim}{\Pi}^{\bullet}$ is injective in $X^{\bullet}$. It follows that $e_{\underline{A}}$ is onto as $e_{\underline{A}} \circ k = E^{\bullet}D^{\bullet}(k) \circ e_{\underline{F}}$ and both $E^{\bullet}D^{\bullet}(k)$ and $e_{\underline{F}}$ are onto.■

2. **Piggyback dualities revisited.** If each algebra $\underline{A} \in A$ has a reduct $\underline{\underline{A}}$ in some other prevariety D for which we have already established a duality H: $D \to E$ and K: $E \to D$, then we can obtain a duality for A by restricting the functor H to (the reducts of) the algebras and morphisms in A. Unfortunately the resulting *restricted D-E-duality* need not be a natural duality even when the duality between D and E is. In B. A. Davey and H. Werner [12,13], a method was given whereby a natural duality for A could sometimes be obtained from the restricted D-E-duality. Their results applied only when A was a prevariety generated by a single algebra, and although applicable to a large number of examples did not always give rise to a duality even when a natural duality was known to exist (in the case of Kleene algebras for example). We now extend the Piggyback-Duality Theorem of [12,13] to cover examples like Kleene algebras and to make it applicable to $\underline{\Pi}$-indexed structures.

Let $A = \mathbf{ISP}(\underline{\Pi})$ and $X = \mathbf{IS_cP}(\underset{\sim}{\Pi})$ and suppose D: $A \to X$, E: $X \to A$ is a natural proto-duality, as described in the previous section, with

$$\underset{\sim}{\Pi} = (\dot{\cup}(P|\underline{P} \in \underline{\Pi}); G, H, R, \tau).$$

Assume that A has a reduct in another prevariety D; that is, assume there is a family F_0 of terms in the language of A with respect to which every algebra $\underline{A} = (A;F)$ becomes an algebra $\underline{\underline{A}} = (A;F_0)$ in D. (The most important consequence of this is that if g: $\underline{A} \to \underline{B}$ is an A-homomorphism, then g: $\underline{\underline{A}} \to \underline{\underline{B}}$ is also a D-homomorphism.) Moreover, assume that $D = \mathbf{ISP}(\underline{\underline{\Delta}})$ and that there is a $\underline{\underline{\Delta}}$-indexed structure

$$\underset{\approx}{\Delta} = (\dot{\cup}(D|\underline{\underline{D}} \in \underline{\underline{\Delta}}); G_0, H_0, R_0, \tau_0)$$

such that the natural proto-duality H: $D \to E$, K: $E \to D$, with $E = \mathbf{IS_cP}(\underset{\approx}{\Delta})$, is a duality.

For every $\underline{A} \in A$ we have two evaluation maps: namely $e_{\underline{A}}: \underline{A} \longrightarrow ED(\underline{A})$ which is an embedding in A and $k_{\underline{\underline{A}}}: \underline{\underline{A}} \longrightarrow KH(\underline{\underline{A}})$ which is an isomorphism in D. Clearly $e_{\underline{A}}$ will be onto (and therefore an A-isomorphism) if and only if we can define a one-to-one map d: $ED(\underline{A}) \to KH(\underline{\underline{A}})$ which commutes with the evaluations.

$$\begin{array}{ccc} & & \mathrm{ED}(\underline{A}) \\ & \overset{e_{\underline{A}}}{\nearrow} & \Big\downarrow d \\ A & \underset{k_{\underline{\underline{A}}}}{\searrow} & \\ & & \mathrm{KH}(\underline{\underline{A}}) \end{array}$$

A natural way to define d is via $\mathcal{D}$-homomorphisms $\alpha\colon \underline{\underline{P}} \to \underline{\underline{D}}$ which connect algebras $\underline{P} \in \underline{\Pi}$ and $\underline{\underline{D}} \in \underline{\underline{\Delta}}$. Given an $\mathcal{X}$-morphism $\phi\colon \mathrm{D}(\underline{A}) \to \underset{\sim}{\Pi}$ we must define $\mathrm{d}(\phi)\colon \mathrm{H}(\underline{\underline{A}}) \to \underset{\approx}{\Delta}$. If the induced maps

$$\Phi_\alpha := \alpha \circ -\colon \mathcal{A}(\underline{A},\underline{P}) \to \mathcal{D}(\underline{\underline{A}},\underline{\underline{D}})$$

are jointly onto, then for each $y \in \mathcal{D}(\underline{\underline{A}},\underline{\underline{D}})$ there exists an $\alpha\colon \underline{\underline{P}} \to \underline{\underline{D}}$ and $x \in \mathcal{A}(\underline{A},\underline{P})$ such that $\Phi_\alpha(x) = \alpha \circ x = y$, and we can attempt to define $\mathrm{d}(\phi)$ by

$$\mathrm{d}(\phi)(y) = \alpha(\phi(x))$$

thereby making the diagram below commute.

$$\begin{array}{ccc} \mathcal{A}(\underline{A},\underline{P}) & \xrightarrow{\quad\phi\quad} & P \\ \Phi_\alpha \Big\downarrow & & \Big\downarrow \alpha \\ \mathcal{D}(\underline{\underline{A}},\underline{\underline{D}}) & \xrightarrow[\mathrm{d}(\phi)]{\quad\quad} & D \end{array}$$

Of course we need to know that $\mathrm{d}(\phi)$ is well-defined (independent of the choice of α) and is an $\mathcal{X}$-morphism, and that d is one-to-one.

Note that if $U \subseteq P_1 \times \cdots \times P_n$ and $h\colon U \to P_{n+1}$ is a map, then U is an $\mathcal{A}$-subalgebra of $\underline{P}_1 \times \cdots \times \underline{P}_n$ and h is an $\mathcal{A}$-homomorphism if and only if the graph of h, that is,

$$r_h := \{(a_1,\ldots,a_n,a_{n+1}) \in \underline{P}_1 \times \cdots \times \underline{P}_{n+1} \mid (a_1,\ldots,a_n) \in U \text{ and } h(a_1,\ldots,a_n) = a_{n+1}\},$$

is an $\mathcal{A}$-subalgebra of $\underline{P}_1 \times \cdots \times \underline{P}_{n+1}$. Moreover, if $X \subseteq \underset{\sim}{\Pi}^S$ is closed under h, then a map $\phi\colon X \to \underset{\sim}{\Pi}$ preserves h precisely when it preserves r_h. Our results are most easily stated if we replace the (partial) operations in $H \cup G$ and in $H_0 \cup G_0$ by their graphs. To this end, define R^+ by

$$R^+ := R \cup \{r_h \mid h \in H \cup G\}$$

and define R_0^+ similarly.

The proof of the result below follows the same lines as that of the corresponding theorem in B. A. Davey and H. Werner [13].

2.1 PROPOSITION. *Let A, X, D and E be as above and assume that for all* $\underline{P} \in \underline{\Pi}$, $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *there is a (possibly empty) set* $\Omega_{\underline{P},\underline{\underline{D}}}$ *of D-homomorphisms* $\alpha: \underline{\underline{P}} \to \underline{\underline{D}}$ *such that for all* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *the induced maps*

$$\Phi_\alpha := \alpha \circ -: A(\underline{A},\underline{P}) \to D(\underline{\underline{A}},\underline{\underline{D}}) \text{ with } \alpha \in \cup(\Omega_{\underline{P},\underline{\underline{D}}} | P \in \underline{\Pi})$$

are jointly onto. If ϕ: $D(\underline{A}) \to \underset{\sim}{\Pi}$ *is a map such that* $\phi(A(\underline{A},\underline{P})) \subseteq P$ *for all* $\underline{P} \in \underline{\Pi}$, *then*

(i) *there is a (necessarily unique) map* $d(\phi)$: $H(\underline{\underline{A}}) \to \underset{\approx}{\Delta}$ *such that the square above commutes for all* $\alpha \in \Omega_{\underline{P},\underline{\underline{D}}}$, *provided that for all* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *and all* $\alpha \in \Omega_{\underline{P},\underline{\underline{D}}}$, $\beta \in \Omega_{\underline{Q},\underline{\underline{D}}}$ *each A-subalgebra of* $\underline{P} \times \underline{Q}$ *maximal in*

$$\ker(\alpha,\beta) := \{(a,b) \in P \times Q | \alpha(a) = \beta(b)\}$$

is preserved by ϕ;

(ii) $d(\phi)$ *preserves an n-ary relation* $r_0 \leqslant \underline{\underline{D}}_1 \times \cdots \times \underline{\underline{D}}_n$ *in* R_0^+ *provided that for all* $\alpha_1 \in \Omega_{\underline{P}_1,\underline{\underline{D}}_1}, \ldots, \alpha_n \in \Omega_{\underline{P}_n,\underline{\underline{D}}_n}$ *each A-subalgebra of* $\underline{P}_1 \times \cdots \times \underline{P}_n$ *maximal in*

$$(\alpha_1,\ldots,\alpha_n)^{-1}(r_0) := \{(a_1,\ldots,a_n) \in P_1 \times \cdots \times P_n | (\alpha_1(a_1),\ldots,\alpha_n(a_n)) \in r_0\},$$

is preserved by ϕ;

(iii) $d(\phi)$ *is continuous provided that* ϕ *is continuous, each* $\alpha: \underline{\underline{P}} \to \underline{\underline{D}}$ *is continuous, and* $\cup(\Omega_{\underline{P},\underline{\underline{D}}} | \underline{P} \in \underline{\Pi})$ *is finite for all* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$;

(iv) $d(\phi) = d(\psi)$ *implies* $\phi = \psi$ *provided that for all* $\underline{P} \in \underline{\Pi}$ *and* $a \neq b$ *in* P *there exists an A-homomorphism* u: $\underline{P} \to \underline{Q}$, *preserved by both* ϕ *and* ψ, *and* $\alpha \in \Omega_{\underline{Q},\underline{\underline{D}}}$ *such that* $(\alpha \circ u)(a) \neq (\alpha \circ u)(b)$.

PROOF. (i) Let $x \in A(\underline{A},\underline{P})$, $z \in A(\underline{A},\underline{Q})$ and $\alpha \in \Omega_{\underline{P},\underline{\underline{D}}}$, $\beta \in \Omega_{\underline{Q},\underline{\underline{D}}}$ with $\Phi_\alpha(y) = \Phi_\beta(z)$, that is, $\alpha \circ x = \beta \circ z$. We must show that $\alpha(\phi(x)) = \beta(\phi(z))$. Since x and z are A-homomorphisms and since $\alpha \circ x = \beta \circ z$, there exists an A-subalgebra, say r, of $\underline{P} \times \underline{Q}$ which contains the subalgebra $\{(x(a),z(a)) | a \in A\}$ and is maximal in $\ker(\alpha,\beta)$. But $(x,z) \in r$ on $D(\underline{A})$ and hence $(\phi(x),\phi(z)) \in r$ on $\underset{\sim}{\Pi}$ since ϕ preserves r. Hence $\alpha(\phi(x)) = \beta(\phi(z))$ since $r \subseteq \ker(\alpha,\beta)$.

(ii) Let $r_0 \in R_0^+$ with $r_0 \leqslant \underline{\underline{D}}_1 \times \cdots \times \underline{\underline{D}}_n$ and assume $y_i \in D(\underline{\underline{A}},\underline{\underline{D}}_i)$ with $(y_1,\ldots,y_n) \in r_0$ on $H(\underline{\underline{A}})$. Since the maps Φ_α are jointly onto, we can find $\alpha_i \in \Omega_{\underline{P}_i,\underline{\underline{D}}_i}$

and $x_i \in A(\underline{A},\underline{P}_i)$ such that $\alpha_i \circ x_i = y_i$. Since the maps x_i are A-homomorphisms and since $(\alpha_1 \circ x_1,...,\alpha_n \circ x_n) \in r_0$, there exists an A-subalgebra, say r, of $\underline{P}_1 \times \cdots \times \underline{P}_n$ which contains $\{(x_1(a),...,x_n(a)) | a \in A\}$ and is maximal in $(\alpha_1,...,\alpha_n)^{-1}(r_0)$. Now $(x_1,...,x_n) \in r$ on $D(\underline{A})$ and hence $(\phi(x_1),...,\phi(x_n)) \in r$ on $\underset{\sim}{\Pi}$ since ϕ preserves r. Thus

$$(d(\phi)(y_1),...,d(\phi)(y_n)) = (\alpha(\phi(x_1)),...,\alpha(\phi(x_n))) \in r_0$$

as $r \subseteq (\alpha_1,...,\alpha_n)^{-1}(r_0)$.

(iii) To prove that $d(\phi)$ is continuous it suffices to show that $d(\phi)^{-1}(V)$ is closed in $D(\underline{\underline{A}},\underline{\underline{D}})$ for every closed set V in $\underline{\underline{D}}$. Now

$$d(\phi)^{-1}(V) = \cup(\Phi_\alpha(\phi^{-1}(\alpha^{-1}(V))) | \alpha \in \cup(\Omega_{\underline{P},\underline{\underline{D}}} | \underline{P} \in \underline{\Pi})).$$

The space $A(\underline{A},\underline{P})$ is compact since it is a closed subspace of $\underline{P}^A$ and the topology on $\underline{P}$ is compact; hence each Φ_α is a closed map. Thus $d(\phi)^{-1}(V)$ is closed as the union is finite.

(iv) Suppose that $\phi,\psi\colon D(\underline{A}) \to \underset{\sim}{\Pi}$ with $\phi \neq \psi$. Then there exist $\underline{P} \in \underline{\Pi}$ and $x \in A(\underline{A},\underline{P})$ such that $\phi(x) \neq \psi(x)$. If there is an A-homomorphism $u\colon \underline{P} \to \underline{Q}$, preserved by ϕ and ψ, and an $\alpha \in \Omega_{\underline{Q},\underline{\underline{D}}}$ satisfying $(\alpha \circ u)(\phi(x)) \neq (\alpha \circ u)(\psi(x))$, then $(\alpha \circ \phi)(u \circ x) \neq (\alpha \circ \psi)(u \circ x)$. Thus

$$d(\phi) \circ \Phi_\alpha = \alpha \circ \phi \neq \alpha \circ \psi = d(\psi) \circ \Phi_\alpha,$$

and hence $d(\phi) \neq d(\psi)$.∎

The strength of this result is that rather than simply giving conditions which the indexed structure $\underset{\sim}{\Pi}$ must satisfy, it actually indicates what the sets R, G and H of relations, operations and partial operations on $\underset{\sim}{\Pi}$ should be.

2.2 PIGGYBACK-DUALITY THEOREM. *Let* $D\colon A \to X$, $E\colon X \to A$ *be a natural proto-duality given by a* $\underline{\Pi}$*-indexed structure* $\underset{\sim}{\Pi}$ *and assume that* $A := \mathbb{ISP}(\underline{\Pi})$ *has a reduct in a prevariety* $D := \mathbb{ISP}(\underline{\underline{\Delta}})$ *which has a natural duality* $H\colon D \to E$, $K\colon E \to D$ *given by a* $\underline{\underline{\Delta}}$*-indexed structure* $\underset{\approx}{\Delta}$. *Then* D,E *is a duality between* A *and* X *provided that for each* $\underline{P} \in \underline{\Pi}$, $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *there is a (possibly empty) family* $\Omega_{\underline{P},\underline{\underline{D}}}$ *of continuous* D*-homomorphisms* $\alpha\colon \underline{P} \to \underline{\underline{D}}$, *with* $\cup(\Omega_{\underline{P},\underline{\underline{D}}} | \underline{P} \in \underline{\Pi})$ *finite for each* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$, *such that*

(0) *for all* $\underline{A} \in A$ *and for all* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *the maps* $\Phi_\alpha\colon A(\underline{A},\underline{P}) \to D(\underline{\underline{A}},\underline{\underline{D}})$ *with* $\alpha \in \cup(\Omega_{\underline{P},\underline{\underline{D}}} | \underline{P} \in \underline{\Pi})$ *are jointly onto*;

(A) *for all* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *and all* $\alpha \in \Omega_{\underline{P},\underline{\underline{D}}}$, $\beta \in \Omega_{\underline{Q},\underline{\underline{D}}}$ *each* A*-subalgebra of* $\underline{P} \times \underline{Q}$

maximal in $\ker(\alpha,\beta)$ *is in* R^+;

(B) *for every* n*-ary relation* $r_0 \leqslant \underline{D}_1 \times\cdots\times \underline{D}_n$ *in* R_0^+ *and for all* $\alpha_1 \in \Omega_{\underline{P}_1,\underline{D}_1},\ldots,\alpha_n \in \Omega_{\underline{P}_n,\underline{D}_n}$, *each A-subalgebra of* $\underline{P}_1 \times\cdots\times \underline{P}_n$ *maximal in* $(\alpha_1,\ldots,\alpha_n)^{-1}(r_0)$ *is in* R^+;

(C) *for all* $\underline{P} \in \underline{\Pi}$ *and* $a \neq b$ *in* P *there is a unary operation* $u\colon \underline{P} \to \underline{Q}$ *in* G *and* $\alpha \in \Omega_{\underline{Q},\underline{D}}$ *such that* $(\alpha\circ u)(a) \neq (\alpha\circ u)(b)$.■

Since they allow us to lift the restricted *D-E*-duality up to a natural duality between *A* and *X*, we refer to the maps $\alpha\colon \underline{P} \to \underline{D}$ of the Piggyback-Duality Theorem as *carriers*.

2.3 REMARKS. (a) Although the Piggyback-Duality Theorem indicates that certain relations should be in R^+ and certain operations be in G, it is enough to know that these are generated by the relations in R^+ in the sense indicated in Section 1.

(b) When considering the case $\underline{P} = \underline{Q}$ and $\alpha = \beta$ in Condition (A) of the theorem, we can always omit subalgebras of the diagonal of $\underline{P} \times \underline{P}$. They arise in the proof only when $x = z$, in which case $\alpha(\phi(x)) = \beta(\phi(z))$ holds trivially.

The Piggyback-Duality Theorem has a most important feature. Although it produces a global result, all the conditions, with the exception of (0), are local in that they involve only the algebras in $\underline{\Pi}$ and $\underline{\Delta}$ and the carriers between them. Fortunately, just as in the single-carrier version considered in [12,13], Condition (0) can be dispensed with in some important cases because of its intimate relationship with Condition (C).

2.4 PROPOSITION. *Let A, X, D and E be as in the Piggyback-Duality Theorem, and for each* $\underline{P} \in \underline{\Pi}$, $\underline{D} \in \underline{\Delta}$ *let* $\Omega_{\underline{P},\underline{D}}$ *be a family of D-homomorphisms* $\alpha\colon \underline{P} \to \underline{D}$.

(i) *If the induced maps* Φ_α *are jointly onto, then*

> (C1) *for all* $\underline{P} \in \underline{\Pi}$ *and* $a \neq b$ *in* P *there is an A-homomorphism* $u\colon \underline{P} \to \underline{Q}$ *and* $\alpha \in \Omega_{\underline{Q},\underline{D}}$ *such that* $(\alpha\circ u)(a) \neq (\alpha\circ u)(b)$.

(ii) *Suppose that the duality between D and E is full and that* $\underset{\approx}{\Delta}$ *is injective in E, and assume that* (C1) *holds. Then the induced maps* Φ_α *are jointly onto if and only if the image of* $D(\underline{A})$ *under these maps is an E-subalgebra of* $H(\underline{A})$.

PROOF. (i) Let $\underline{P} \in \underline{\Pi}$ and $a \neq b$ in P. Since $\underline{P} \in D = \mathbf{ISP}(\underline{\Delta})$, there exists $\beta \in D(\underline{P},\underline{D})$ for some $\underline{D} \in \underline{\Delta}$ with $\beta(a) \neq \beta(b)$. Since the maps Φ_α are jointly onto there exists $\underline{Q} \in \underline{\Pi}$ and morphisms $u \in \mathcal{A}(\underline{P},\underline{Q})$ and $\alpha \in \Omega_{\underline{Q},\underline{\underline{D}}}$ such that $\alpha \circ u = \Phi_\alpha(u) = \beta$. Thus $(\alpha \circ u)(a) = \beta(a) \neq \beta(b) = (\alpha \circ u)(b)$, as required.

(ii) Assume that the set

$$\Phi(D(\underline{A})) := \cup(\Phi_\alpha(\mathcal{A}(\underline{A},\underline{P})) | \underline{P} \in \underline{\Pi},\ \underline{\underline{D}} \in \underline{\underline{\Delta}},\ \alpha \in \Omega_{\underline{P},\underline{\underline{D}}})$$

is an E-subalgebra of $H(\underline{\underline{A}})$. Condition (C1) says exactly, that $\Phi(D(\underline{A}))$ separates $\underline{\underline{A}}$ and hence by Proposition 1.11 we have $\Phi(D(\underline{A})) = H(\underline{\underline{A}})$, as required.■

If the structure on $\underset{\approx}{\Delta}$ is purely relational (that is, $G_0 = H_0 = \emptyset$), then every subset of $H(\underline{\underline{A}})$ is an E-subalgebra. Since (C) implies (C1) our next result follows.

2.5 THEOREM. *Assume that there is a natural full duality between* **D** *and* **E** *given by a purely relational* $\underline{\underline{\Delta}}$*-indexed structure* $\underset{\approx}{\Delta}$*, and assume that* $\underset{\approx}{\Delta}$ *is injective in* **E**. *Then, under the assumptions of the Piggyback-Duality Theorem,* D,E *is a duality between* A *and* X *provided Conditions* (A), (B) *and* (C) *hold.*

This theorem will apply whenever A is a class of bounded-distributive-lattice-ordered algebras, in which case **D** is the variety of bounded distributive lattices and E is the category of compact totally order-disconnected spaces.

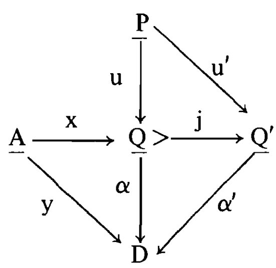
If the maps Φ_α are not jointly onto, all is not lost. Suppose we can establish (A), (B) and (C) but not (0). Then for every X-morphism ϕ: $D(\underline{A}) \to \underset{\sim}{\Pi}$ we have a map $d(\phi)$: $\Phi(D(\underline{A})) \to \underset{\approx}{\Delta}$ which preserves all the relations in R_0^+, and moreover d is one-to-one. If we can establish the following restricted injectivity condition

(I) *the map* $d(\phi)$: $\Phi(D(\underline{A})) \to \underset{\approx}{\Delta}$ *can be extended to an E-morphism defined on* $H(\underline{\underline{A}})$,

then the proof of 2.4(ii) shows that the extension is unique and so can also be denoted by $d(\phi)$: $H(\underline{\underline{A}}) \to \underset{\approx}{\Delta}$. Thus the uniqueness guarantees that d: $ED(A) \to KH(A)$ is a function; clearly d is one-to-one. Since $d \circ e_{\underline{A}}$ and $k_{\underline{\underline{A}}}$ agree on $\phi(D(\underline{A}))$ we have $d \circ e_{\underline{A}} = k_{\underline{\underline{A}}}$ and hence $e_{\underline{A}}$ is an isomorphism for all $A \in \mathcal{A}$. This approach was used for certain prevarieties of relatively pseudocomplemented semilattices in [12,13].

Naturally we wish the category X to be as simple as possible. Consequently we attempt to minimize the size of $\underline{\Pi}$ and minimize the number of carriers.

2.6 REMARK. If $\underline{Q},\underline{Q}' \in \underline{\Pi}$ and $\underline{Q}$ can be embedded into $\underline{Q}'$, it is natural to try to delete $\underline{Q}$ from $\underline{\Pi}$. This could not affect Conditions (A) and (B) but to fulfill Conditions (0) and (C) it may be necessary to add new carriers and operations to replace those involving $\underline{Q}$. We can successfully delete $\underline{Q}$ provided there is some embedding j: $\underline{Q} \longrightarrow \underline{Q}'$ such that for all $\underline{\underline{D}} \in \underline{\underline{\Delta}}$, every $\alpha \in \Omega_{\underline{Q},\underline{\underline{D}}}$ extends along j to some $\alpha' \in \Omega_{\underline{Q}',\underline{\underline{D}}}$. If α extends to a $\mathcal{D}$-homomorphism α': $\underline{\underline{Q}}' \to \underline{\underline{D}}$ which is not in $\Omega_{\underline{Q}',\underline{\underline{D}}'}$, then we would have to add it. If $\underline{\underline{D}}$ is injective in $\mathcal{D}$, at least with respect to embeddings in $\underline{\Pi}$, then such a homomorphism α': $\underline{\underline{Q}}' \to \underline{\underline{D}}$ will always exist. The diagram below shows that Conditions (0) and (C) will still hold.



Quasiorder $\underline{\Pi}$ by $\underline{P} \leqslant \underline{Q}$ if and only if $\underline{P}$ can be embedded into $\underline{Q}$. Define a subset $\underline{\Pi}'$ of $\underline{\Pi}$ to be a *maximal set in* $(\underline{\Pi};\leqslant)$ if every algebra in $\underline{\Pi}$ is dominated by some member of $\underline{\Pi}'$ but no member of $\underline{\Pi}'$ dominates another. If $\underline{\Pi}$ is a finite set of finite algebras, then up to isomorphism there is a unique maximal set in $(\underline{\Pi};\leqslant)$ while if $\underline{\Pi}$ is infinite it may have none. Note that every maximal set in $(\underline{\Pi};\leqslant)$ is cofinal and that if $\underline{\Pi}'$ is cofinal in $(\underline{\Pi};\leqslant)$, then

$$\mathcal{A} := \mathbf{ISP}(\underline{\Pi}) = \mathbf{ISP}(\underline{\Pi}').$$

2.7 LEMMA. *Suppose that* $\underline{\Pi}'$ *is cofinal in* $(\underline{\Pi};\leqslant)$, *in particular suppose that* $\underline{\Pi}'$ *is a maximal set in* $(\underline{\Pi};\leqslant)$, *and assume that each* $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ *is injective with respect to embeddings in* $\underline{\Pi}$. *If the Piggyback-Duality Theorem applies to yield a duality for* $\mathcal{A}$ *based on* $\underline{\Pi}$, *then it also applies to yield a duality for* $\mathcal{A}$ *based on* $\underline{\Pi}'$.

While deleting an algebra from $\underline{\Pi}$ reduces by one the number of components in $\mathrm{D}(\underline{A})$, deletion of a carrier results in a drastic drop in the number of relations required to satisfy Conditions (A) and (B). For this reason, in assuring that the separation condition (C) is satisfied it is best to add an extra unary operation to G rather than add an extra carrier. This interplay between carriers and unary operations will be illustrated in the next section. Although we wish to use the smallest possible number of carriers needed to satisfy Condtion (C), brute force will always work when $\underline{\Pi}$ is a finite set of finite algebras. In the result below, this finiteness condition is required only to guarantee that for each $\underline{P} \in \underline{\Pi}$ there are enough continuous homomorphisms $\alpha: \underline{\underline{P}} \to \underline{\underline{D}}$ ($\underline{\underline{D}} \in \underline{\underline{\Delta}}$) to separate the points of $\underline{P}$.

2.8 LEMMA. *Let A, X, D and E be as in the Piggyback-Duality Theorem and assume that* $\underline{\Pi}$ *is a finite set of finite algebras. If we choose* $\Omega_{\underline{P},\underline{\underline{D}}} = \mathcal{D}(\underline{\underline{P}},\underline{\underline{D}})$ *for all* $\underline{P} \in \underline{\Pi}$, $\underline{\underline{D}} \in \underline{\underline{\Delta}}$, *then Condition* (C) *can be deleted from the statement of the Piggyback Duality Theorem.*

PROOF. Let $\underline{P} \in \underline{\Pi}$ and let $a \neq b$ in $\underline{P}$. Since $\underline{\underline{P}} \in \underline{\underline{\mathcal{D}}} = \mathbf{ISP}(\underline{\underline{\Delta}})$ there exists $\underline{\underline{D}} \in \underline{\underline{\Delta}}$ and $\alpha \in \mathcal{D}(\underline{\underline{P}},\underline{\underline{D}})$ such that $\alpha(a) \neq \alpha(b)$; whence $(\alpha \circ u)(a) \neq (\alpha \circ u)(b)$ with $u = id_{\underline{P}}$. The only reason for insisting in Condition (C) that $u \in G$ is to ensure that every $\mathcal{X}$-morphism preserves u. Since every $\mathcal{X}$-morphism preserves $id_{\underline{P}}$, it follows that if $\Omega_{\underline{P},\underline{\underline{D}}} = \mathcal{D}(\underline{\underline{P}},\underline{\underline{D}})$, then Condition (C) is redundant.■

We can combine this lemma with Theorem 2.5 to yield a general set of sufficient conditions on the prevariety $\boldsymbol{A}$ for it to have a natural duality. It should be noted in this context that the variety $\boldsymbol{I}$ of implication algebras, which is generated as a prevariety by the two-element implication algebra, has no natural duality at all (see [11], pages 148-151).

2.9 THEOREM. *Assume that there is a natural full duality between* **D** *and* **E** *given by a purely relational* $\underline{\underline{\Delta}}$*-indexed structure* $\underset{\approx}{\Delta}$, *and assume that* $\underset{\approx}{\Delta}$ *is injective in* **E**. *Suppose that* $\underline{\Pi}$ *is a finite set of finite algebras and that the prevariety* **A** := $\mathbf{ISP}(\underline{\Pi})$ *has a reduct in D. Then there is a natural duality for A. If the set* R_0 *of relations on* $\underset{\approx}{\Delta}$ *is finite, then the* $\underline{\Pi}$*-indexed structure* $\underset{\sim}{\Pi}$ *giving rise to the duality between A and X can be chosen to be of finite type (that is, with* $G \cup H \cup R$ *finite).*

Although the variety of implication algebras has no natural duality, it has a reduct in sets which has a natural full duality (which, of course, is not purely relational; see [11]). Thus some restriction on the structure $\underset{\approx}{\Delta}$ in this theorem is necessary.

In our applications $\underline{\Pi}$ will be finite and $\underline{\underline{\Delta}}$ will consist of a single algebra $\underline{\underline{D}}$. Consequently we shall write $\underline{\Pi} = \{\underline{P}_0,\ldots,\underline{P}_{n-1}\}$ and shall abbreviate $\Omega_{\underline{P}_k,\underline{\underline{D}}}$ to Ω_k. To simplify the notation further, with the exception of the algebras $\underline{P}_0,\ldots,\underline{P}_{n-1}$ and $\underline{\underline{D}}$ we shall identify an algebra in $\boldsymbol{A}$ or $\boldsymbol{D}$ with its underlying set, and similarly for spaces in $\boldsymbol{X}$ or $\boldsymbol{E}$.

3. **Varieties and prevarieties of Ockham algebras.** Natural dualities for the variety $\boldsymbol{O}$ of Ockham algebras and for its subvarieties $\boldsymbol{P}_{m,n}$ $(m > n \geqslant 0)$ were constructed by B. A. Davey and H. Werner [12,13]. This theory does not apply to every variety generated by a single finite subdirectly algebra $\underline{P}$: in general $\mathbf{ISP}(\underline{P}) \neq \mathbf{HSP}(\underline{P})$. Consequently the generalised Piggyback-Duality Theorem 2.2, or the version of it presented in Theorem 2.5, is required.

Throughout this section we take $\boldsymbol{D} = \mathbf{ISP}(\underline{\underline{2}})$ to be the variety of bounded distributive lattices and $\boldsymbol{E}$ to be $\boldsymbol{P}$, the category of compact totally order-disconnected spaces. The resulting natural duality is just Priestley duality for $\boldsymbol{D}$. The set $\underline{\underline{\Delta}}$ consists of the single algebra $\underline{\underline{2}}$, while $\underset{\approx}{2}$ is the 2-element ordered space, $0 < 1$, with the discrete topology.

We recall that an *Ockham algebra* $(A;\wedge,\vee,0,1,\sim)$ is a bounded distributive lattice $(A;\wedge,\vee,0,1)$ with a unary operation $\sim$ satisfying

$$\sim(a \wedge b) = \sim a \vee \sim b,\ \sim(a \vee b) = \sim a \wedge \sim b,\ \sim 0 = 1,\ \sim 1 = 0.$$

The variety $\boldsymbol{O}$ of Ockham algebras and its subvarieties have been extensively studied by A. Urquhart [17] and M. S. Goldberg [15], with the aid of the restricted $\boldsymbol{D}$-$\boldsymbol{P}$ duality. Under this duality, $\boldsymbol{O}$ is dually equivalent to the category $\boldsymbol{S}$ of *Ockham spaces*: an object in $\boldsymbol{S}$ consists of a pair, $(Y;g)$, where $Y \in \boldsymbol{P}$ and $g\colon Y \to Y$ is a continuous order-reversing map; morphisms in $\boldsymbol{S}$ are $\boldsymbol{P}$-morphisms commuting with g. Each $A \in \boldsymbol{O}$ is isomorphic to $\mathrm{KH}(A) = \boldsymbol{P}(\boldsymbol{D}(A,\underline{\underline{2}}),\underset{\approx}{2})$; $\sim$ is defined on $\mathrm{KH}(A)$ as follows: for $\phi \in \mathrm{KH}(A)$,

$$(\forall y \in \boldsymbol{D}(A,\underline{\underline{2}}))\quad (\sim\phi)(y) = 1 \Leftrightarrow \phi(g(y)) = 0.$$

For $m > n \geqslant 0$, the subvariety $\boldsymbol{P}_{m,n}$ is $\mathbf{ISP}(\mathrm{L}_{m,n})$, where $\mathrm{L}_{m,n}$ is the algebra whose dual in $\boldsymbol{S}$ is $(\mathbf{Z}_m;\gamma_n)$, where $\mathbf{Z}_m = \{0,1,\ldots,m-1\}$ has the discrete order and $\gamma_n\colon \mathbf{Z}_m \to \mathbf{Z}_m$ is defined by $\gamma_n(k) = k+1$ for $0 \leqslant k < m-1$ and $\gamma_n(m-1) = n$; the equational characterisation of $\boldsymbol{P}_{m,n}$ can be found in [15] (Proposition 2.7).

The Piggyback-Duality Theorem was applied to the varieties $\boldsymbol{P}_{m,n}$ in [13]. It yields a duality based on 2^m where

$$\underset{-}{2}^m = (2^m; \wedge, \vee, 0, 1, \sim)$$

has pointwise-defined lattice structure and negation, $\sim$, given by

$$(\forall k < m) \quad (\sim a)(k) = 1 \Leftrightarrow a(\gamma_n^{k+1}(0)) = 0,$$

and

$$\underset{\sim}{2}^m = (2^m; \tau, g, \preccurlyeq)$$

has discrete topology τ, g-map given by

$$(\forall a \in 2^m)(\forall k < m) \quad g(a)(k) = a(\gamma_n(k))$$

and alternating order $\preccurlyeq$ given by

$$a \preccurlyeq b \Leftrightarrow (\forall k \geqslant 0) \begin{cases} a(\gamma_n^k(0)) \leqslant b(\gamma_n^k(0)) & (\text{k even}), \\ a(\gamma_n^k(0)) \geqslant b(\gamma_n^k(0)) & (\text{k odd}). \end{cases}$$

A finite algebra in $\mathbf{O}$ is subdirectly irreducible if and only if it is a subalgebra of $L_{m,n}$ for some $m > n \geqslant 0$.

We now let $\underline{P}_0$ be a fixed finite nontrivial subdirectly irreducible Ockham algebra and let $\mathcal{A} = \mathbf{HSP}(\underline{P}_0)$. By Jónsson's Lemma, $\mathcal{A} = \mathbf{ISP}(\underline{\Pi})$, where $\underline{\Pi} = \mathbf{HS}(\underline{P}_0)$. We regard $\underline{P}_0$ as $K(W_0) = \mathcal{P}(W_0, \underset{\approx}{2})$; the Ockham space W_0 is of the form $(\mathbf{Z}_m; \gamma, \leqslant)$ where $\gamma = \gamma_n$ for some $n < m$ and $\leqslant$ is an order with respect to which γ is order-reversing. We denote the γ-closed subset $\{\gamma^k(j) | k \geqslant 0\}$ of W_0 by W_j. If $0 \leqslant j \leqslant n$, then $W_j = \{j, j+1, ..., m-1\}$ and if $n \leqslant j < m$, then $W_j = \{n, n+1, ..., m-1\}$ (this set is called the *loop* of W_0).

We let $\mathcal{Y} = \{Y \in \mathcal{S} | K(Y) \in \mathcal{A}\}$. The description of $\mathcal{Y}$ given by M. S. Goldberg in [15], Theorem 2.11, has been refined by B. A. Davey and H. A. Priestley, as follows.

3.1 THEOREM. ([10], Theorem 3.15.) *An Ockham space* Y *lies in* $\mathcal{Y}$ *if and only if it is the (not necessarily disjoint) union of spaces*

$$Y^0 := \{y \in Y | (\exists \phi \in \mathcal{S}(W_0, Y)) \ \phi(0) = y\}$$

and

$$Y^1 := \{y \in Y | (\exists \phi \in \mathcal{S}(W_1, Y)) \ \phi(1) = y\}.$$

Equivalently,

$$Y^i = \{y \in Y | Y \models \sigma_i(y)\},$$

where

$$\sigma_i(y) = \underset{0\leqslant k,\ell<m}{\&} (g^k(y) \leqslant g^\ell(y) | \gamma^k(i) \leqslant \gamma^\ell(i)),$$

and $Y \in \boldsymbol{Y}$ *if and only if* $Y \models (\forall y)\sigma(y)$, *where* $\sigma(y) = \sigma_0(y) \vee \sigma_1(y)$.

3.2 COROLLARY. $\boldsymbol{A} = \mathbf{ISP}(\underline{P}_0,\underline{P}_1)$, *where* $\underline{P}_0 = K(W_0)$ *and* $\underline{P}_1 = K(W_1)$.

For $i = 0,1$, we take Ω_i to consist of the single homomorphism $\alpha_i = \underline{\underline{P}}_i \to \underline{\underline{2}}$, where $(\forall \phi \in \underline{\underline{P}}_i)\ \alpha_i(\phi) = \phi(i)$. We write Φ_i in place of Φ_{α_i}.

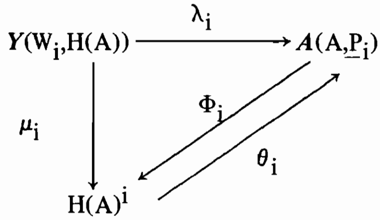
3.3 LEMMA. *For* $A \in \boldsymbol{A}$ *and* $i = 0,1$, *there exist natural bijections between* (i) $\boldsymbol{A}(A,\underline{P}_i)$ *and* $\boldsymbol{Y}(W_i,H(A))$ *(induced by* H*) and* (ii) $\boldsymbol{Y}(W_i,H(A))$ *and* $H(A)^i$ *(given by evaluation at* i*). These bijections combine to produce a pair of mutually inverse maps from* $\boldsymbol{A}(A,\underline{P}_i)$ *onto* $H(A)^i$ *and from* $H(A)^i$ *onto* $\boldsymbol{A}(A,\underline{P}_i)$; *the former map is* Φ_i, *the latter is* $\theta_i = k_A(\text{-})\circ f_i$, *where*

$$(\forall y \in H(A)^i)(\forall k \geqslant 0)\quad f_i(y)(\gamma^k(i)) = g^k(y).$$

PROOF. We denote by λ_i the natural bijection from $\boldsymbol{Y}(W_i,H(A))$ to $\boldsymbol{A}(A,\underline{P}_i)$ and by μ_i that from $\boldsymbol{Y}(W_i,H(A))$ to $H(A)^i$:

$$(\forall \phi \in \boldsymbol{Y}(W_i,H(A)))\quad \lambda_i(\phi) = k_A(\text{-})\circ\phi \text{ and } \mu_i(\phi) = \phi(i).$$

It is now enough to show that $\Phi_i\circ\lambda_i = \mu_i$ and $\theta_i\circ\mu_i = \lambda_i$ where θ_i is as in the statement of the lemma.



For $\phi \in \boldsymbol{Y}(W_i,H(A))$ and $k \geqslant 0$,

$$(f_i\circ\mu_i)(\phi)(\gamma^k(i)) = f_i(\phi(i))(\gamma^k(i)) = g^k(\phi(i)) = \phi(\gamma^k(i)).$$

Hence $(f_i\circ\mu_i)(\phi) = \phi$ and consequently $\theta_i\circ\mu_i = \lambda_i$. Also for $a \in A$, $\phi \in \boldsymbol{Y}(W_i,H(A))$ and $k \geqslant 0$,

$$(\Phi_i\circ\lambda_i)(\phi)(a) = \alpha_i(\lambda_i(\phi)(a)) = (\lambda_i(\phi)(a))(i) = \phi(i)(a) = \mu_i(\phi)(a),$$

whence $\Phi_i\circ\lambda_i = \mu_i$, as required.■

From Lemma 3.3 we deduce an alternative description of the subspaces $H(A)^i$:

$$H(A)^i = \{y \in H(A) | (\exists x \in \boldsymbol{A}(A,\underline{P}_i))\ \alpha_i\circ x = y\}.$$

It is clear from this that Φ_0 and Φ_1 are jointly onto. A further consequence is that, for $A \in \mathcal{A}$ and $i = 0,1$, $H(A)^i$ is closed. This can be seen more directly from the fact that for any $Y \in \mathcal{Y}$,

$$Y^i = \cap_{k,\ell}\{(g^k \times g^\ell)^{-1}(\leqslant) \mid \gamma^k(i) \leqslant \gamma^\ell(i)\},$$

which is closed since $\leqslant$ is a closed relation and g is continuous. In obtaining a duality theorem for $\mathcal{A}$ based on the algebras $\underline{P}_0$ and $\underline{P}_1$ and the carriers α_0 and α_1, it is in fact not necessary to verify directly that Φ_0 and Φ_1 are jointly onto because this is a consequence of Condition (C) (which we do have to ensure is satisfied); see Proposition 2.4 and Theorem 2.5. Lemma 3.4 handles Condition (C).

3.4 LEMMA. *Let* $g_0: \underline{P}_0 \to \underline{P}_1$ *and* $g_1: \underline{P}_1 \to \underline{P}_0$ *be defined by*

$$(\forall \phi \in \underline{P}_0) g_0(\phi) = \phi\restriction_{W_1} \text{ and } (\forall \phi \in W_1)\, g_1(\phi) = \phi \circ \gamma^2.$$

Suppose $\phi \neq \psi$ *in* $\underline{P} \in \underline{\Pi} = \{\underline{P}_0, \underline{P}_1\}$. *Then there exists* $u: \underline{P} \to \underline{Q} \in \underline{\Pi}$, *where* u *is a* k-*fold composite of* g_0 *and* g_1, *and* $\alpha \in \Omega_Q$ *such that* $(\alpha \circ u)(\phi) \neq (\alpha \circ u)(\psi)$.

PROOF. Suppose $\underline{P} = \underline{P}_0$ and $\phi \neq \psi$ in $\underline{P}$. Then there exists a smallest integer $k \geqslant 0$ such that $\phi(k) \neq \psi(k)$. If k is even, $k = 2j$, say, then $(\alpha_0 \circ (g_1 \circ g_0)^j)(\phi) \neq (\alpha_0 \circ (g_1 \circ g_0)^j)(\psi)$. On the other hand, if k is odd, $k = 2j + 1$, say, $(\alpha_1 \circ g_0 \circ (g_1 \circ g_0)^j)(\phi) \neq (\alpha_1 \circ g_0 \circ (g_1 \circ g_0)^j)(\psi)$. A similar argument works when $\underline{P} = \underline{P}_1$. ■

The next lemma shows how we can construct the maximal Ockham subalgebras required to meet Conditions (A) and (B) in the Piggyback-Duality Theorem.

3.5 LEMMA. *Let* $(A; \wedge, \vee, S)$ *be an algebra such that*

(i) $(A; \wedge, \vee)$ *is a lattice*;

(ii) S *is a family of unary operations which are either endomorphisms or dual endomorphisms of* $(A; \wedge, \vee)$. *For a subset* $B \subseteq A$, *define* $B_0 = B$ *and*

$$(\forall n \geqslant 0) \quad B_{n+1} = B_n \backslash \{a \in B_n \mid (\exists s \in S)\ s(a) \notin B_n\},$$

and let $B^o = \cap\{B_n \mid n \geqslant 0\}$. *Provided* B *is a sublattice of* A, B^o *is a subalgebra of* A *and is the largest subalgebra of* A *contained in* B.

PROOF. Note that for all $s \in S$ we have

$$a \in B_{n+1} \Rightarrow s(a) \in B_n.$$

It follows at once that B^o is closed under s for each $s \in S$. In fact we have

$$a \in B_{n+1} \Leftrightarrow a \in B_n \text{ and } (\forall s \in S)\ s(a) \in B_n.$$

We now prove by induction that B_n is a sublattice for every $n \geqslant 0$. Firstly we note that B_0 is a sublattice by assumption. Assume B_n is a sublattice. Then

$$\begin{aligned} a,b \in B_{n+1} &\Rightarrow a,b \in B_n \text{ and } (\forall s \in S)\ s(a),s(b) \in B_n \\ &\Rightarrow a \wedge b, a \vee b \in B_n \text{ and } (\forall s \in S)\ s(a) \wedge s(b), s(a) \vee s(b) \in B_n \\ &\Rightarrow a \wedge b, a \vee b \in B_n \text{ and } (\forall s \in S)\ s(a \wedge b), s(a \vee b) \in B_n \\ &\Rightarrow a \wedge b, a \vee b \in B_{n+1} \end{aligned}$$

(the penultimate implication being valid because each $s \in S$ is an endomorphism or a dual endomorphism).■

It follows, since $\sim$ is a dual endomorphism, that any sublattice B of an Ockham algebra A has a unique maximal Ockham subalgebra denoted B^{o}. We remark that if θ is a lattice congruence on A, then θ^{o} is a congruence and is the largest Ockham congruence contained in θ.

It will not be necessary to include in R^+ all the subalgebras of $\underline{P} \times \underline{Q}$ ($\underline{P},\underline{Q} \in \underline{\Pi}$) listed in Conditions (A) and (B) of Theorem 2.2; see Remarks 2.3. In particular, if $r \leqslant \underline{P} \times \underline{Q}$ is in R^+ then $r^{-1} \leqslant \underline{Q} \times \underline{P}$ is not needed. Thus Lemma 3.6 describes only one from each pair r,r^{-1} of maximum subalgebras.

3.6 LEMMA. *Let* $\underline{P}_0$, $\underline{P}_1$, α_0 *and* α_1 *be as above. Then*

(a) *for* $i = 0,1$, $\ker(\alpha_i)^{\text{o}}$ *is the equality relation on* $\underline{P}_i$;

(b) $\ker(\alpha_1,\alpha_0)^{\text{o}} = \{(\psi,\phi) \in \underline{P}_1 \times \underline{P}_0 \mid \phi = \psi\circ\gamma\}$ *and is the graph of a partial map from* $\underline{P}_1$ *to* $\underline{P}_0$;

(c) $\alpha_0^{-1}(\leqslant)^{\text{o}}$ *is the alternating order,* $\preccurlyeq$, *on* $\underline{P}_0$, *viz. for* $\phi,\psi \in \underline{P}_0$,

$$\phi \preccurlyeq \psi \Leftrightarrow (\forall k \geqslant 0) \begin{cases} \phi(\gamma^k(0)) \leqslant \psi(\gamma^k(0)) & (k \textit{ even}), \\ \phi(\gamma^k(0)) \geqslant \psi(\gamma^k(0)) & (k \textit{ odd}), \end{cases}$$

and $\alpha_1^{-1}(\leqslant)^{\text{o}} = \{(\phi,\psi) \in \underline{P}_1 \times \underline{P}_1 \mid \psi\circ\gamma \preccurlyeq \psi\circ\gamma\}$, *where* $\preccurlyeq$ *is the alternating order above*;

(d) $(\alpha_0,\alpha_1)^{-1}(\leqslant)^{\text{o}} = \{(\phi,\psi) \in \underline{P}_0 \times \underline{P}_1 \mid \phi \preccurlyeq \psi\circ\gamma\}$ *and* $(\alpha_1,\alpha_0)^{-1}(\leqslant)^{\text{o}} = \{(\psi,\phi) \in \underline{P}_1 \times \underline{P}_0 \mid \psi\circ\gamma \preccurlyeq \phi\}$, *where* $\preccurlyeq$ *is the alternating order.*

PROOF. We prove only (b) and (c). The remaining assertions are proved analogously.

Firstly

$$\begin{aligned}\ker(\alpha_1,\alpha_0)^o &= \{(\psi,\phi)\in \underline{P}_1\times \underline{P}_0 \mid \alpha_1(\psi)=\alpha_0(\phi)\}^o\\ &= \{(\psi,\phi)\in \underline{P}_1\times \underline{P}_0 \mid \psi(1)=\phi(0)\}^o\\ &= \{(\psi,\phi)\in \underline{P}_1\times \underline{P}_0 \mid (\forall k\geqslant 0)\ \psi(\gamma^k(1))=\phi(\gamma^k(0))\},\end{aligned}$$

and this set is clearly the Ockham subalgebra B^o associated with the sublattice

$$B=\{(\psi,\phi)\in \underline{P}_1\times \underline{P}_0 \mid \psi(1)=\phi(0)\}$$

of $\underline{P}_1 \times \underline{P}_0$, as in Lemma 3.5. Hence

$$\ker(\alpha_1,\alpha_0)^o=\{(\psi,\phi)\in \underline{P}_1\times \underline{P}_0 \mid \phi=\psi\circ\gamma\}$$

as required.

Similarly,

$$\begin{aligned}\alpha_0^{-1}(\leqslant)^o &= \{(\phi,\psi)\in \underline{P}_0\times \underline{P}_0 \mid \alpha_0(\phi)\leqslant \alpha_0(\psi)\}^o\\ &= \{(\phi,\psi)\in \underline{P}_0\times \underline{P}_0 \mid (\forall k\geqslant 0)\\ &\qquad \phi(\gamma^k(0))\leqslant \psi(\gamma^k(0))\ (k \text{ even}),\\ &\qquad \phi(\gamma^k(0))\geqslant \psi(\gamma^k(0))\ (k \text{ odd})\}\\ &= \{(\phi,\psi)\in \underline{P}_0\times \underline{P}_0 \mid \phi \leqslant \psi\},\end{aligned}$$

while

$$\begin{aligned}\alpha_1^{-1}(\leqslant)^o &= \{(\phi,\psi)\in \underline{P}_1\times \underline{P}_1 \mid \alpha_1(\phi)\leqslant \alpha_1(\psi)\}^o\\ &= \{(\phi,\psi)\in \underline{P}_1\times \underline{P}_1 \mid (\forall k\geqslant 1)\ \phi(\gamma^k(0))\leqslant \psi(\gamma^k(0))\ \ (k \text{ odd}),\\ &\qquad\qquad \phi(\gamma^k(0))\geqslant \psi(\gamma^k(0))\ \ (k \text{ even})\}\\ &= \{(\phi,\psi)\in \underline{P}_1\times \underline{P}_1 \mid (\forall \ell\geqslant 0)\ (\phi\circ\gamma)(\gamma^\ell(0))\ \leqslant (\psi\circ\gamma)(\gamma^\ell(0))\\ &\qquad (\phi\circ\gamma)(\gamma^\ell(0))\leqslant(\psi\circ\gamma)(\gamma^\ell(0))\ (\ell \text{ even}),\\ &\qquad (\phi\circ\gamma)(\gamma^\ell(0))\geqslant(\psi\circ\gamma)(\gamma^\ell(0))\ (\ell \text{ odd})\}\\ &= \{(\phi,\psi)\in \underline{P}_1\times \underline{P}_1 \mid \phi\circ\gamma \leqslant \psi\circ\gamma\}.\blacksquare\end{aligned}$$

We note that $\ker(\alpha_i)^o$ $(i=0,1)$ will not be needed as a member of R^+ and that $\ker(\alpha_1,\alpha_0)^o$, as the intersection of $((\alpha_0,\alpha_1)^{-1}(\leqslant)^o)^{-1}$ and $(\alpha_1,\alpha_0)^{-1}(\leqslant)^o$, is redundant once $(\alpha_1,\alpha_0)^{-1}(\leqslant)^o$ and $(\alpha_0,\alpha_1)^{-1}(\leqslant)^o$ are included.

From the preceding lemmas and Theorem 2.5 we obtain a duality theorem for A.

3.7 THEOREM. *Let* $A = \mathrm{ISP}(\underline{\Pi})$ *be the variety of Ockham algebras generated by a finite subdirectly irreducible algebra* $\underline{P}_0$ *(where* $\underline{\Pi} = \{\underline{P}_0,\underline{P}_1\}$*). Let*

$$\underset{\sim}{\Pi} = (P_0 \dot{\cup} P_1;G,R,\tau);$$

here τ *is the discrete topology on* $P_0 \dot{\cup} P_1$,

$$G = \{g_0\colon \underline{P}_0 \to \underline{P}_1,\ g_1\colon \underline{P}_1 \to \underline{P}_0\}$$

(where g_0 *and* g_1 *are the homomorphisms defined in Lemma* 3.4*), and*

$$R = \{\leqslant_0, \leqslant_1, \leftarrow, \succ\!\!-\}$$

where

$$\leqslant_0 = \alpha_0^{-1}(\leqslant)^o,\ \leqslant_1 = \alpha_1^{-1}(\leqslant)^o,\ \leftarrow = (\alpha_0,\alpha_1)^{-1}(\leqslant)^o$$

and

$$\succ\!\!- = (\alpha_1,\alpha_0)^{-1}(\leqslant)^o.$$

Let $X = \mathrm{IS_cP}(\underset{\sim}{\Pi})$. *Then the functors* D: $A \to X$ *and* E: $X \to A$ *yield a natural duality.*

We now have two dualities for A: the restricted ***D-P*** duality and the natural duality described in Theorem 3.7. There is a procedure for translating from one duality to the other. The formal statement of the translation is quite involved, but the underlying ideas are extremely simple. To obtain D(A) from H(A) we separate the portions $H(A)^0$ and $H(A)^1$, while to obtain H(A) from $D(A) = X^0 \dot{\cup} X^1$ (where $X^i = A(A,\underline{P}_i)$) we paste X^0 and X^1 together appropriately. The relations of the $\underline{\Pi}$-indexed structure $\underset{\sim}{\Pi}$ are exactly those needed to provide the instructions for re-creating the Y-object H(A). The effect of decomposing H(A) to give D(A) is to produce a structure with algebraic relations from one whose relations are conveniently simple, but non-algebraic in general.

3.8 THEOREM. *Let* $A = \mathrm{ISP}(\underline{\Pi})$ *be the variety of Ockham algebras generated by the finite subdirectly irreducible algebra* $\underline{P}_0$ *(where* $\underline{\Pi} = \{\underline{P}_0,\underline{P}_1\}$*). Let* $\underset{\sim}{\Pi}$ *be as in Theorem* 3.7 *and for* $A \in A$, *let* $X^i := A(A,\underline{P}_i)$ *and*

$$D(A) := X = (X^0 \dot{\cup} X^1) \leqslant \underset{\sim}{\Pi}^A.$$

Let $Y = H(A)$ *and let* Y^0 *and* Y^1 *be as in Theorem* 3.1. *From the Ockham space*

$(Y;\tau,\leqslant,g)$ *form the* $\underline{\Pi}$*-indexed structure*

$$\underset{\sim}{Y} = (Y^0 \dot{\cup} Y^1; G', R', \tau')$$

as follows:

(i) τ' *is the sum of the topologies induced by* τ *on* Y^0 *and* Y^1;

(ii) $G' = \{g\restriction_{Y^0}, g\restriction_{Y^1}\}$;

(iii) $R' = \{\leqslant \cap (Y^0 \times Y^0), \leqslant \cap (Y^1 \times Y^1), \leqslant \cap (Y^0 \times Y^1), \leqslant \cap (Y^1 \times Y^0)\}$.

Then D(A) *is isomorphic to* $\underset{\sim}{Y}$.

Conversely, from D(A) *define* $(Z;\tau_Z,\leqslant_Z,g_Z)$ *as follows*:

(i)* $Z = X/\rho$ *where* ρ *is the equivalence relation on* X *given by* $x\rho y$ *if and only if* $x = y$ *or* $x = I(y)$ *or* $y = I(x)$, *where* I *is the pointwise extension to* D(A) *of the partial function whose graph is*

$$\ker(\alpha_1,\alpha_0)^0 = \{(\psi,\phi) \in \underline{P}_1 \times \underline{P}_0 \mid (\phi,\psi) \in \leftarrow \text{ and } (\psi,\phi) \in \rightarrowtail\};$$

(ii)* τ_Z *is the finest topology such that the maps* $q_i = \pi \circ j_i$ $(i = 0,1)$ *are continuous (where* $\pi\colon X \to Z$ *is the canonical projection and* $j_i\colon X^i \to X$ $(i = 0,1)$ *are the canonical injections). Identify* X^i *with* $\operatorname{Im} q_i \subseteq Z$ $(i = 0,1)$.

(iii)* $\leqslant_Z = \leqslant_0 \cup \leqslant_1 \cup \leftarrow \cup \rightarrowtail$;

(iv)* g_Z *is the function whose restriction to* X^i *is* g_i $(i = 0,1)$.

Then $(Z;\tau_Z,\leqslant_Z,g_Z)$ *is a well-defined Ockham space isomorphic to* H(A). [*Each relation on* $\underset{\sim}{\Pi}^A$ *is denoted by the same symbol as the relation on* $\underset{\sim}{\Pi}$ *of which it is the pointwise extension, and similarly for operations.*]

PROOF. Lemma 3.3 provides a bijection $\theta_i\colon Y^i = H(A)^i \to X^i = \mathcal{A}(A,\underline{P}_i)$ inverse to Φ_i. We first show how the maps θ_i treat the relations and operations of $G' \cup R'$.

For $x,y \in Y^0$,

$$\theta_0(x) \leqslant_0 \theta_0(y) \Leftrightarrow (\forall a \in A)\,(\forall k \geqslant 0)\begin{cases}\theta_0(x)(a)(\gamma^k(0)) \leqslant \theta_0(y)(a)(\gamma^k(0)) & (k \text{ even})\\ \theta_0(x)(a)(\gamma^k(0)) \geqslant \theta_0(y)(a)(\gamma^k(0)) & (k \text{ odd})\end{cases}$$

$$\Leftrightarrow (\forall a \in A)(\forall k \geqslant 0)\begin{cases}g^k(x)(a) \leqslant g^k(y)(a) & (k \text{ even})\\ g^k(x)(a) \geqslant g^k(y)(a) & (k \text{ odd})\end{cases}$$

$$\Leftrightarrow (\forall a \in A) \quad x(a) \leqslant y(a) \quad (\text{since } g \text{ is order-reversing})$$

$$\Leftrightarrow x \leqslant y \quad \text{in } H(A).$$

For $x,y \in Y^1$,

$$\theta_1(x) \leqslant_1 \theta_1(y) \Leftrightarrow (\forall a \in A)(\forall k \geqslant 0) \begin{cases} \theta_1(x)(a)(\gamma^k(1)) \leqslant \theta_1(y)(a)(\gamma^k(1)) & (k \text{ even}) \\ \theta_1(x)(a)(\gamma^k(1)) \geqslant \theta_1(y)(a)(\gamma^k(1)) & (k \text{ odd}) \end{cases}$$

$$\Leftrightarrow (\forall a \in A)(\forall k \geqslant 0) \begin{cases} g^k(x)(a) \leqslant g^k(y)(a) & (k \text{ even}) \\ g^k(x)(a) \geqslant g^k(y)(a) & (k \text{ odd}) \end{cases}$$

$$\Leftrightarrow x \leqslant y \qquad \text{in } H(A).$$

Similarly, for $x \in Y^0$, $y \in Y^1$, $\theta_0(x) \leftarrow \theta_1(y) \Leftrightarrow x \leqslant y$ and, for $x \in Y^1$, $y \in Y^0$, $\theta_1(x) \succ\!\!\!\!- \theta_0(y) \Leftrightarrow x \leqslant y$. Further, for $x \in Y^0$, $y \in Y^1$,

$$\begin{aligned} g_0(\theta_0(x)) = \theta_1(y) &\Leftrightarrow (\forall a \in A)\ g_0(\theta_0(x)(a)) = \theta_1(y)(a) \\ &\Leftrightarrow (\forall a \in A)(\forall k \geqslant 0)\ \theta_0(x)(a)(\gamma^k(1)) = \theta_1(y)(a)(\gamma^k(1)) \\ &\Leftrightarrow (\forall a \in A)(\forall k \geqslant 0)\ g^{k+1}(x)(a) = g^k(y)(a) \\ &\Leftrightarrow g(x) = y. \end{aligned}$$

Similarly, for $x \in Y^1$ and $y \in Y^0$, $g_1(\theta_1(x)) = \theta_0(y)$. Further, since Φ_0 and Φ_1 are continuous, X^i is homeomorphic to $Y^i = H(A)^i$ for $i = 0,1$. It follows that D(A) and $\underset{\sim}{Y}$ are isomorphic structures.

The equivalence relation ρ on X is a convenient way to formalise the process by which X^0 and X^1 are glued together. Less formally, we identify each point x in the domain of I with its image I(x). On $Z = X/\rho$, $\leqslant_Z$ (as given in (iii)*) is a well-defined partial order, since $\leqslant$ is a partial order on Y, and g_Z (as given in (iv)*) is a well-defined order-reversing map, since $g\colon Y \to Y$ is an order-reversing map. (These facts could, of course, be checked directly.) To see that the topologies match up correctly, it is sufficient to note that the map q_i is a homeomorphism from X^i onto $q_i(X^i) \subseteq Z$ $(i = 0,1)$ (see N. Bourbaki [4], I.2.5). We deduce that $(Z;\tau_Z,\leqslant_Z,g_Z)$ is isomorphic to $\underset{\sim}{Y} = H(A)$.■

We have, for each $A \in \boldsymbol{A}$,

$$A \cong ED(A) = \boldsymbol{X}(D(A),\underset{\sim}{\Pi})$$

and

$$A \cong KH(A) = \boldsymbol{P}(H(A),\underset{\approx}{2}).$$

The isomorphism between $\boldsymbol{X}(D(A),\underset{\sim}{\Pi})$ and $\boldsymbol{P}(H(A),\underset{\approx}{2})$ is established by maps s and t defined as follows:

$$(\forall \phi \in \mathcal{P}(\mathrm{H}(\mathrm{A}),\underset{\approx}{2}))\quad s(\phi)\upharpoonright_{\mathcal{A}(\mathrm{A},\underline{\mathrm{P}}_{\mathrm{i}})} = \phi \circ \Phi_{\mathrm{i}} \quad (\mathrm{i} = 0,1)$$

and

$$(\forall \psi \in \mathcal{X}(\mathrm{D}(\mathrm{A}),\underset{\sim}{\Pi}))\quad t(\psi)\upharpoonright_{\mathrm{H}(\mathrm{A})^{\mathrm{i}}} = \alpha_{\mathrm{i}} \circ \psi \circ \theta_{\mathrm{i}} \quad (\mathrm{i} = 0,1)$$

where Φ_{i} and θ_{i} are as in Lemma 3.3.

It may happen that the prevariety **ISP**($\underline{\mathrm{P}}_0$) (which we shall henceforth denote by $\mathcal{A}^{\bullet}$) coincides with the variety $\mathcal{A}$ = **HSP**($\underline{\mathrm{P}}$) = **ISP**($\underline{\mathrm{P}}_0,\underline{\mathrm{P}}_1$). This occurs precisely when there exists an embedding j: $\underline{\mathrm{P}}_1 \to \underline{\mathrm{P}}_0$. In this case Remark 2.6 applies: it is possible to dispense with the algebra $\underline{\mathrm{P}}_1$, so long as extra carriers are introduced so that Condition (C) is met. A single extra carrier, β_0, suffices, defined by

$$(\forall \phi \in \underline{\mathrm{P}}_0)\quad \beta_0(\phi) = \phi(1).$$

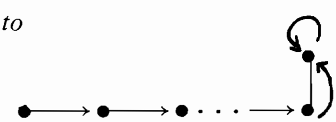
(This can be seen by adapting Lemma 3.4.)

The prevariety $\mathcal{A}^{\bullet}$ coincides with the variety $\mathcal{A}$ if and only if $\underline{\mathrm{P}}_1$ belongs to $\mathcal{A}^{\bullet}$, or equivalently if and only if there exists an $\mathcal{S}$-morphism of W_0 onto W_1. Lemma 3.9 shows when, and how, this happens.

3.9 LEMMA. (i) *The map* $\gamma\colon \mathrm{W}_0 \to \mathrm{W}_1$ *is order-preserving in the following cases*:

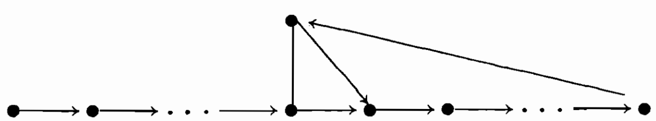
(a) W_0 *is an antichain*;

(b) W_0 *is isomorphic to*



or its order-theoretic dual;

(c) W_0 *is isomorphic to*



or its order-theoretic dual.

(ii) *The order-preserving map* $\gamma^2\colon \mathrm{W}_0 \to \mathrm{W}_1$ *is onto if*

(d) W_1 *is a loop.*

Only if (at least) one of (a) - (d) *holds does there exist an* $\mathcal{S}$-*morphism from* W_0 *onto*

W_1. *It is only possible for* (d) *to hold without one of* (a) - (c) *holding when* m (= $|W_0|$) *is odd.*

PROOF. M. S. Goldberg has proved the final assertion ([15], Remarks following Corollary 2.15). It is obvious that, in cases (a) - (c), γ provides the required $\mathcal{S}$-morphism and that in case (d), γ^2 does.■

3.10 THEOREM. *Let* $\mathcal{A} = \mathbf{ISP}(\underline{P}_0,\underline{P}_1)$. *The following are equivalent*:

(i) $\gamma: W_0 \to W_1$ *is order-preserving*;

(ii) $j_1: P_1 \to P_0$ *given by* $j_1(\phi) = \phi \circ \gamma$ *is a well-defined embedding*;

(iii) *for all* $A \in \mathcal{A}$, $H(A)^1 \subseteq H(A)^0$.

The following are also equivalent:

(i)* $\gamma^2: W_0 \to W_1$ *is surjective*;

(ii)* *the homomorphism* $j_2: \underline{P}_1 \to \underline{P}_0$ *given by* $j_2(\phi) = \phi \circ \gamma^2$ *is an embedding*;

(iii)* *if* m *is odd, then* $(\forall A \in \mathcal{A})\ H(A)^1 \subseteq g(H(A)^0)$ *and if* m *is even then* $(\forall A \in \mathcal{A})\ H(A)^1 \subseteq H(A)^0$.

The variety $\mathcal{A} = \mathbf{HSP}(\underline{P}_0)$ *coincides with the prevariety* $\mathcal{A}^{\cdot} = \mathbf{ISP}(\underline{P}_0)$ *if and only if either* $(\forall A \in \mathcal{A})\ H(A)^1 \subseteq H(A)^0$ *or* $(\forall A \in \mathcal{A})\ H(A)^1 = g(H(A)^0)$ *(or both)*.

PROOF. Conditions (i) and (ii) are dual, as are (i)* and (ii)*.

Suppose $\gamma \in \mathcal{S}(W_0,W_1)$. Take $A \in \mathcal{A}$ and $y \in H(A)^1$, so there exists $\phi \in \mathcal{S}(W_1,H(A))$ with $\phi(1) = y$. Then $\phi\circ\gamma \in \mathcal{S}(W_0,H(A))$ and $(\phi\circ\gamma)(0) = y$, whence $y \in H(A)^0$. Conversely suppose (iii) holds. Let $A = \underline{P}_0$ and identify H(A) with W_0. Certainly $1 \in (W_0)^1$ so there exists by hypothesis $\phi \in \mathcal{S}(W_0,W_0)$ with $\phi(0) = 1$. Necessarily $\phi = \gamma$, so γ is order-preserving. Hence (i) and (iii) are equivalent.

Now assume (i)* holds. This forces W_1 to be a loop. Take $A \in \mathcal{A}$ and $y \in H(A)^1$ and let $\phi \in \mathcal{S}(W_1,H(A))$ map 1 to y. Suppose first m is odd. Then $1 = \gamma(0) = \gamma^m(0)$. Hence $(\phi\circ\gamma^{m-1})(0) = z$ is such that $g(z) = g(\phi(\gamma^{m-1}(0))) = \phi(\gamma^m(0)) = y$ and (since γ^{m-1} is a power of γ^2) $\phi\circ\gamma^{m-1} \in \mathcal{S}(W_0,H(A))$. We thus have $y \in g(H(A)^0)$. Now suppose m is even. In this case W_1 is a loop of odd length and so the order on it is discrete ([17], Theorem 11). This implies that $\gamma\restriction_{W_1}$ is order-preserving. We have $1 = \gamma^2(k)$ for some $k > 0$, whence $1 = ((\gamma\restriction W_1)^j\circ\gamma^2)(0)$ for some $j > 0$. It follows that $y = (\phi\circ(\gamma\restriction W_1)^j\circ\gamma^2)(0)$ and so $y \in H(A)^0$.

Conversely assume (iii)* holds, with m odd and $(\forall A \in \mathcal{A})\ H(A)^1 \subseteq g(H(A)^0)$

(the other case was handled in the proof above that (iii) implies (i)). Take $A = \underline{P}_1$ and identify H(A) with W_1. Clearly $1 \in (W_1)^1$, so by hypothesis there exists $\phi \in S(W_0, W_1)$ such that $\phi(0) = j$ and $1 = \gamma(j)$ for some $j \geqslant 1$. Then $\phi(1) = \phi(\gamma(0)) = \gamma(\phi(0)) = \gamma(j) = 1$, whence ϕ restricted to W_1 is the identity map and $\phi\colon W_0 \to W_1$ is surjective. But $\phi = \gamma^j$. If $j \geqslant 2$, γ^2 is surjective, while if $j = 1$, $|W_1| = 1$ and $\gamma^2 = \gamma = \phi$ is surjective.

The last part of the theorem is now immediate from Lemma 3.9 since $A = A^{\bullet}$ if and only if W_1 is an S-morphic image of W_0.■

3.11 COROLLARY. *The variety* A = **HSP**($\underline{P}_0$) *has a natural duality based on the single algebra* $\underline{P}_0$ *(and in general two carriers,* α_0 *and* β_0*) if and only if any of the equivalent conditions* (i) - (iii) *holds or any of conditions* (i)* - (iii)* *holds.*

The variety A *has a natural duality based on the single algebra* $\underline{P}_0$ *and a single carrier,* α_0, *if and only if any of conditions* (i) - (iii) *holds. (The conditions referred to are those in Theorem* 3.10.*)*

PROOF. The first part is immediate from Theorem 3.10. The second deals with the situation considered by B. A. Davey and H. Werner in [12], [13]; note that $H(A)^1 \subseteq H(A)^0$ for all $A \in A$ if and only if Φ_0 is onto.■

The prevariety $A^{\bullet}$ may be considered in its own right. Theorem 2.5 implies that we obtain a duality for $A^{\bullet}$ by taking

$$\underset{\sim}{P}_0 = (P_0; R^{\bullet +}, \tau^{\bullet})$$

where $\tau^{\bullet}$ is discrete and $R^{\bullet +}$ contains

$$\ker(\alpha_0)^o,\ \ker(\beta_0)^o,\ \alpha_0^{-1}(\leqslant)^o,\ \beta_0^{-1}(\leqslant)^o,\ (\alpha_0,\beta_0)^{-1}(\leqslant)^o,\ (\beta_0,\alpha_0)^{-1}(\leqslant)^o$$

and the graph of the endomorphism dual to γ^2. Of these, $\ker(\alpha_0)^o$ is just the diagonal. We may discard it and shall assume that $R^{\bullet +}$ consists of the remaining six relations. We have functors

$$D^{\bullet}\colon A \to X^{\bullet} := \mathbf{IS_cP}(\underset{\sim}{P}_0)$$

and

$$E^{\bullet}\colon X^{\bullet} \to A^{\bullet}$$

given by $D^{\bullet}(-) = A(-, \underline{P}_0)$ and $E^{\bullet}(-) = X^{\bullet}(-, \underset{\sim}{P}_0)$. For every $A \in A^{\bullet}$, A is isomorphic to $E^{\bullet}D^{\bullet}(A)$. For $A \in A \backslash A^{\bullet}$, A cannot be isomorphic to $E^{\bullet}D^{\bullet}(A)$.

Let $\underset{\sim}{\Pi} = (P_0 \dot{\cup} P_1; R^+, \tau)$ be the $\underset{\sim}{\Pi}$-indexed structure considered in Theorem 3.7 (for convenience we replace operations by their graphs). Take $r \in R^+$.

(i) If $r \in R$, define an associated relation $\bar{r}$ by replacing any occurrence of α_1 by β_0;

(ii) if $r = r_{g_0}$, take $\bar{r} = \ker(\beta_0)^{\circ}$, which is the graph of the restriction $\restriction: P_0 \to P_1$, and if $r = r_{g_1}$, take $\bar{r}$ to be the graph of the endomorphism dual to γ^2.

Then $\{\bar{r} | r \in R^+\} = R^{\bullet +}$.

Take $A \in \mathcal{A}$ and $X^i = \mathcal{A}(A, \underline{P}_i)$ $(i = 0,1)$. Define $\Gamma: X^0 \to X^1$ by $\Gamma = \restriction \circ -$. Given $x,y \in X^0$,

$$\Gamma(x) = \Gamma(y) \Leftrightarrow (\forall a \in A)\ (x(a),y(a)) \in \ker(\beta_0)^{\circ}.$$

Let θ be the equivalence relation on X^0 obtained by extending $\ker(\beta_0)^{\circ}$ to X_0. Then the set X^0/θ can be identified with $\operatorname{Im}\Gamma \subseteq X^1$. Let

$$\bar{\underset{\sim}{\Pi}} = (P_0 \dot{\cup} \Gamma(P_0); \bar{R}^+, \bar{\tau})$$

where $\bar{\tau}$ is discrete and $\bar{R}^+$ is obtained by restricting each of the relations in R^+. Using the same constructions as in Theorem 3.7, but with $\bar{\underset{\sim}{\Pi}}$ in place of $\underset{\sim}{\Pi}$, we can derive from $\bar{X} = (X^0 \dot{\cup} \Gamma(X^0), \bar{R}^+, \bar{\tau})$ an Ockham space $Y^{\bullet}$ isomorphic to the subspace $Y^0 \cup g(Y^0)$ of $Y = H(A)$ and conversely can reconstruct $\bar{X}$ given $Y^{\bullet}$. Given the relationship between $\Gamma(X^0)$ and X^0/θ and between the sets R^+, $\bar{R}^+$ and $R^{\bullet +}$, $Y^{\bullet}$ is also determined by, and determines, $D^{\bullet}(A)$.

Specifically, we let $j: X^{0\prime} \to X^0$ be the internal homomorphism given by $(\alpha_0,\beta_0)^{-1}(\leqslant) \cap ((\beta_0,\alpha_0)^{-1}(\leqslant))^{-1}$ and note that $\ker(j) \subseteq \theta$. (We remark in passing that $\ker(j)$ is the congruence $\{((\sim a)', a) \colon a \in C(A)\}$, where $C(A)$ is the centre of A and $'$ denotes Boolean complement.) We take $X^0 \cup X^0/\theta$ and for every $x \in X^{0\prime}$, identify $j(x)$ with x/θ and denote the resulting set by Z. We define $\leqslant$ on Z by

$$(\forall x,y \in X^0)\ \ x \leqslant y \Leftrightarrow (x,y) \in \alpha_0^{-1}(\leqslant)^{\circ},$$

$$(\forall x/\theta, y/\theta \in X^0/\theta)\ \ x/\theta \leqslant y/\theta \Leftrightarrow (x,y) \in \beta_0^{-1}(\leqslant)^{\circ},$$

$$(\forall x \in X^0, y/\theta \in X^0/\theta)\ \ x \leqslant y/\theta \Leftrightarrow (x,y) \in (\alpha_0,\beta_0)^{-1}(\leqslant)^{\circ},$$

$$(\forall x/\theta \in X^0, y \in X^0)\ \ x/\theta \leqslant y \Leftrightarrow (x,y) \in (\beta_0,\alpha_0)^{-1}(\leqslant)^{\circ},$$

and g by

$$(\forall x \in X) \quad g(x) = x/\theta$$

$$(\forall x/\theta \in X^0/\theta) \quad g(x/\theta) = e(x),$$

where e is the endomorphism dual to γ^2. The set Z is topologised by pasting together the topology on X^0 and the quotient topology on X^0/θ in the expected way. The resulting space Z is isomorphic to $Y^{\cdot}$.

The isomorphism between the algebras $\boldsymbol{P}(Y^{\cdot},\underset{\approx}{2})$ and $\boldsymbol{X}^{\cdot}(Y^0,\underset{\sim}{P}_0)$ is established by mutually inverse maps $s^{\cdot}: \boldsymbol{P}(Y^{\cdot},\underset{\approx}{2}) \to \boldsymbol{X}^{\cdot}(Y^0,\underset{\sim}{P}_0)$ and $t^{\cdot}: \boldsymbol{X}^{\cdot}(Y^0,\underset{\sim}{P}_0) \to \boldsymbol{P}(Y^{\cdot},\underset{\approx}{2})$ defined by

$$(\forall \phi \in \boldsymbol{P}(Y^{\cdot},\underset{\approx}{2}))(\forall y \in Y^0)(\forall k \geqslant 0) \quad s^{\cdot}(\phi)(y)(k) = \phi(g^k(y))$$

and

$$(\forall \psi \in \boldsymbol{X}^{\cdot}(Y^0,\underset{\sim}{P}_0))\ (\forall y \in Y^0) \begin{cases} t^{\cdot}(\psi)(y) = \alpha_0(\psi(y)) \\ t^{\cdot}(\psi)(g(y)) = \beta_0(\psi(y)) \end{cases}$$

(cf. the remarks following Theorem 3.8).

For any $Y \in \boldsymbol{Y}$, $Y \backslash Y^{\cdot}$ may be thought of as a collection of "isolated tails." If $y \in Y$, then $\{g^k(y) | 0 \leqslant k \leqslant m-1\}$ is the smallest closed g-subset containing y and $\{g^k(y) | 1 \leqslant k \leqslant m-1\} \subseteq Y^{\cdot}$, so that y is the only point of $\{g^k(y) | 0 \leqslant k \leqslant m-1\}$ which may lie in $Y \backslash Y^{\cdot}$. It is the isolated tails which are lost when we consider $D^{\cdot}(A) = \boldsymbol{A}(A,\underline{P}_0)$ instead of $D(A) = \boldsymbol{A}(A,\underline{P}_0) \dot{\cup} \boldsymbol{A}(A,\underline{P}_1)$ and translate to the $\boldsymbol{D}$-$\boldsymbol{P}$ duality.

Proposition 3.12 sheds further light on the set $Y^{\cdot} = Y^0 \cup g(Y^0)$. The proposition is closely related to criteria for $\boldsymbol{A}$ to equal $\boldsymbol{A}^{\cdot}$ (viz. $(\forall y \in \boldsymbol{Y})\ Y^1 \subseteq Y^0$ or $(\forall y \in \boldsymbol{Y})\ Y^1 \subseteq g(Y^0)$); Theorem 3.10 however analyses the case $\boldsymbol{A} = \boldsymbol{A}^{\cdot}$ in more detail.

3.12 PROPOSITION. *Let* $A \in \boldsymbol{A} = \mathbf{HSP}(\underline{P}_0)$. *Then* $A \in \boldsymbol{A}^{\cdot} = \mathbf{ISP}(\underline{P}_0)$ *if and only if* $Y = Y^0 \cup g(Y^0)$ *where* $Y = H(A)$.

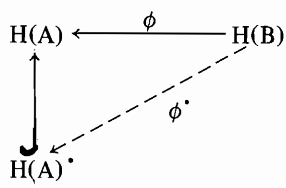
PROOF. The algebra $A \in \boldsymbol{A}$ lies in $\boldsymbol{A}^{\cdot}$ if and only if $\boldsymbol{A}(A,\underline{P}_0)$ separates the points of A. We denote the evaluation map from Y to HK(Y) by κ_Y. We have, given $a,b \in A$,

$$\begin{aligned}(\forall x \in \boldsymbol{A}(A,\underline{P}_0))\ x(a) = x(b) &\Leftrightarrow (\forall x \in \boldsymbol{A}(A,\underline{P}_0))(\forall j \geqslant 0) x(a)(j) = x(b)(j) \\ &\Leftrightarrow (\forall x \in \boldsymbol{A}(A,\underline{P}_0))(\forall j \geqslant 0)\ (\kappa_{W_0}(j) \circ x)(a) \\ &\qquad = (\kappa_{W_0}(j) \circ x)(b)\end{aligned}$$

$$\Leftrightarrow (\forall x \in \mathcal{A}(A,\underline{P}_0))(\forall j \geqslant 0)\ (H(x)\circ\kappa_{W_0})(j)(a)$$
$$= (H(x)\circ\kappa_{W_0})(j)(b)$$
$$\Leftrightarrow (\forall \phi \in \mathcal{S}(W_0,H(A)))(\forall j \geqslant 0)\ \phi(j)(a) = \phi(j)(b).$$

Suppose $a \neq b$ in A and $\phi \in \mathcal{S}(W_0,H(A))$ and $j \geqslant 0$ are such that $\phi(j)(a) = \phi(j)(b)$. Let $z = \phi(j)$ if j is even and $z = \phi(j-1)$ if j is odd. Note that $z \in H(A)^0$ and that in the case j is odd $g(z) = g(\phi(j\text{-}1)) = \phi(\gamma(j\text{-}1)) = \phi(j)$. We have proved that $\mathcal{A}(A,\underline{P}_0)$ separates the points of A if and only if $Z = H(A)^0 \cup g(H(A)^0)$ separates the points of A. Since $H(A)^0$ is closed and g is a closed map, Z is closed. Proposition 1.11 implies that Z separates the points of Y if and only if $Z = H(A)$.■

Fix $A \in \mathbf{HSP}(\underline{P}_0)$. For any $B \in \mathbf{ISP}(\underline{P}_0)$ and any $\mathcal{S}$-morphism ϕ: $H(B) \to H(A)$, ϕ factors through $H(A)^{\bullet}$:



This is because ϕ is an $\mathcal{S}$-morphism and so satisfies $\phi(H(B)^0) \subseteq H(A)^0$ and $\phi(g(H(A)^0)) = g(\phi(H(A)^0)) \subseteq g(H(A)^0)$. Consequently for $A \in \mathbf{HSP}(\underline{P}_0)$, $K(H(A)^{\bullet})$ is the maximum homomorphic image, $A^{\bullet}$, of A in $\mathbf{ISP}(\underline{P}_0)$.

It follows from Proposition 1.12 that there is a dual isomorphism between the lattice of lattice congruences of A and the lattice of closed subsets of $Y = H(A)$. This is given by $\theta \leftrightarrow Y_\theta$ where

$$(\forall a,b \in A)\ (a \equiv b(\theta) \Leftrightarrow Y_\theta \subseteq \{y \in Y \mid y(a) = y(b)\}).$$

Further, Ockham algebra congruences correspond to closed g-subsets ([17], Theorem 1) and if θ is a lattice congruence on A then the smallest closed g-subset containing Y_θ corresponds to the maximum Ockham congruence majorised by θ. We denote by $\theta_{\mathrm{Lat}}(a,b)$ the smallest lattice congruence identifying $a,b \in A$.

3.13 PROPOSITION. *The maximum homomorphic image of* $A \in \mathbf{HSP}(\underline{P}_0)$ *in* $\mathbf{ISP}(\underline{P}_0)$ *is* A/θ *where* θ *is the maximum Ockham congruence contained in*

$$\eta = \bigvee_{k,\ell} \{\eta_{k\ell} \mid \gamma^k(0) \leqslant \gamma^\ell(0) \text{ in } W_0\},$$

where

$$\eta_{k\ell} = \bigwedge_{a\in A} \theta_{\mathrm{Lat}}(f_{k\ell}(a), g_{k\ell}(a))$$

and

$$f_{k\ell}(a) = \begin{cases} \sim^k a \vee \sim^\ell a \text{ if } k \text{ is odd,} \\ \sim^k a \wedge \sim^\ell a \text{ if } k \text{ is even,} \end{cases}$$

$$g_{k\ell}(a) = \begin{cases} \sim^\ell a \text{ if } k \text{ and } \ell \text{ are even,} \\ \sim^k a \text{ if } k \text{ and } \ell \text{ are odd,} \\ 0 \quad \text{if } k \text{ is even and } \ell \text{ is odd,} \\ 1 \quad \text{if } k \text{ is odd and } \ell \text{ is even.} \end{cases}$$

PROOF. The set $H(A)^{\bullet} = H(A)^0 \cup g(H(A)^0)$ is the smallest closed g-subset containing $H(A)^0$ and this in turn is the intersection of the closed sets $Z_{k\ell} :=$ $\{y \in H(A) | g^k(y) \leqslant g^\ell(y)\}$, taken over those values of k and ℓ for which $\gamma^k(0) \leqslant \gamma^\ell(0)$. It remains to show that $Z_{k\ell}$ is the closed set associated with the intersection of principal congruences $\eta_{k\ell}$. This was first proved by P. R. Fowler in [14]. For completeness we outline a proof. We have

$$y \in Z_{k\ell} \Leftrightarrow (\forall a \in A) \;\; g^k(y)(a) \leqslant g^\ell(y)(a).$$

For k even, $g^k(y)(a) = y(\sim^k a)$, while for k odd, $g^k(y)(a) = (y(\sim^k a))'$ where $'$ denotes Boolean complement. Thus

$$g^k(y)(a) \leqslant g^\ell(y)(a) \Leftrightarrow \begin{cases} y(\sim^k a \wedge \sim^\ell a) = y(\sim^k a) & (k,\ell \text{ even}), \\ y(\sim^k a \vee \sim^\ell a) = y(\sim^k a) & (k,\ell \text{ odd}), \\ y(\sim^k a \wedge \sim^\ell a) = y(0) & (k \text{ even}, \ell \text{ odd}), \\ y(\sim^k a \vee \sim^\ell a) = y(1) & (k \text{ odd}, \ell \text{ even}). \end{cases}$$

These relations yield the required principal congruences.■

As already noted, an advantage of a natural duality is the way in which it behaves as regards free algebras and, more generally, free products. We again consider the variety $\mathcal{A} = \mathrm{ISP}(\underline{P}_0,\underline{P}_1) = \mathrm{ISP}(\underline{\Pi})$. The dual of the free object on one generator, $D(F\mathcal{A}(1))$, is, by definition, $\mathcal{A}(F\mathcal{A}(1),\underline{P}_0) \,\dot\cup\, \mathcal{A}(F\mathcal{A}(1),\underline{P}_1) \leqslant \underset{\sim}{\Pi}^{F\mathcal{A}(1)}$. We have natural bijections from $\mathcal{A}(F\mathcal{A}(1),\underline{P}_i)$ onto P_i ($i = 0,1$), and may identify $D(F\mathcal{A}(1))$ with $\underset{\sim}{\Pi}$, which is $P_0 \,\dot\cup\, P_1$ with a relational structure specified in terms of the structure of the Ockham space $\underset{\sim}{2}^m = (2^m;\tau,g,\leqslant)$ (see Lemma 3.6). Proposition 3.14 shows how $\underline{P}_0$

inherits the algebraic structure from the Ockham algebra $\underline{2}^m = (2^m;\wedge,\vee,0,1,\sim)$ and describes $\underline{P}_1$ similarly. Since $\mathrm{F}\mathcal{A}(1)$ belongs to the prevariety $\mathcal{A}^{\bullet}$, $\mathrm{D}(\mathrm{F}\mathcal{A}(1))$, and thence $\mathrm{H}(\mathrm{F}\mathcal{A}(1))$, can be determined without reference to $\underline{P}_1$. However we shall use the translation process given in Theorem 3.8 and so involve $\underline{P}_1$.

3.14 PROPOSITION.

$$\underline{P}_0 = \{\phi\colon W_0 \to 2 \mid \underset{\sim}{2}^{W_0} \models \sigma_0(\phi)\} \leqslant \underline{2}^{W_0},$$

where $\underset{\sim}{2}^{W_0} = \underset{\sim}{2}^m$ *and* $\underline{2}^{W_0} = \underline{2}^m$. *If* W_0 *is not a loop,*

$$\underline{P}_1 = \{\phi\colon W_1 \to 2 \mid \underset{\sim}{2}^{W_1} \models \sigma_1(\phi)\} \leqslant \underline{2}^{W_1}.$$

(Here 2^{W_1} *consists of functions from* $\{1,2,\ldots,m-1\}$ *to* $\{0,1\}$, $\underline{2}^{W_1}$ *has Ockham algebra structure defined by restriction from* $\underline{2}^{W_0}$ *and* $\underset{\sim}{2}^{W_1}$ *carries the alternating order and has g-map defined by restriction.) When* W_0 *is a loop,*

$$\underline{P}_1 \cong \{\phi\colon W_0 \to 2 \mid \underset{\sim}{2}^{W_0} \models \sigma_1(\phi)\} \leqslant \underline{2}^{W_0}.$$

PROOF. We consider $\underline{P}_0$. Suppose $\phi\colon W_0 \to 2$ is order-preserving. Let $\gamma^k(0) \leqslant \gamma^\ell(0)$ in W_0. Then

$$(\forall s \geqslant 0)\begin{cases}\gamma^k(s) \leqslant \gamma^\ell(s) & (s \text{ even}),\\ \gamma^k(s) \geqslant \gamma^\ell(s) & (s \text{ odd}),\end{cases}$$

so that

$$(\forall s \geqslant 0)\begin{cases}\phi(\gamma^k(s)) \leqslant \phi(\gamma^\ell(s)) & (s \text{ even}),\\ \phi(\gamma^k(s)) \geqslant \phi(\gamma^\ell(s)) & (s \text{ odd}).\end{cases}$$

Hence (by definition of g and induction),

$$(\forall s \geqslant 0)\begin{cases}g^k(\phi)(s) \leqslant g^\ell(\phi)(s) & (s \text{ even}),\\ g^k(\phi)(s) \geqslant g^\ell(\phi)(s) & (s \text{ odd}),\end{cases}$$

that is, $g^k(\phi) \preccurlyeq g^\ell(\phi)$. It follows that $\underset{\sim}{2}^{W_0} \models \sigma_0(\phi)$. Conversely, suppose $\phi\colon W_0 \to 2$ is such that $\underset{\sim}{2}^{W_0} \models \sigma_0(\phi)$ and let $\gamma^k(0) \leqslant \gamma^\ell(0)$. Then $\phi(\gamma^k(0)) = g^k(\phi)(0) \leqslant g^\ell(\phi)(0) = \phi(\gamma^\ell(0))$, so ϕ is order-preserving.

It remains to show that $\underline{P}_0$ inherits its Ockham algebra structure from $\underline{2}^{W_0}$. Since $\underline{P}_0$ and $\underline{2}^{W_0}$ both carry the pointwise order from $\underline{2}$, $\underline{P}_0$ is a sublattice of $\underline{2}^{W_0}$. Suppose $\phi \in \underline{P}_0$. Then $(\sim\phi)$ is given by

$$(\forall s \geqslant 0)\ (\sim\phi)(\gamma^s(0)) = 1 \Leftrightarrow \phi(\gamma^{s+1}(0)) = 0.$$

But this is how negation is defined in $\underset{\sim}{2}^{W_0}$.

The characterisation of $\underline{P}_1$ is obtained similarly. (In the loop case it is convenient to think in terms of functions from $\{0,1,\ldots,m\text{-}1\}$ to $\{0,1\}$ rather than in terms of functions from $\{1,2,\ldots,m\text{-}1,0\}$, whence the isomorphism occurring in the final assertion of the proposition.)■

Free objects in $\mathbf{HSP}(\underline{P}_0)$ were described in terms of the restricted $\boldsymbol{D}$-$\boldsymbol{P}$ duality in [15], Corollary 3.2, with a more economical description being given for the case $A = \boldsymbol{P}_{m,n}$, m-n even, in [15], Theorem 3.8, and extended to all varieties $\boldsymbol{P}_{m,n}$ in [13]. Free products have been considered before in particular varieties $A = \mathbf{HSP}(\underline{P}_0)$ (notably the variety of Kleene algebras; see [5] and [8]) and in general [14]. Theorems 3.15 and 3.16 encompass, and somewhat simplify, these earlier results.

3.15 THEOREM. *Let* $A = \mathbf{HSP}(\underline{P}_0)$. *Then*

$$FA(1) \cong K(\{y \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma(y)\}) \cong K(\{y, g(y) \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma_0(y)\}).$$

PROOF. We apply the translation process of Theorem 3.8 to $\underset{\sim}{\Pi}$ to obtain $H(FA(1))$. The identification of points of $\underline{P}_0$ with points of $\underline{P}_1$ is determined by

$$\begin{aligned} \ker(\alpha_1,\alpha_0)^o &= \{(\psi,\phi) \in \underline{P}_1 \times \underline{P}_0 | \phi = \psi \circ \gamma\} \\ &= \{(\psi,\phi) \in \underline{2}^{W_0} \times \underline{2}^{W_1} | \phi \\ &= \psi\circ\gamma, \underline{2}^{W_0} \models \sigma_0(\phi), \underline{2}^{W_1} \models \sigma_1(\psi)\}. \end{aligned}$$

The map $\psi \to \psi\circ\gamma$ sets up a bijection between $P_1 \subseteq 2^{W_1}$ and $g(P_0) \subseteq 2^{W_0}$; this takes $(x_1,\ldots,x_{m-1}) \in 2^{m-1}$ to $(x_1,x_2,\ldots,x_{m-1},x_n) \in 2^m$. Further, for $\psi \in \underline{P}_1$, $\underset{\sim}{2}^{W_1} \models \sigma_1(\psi)$ if and only if $\underset{\sim}{2}^{W_0} \models \sigma_1(\psi\circ\gamma)$. We may thus regard $P_0 \dot{\cup} P_1$ as the disjoint union of $\{y \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma_0(y)\}$ and $\{g(y) \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma_1(g(y))\} = \{z \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma_1(z)\}$. The identification prescribed above simply recombines these to form $\{y \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma(y)\}$ (or equivalently $\{y, g(y) \in \underset{\sim}{2}^m | \underset{\sim}{2}^m \models \sigma_0(y)\}$). It only remains to check that the Ockham space structure on $H(FA(1))$ produced by $\underset{\sim}{\Pi}$ coincides with that induced from $\underset{\sim}{2}^m$ on our copy of $H(FA(1))$ in $\underset{\sim}{2}^m$. It is clear that the bijection between P_1 and $g(P_0)$ is such that this is indeed the case.■

3.16 THEOREM. *Let* $\{A_j | j \in J\}$ *be a family of algebras in* $A = \mathbf{HSP}(\underline{P}_0)$ *and let* $Y = \prod\{H(A_j) | j \in J\}$ *(equipped with the product topology and pointwise-defined order* *and* g-*map). Then*

$$\coprod_{\mathcal{A}} \{A_j | j \in J\} \cong K(\{y \in Y | Y \models \sigma(y)\}).$$

In particular $F\mathcal{A}(\kappa) \cong K(\{y \in (\underset{\sim}{2}^m)^\kappa | (\underset{\sim}{2}^m)^\kappa \models \sigma_0(y)\})$

$$\cong K(\{y, g(y) \in (\underset{\sim}{2}^m)^\kappa | (\underset{\sim}{2}^m)^\kappa \models \sigma(y)\}).$$

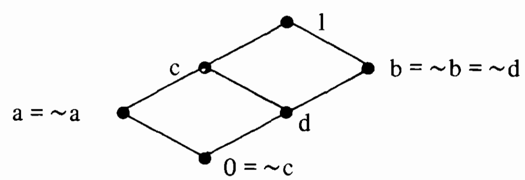
PROOF. Let $A = \coprod_{\mathcal{A}} \{A_j | j \in J\}$. Since the Ockham space structure on Y is defined pointwise, $Y^i = \prod\{H(A_j)^i | j \in J\}$ (for $i = 0,1$). For each $j \in J$, $D(A_j)$ is isomorphic to $H(A_j)^0 \dot{\cup} H(A_j)^1$, where this set carries a relational structure derived from $H(A_j)$, as described in Theorem 3.8. Hence D(A) is isomorphic to $\prod_j H(A_j)^0 \dot{\cup} \prod_j H(A_j)^1$, structured pointwise. Translating back to the ***D-P*** duality we get (up to isomorphism)

$$H(A) = \{y \in Y | Y \models \sigma(y)\}.$$

The first description of $F\mathcal{A}(\kappa)$ is a special case of the description of free products and the second follows because $F\mathcal{A}(\kappa)$ lies in $\mathbf{ISP}(\underset{\sim}{P}_0)$.■

We conclude this section with examples.

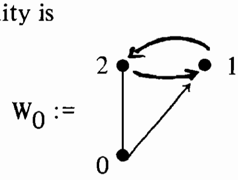
3.17 EXAMPLES. A study of the variety of MS-algebras was initiated by T. S. Blyth and J. C. Varlet in [2] and [3]. It is the smallest Ockham variety generated by a subdirectly irreducible algebra and containing the de Morgan algebras and the Stone algebras, and is generated by the algebra



and characterised equationally by

$$a \leqslant \sim^2 a, \sim(a \wedge b) = \sim a \vee \sim b, \sim(a \vee b) = \sim a \wedge \sim b, \sim 0 = 1, \sim 1 = 0.$$

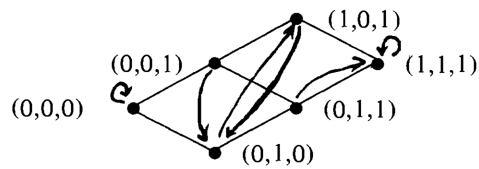
The starting point for our duality is



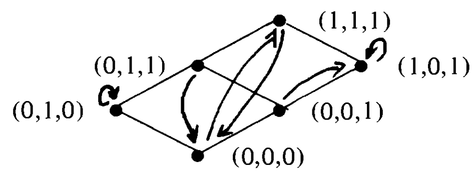
This space is of the type described in Lemma 3.9(i)(c) and gives

$$\sigma_0(y) = (y \leqslant g^2(y) \text{ and } g(y) = g^3(y)),\ \sigma_1(y) = (y = g^2(y)).$$

We may base our duality on a single algebra and a single carrier (see Theorem 3.11). Proposition 3.14 gives



$\underset{\sim}{P} = (P;\tau,g,\leqslant) \leqslant \underset{\sim}{2}^3$



$\underline{P} = (P;\wedge,\vee,0,1,\sim) \leqslant \underline{2}^3$

We next consider $A = \mathbf{ISP}(\underline{P}_0)$, where $\underline{P}_0$ is simple. The algebra $\underline{P}_0$ is simple if and only if it is of the form $K(W_0)$, where W_0 is a loop ([15], Corollary 2.4). In this case it makes no difference whether we base our duality on two algebras (as in Theorem 3.7) or one algebra and two carriers (by applying Theorem 2.5 to $\mathbf{ISP}(\underline{P}_0)$ (= $\mathbf{HSP}(\underline{P}_0)$)). We opt for the former, but take

$$\underline{P}_i = \{\phi\colon W_0 \to 2 | \underset{\sim}{2}^{W_0} \models \sigma_i(\phi)\} \leqslant \underline{2}^{W_0}$$

(see Proposition 3.14). The possible orders on W_0 were described by A. Urquhart in [17] and analysed in greater detail by M. S. Goldberg [15]. Let $S(m) = \{k \in \mathbf{Z}_m | k \text{ odd and } 2k \leqslant m\}$. Each subset S of S(m) gives rise to an order on W_0, *viz.* the weakest order on W_0 such that $(\forall k \in S)\ 0 < \gamma^k(0)$ and such that γ is order-reversing. Every allowable order on W_0 is an order of this type or is the order-theoretic dual of such an order. Given $S \subseteq S(m)$, the associated σ_0 and σ_1 are specified by

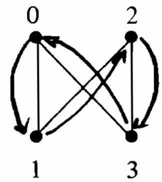
$$\sigma_0(y) = \&(y \leqslant g^k(y) | k \in S \text{ or } m\text{-}k \in S)$$

and

$$\sigma_1(y) = \&(y \geqslant \bar{g}^k(y) \mid k \in S \text{ or } m\text{-}k \in S).$$

The duality can now be constructed in any required case. We illustrate by taking a 4-element loop:

W_0 :=

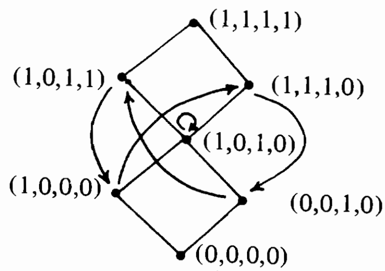


In this case

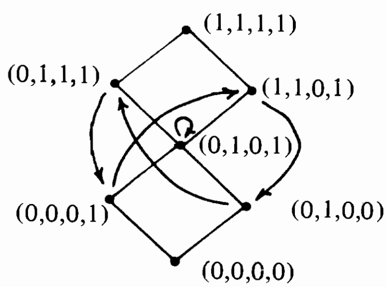
$$\underline{P}_0 = \{(y_0,y_1,y_2,y_3) \mid (\forall i)\ y_i = 0 \text{ or } 1 \text{ and } y_0,y_2 \geqslant y_1,y_3\}$$

and

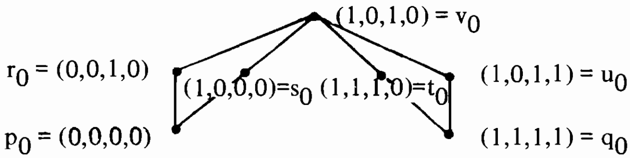
$$\underline{P}_1 = \{(y_0,y_1,y_2,y_3) \mid (\forall i)\ y_i = 0 \text{ or } 1 \text{ and } y_0,y_2 \leqslant y_1,y_3\}.$$



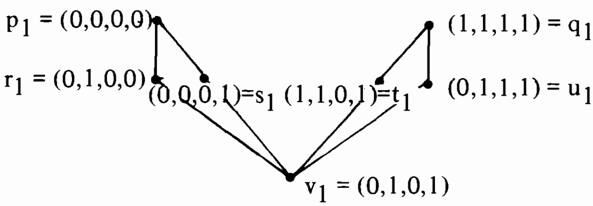
$\underline{P}_0 \leqslant \underline{2}^4$



$\underline{P}_1 \leqslant \underline{2}^4$



$(P_0, \preccurlyeq_0)$



$(P_1, \preccurlyeq_1)$

$$r_{g_0} = \{(p_0,p_1),(q_0,q_1),(r_0, r_1),(s_0,s_1),(t_0,t_1),(u_0,u_1),(v_0,v_1)\} \subseteq \underline{P}_0 \times \underline{P}_1,$$

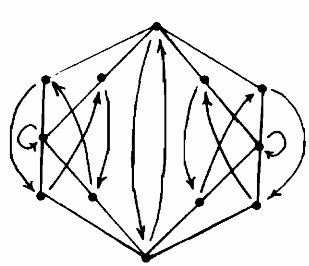
$$r_{g_1} = \{(p_1,p_0),(q_1,q_0),(r_1,s_0),(s_1,r_0),(t_1,u_0),(u_1,t_0),(v_1,v_0)\} \subseteq \underline{P}_1 \times \underline{P}_0$$

$$\leftarrow = \{(p_0,p_1),(q_0,q_1)\} \subseteq \underline{P}_0 \times \underline{P}_1$$

and

$$\succ\!\!\!- = \{(y_1,x_0) | y_1 \in \{p_1,r_1,s_1,v_1\}, x_0 \in \{p_0,v_0,s_0,v_0\}\}$$
$$\cup \{(y_1,x_0) | y_1 \in \{q_1,t_1,u_1,v_1\}, x_0 \in \{q_0,t_0,u_0,v_0\}\} \subseteq \underline{P}_1 \times \underline{P}_0.$$

Applying the translation process, we find

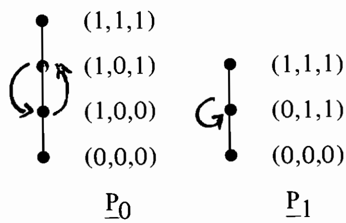
$H(FA(1)) =$ 

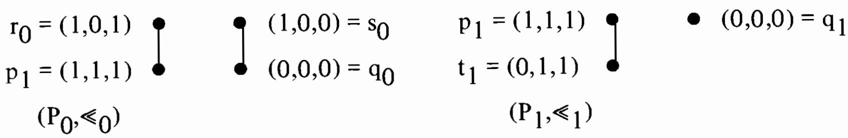
as predicted by Theorem 3.15.

Our final example is one in which the prevariety and the variety are different. We take $\mathcal{A}$ = **HSP**($\underline{P}_0$), where $\underline{P}_0 = K(W_0)$ and

W_0 :=

Then $\sigma_0(y) = (y \geqslant g^2(y) = g^3(y) \geqslant g(y))$ and $\sigma_1(y) = (y \leqslant g(y) = g^2(y))$. We find





$$r_{g_0} = \{(p_0,p_1),(q_0,q_1),(r_0,t_1),(s_0,q_1)\},$$

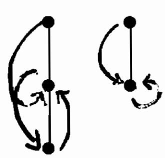
$$r_{g_1} = \{(p_1,p_0),(q_1,q_0),(t_1,p_0)\},$$

$$\leftarrow = \{(p_0,p_1),(q_0,q_1)\},$$

$$\succ\!\!- = \{(y_1,x_0) \mid y_1 \in \{p_1,t_1\}, x_0 \in \{p_0,r_0\}\} \cup \{(y_1,x_0) \mid y_1 = q_1, x_0 \in \{q_0,s_0\}\}.$$

Thence

H(F$\mathcal{A}$(1)) =



Suppose $A \in \mathcal{A}$ and let $Y = H(A)$. As a subset of Y, Y^0 is $\{y \mid y \geqslant g^3(y)\}$, since the remaining inequalities involved in $\sigma_0(y)$ hold automatically on this subset. The lattice congruence associated with the closed set Y^0 is $\eta = \bigwedge(\eta_c \mid c \in A)$ where $\eta_c = \theta_{Lat}(c \vee \sim^3 c, 1)$. The corresponding Ockham congruence, θ, is that associated with the smallest closed g-subset of Y containing Y^0, *viz.* $Y^0 \cup g(Y^0) = Y^{\bullet}$, and is $(\bigwedge(\eta_c | c \in A))^0$ (regarded as a subalgebra of A × A). In the notation of Lemma 3.5,

$$\eta = \{(a,b) \in A \times A | (\forall c)\ a \equiv b(\eta_c)\}$$

and, for all n,

$$\eta_n = \{(a,b) \in A \times A | (\forall c)(\forall j \leqslant n) \sim^j a \equiv \sim^j b(\eta_c)\}.$$

Because $A \in \boldsymbol{P}_{3,2}$, $\sim^3 a = (\sim^2 a)'$ for every $a \in A$ ([15], Proposition 2.7). Hence, for $a \in A$ and $k \geqslant 1$, $\sim^{2k} a = \sim^2 a$ and $\sim^{2k+1} a = \sim^3 a$. It follows that

$$\theta = \{(a,b) \in A \times A | (\forall c) \sim^j a \equiv \sim^j b(\eta_c) \text{ for } j = 0,1,2,3\}.$$

Hence, by definition of η_c,

$$\theta = \{(a,b) \in A \times A | (\forall c) \sim^j a \wedge (c \vee \sim^3 c) = \sim^j b \wedge (c \vee \sim^3 c) \text{ for } j = 0,1,2,3\}.$$

The maximum homomorphic image of A in the prevariety $\boldsymbol{A}^{\bullet}$ is A/θ. An algebra $A \in \boldsymbol{A}$ lies in the prevariety if and only if

$$(\forall a,b \in A)(((\forall c \in A)(\forall j = 0,1,2,3) \sim^j a \wedge (c \vee \sim^3 c) = \sim^j b \wedge (c \vee \sim^3 c)) \rightarrow a = b).$$

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