

Endoprimal algebras

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Abstract. An algebra \mathbf{A} is endoprimal if, for all $k \in \mathbb{N}$, the only maps from A^k to A which preserve the endomorphisms of \mathbf{A} are its term functions. One method for finding finite endoprimal algebras is via the theory of natural dualities since an endodualisable algebra is necessarily endoprimal. General results on endoprimality and endodualisability are proved and then applied to the varieties of sets, vector spaces, distributive lattices, Boolean algebras, Stone algebras, Heyting algebras, semilattices and abelian groups. In many classes the finite endoprimal algebras turn out to be endodualisable. We show that this fails in general by proving that $2^2 \oplus 1$, regarded as either a bounded semilattice or upper-bounded semilattice is dualisable, endoprimal but not endodualisable.

0. Introduction

For each $k \in \mathbb{N}$, an algebra \mathbf{A} is k -**endoprimal** if the only functions from A^k to A which preserve the endomorphisms of \mathbf{A} are the k -ary term functions of \mathbf{A} . If \mathbf{A} is k -endoprimal for all $k \in \mathbb{N}$, then we say that \mathbf{A} is **endoprimal**. Up to term equivalence, every endoprimal algebra \mathbf{A} may be constructed by taking a unary algebra $\mathbf{A}' = \langle A; G \rangle$ and forming the algebra $\mathbf{A} := \langle A; F \rangle$, where F is the hom-clone of \mathbf{A}' , that is, F is the set of all finitary maps on A which commute with the operations in G . Here we are concerned with ‘naturally occurring’ examples of endoprimal algebras. Before 1994, very few natural examples of endoprimal algebras were known. The first examples were certain quasi-primal algebras. For example, if \mathbf{A} is quasi-primal and has no proper non-trivial subalgebras, then \mathbf{A} is endoprimal. In 1976, the first author published, in [4], a proof that every finite chain is endoprimal as a Heyting algebra. In the mid-1980s, Davey and Werner [14][15] proved that the Heyting algebra $2^2 \oplus 1$ is also endoprimal. No further examples of endoprimal algebras appeared in the literature until 1993 when Márki and Pöschel [17] proved that a distributive lattice is endoprimal if and only if it is not relatively complemented.

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One natural way to prove that a finite algebra is endoprimal is to prove somewhat more and show that it is endodualisable (see the discussion below, or Davey [7], Davey [6] or Clark and Davey [2] for increasingly more comprehensive discussions of endodualisability in particular and natural dualities in general). Indeed, the endoprimality of the Heyting algebras mentioned above was proved by showing that they are endodualisable, and in Davey, Haviar and Priestley [8] it is proved that a finite endoprimal distributive lattice is necessarily endodualisable. The paucity of examples at the time led the authors of [8] to remark that examples of endoprimal algebras are “rather scarce” and that endodualisability is a “rather rare phenomenon”. Inspired in part by the results in [8], the first author [7] developed some general sufficient conditions for endodualisability and applied these to produce a plethora of endodualisable, and therefore endoprimal, algebras.

This paper is a companion to [7]. Here we refine and extend the results of [7], develop general results which yield infinite as well as finite endoprimal algebras and answer a problem posed at the conclusion of [7] by proving that $2^2 \oplus \mathbf{1}$, regarded as either a bounded semilattice or upper-bounded semilattice is dualisable, endoprimal but not endodualisable. To illustrate the efficacy of our methods we apply them to yield descriptions of endoprimal and endodualisable sets, vector spaces, distributive lattices, Boolean algebras, Stone algebras, Heyting algebras, semilattices and abelian groups. In the Heyting algebra and abelian group cases our results are not complete: we give large classes within which the endoprimal and endodualisable algebras are known but we do not have a completely general description.

Since we wish to use endodualisability as a tool for producing endoprimal algebras, it is necessary to say a little about natural dualities. We shall do this in the restricted context which is relevant to endodualisability. In particular, we shall restrict our attention to unary operations and we shall not consider partial operations at all.

Consider a finite algebra \mathbf{M} and let $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$ be the quasi-variety it generates. If \mathbf{M} is endoprimal, then for all $k \in \mathbb{N}$ there is a finite set X equipped with an action of the monoid $\text{End } \mathbf{M}$ (namely M^k with the pointwise action) such that the maps from X to M which preserve the action of $\text{End } \mathbf{M}$ form a subalgebra of \mathbf{M}^X isomorphic to the free algebra $\mathbf{F}_{\mathcal{A}}(k)$. It is not difficult to see this can be extended to arbitrary free algebras in \mathcal{A} . Endow M with the discrete topology and for any non-empty set I , equip $X := M^I$ with the product topology and the pointwise action of $\text{End } \mathbf{M}$. If \mathbf{M} is endoprimal, then the continuous maps from X to M which preserve the action of $\text{End } \mathbf{M}$ form a subalgebra of \mathbf{M}^X isomorphic to the free algebra $\mathbf{F}_{\mathcal{A}}(I)$. Roughly speaking, we say that \mathbf{M} is endodualisable if this representation can be extended from the free algebras in \mathcal{A} to every algebra $\mathbf{A} \in \mathcal{A}$. We want a set X equipped with a Boolean topology and a continuous action of $\text{End } \mathbf{M}$ such that the continuous maps from X to M which preserve the action of $\text{End } \mathbf{M}$

form a subalgebra of \mathbf{M}^X isomorphic to \mathbf{A} . There is a canonical choice for both X and for the isomorphism:

- (a) let X equal $\mathcal{A}(\mathbf{A}, \mathbf{M})$, the set of homomorphisms from \mathbf{A} into \mathbf{M} , with the subspace topology from the power M^A and the action of $e \in \text{End } \mathbf{M}$ defined by composition ($e(x) := e \circ x$ for all $x \in \mathcal{A}(\mathbf{A}, \mathbf{M})$),
- (b) let \mathbf{B} be the subalgebra of \mathbf{M}^X consisting of all continuous maps which preserve the action of $\text{End } \mathbf{M}$, and
- (c) define $\eta_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{B}$ by $\eta_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and all $x \in X = \mathcal{A}(\mathbf{A}, \mathbf{M})$.

If $\eta_{\mathbf{A}}$ is an isomorphism, we say that the structure $\underline{\mathbf{M}} = \langle M; \text{End } \mathbf{M}, \mathcal{T} \rangle$ **yields a duality** on \mathbf{A} . If $\eta_{\mathbf{A}}$ is an isomorphism, for all $\mathbf{A} \in \mathcal{A}$, then we say that \mathbf{M} is **endodualisable**. It is easily seen that $\mathcal{A}(\mathbf{F}_{\mathcal{A}}(I), \mathbf{M})$, with the action of $\text{End } \mathbf{M}$ defined by composition, is isomorphic to M^I with the action of $\text{End } \mathbf{M}$ defined pointwise. It follows that, for all $k \in \mathbb{N}$, the structure $\underline{\mathbf{M}} = \langle M; \text{End } \mathbf{M}, \mathcal{T} \rangle$ yields a duality on $\mathbf{F}_{\mathcal{A}}(k)$ if and only if \mathbf{M} is k -endoprimal. Consequently, if \mathbf{M} is endodualisable, then it is necessarily endoprimal. The observations above concerning $\mathbf{F}_{\mathcal{A}}(I)$ show that if \mathbf{M} is endoprimal, then $\underline{\mathbf{M}} = \langle M; \text{End } \mathbf{M}, \mathcal{T} \rangle$ yields a duality on $\mathbf{F}_{\mathcal{A}}(I)$ for every non-empty set I .

We are also interested in a more general situation. If a k -ary relation $r \subseteq M^k$ on M is the underlying set of a subalgebra of \mathbf{M}^k , then we say that r is **algebraic** over \mathbf{M} and denote the corresponding algebra by \mathbf{r} . Let R be a set of finitary relations which are algebraic over \mathbf{M} . For all $\mathbf{A} \in \mathcal{A}$, each $r \in R$ may be extended pointwise to $X := \mathcal{A}(\mathbf{A}, \mathbf{M})$. The fact that the relations are algebraic guarantees, among other things, that the set B of all $\underline{\mathbf{M}}$ -morphisms from X to M (that is, continuous maps from X to M which preserve the action of $\text{End } \mathbf{M}$ and also preserve the relations in R) forms a subalgebra of \mathbf{M}^X . If the map $\eta_{\mathbf{A}}$ is an isomorphism between \mathbf{A} and \mathbf{B} for all $\mathbf{A} \in \mathcal{A}$, we say that $\underline{\mathbf{M}} = \langle M; \text{End } \mathbf{M}, R, \mathcal{T} \rangle$ **yields a duality** on \mathcal{A} or that \mathbf{M} is **dualised by** $\underline{\mathbf{M}}$. More generally still, we could allow homomorphisms $g: \mathbf{M}^n \rightarrow \mathbf{M}$ of arbitrary finite arity to be added to the type of $\underline{\mathbf{M}}$ (see [2][6]).

It has become customary when discussing natural dualities to denote the algebra by $\underline{\mathbf{M}}$ and the topological structure on the same underlying set by $\underline{\mathbf{M}}$. Since, in the present context, we regard a natural duality as a tool for proving that \mathbf{M} is endoprimal rather than an end in itself, we shall be less evenhanded and drop the bar from below $\underline{\mathbf{M}}$.

A subset X of M^I is called **term-closed** if it is an intersection of equalizers of I -ary term functions of the algebra \mathbf{M} . Term-closed sets play a central role in natural duality theory. For example, the Third Duality Theorem (see [2][6]) states that $\underline{\mathbf{M}}$ yields a duality on $\text{ISP } \mathbf{M}$ if and only if, for every non-empty set I and term-closed subset X of M^I , each $\underline{\mathbf{M}}$ -morphism from X to M extends to an I -ary term function on \mathbf{M} . We shall use this fact in Section 5 below.

1. General theory

Let \mathcal{C} be a class of algebras. An algebra $\mathbf{C} \in \mathcal{C}$ is **injective** in \mathcal{C} if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ with an embedding $\varphi: \mathbf{A} \hookrightarrow \mathbf{B}$, every homomorphism $\alpha: \mathbf{A} \rightarrow \mathbf{C}$ extends to a homomorphism $\beta: \mathbf{B} \rightarrow \mathbf{C}$ such that $\alpha = \beta \circ \varphi$. An algebra $\mathbf{B} \in \mathcal{C}$ is a **retract** of $\mathbf{A} \in \mathcal{C}$ if there exist homomorphisms $\alpha: \mathbf{A} \rightarrow \mathbf{B}$ and $\beta: \mathbf{B} \rightarrow \mathbf{A}$ such that $\alpha \circ \beta = \text{id}_{\mathbf{B}}$. The map α is called a **retraction**, and β is the associated **coretraction**. We define an algebra $\mathbf{B} \in \mathcal{C}$ to be a **subretract** of $\mathbf{A} \in \mathcal{C}$ if \mathbf{B} is a subalgebra of \mathbf{A} and there is a homomorphism $\alpha: \mathbf{A} \rightarrow \mathbf{B}$ such that $\alpha \upharpoonright_{\mathbf{B}} = \text{id}_{\mathbf{B}}$. In this case, the map $\alpha: \mathbf{A} \rightarrow \mathbf{B}$ is a retraction, with the inclusion map $\iota: \mathbf{B} \rightarrow \mathbf{A}$ serving as the coretraction.

The following result generalises a lemma of Márki and Pöschel [17], which was used to help classify endoprimal distributive lattices.

THEOREM 1.1. *Let \mathbf{A} be an algebra and let $\mathbf{B} \in \mathbb{ISP} \mathbf{A}$ such that \mathbf{A} is a subretract of \mathbf{B} . Let $f: B^k \rightarrow B$ preserve the endomorphisms of \mathbf{B} . Then f preserves A and for all endomorphism-preserving maps $g: B^k \rightarrow B$ such that $g \upharpoonright_{A^k} = f \upharpoonright_{A^k}$ we have $g = f$.*

Proof. Let $\alpha: \mathbf{B} \rightarrow \mathbf{A}$ be a retraction which fixes A . Then α is an endomorphism of \mathbf{B} . For all $a_1, \dots, a_k \in A$ we have

$$f(a_1, \dots, a_k) = f(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(f(a_1, \dots, a_k)) \in A.$$

So f preserves A . Now assume that $g: B^k \rightarrow B$ preserves the endomorphisms of \mathbf{B} and $g \upharpoonright_{A^k} = f \upharpoonright_{A^k}$. Let $b_1, \dots, b_k \in B$. Every homomorphism $\gamma: \mathbf{B} \rightarrow \mathbf{A}$ is an endomorphism of \mathbf{B} , and so

$$\begin{aligned} \gamma(g(b_1, \dots, b_k)) &= g(\gamma(b_1), \dots, \gamma(b_k)) = f(\gamma(b_1), \dots, \gamma(b_k)) \\ &= \gamma(f(b_1, \dots, b_k)). \end{aligned}$$

Since $\mathbf{B} \in \mathbb{ISP} \mathbf{A}$, it follows that the points of \mathbf{B} are separated by homomorphisms into \mathbf{A} , and so $g(b_1, \dots, b_k) = f(b_1, \dots, b_k)$. Hence $g = f$. \square

THEOREM 1.2. *Let \mathbf{A} be a k -endoprimal algebra. If $\mathbf{B} \in \mathbb{ISP} \mathbf{A}$ and \mathbf{A} is a retract of \mathbf{B} , then \mathbf{B} is k -endoprimal.*

Proof. An isomorphic copy of a k -endoprimal algebra is k -endoprimal. So we can assume that \mathbf{A} is a subretract of \mathbf{B} , via the retraction $\alpha: \mathbf{B} \rightarrow \mathbf{A}$. Let $f: B^k \rightarrow B$ preserve the endomorphisms of \mathbf{B} . Then f preserves A by Theorem 1.1. To see that $f \upharpoonright_{A^k}: A^k \rightarrow A$ preserves the endomorphisms of \mathbf{A} , let $e \in \text{End} \mathbf{A}$ and let $a_1, \dots, a_k \in A$. Then $e \circ \alpha \in \text{End} \mathbf{B}$, and therefore

$$\begin{aligned} e(f(a_1, \dots, a_k)) &= e \circ \alpha(f(a_1, \dots, a_k)) = f(e \circ \alpha(a_1), \dots, e \circ \alpha(a_k)) \\ &= f(e(a_1), \dots, e(a_k)). \end{aligned}$$

So $f \upharpoonright_{A^k}$ preserves End \mathbf{A} . Thus $f \upharpoonright_{A^k} = t \upharpoonright_{A^k}$ for some k -ary term function t of \mathbf{B} , as \mathbf{A} is k -endoprimal. By Theorem 1.1 we have $f = t$, whence \mathbf{B} is k -endoprimal. \square

We close this discussion of endoprimality with a result which guarantees the endoprimality of almost all free algebras in some familiar varieties.

THEOREM 1.3. *Let \mathcal{A} be a class of algebras of type F such that $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P} \mathbf{F}_{\mathcal{A}}(1)$. If*

- (i) *there is a binary operation $+$ in F and a nullary operation $0 \in F$,*
- (ii) *the identity $0 + x \approx x + 0 \approx x$ holds for each $\mathbf{A} \in \mathcal{A}$,*
- (iii) *$+$: $\mathbf{A}^2 \rightarrow \mathbf{A}$ is a homomorphism for all $\mathbf{A} \in \mathcal{A}$,*
- (iv) *every operation in F preserves 0 ,*

then $\mathbf{F}_{\mathcal{A}}(2)$, and more generally, every free algebra with the possible exception of $\mathbf{F}_{\mathcal{A}}(1)$, is endoprimal.

Proof. Note that it follows from (i)–(iii) that $+$ is both associative and commutative. Since every free algebra other than $\mathbf{F}_{\mathcal{A}}(1)$ has $\mathbf{F}_{\mathcal{A}}(2)$ as a retract, it suffices, by Theorem 1.2, to show that $\mathbf{F}_{\mathcal{A}}(2)$ is endoprimal. Let a and b be the generators of $\mathbf{F}_{\mathcal{A}}(2)$ and let F_2 denote the underlying set of $\mathbf{F}_{\mathcal{A}}(2)$. We will regard $\mathbf{F}_{\mathcal{A}}(1)$ as the subalgebra of $\mathbf{F}_{\mathcal{A}}(2)$ generated by a with the underlying set F_1 . For all $x, y \in F_2$ there is a unique endomorphism e_{xy} of $\mathbf{F}_{\mathcal{A}}(2)$ satisfying $a \mapsto x$ and $b \mapsto y$.

Let $f: F_2^k \rightarrow F_2$ preserve the endomorphisms of $\mathbf{F}_{\mathcal{A}}(2)$. Then f preserves F_1 by Theorem 1.1. We will show that $f \upharpoonright_{F_1^k}: F_1^k \rightarrow F_1$ preserves $+$.

Let $x, y \in F_1^k$ and define $z \in F_2^k$ by $z := x + e_{b0}(y)$. Note that (ii) and (iii) imply that $e_{aa} = e_{a0} + e_{0a}$. It follows by (ii) and (iv) that $x = e_{a0}(z)$ and $y = e_{0a}(z)$. So

$$\begin{aligned} f(x + y) &= f(e_{aa}(z)) = e_{aa}(f(z)) = (e_{a0} + e_{0a})(f(z)) \\ &= e_{a0}(f(z)) + e_{0a}(f(z)) = f(e_{a0}(z)) + f(e_{0a}(z)) \\ &= f(x) + f(y). \end{aligned}$$

Thus $f \upharpoonright_{F_1^k}$ preserves $+$.

Now let $x \in F_1^k$ and let $n \in \{1, \dots, k\}$. Define $a_n \in F_1^k$ and $x_n \in F_1^k$ by

$$a_n(i) = \begin{cases} a & \text{if } i = n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad x_n(i) = \begin{cases} x(i) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

As f preserves F_1 , we have $f(a_n) = t_n(a)$ for some unary term function t_n . We have

$$\begin{aligned} f(x_n) &= f(e_{x(n)0}(a_n)) = e_{x(n)0}(f(a_n)) = e_{x(n)0}(t_n(a)) \\ &= t_n(e_{x(n)0}(a)) = t_n(x(n)). \end{aligned}$$

Since $x = x_1 + \cdots + x_k$, we conclude that

$$\begin{aligned} f(x) &= f(x_1 + \cdots + x_k) = f(x_1) + \cdots + f(x_k) \\ &= t_1(x(1)) + \cdots + t_k(x(k)) = (t_1 \circ \pi_1 + \cdots + t_k \circ \pi_k)(x). \end{aligned}$$

Thus $f \upharpoonright_{F_1^k} = t_1 \circ \pi_1 + \cdots + t_k \circ \pi_k \upharpoonright_{F_1^k}$. By Theorem 1.1, it follows that f is a term function. Hence $\mathbf{F}_{\mathcal{A}}(2)$ is endoprimal. \square

A careful reading of the proof shows that we may replace (iii) in the lemma above by the weaker assumption that \mathcal{A} satisfies the identity $t(x, 0) + t(0, x) \approx t(x, x)$ for every binary term t of type F ; in particular, the associativity and commutativity of $+$ are not required.

We turn now to endodualisability. The following result is an easy consequence of Corollary 3.2 of [7].

THEOREM 1.4. *Let \mathbf{M} be a finite algebra in $\mathcal{D} := \mathbb{ISP} \mathbf{D}$, such that \mathbf{D} is a subretract of \mathbf{M} . If $\mathbf{D} = \langle D; \text{End } \mathbf{D}, R, \mathcal{T} \rangle$ yields a duality on \mathcal{D} , then $\mathbf{M} = \langle M; \text{End } \mathbf{M}, R, \mathcal{T} \rangle$ yields a duality on \mathcal{D} .*

Proof. Let \mathbf{D} be a subretract of \mathbf{M} and let $\alpha: \mathbf{M} \rightarrow \mathbf{D}$ denote the retraction. Assume that $\mathbf{D} = \langle D; \text{End } \mathbf{D}, R, \mathcal{T} \rangle$ yields a duality on \mathcal{D} . By Corollary 3.2 of [7], $\mathbf{M}' = \langle M; \text{End } \mathbf{M}, \text{End } \mathbf{D}, R, \mathcal{T} \rangle$ yields a duality on \mathcal{D} , where $\text{End } \mathbf{D}$ is now regarded as a set of partial endomorphisms of \mathbf{M} . A simple application of entailment (see Davey, Haviar and Priestley [9] or Clark and Davey [2]) now shows that we may remove $\text{End } \mathbf{D}$ from the structure on \mathbf{M}' without destroying the duality: since $\alpha \in \text{End } \mathbf{M}$, the unary relation $D = \text{fix}(\alpha)$ is entailed by $\text{End } \mathbf{M}$; for each $e \in \text{End } \mathbf{D}$, we have $e \circ \alpha \in \text{End } \mathbf{M}$ with $e = e \circ \alpha \upharpoonright_D$, and so $\text{End } \mathbf{D}$ is entailed by $\text{End } \mathbf{M}$. Consequently, $\mathbf{M} = \langle M; \text{End } \mathbf{M}, R, \mathcal{T} \rangle$ yields a duality on \mathcal{D} . \square

The case where $|R| \leq 1$ is particularly important here. We say that \mathbf{M} is **almost endodualisable** if there is some algebraic relation s on \mathbf{M} such that $\mathbf{M} = \langle M; \text{End } \mathbf{M}, s, \mathcal{T} \rangle$ yields a duality on $\mathbb{ISP} \mathbf{M}$. Part (i) of the following result was obtained independently by M. Saramago (private communication).

THEOREM 1.5. *Let \mathbf{M} be a finite algebra in $\mathbb{ISP} \mathbf{D}$ and assume that \mathbf{D} is a retract of \mathbf{M} .*

- (i) *If \mathbf{D} is endodualisable, then so is \mathbf{M} .*
- (ii) *If \mathbf{D} is almost endodualisable via $\mathbf{D} = \langle D; \text{End } \mathbf{D}, s, \mathcal{T} \rangle$, with s an n -ary relation, then \mathbf{M} is almost endodualisable via $\mathbf{M} = \langle M; \text{End } \mathbf{M}, t, \mathcal{T} \rangle$ where t is an n -ary relation on M with \mathbf{t} isomorphic to \mathbf{s} .*

Proof. Assume that \mathbf{D} is a retract of \mathbf{M} . There is some $\mathbf{M}' \in \mathcal{D} := \mathbb{ISP} \mathbf{D}$ such that \mathbf{M}' is isomorphic to \mathbf{M} and \mathbf{D} is a subretract of \mathbf{M}' . If \mathbf{D} is endodualisable, Theorem 1.4 with $R = \emptyset$ implies that \mathbf{M}' is endodualisable and consequently \mathbf{M} is endodualisable. If \mathbf{D} is almost endodualisable via $\mathbf{D} = \langle D; \text{End } \mathbf{D}, s, \mathcal{T} \rangle$, Theorem 1.4 with $R = \{s\}$ implies that \mathbf{M}' is almost endodualisable via $\mathbf{M}' = \langle M'; \text{End } \mathbf{M}', s, \mathcal{T} \rangle$, and consequently \mathbf{M} is almost endodualisable via $\mathbf{M} = \langle M; \text{End } \mathbf{M}, t, \mathcal{T} \rangle$ where \mathbf{t} is the subalgebra of \mathbf{M}'^n corresponding to the subalgebra \mathbf{s} of $(\mathbf{M}')^n$ under the isomorphism between \mathbf{M}' and \mathbf{M} . \square

We can now present an improved version of Theorem 4.5 of [7].

THEOREM 1.6. *Assume that \mathbf{D} is almost endodualisable via the structure $\mathbf{D} = \langle D; \text{End } \mathbf{D}, s, \mathcal{T} \rangle$ and let \mathbf{M} be a finite algebra in $\mathbb{ISP} \mathbf{D}$. If both \mathbf{D} and \mathbf{s} are retracts of \mathbf{M} , then \mathbf{M} is endodualisable. In particular, if \mathbf{D} is a retract of \mathbf{M} and \mathbf{M} is isomorphic to \mathbf{s} , then \mathbf{M} is endodualisable.*

Proof. The proof is an easy modification of the proof of Theorem 4.5 in [7]. We sketch the details. If \mathbf{D} is almost endodualisable via $\mathbf{D} = \langle D; \text{End } \mathbf{D}, s, \mathcal{T} \rangle$ and \mathbf{D} is a retract of \mathbf{M} , then, by Corollary 1.5(ii), \mathbf{M} is almost endodualisable via $\mathbf{M} = \langle M; \text{End } \mathbf{M}, t, \mathcal{T} \rangle$ where \mathbf{t} is isomorphic to \mathbf{s} . Since \mathbf{s} is a retract of \mathbf{M} so is \mathbf{t} . By Lemma 4.3 of [7], $\text{End } \mathbf{M}$ entails t and hence t can be deleted from \mathbf{M} without destroying the duality. Hence \mathbf{M} is endodualisable. \square

In Davey [7], a technique was used which, in certain circumstances, allowed the almost endodualisability of \mathbf{D} to be lifted up to a characterisation of the endodualisable (finite) algebras in $\mathcal{D} = \mathbb{ISP} \mathbf{D}$. We are now able to extend the technique to obtain a corresponding characterisation of the endoprimal algebras (finite or infinite).

Assume that \mathbf{E} is a finite algebra in \mathcal{D} and that the following two steps have been carried out.

- (1) Find an internal property, say (\mathcal{E}) , such that, for all $\mathbf{M} \in \mathcal{D}$, the algebra \mathbf{E} is a retract of \mathbf{M} if and only if \mathbf{M} satisfies (\mathcal{E}) .

- (2) Prove that, for some $k \in \mathbb{N}$, if $\mathbf{M} \in \mathcal{D}$ has \mathbf{D} as a retract and does not satisfy (\mathcal{E}) , then \mathbf{M} is not k -endoprimal.

(It is important in both steps that the arguments work for all $\mathbf{M} \in \mathcal{D}$ not just for finite algebras.)

Once these steps have been completed we have the following characterisation of endoprimality.

THEOREM 1.7. *Let \mathbf{D} be almost endodualisable via $\langle D; \text{End } \mathbf{D}, s, \mathcal{F} \rangle$. Assume that Steps (1) and (2) have been completed for some $k \in \mathbb{N}$ with \mathbf{E} isomorphic to \mathbf{s} . Assume that \mathbf{D} is a retract of \mathbf{E} . Then, for any $\mathbf{M} \in \mathcal{D} = \mathbb{I}\mathbb{S}\mathbb{P} \mathbf{D}$ which has \mathbf{D} as a retract, the following are equivalent:*

- (i) \mathbf{M} is endoprimal;
- (ii) \mathbf{M} is k -endoprimal;
- (iii) \mathbf{E} is a retract of \mathbf{M} ;
- (iv) \mathbf{M} satisfies (\mathcal{E}) .

Moreover, if \mathbf{M} is finite, then these conditions are also equivalent to

- (v) \mathbf{M} is endodualisable.

Proof. By Steps (1) and (2), we have (ii) \Rightarrow (iv) \Leftrightarrow (iii) and, of course, (i) \Rightarrow (ii) is trivial. By Theorem 1.6, \mathbf{E} is endodualisable and therefore endoprimal. Hence, by Theorem 1.2, if \mathbf{E} is a retract of \mathbf{M} , then \mathbf{M} is endoprimal, that is (iii) implies (i). If \mathbf{M} is finite, then certainly (v) \Rightarrow (i). Finally, since \mathbf{E} is endodualisable, if \mathbf{E} is a retract of \mathbf{M} , then, by Theorem 1.6 (or by Theorem 1.5(i) with \mathbf{D} replaced by \mathbf{E}), the algebra \mathbf{M} is endodualisable, whence (iii) implies (v). \square

2. Sets and vector spaces

Sets and vector spaces may seem like unlikely bed fellows, but these varieties share the rather unusual property that every algebra is free. This property will allow us to lift the endoprimality of a finite algebra up to endodualisability.

Sets. Note that if \mathbf{S} is a non-trivial set, then $\mathbb{I}\mathbb{S}\mathbb{P} \mathbf{S}$ is the class of all non-empty sets. In [17], Márki and Pöschel stated (without proof) that every set is endoprimal. In fact, this is true of all non-empty sets other than those with exactly two elements. We now show how to obtain this result from the general results of the previous section. It then follows easily that every finite set with more than two elements is endodualisable.

The results for arbitrary sets hinge on the fact that a three-element set is endoprimal.

LEMMA 2.1. *Let $\mathbf{S} = \langle \{0, 1, a\}; \emptyset \rangle$ be a three-element set. Then \mathbf{S} is endoprimal; indeed, for all $k \geq 1$, a map $f: S^k \rightarrow S$ is a projection (that is, a term function of \mathbf{S}) if and only if f preserves every map $e: S \rightarrow \{0, 1\}$.*

Proof. For all $x, y, z \in S$, define the map $e_{xyz}: S \rightarrow S$ such that $0 \mapsto x$, $1 \mapsto y$ and $a \mapsto z$. As \mathbf{S} is a set, the map e_{xyz} is an endomorphism of \mathbf{S} . We shall require only the set G consisting of all endomorphisms e_{xyz} with $x, y, z \in \{0, 1\}$.

Let $f: S^k \rightarrow S$ preserve the endomorphisms in G . For each $n \in \{0, \dots, k\}$, define $c_{\downarrow n} \in \{0, 1\}^k$ by

$$c_{\downarrow n}(i) = \begin{cases} 0 & \text{if } i \leq n, \\ 1 & \text{otherwise.} \end{cases}$$

The map $e_{010} \in G$ is a retraction from \mathbf{S} onto $\{0, 1\}$, so $f(c_{\downarrow n}) \in \{0, 1\}$ for all $n \in \{0, \dots, k\}$ by Theorem 1.1. We have $f(c_{\downarrow k}) = f(e_{000}(c_{\downarrow k})) = e_{000}(f(c_{\downarrow k})) = 0$ and, similarly, $f(c_{\downarrow 0}) = 1$. Therefore we can define $m := \min\{n \in \{0, \dots, k\} \mid f(c_{\downarrow n}) = 0\}$.

We aim to show that $f \upharpoonright_{\{0,1\}^k} = \pi_m \upharpoonright_{\{0,1\}^k}$.

Let $x \in \{0, 1\}^k$ and define $z_1, z_2 \in S^k$ by

$$z_1(i) = \begin{cases} 0 & \text{if } i < m, \\ 1 & \text{if } i = m, \\ a & \text{otherwise,} \end{cases} \quad \text{and} \quad z_2(i) = \begin{cases} 0 & \text{if } i \neq m \text{ and } x(i) = 0, \\ 1 & \text{if } i \neq m \text{ and } x(i) = 1, \\ a & \text{otherwise.} \end{cases}$$

We have

$$e_{001}(f(z_1)) = f(e_{001}(z_1)) = f(c_{\downarrow m}) = 0,$$

and

$$e_{011}(f(z_1)) = f(e_{011}(z_1)) = f(c_{\downarrow m-1}) = 1,$$

so $f(z_1) = 1$. Since

$$e_{001}(f(z_2)) = f(e_{001}(z_2)) = f(e_{010}(z_1)) = e_{010}(f(z_1)) = e_{010}(1) = 1,$$

it follows that $f(z_2) = a$. This gives us

$$f(x) = f(e_{01x(m)}(z_2)) = e_{01x(m)}(f(z_2)) = e_{01x(m)}(a) = x(m).$$

Thus $f \upharpoonright_{\{0,1\}^k} = \pi_m \upharpoonright_{\{0,1\}^k}$. It follows by Theorem 1.1 that $f = \pi_m$. Hence a k -ary map on S is a projection if and only if it preserves all the maps in G and consequently \mathbf{S} is endoprimal. \square

THEOREM 2.2. *Let \mathbf{S} be a set. Then \mathbf{S} is endoprimal if and only if $|S| \neq 2$. If \mathbf{S} is finite, then \mathbf{S} is endodualisable if and only if \mathbf{S} is endoprimal.*

Proof. Assume that \mathbf{S} is a two-element set. The 3-ary majority function on S preserves $\text{End } \mathbf{S}$ but is not a projection. So \mathbf{S} is not 3-endoprimal.

Now assume $|S| \geq 3$; then \mathbf{S} has a retraction onto a three-element set. So \mathbf{S} is endoprimal by Lemma 2.1 and Theorem 1.2.

If \mathbf{S} is a finite endoprimal set, then $\mathfrak{S} = \langle S; \text{End } \mathbf{S}, \mathcal{F} \rangle$ yields a duality on the free algebras in $\mathbb{I}\mathbb{S}\mathbb{P} \mathbf{S}$. As every set is free, it follows immediately that \mathbf{S} is endodualisable. \square

Most natural dualities have been proved by establishing an interpolation condition (IC): see Clark and Davey [2], Davey [6] or Davey and Werner [13]. In the present setting, we would say that the algebra \mathbf{M} and the structure $\mathfrak{M} = \langle M; \text{End } \mathbf{M}, \mathcal{F} \rangle$ satisfy (IC) if, for each $k \in \mathbb{N}$ and each subset X of M^k which is closed under the action of $\text{End } \mathbf{M}$, every map $\alpha: X \rightarrow M$ which preserves the action of $\text{End } \mathbf{M}$ extends to a term function $t: M^k \rightarrow M$ of \mathbf{M} . The next lemma demonstrates that (IC) is not a necessary condition for dualisability. For a discussion of full dualities we refer to Clark and Davey [2] or Davey [6].

LEMMA 2.3. *Let \mathbf{S} be a non-trivial finite set and define $\mathfrak{S} = \langle S; \text{End } \mathbf{S}, \mathcal{F} \rangle$.*

- (i) (IC) does not hold with respect to \mathbf{S} and \mathfrak{S} .
- (ii) \mathfrak{S} does not yield a full duality on $\mathbb{I}\mathbb{S}\mathbb{P} \mathbf{S}$.

Proof. Define the set $X = \text{eq}(\pi_1, \pi_2) \cup \text{eq}(\pi_2, \pi_3) \subseteq S^3$. Then

$$X = \{(a, a, b) \mid a, b \in S\} \cup \{(b, a, a) \mid a, b \in S\},$$

and X is closed under the endomorphisms of \mathbf{S} . Define $\alpha: X \rightarrow S$ by

$$\alpha(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } b = c, \end{cases}$$

for all $(a, b, c) \in X$. It is easy to check that α is well-defined and preserves the endomorphisms of \mathbf{S} . As \mathbf{S} is non-trivial, there are distinct elements $a, b \in S$. We

have $\alpha(a, a, b) = b$ and $\alpha(b, a, a) = b$, so α cannot be extended to a 3-ary term function of \mathbf{S} . Thus (IC) does not hold with respect to \mathbf{S} and \mathfrak{S} .

A term-closed subset of \mathfrak{S}^3 is an intersection of equalisers of projections, and is therefore isomorphic to \mathfrak{S} , \mathfrak{S}^2 or \mathfrak{S}^3 . Define $n := |S|$. Since we have $|X| = 2n^2 - n$ and $n \neq 1$, the set X determines a substructure of \mathfrak{S}^3 which is not isomorphic to a term-closed set. So \mathfrak{S} does not yield a full duality on $\mathbb{I}\mathbb{S}\mathbb{P}\mathbf{S}$ by the Full Duality Theorem [2]. \square

We have shown that $\mathfrak{S} = \langle S; \text{End } \mathbf{S}, \mathcal{T} \rangle$ dualises the three-element set \mathbf{S} . However, \mathbf{S} and \mathfrak{S} do not satisfy (IC) by Lemma 2.3, and so (IC) is not a necessary condition for dualisability.

Vector spaces. In Davey [7], it is proved that a finite vector space \mathbf{V} over a finite field is endoprimal if and only if it is endodualisable if and only if its dimension is not 1. Here we shall extend this characterisation of endoprimality to arbitrary vector spaces over arbitrary fields and then use this to give a different proof of endodualisability in the finite case.

Let \mathcal{V} denote the variety of vector spaces over the field \mathbb{F} and let $\mathbf{F} := \langle F; +, \{\lambda_a \mid a \in F\} \rangle$ be the one-dimensional vector space over \mathbb{F} , where λ_a is the unary operation given by scalar multiplication by a .

THEOREM 2.4. *Let \mathbf{V} be a vector space; then \mathbf{V} is endoprimal if and only if $\dim \mathbf{V} \neq 1$. If \mathbf{V} is finite, then \mathbf{V} is endodualisable if and only if \mathbf{V} is endoprimal.*

Proof. The map $f: F^2 \rightarrow F$, defined by

$$f(x, y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{otherwise,} \end{cases}$$

for all $x, y \in F$, is used in Davey [7] to show that the one-dimensional vector space \mathbf{F} is not 2-endoprimal.

Assume that $\dim \mathbf{V} > 1$; then \mathbf{V} is isomorphic to $\mathbf{F}_{\mathcal{V}}(S)$ for some set S with $|S| > 1$. It follows by Theorem 1.3 that \mathbf{V} is endoprimal.

Assume that \mathbf{V} is a finite endoprimal vector space. Then $\mathfrak{V} = \langle V; \text{End } \mathbf{V}, \mathcal{T} \rangle$ yields a duality on every free algebra in \mathcal{V} . Hence \mathbf{V} is endodualisable, since every vector space is free.

3. Distributive lattices, Boolean algebras and Stone algebras

Distributive lattices. In [17], [8] and [7], Steps (1) and (2), required for Theorem 1.7, are carried out for the class of distributive lattices with or without bounds. Define $\mathbf{2}_K = \langle \{0, 1\}; \vee, \wedge, K \rangle$ to be the two-element distributive lattice with the bounds $K \subseteq \{0, 1\}$. Denote the variety of K -bounded distributive lattices by \mathcal{D}_K , then $\mathcal{D}_K = \mathbb{I}\mathbb{S}\mathbb{P} \mathbf{2}_K$ and a simple application of Priestley duality for \mathcal{D}_K (see Priestley [18] and Davey and Werner [13]) shows that $\mathbf{2}_K$ is almost endodualisable via the structure $\mathbf{2}_{K^c} := \langle \{0, 1\}; \text{End } \mathbf{2}_K, \leq, \mathcal{T} \rangle$. Since $\mathbf{2}_K$ is injective in \mathcal{D}_K , it is a retract of every non-trivial algebra in \mathcal{D}_K . As a sublattice of $\mathbf{2}_{K^c}^2$, the relation \leq is isomorphic to the three-element chain $\mathbf{3}_K$. Thus we are set up to apply Theorem 1.7 with $\mathbf{E} = \mathbf{3}_K$.

- (1) It is well known (and straightforward to prove) that a distributive lattice $\mathbf{L} \in \mathcal{D}_K$ has $\mathbf{3}_K$ as a retract if and only if \mathbf{L} has a pair of comparable prime filters if and only if \mathbf{L} is not relatively complemented.
- (2) If $\mathbf{L} \in \mathcal{D}_K$ is non-trivial and relatively complemented, then \mathbf{L} is not k -endoprimal, where $k = 3 - |K|$. (If $K = \emptyset$, then an endomorphism-preserving map $f: L^3 \rightarrow L$ which is not a term function is obtained by declaring $f(a, b, c)$ to be the complement of a in $[a \wedge b \wedge c, a \vee b \vee c]$; if $K = \{0\}$, then define $f: L^2 \rightarrow L$ by declaring $f(a, b)$ to be the complement of a in the interval $[0, a \vee b]$; if $K = \{0, 1\}$, then define $f: L \rightarrow L$ by declaring $f(a)$ to be the complement of a .)

Theorem 1.7 now yields the characterisations of endoprimal and endodualisable algebras in \mathcal{D}_K given in [17], [8] and [7]: for all choices of $K \subseteq \{0, 1\}$, the endoprimals are those $\mathbf{L} \in \mathcal{D}_K$ which are not relatively complemented and the endodualisables are simply the finite endoprimals.

Boolean algebras. For Boolean algebras the situation is particularly simple. Since the two-element Boolean algebra $\mathbf{2}$ is dualised by $\mathbf{2} := \langle \{0, 1\}; \mathcal{T} \rangle$ and we can add endomorphisms without destroying the duality, $\mathbf{2}$ is certainly endodualisable. Since every non-trivial Boolean algebra has $\mathbf{2}$ as a retract, Theorem 1.2 and Theorem 1.5(i) imply that every Boolean algebra is endoprimal and every finite Boolean algebra is endodualisable.

Stone algebras. The variety of Stone algebras equals $\mathbb{I}\mathbb{S}\mathbb{P} \mathbf{3}$ where $\mathbf{3} = \langle \{0, a, 1\}; \vee, \wedge, *, 0, 1 \rangle$, with $0^* = 1$ and $a^* = 1^* = 0$. It was first proved in Davey [3] that the Stone algebra $\mathbf{3}$ is almost endodualisable – see [5], [13] and [2] for progressively simpler proofs: choose $\mathbf{3} = \langle \{0, a, 1\}; \text{End } \mathbf{3}, \leq, \mathcal{T} \rangle$ where \leq is the

order relation $\{(0, 0), (a, a), (a, 1), (1, 1)\}$. It is easy to check that, as a Stone algebra, \preceq is isomorphic to the four-element chain $\mathbf{4}$.

Endodualisable Stone algebras were characterised in Davey [7]. In the process, Steps (1) and (2) required for Theorem 1.7 were performed (with $\mathbf{E} = \mathbf{4}$). Note that, since $\mathbf{3}$ is injective as a Stone algebra, it is easily seen that $\mathbf{3}$ is a retract of a Stone algebra \mathbf{S} if and only if \mathbf{S} is non-boolean.

- (1) It is well known and relatively straightforward to prove the equivalence of the following conditions on a Stone algebra \mathbf{S} :
 - (i) $\mathbf{4}$ is a retract of \mathbf{S} ;
 - (ii) there is a chain $F \subset G \subset M$ of prime filters of \mathbf{S} with M a maximal filter;
 - (iii) the dense set, $d(\mathbf{S}) := \{a \in S \mid a^* = 0\}$, is not relatively complemented.
- (2) If \mathbf{S} is a non-boolean Stone algebra and $d(\mathbf{S})$ is relatively complemented, then \mathbf{S} is not 2-endoprimal. (Define $f_1: S^2 \rightarrow d(\mathbf{S})^2$ by $f_1(a, b) = (a \vee a^*, b \vee b^*)$, for all $a, b \in S$, and define $f_2: d(\mathbf{S})^2 \rightarrow d(\mathbf{S})$ by declaring $f_2(c, d)$ to be the complement of c in the interval $[c \wedge d, 1]$ for all $c, d \in d(\mathbf{S})$. Then the map $f := f_2 \circ f_1: S^2 \rightarrow S$ preserves the endomorphisms of \mathbf{S} but is not a term function – see [7].)

Theorem 1.7 yields the characterisation of endodualisable Stone algebras given in [7] and provides a (new) characterisation of endoprimal algebras: a Stone algebra \mathbf{S} is endoprimal if and only if it is boolean or its dense set $d(\mathbf{S})$ is not relatively complemented and a finite Stone algebra is endodualisable if and only if it is endoprimal.

4. Heyting algebras

Let \mathbf{n} denote the n -element Heyting algebra chain, and define $\mathcal{L}_n := \mathbb{I}\mathbb{S}\mathbb{P} \mathbf{n}$. Proofs of the fact that \mathbf{n} is endodualisable have been seminal in the development of the theory of natural dualities. The first, bare-hands proof was in Davey [4]. This proof was generalised and simplified in Davey [5]. Next, Davey and Werner [13] gave a proof via the NU Duality Theorem. Based on an analysis of the proofs in [4] and [5], Davey and Werner [14], [15] developed the piggyback approach to obtaining natural dualities and used it to give a new proof of the endodualisability of \mathbf{n} . Finally, Davey and Priestley [12] used the test algebra lemma (see [11] and [9]) to give a particularly simple proof.

In Davey [7], it was proved that every finite algebra in \mathcal{L}_3 is endodualisable. The proof used the injectivity of $\mathbf{3}$ in \mathcal{L}_3 and hence did not extend to \mathcal{L}_n for $n \geq 4$. By using Theorem 1.2 and Theorem 1.5(i), we can prove a much stronger result.

THEOREM 4.1. *For all $n \geq 2$, every Heyting algebra in \mathcal{L}_n is endoprimal, and every finite Heyting algebra in \mathcal{L}_n is endodualisable.*

Proof. If $\mathbf{L} \in \mathcal{L}_2$ then \mathbf{L} is term equivalent to a Boolean algebra and is therefore endodualisable by the discussion in Section 3.

Now let $n \geq 3$. If $\mathbf{L} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ then there is a homomorphism from \mathbf{L} onto \mathbf{n} . The chain \mathbf{n} is a projective Heyting algebra by Balbes and Horn [1], and therefore \mathbf{n} is a retract of \mathbf{L} . Since the chain \mathbf{n} is endodualisable and therefore endoprimal, \mathbf{L} is endoprimal, by Theorem 1.2. If \mathbf{L} is finite, then \mathbf{L} is endodualisable by Theorem 1.5(i). The required result now follows by induction. \square

The above result characterises the finite endoprimal and endodualisable algebras in a subvariety of the class of Heyting algebras. A **relative Stone algebra** is a Heyting algebra satisfying the additional identity

$$(x \rightarrow y) \vee (y \rightarrow x) \approx 1.$$

It is shown in Hecht and Katriňák [16] that every relative Stone algebra belongs to $\mathbb{ISP} \mathbf{A}$ for some Heyting algebra chain \mathbf{A} . If \mathbf{S} is a finite relative Stone algebra, then $\mathbf{S} \in \mathcal{L}_n$ for some $n \in \mathbb{N}$. Therefore \mathbf{S} is endodualisable by Theorem 4.1.

In Davey and Werner [14][15], it was proved that the Heyting algebra $\mathbf{2}^2 \oplus \mathbf{1}$ is endodualisable and in Davey [7] it was proved that every finite algebra in $\mathbb{ISP}(\mathbf{2}^2 \oplus \mathbf{1})$ is endodualisable. Our techniques allow us to extend this result.

THEOREM 4.2. *Every Heyting algebra in $\mathbb{ISP}(\mathbf{2}^2 \oplus \mathbf{1})$ is endoprimal, and every finite Heyting algebra in $\mathbb{ISP}(\mathbf{2}^2 \oplus \mathbf{1})$ is endodualisable.*

Proof. Let $\mathbf{L} \in \mathbb{ISP}(\mathbf{2}^2 \oplus \mathbf{1})$. If $\mathbf{L} \in \mathcal{L}_3$ then we can use Theorem 4.1. If we have $\mathbf{L} \notin \mathcal{L}_3$ then there is a homomorphism from \mathbf{L} onto $\mathbf{2}^2 \oplus \mathbf{1}$. So $\mathbf{2}^2 \oplus \mathbf{1}$ is a retract of \mathbf{L} , as $\mathbf{2}^2 \oplus \mathbf{1}$ is a projective Heyting algebra [1]. Since $\mathbf{2}^2 \oplus \mathbf{1}$ is endodualisable, it follows by Theorem 1.2 that \mathbf{L} is endoprimal. If \mathbf{L} is finite, then \mathbf{L} is endodualisable by Theorem 1.5(i). \square

It is shown in Davey and Priestley [12] that $\mathbf{2}^2 \oplus \mathbf{1}$ and the chains \mathbf{n} are the only endodualisable subdirectly irreducible Heyting algebras. It now follows that $\text{Var } \mathbf{2}^2 \oplus \mathbf{1}$ and $\text{Var } \mathbf{n}$ for $n \geq 2$ are the only varieties of Heyting algebras generated by a single subdirectly irreducible in which every finite algebra is endodualisable.

5. Semilattices

Define $\mathbf{D}_K := \langle \{0, 1\}; \vee, K \rangle$ to be the two-element semilattice with the bounds $K \subseteq \{0, 1\}$, and denote the variety of K -bounded semilattices by \mathcal{S}_K . Define the topological semilattices

$$\mathbf{D} := \langle \{0, 1\}; \vee, 0, 1, \mathcal{T} \rangle, \quad \mathbf{D}_0 := \langle \{0, 1\}; \vee, 0, \mathcal{T} \rangle,$$

$$\mathbf{D}_1 := \langle \{0, 1\}; \vee, 1, \mathcal{T} \rangle, \quad \mathbf{D}_{01} := \langle \{0, 1\}; \vee, \mathcal{T} \rangle.$$

It is shown in Davey and Werner [13] that \mathbf{D}_K yields a duality on $\mathcal{S}_K = \mathbb{I}\mathbb{S}\mathbb{P} \mathbf{D}_K$. Consequently, for all choices of $K \subseteq \{0, 1\}$, the semilattice \mathbf{D}_K is almost endodualisable via $\mathbf{D}'_K := \langle \{0, 1\}; \text{End } \mathbf{D}_K, s, \mathcal{T} \rangle$, where $s = \text{graph}(\vee) := \{(a, b, a \vee b) \mid a, b \in \{0, 1\}\}$. Once again, we can apply Theorem 1.7 to obtain and extend the characterisation from [7] of endodualisability in \mathcal{S}_0 and \mathcal{S} . Since \mathbf{s} is isomorphic to \mathbf{D}_K^2 , we choose \mathbf{E} to be \mathbf{D}_K^2 . Steps (1) and (2) were performed in [7]. Since \mathbf{D}_K is injective in \mathcal{S}_K , it is a retract of every non-trivial semilattice in \mathcal{S}_K and \mathbf{D}_K^2 is a retract of $\mathbf{S} \in \mathcal{S}_K$ if and only if \mathbf{S} has a subalgebra isomorphic to \mathbf{D}_K^2 .

- (1) A semilattice \mathbf{S} in \mathcal{S}_0 [in \mathcal{S}] has \mathbf{D}_0^2 [\mathbf{D}^2] as a retract if and only if \mathbf{S} is not a chain [is not a tree].
- (2) Let \mathbf{S} be a non-trivial semilattice in \mathcal{S}_K . If \mathbf{S} is a chain [a tree], then \mathbf{S} is not 2-endoprimal [is not 3-endoprimal].

We conclude at once from Theorem 1.7 that a semilattice in \mathcal{S}_0 [in \mathcal{S}] is endoprimal if and only if it is not a chain [is not a tree] and that a finite semilattice in either \mathcal{S}_0 or in \mathcal{S} is endodualisable if and only if it is endoprimal.

The varieties \mathcal{S}_1 and \mathcal{S}_{01} were notable in [7] by their absence. This is because the technique used in [7], the precursor of our Theorem 1.7, yields a characterisation of endodualisability only in the case when endodualisability is equivalent to endoprimality. As we are about to see, this fails in both \mathcal{S}_1 and \mathcal{S}_{01} !

LEMMA 5.1. *If $\mathbf{S} \in \mathcal{S}_{01}$ is not a chain, then \mathbf{S} is endoprimal.*

Proof. Assume that \mathbf{S} is not a chain and let $f: S^k \rightarrow S$ preserve the endomorphisms of \mathbf{S} . There are non-comparable elements $a, b \in S$. Define the endomorphisms $e_{01} := \chi_{S \setminus a}$, $e_{10} := \chi_{S \setminus b}$ and $e_{11} := \chi_{S \setminus \{0\}}$. The map e_{01} is a retraction from \mathbf{S} onto $\{0, 1\}$, so f preserves $\{0, 1\}$ by Theorem 1.1. To show that f is a term function of \mathbf{S} , we firstly show that $f \upharpoonright_{\{0, 1\}^k}: \{0, 1\}^k \rightarrow \{0, 1\}$ preserves \vee .

Let $x, y \in \{0, 1\}^k$ and define $z \in S^k$ by

$$z(i) = \begin{cases} 0 & \text{if } x(i) = y(i) = 0, \\ a & \text{if } x(i) = 0 \text{ and } y(i) = 1, \\ b & \text{if } x(i) = 1 \text{ and } y(i) = 0, \\ 1 & \text{if } x(i) = y(i) = 1. \end{cases}$$

There is a retraction from \mathbf{S} onto $A := \{0, a, b, a \vee b, 1\}$, and so $f(z) \in A$ by Theorem 1.1. Since $e_{11} \upharpoonright_A = e_{01} \vee e_{10} \upharpoonright_A$, we have

$$\begin{aligned} f(x \vee y) &= f(e_{11}(z)) = e_{11}(f(z)) = (e_{01} \vee e_{10})(f(z)) \\ &= e_{01}(f(z)) \vee e_{10}(f(z)) = f(e_{01}(z)) \vee f(e_{10}(z)) \\ &= f(x) \vee f(y). \end{aligned}$$

Thus $f \upharpoonright_{\{0,1\}^k}: \{0, 1\}^k \rightarrow \{0, 1\}$ preserves \vee . As $\langle \{0, 1\}; \vee, \mathcal{T} \rangle$ dualises the two-element bounded semilattice $\langle \{0, 1\}; \vee, 0, 1 \rangle$, it follows that $f \upharpoonright_{\{0,1\}^k} = t \upharpoonright_{\{0,1\}^k}$ for some k -ary term function t of \mathbf{S} . So, by Theorem 1.1, f is a term function of \mathbf{S} . \square

The following theorem characterises endoprimal semilattices in \mathcal{S}_K for all four choices of K . The sections on \mathcal{S}_0 and \mathcal{S} provide alternative proofs to the proof via Theorem 1.7 given above.

THEOREM 5.2. *A non-trivial semilattice with bounds $K \subseteq \{0, 1\}$ is endoprimal if and only if it is not a tree. More precisely,*

- (i) *a non-trivial $\mathbf{S} \in \mathcal{S}_0 \cup \mathcal{S}_{01}$ is endoprimal if and only if \mathbf{S} is not a chain,*
- (ii) *a non-trivial $\mathbf{S} \in \mathcal{S} \cup \mathcal{S}_1$ is endoprimal if and only if \mathbf{S} is not a tree.*

Proof. For all K , the ‘only if’ direction follows by **(2)** above. To prove the ‘if’ direction, we consider each choice of $K \subseteq \{0, 1\}$ separately.

Case (a): $K = \{0, 1\}$. This case follows immediately from Lemma 5.1.

Case (b): $K = \{1\}$. Let \mathbf{S} be an upper-bounded semilattice which is not a tree. There is some element $0 \in S$ with non-comparable elements $a, b \in \uparrow 0$. Consider the subsemilattice $\mathbf{A} = \langle \{0, a, b, a \vee b, 1\}; \vee, 1 \rangle$ of \mathbf{S} and the bounded semilattice $\mathbf{A}_{01} = \langle \{0, a, b, a \vee b, 1\}; \vee, 0, 1 \rangle$ which has \mathbf{A} as a term-reduct. There is a retraction from \mathbf{S} onto \mathbf{A} , and so, by Theorem 1.2, it suffices to show that \mathbf{A} is endoprimal.

Let $f: A^k \rightarrow A$ preserve $\text{End } \mathbf{A}$. Then f also preserves $\text{End } \mathbf{A}_{01}$. So f is a term function of \mathbf{A}_{01} by Lemma 5.1. Since $\underline{1} \in \text{End } \mathbf{A}$, we have

$$f(1, \dots, 1) = f(\underline{1}(1), \dots, \underline{1}(1)) = \underline{1}(f(1, \dots, 1)) = 1.$$

Therefore f is a term function of \mathbf{A} .

Case (c): $K = \{0\}$. Let \mathbf{S} be a lower-bounded semilattice which is not a chain. There are non-comparable elements $a, b \in S$, and we can define $1 := a \vee b$. Define $\mathbf{A} = \langle \{0, a, b, 1\}; \vee, 0 \rangle$. Using an argument similar to that given in case (b), we can show that \mathbf{A} , and therefore \mathbf{S} , is endoprimal.

Case (d): $K = \emptyset$. Let $\mathbf{S} \in \mathcal{S}$ such that \mathbf{S} is not a tree. There is some $0 \in S$ with non-comparable elements $a, b \in \uparrow 0$, and we can define $1 := a \vee b$. We can show that \mathbf{S} is endoprimal using the semilattice $\langle \{0, a, b, 1\}; \vee \rangle$. \square

Now that endoprimal semilattices have been classified, it remains to characterise endodualisable bounded and upper-bounded semilattices.

THEOREM 5.3. *A finite non-trivial $\mathbf{S} \in \mathcal{S}_{01}$ is endodualisable if and only if 1 is join-reducible in \mathbf{S} .*

Proof. To show the ‘if’ direction, assume that 1 is join-reducible in \mathbf{S} . Then there exist $a, b \in S \setminus \{1\}$ with $a \vee b = 1$. The set $\{0, a, b, 1\}$ determines a sub-semilattice of \mathbf{S} which is isomorphic to \mathbf{D}_{01}^2 . Therefore \mathbf{D}_{01}^2 is a retract of \mathbf{S} , as \mathbf{D}_{01} is injective in \mathcal{S}_{01} . Since \mathbf{D}_{01} is almost endodualisable via $\langle \{0, 1\}; \text{End } \mathbf{S}, \text{graph}(\vee), \mathcal{T} \rangle$ and since $\mathbf{D}_{01}^2 \cong \text{graph}(\vee)$, it follows by Theorem 1.6 that \mathbf{S} is endodualisable.

To show the ‘only if’ direction, assume that 1 is join-irreducible in \mathbf{S} . Define $\mathfrak{S} := \langle S; \text{End } \mathbf{S}, \mathcal{T} \rangle$. Consider the term-closed set $X := \text{eq}(\pi_1 \vee \pi_2, \underline{1}) \subseteq S^2$. As 1 is join-irreducible, we have

$$X = \{(a, 1) \mid a \in S\} \cup \{(1, a) \mid a \in S\}.$$

Therefore x and y are comparable for all $(x, y) \in X$, and we can define a map $\varphi: X \rightarrow S$ by $\varphi(x, y) = x \wedge y$ for all $(x, y) \in X$. The map φ preserves $\text{End } \mathbf{S}$. Since \mathbf{S} is non-trivial, there is some $a \in S \setminus \{1\}$ with

$$\varphi((a, 1) \vee (1, a)) = \varphi(1, 1) = 1 \neq a = a \vee a = \varphi(a, 1) \vee \varphi(1, a).$$

So φ cannot be extended to a binary term function on \mathbf{S} . By the Third Duality Theorem [2][6], the structure \mathfrak{S} does not yield a duality on $\mathbb{I}\mathbb{S}\mathbb{P} \mathbf{S}$. Hence \mathbf{S} is not endodualisable. \square

THEOREM 5.4. *A finite non-trivial $\mathbf{S} \in \mathcal{S}_1$ is endodualisable if and only if \mathbf{S} has a subalgebra isomorphic to \mathbf{D}_1^2 .*

Proof. The ‘if’ direction follows from Theorem 1.6, as in the previous proof. To show the ‘only if’ direction, assume that \mathbf{D}_1^2 does not embed into \mathbf{S} . Consider the term-closed set $X := \text{eq}(\pi_2 \vee \pi_3, 1) \subseteq S^3$. Let $(x, y, z) \in X$. We have

$$(x \vee y) \vee (x \vee z) = x \vee (y \vee z) = 1.$$

If we suppose that $x \vee y \neq 1$ and $x \vee z \neq 1$, then $\{x, x \vee y, x \vee z, 1\}$ determines a subalgebra of \mathbf{S} which is isomorphic to \mathbf{D}_1^2 , a contradiction. So $x \vee y = 1$ or $x \vee z = 1$. Thus we can define a map $\varphi: X \rightarrow S$ by

$$\varphi(x, y, z) = (x \vee y) \wedge (x \vee z)$$

for all $(x, y, z) \in X$, and φ preserves $\text{End } \mathbf{S}$. As \mathbf{S} is non-trivial, there is some $a \in S \setminus \{1\}$ with

$$\varphi((a, a, 1) \vee (a, 1, a)) = \varphi(a, 1, 1) = 1,$$

but

$$\varphi(a, a, 1) \vee \varphi(a, 1, a) = a \vee a = a.$$

So φ cannot be extended to a term function on \mathbf{S} , and therefore \mathbf{S} is not endodualisable, as in the previous proof. \square

It now follows from Theorems 1.5(ii), 5.2, 5.3 and 5.4 that the semilattice $\mathbf{2}^2 \oplus \mathbf{1}$, regarded as either a bounded semilattice or upper-bounded semilattice, is endoprimal, almost endodualisable, but not endodualisable.

6. Abelian groups

Let \mathbb{Z}_m be the finite cyclic group of order m ; then $\mathcal{A}_m := \text{ISP } \mathbb{Z}_m$ is the variety of abelian groups of exponent m . It is shown in Davey and Werner [13] that $\mathbb{Z}_m := \langle \mathbb{Z}_m; +, -, 0, \mathcal{T} \rangle$ yields a duality on \mathcal{A}_m . It follows that \mathbb{Z}_m is almost endodualisable via $\langle \mathbb{Z}_m; \text{End } \mathbb{Z}_m, \text{graph}(+), \mathcal{T} \rangle$. As an algebra, $\text{graph}(+)$ is isomorphic to \mathbb{Z}_m^2 . Since \mathbb{Z}_m is injective in \mathcal{A}_m , so is \mathbb{Z}_m^2 . Thus \mathbb{Z}_m^2 is a retract of $\mathbf{A} \in \mathcal{A}_m$ if and only if \mathbb{Z}_m^2 embeds into \mathbf{A} . Thus, the lemma below provides Step (2) in preparation for an application of Theorem 1.7 with $\mathbf{E} = \mathbb{Z}_m^2$.

LEMMA 6.1. *Let \mathbf{A} be an abelian group of exponent m . If \mathbb{Z}_m^2 does not embed into \mathbf{A} , then \mathbf{A} is not 2-endoprimal.*

Proof. Assume that \mathbb{Z}_m^2 does not embed into \mathbf{A} . We can assume that $\mathbf{A} = \mathbb{Z}_{p^r} \times \mathbf{B}$ for some $\mathbf{B} \in \mathcal{A}_m$, prime p and $r \in \mathbb{N}$ such that \mathbb{Z}_{p^r} does not embed into \mathbf{B} . Define the map $g: \mathbb{Z}_{p^r}^2 \rightarrow \mathbb{Z}_{p^r}$ by

$$g(a, b) = \begin{cases} p^{r-1}a & \text{if } p \mid (a - b), \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b \in \mathbb{Z}_{p^r}$. Then we can define $f: A^2 \rightarrow A$ by $f((a, b), (c, d)) = (g(a, c), 0)$ for all $(a, b), (c, d) \in A$. To show that f preserves the endomorphisms of \mathbf{A} , let $e \in \text{End } \mathbf{A}$ and define $(k, l) := e(1, 0)$. Let $(a, b), (c, d) \in A$. We will show that $e(f((a, b), (c, d))) = f(e(a, b), e(c, d))$ using the following three claims.

Claim 1. $g(a, c)k = g(ak, ck)$.

Case (a): $p \mid (a - c)$. We have $p \mid (ak - ck)$, so

$$g(a, c)k = p^{r-1}ak = g(ak, ck).$$

Case (b): $p \nmid (a - c)$ and $p \nmid (ak - ck)$. We have

$$g(a, c)k = 0k = 0 = g(ak, ck).$$

Case (c): $p \nmid (a - c)$ and $p \mid (ak - ck)$. As $p \mid (a - c)k$ and p is prime, we must have $p \mid k$. Therefore

$$g(a, c)k = 0k = 0 = p^{r-1}ak = g(ak, ck).$$

Claim 2. $g(a, c)l = 0$.

If $p \nmid (a - c)$ then $g(a, c) = 0$ and $g(a, c)l = 0$. So assume that $p \mid (a - c)$. We have $|e(1, 0)| \mid |(1, 0)|$, so $|(k, l)| \mid p^r$. As $|l| \mid |(k, l)|$, this gives $|l| \mid p^r$. We cannot have $|l| = p^r$ since \mathbb{Z}_{p^r} does not embed into \mathbf{B} . Therefore $|l| \mid p^{r-1}$, which implies that $g(a, c)l = p^{r-1}al = 0$.

Define $(b_1, b_2) := e(0, b)$ and $(d_1, d_2) := e(0, d)$.

Claim 3. $g(ak + b_1, ck + d_1) = g(ak, ck)$.

We will firstly show that $p \mid b_1$ and $p \mid d_1$. Suppose that $p \nmid b_1$. Then $|b_1| = p^r / \gcd(p^r, b_1) = p^r$. We have $|b_1| \mid |(b_1, b_2)|$ and $|(b_1, b_2)| \mid |b|$, so $p^r \mid |b|$. This is a contradiction, since \mathbb{Z}_{p^r} does not embed into \mathbf{B} . Thus $p \mid b_1$, and similarly, $p \mid d_1$.

Case (a): $p \nmid (ak - ck)$. As $p \mid b_1$ and $p \mid d_1$, we have $p \mid ((ak + b_1) - (ck + d_1))$. Therefore

$$g(ak + b_1, ck + d_1) = 0 = g(ak, ck).$$

Case (b): $p \mid (ak - ck)$. We have $p \mid ((ak + b_1) - (ck + d_1))$. Since $p \nmid b_1$ this gives

$$\begin{aligned} g(ak + b_1, ck + d_1) &= p^{r-1}(ak + b_1) = p^{r-1}ak + p^{r-1}b_1 \\ &= p^{r-1}ak = g(ak, ck). \end{aligned}$$

Claims 1 and 2 tell us that

$$\begin{aligned} e(f((a, b), (c, d))) &= e(g(a, c), 0) = g(a, c)e(1, 0) \\ &= (g(a, c)k, g(a, c)l) = (g(ak, ck), 0). \end{aligned}$$

We have

$$\begin{aligned} f(e(a, b), e(c, d)) &= f(e(a, 0) + e(0, b), e(c, 0) + e(0, d)) \\ &= f((ak, al) + (b_1, b_2), (ck, cl) + (d_1, d_2)) \\ &= f((ak + b_1, al + b_2), (ck + d_1, cl + d_2)) \\ &= (g(ak + b_1, ck + d_1), 0). \end{aligned}$$

So it follows by Claim 3 that

$$e(f((a, b), (c, d))) = f(e(a, b), e(c, d)).$$

Thus f preserves the endomorphisms of \mathbf{A} . We have

$$f(((1, 0), (0, 0)) + ((0, 0), (1, 0))) = f((1, 0), (1, 0)) = (p^{r-1}, 0)$$

and

$$f((1, 0), (0, 0)) + f((0, 0), (1, 0)) = (0, 0).$$

So f is not a term function of \mathbf{A} , whence \mathbf{A} is not 2-endoprimal. □

Theorem 1.7 now yields a description of the endoprimalms in \mathcal{A}_m and completes the characterisation, commenced in [7], of the endodualisables in \mathcal{A}_m .

THEOREM 6.2. *Let \mathbf{A} be an abelian group of exponent m . Then the following are equivalent:*

- (i) \mathbf{A} is endoprimal;
- (ii) \mathbf{A} is 2-endoprimal;
- (iii) \mathbb{Z}_m^2 embeds into \mathbf{A} .

Moreover, if \mathbf{A} is finite, these conditions are equivalent to:

- (iv) \mathbf{A} is endodualisable.

We will now describe the endoprimal groups in $\mathbb{ISP} \mathbb{Z}$ and give some information about endoprimal groups in $\mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$.

LEMMA 6.3. *Let $\mathbf{A} \in \mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$ such that $\mathbf{A} \notin \mathcal{A}_m$. If \mathbb{Z}^2 does not embed into \mathbf{A} , then \mathbf{A} is not 2-endoprimal.*

Proof. Let $\mathbf{A} \in \mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$ with $\mathbf{A} \notin \mathcal{A}_m$ and assume that \mathbb{Z}^2 does not embed into \mathbf{A} . We can assume that $\mathbf{A} = \mathbb{Z} \times \mathbf{B}$ for some $\mathbf{B} \in \mathcal{A}_m$.

Define $f: A^2 \rightarrow A$ by

$$f((a, b), (c, d)) = \begin{cases} (ma, 0) & \text{if } a = c, \\ (0, 0) & \text{otherwise,} \end{cases}$$

for all $(a, b), (c, d) \in A$. To show that f preserves the endomorphisms of \mathbf{A} , let $e \in \text{End } \mathbf{A}$. Define $(k, l) := e(1, 0)$ and let $(a, b), (c, d) \in A$. We have $|e(0, b)| \mid |(0, b)|$. So $e(0, b) = (0, b')$ for some $b' \in B$. Similarly, $e(0, d) = (0, d')$ for some $d' \in B$. So

$$\begin{aligned} f(e(a, b), e(c, d)) &= f(e(a, 0) + e(0, b), e(c, 0) + e(0, d)) \\ &= f((ak, al) + (0, b'), (ck, cl) + (0, d')) \\ &= f((ak, al + b'), (ck, cl + d')). \end{aligned}$$

Case (a): $a = c$. We have $ak = ck$. As $\mathbf{B} \in \mathcal{A}_m$, this gives

$$\begin{aligned} e(f((a, b), (c, d))) &= e(ma, 0) = (mak, mal) = (mak, 0) \\ &= f((ak, al + b'), (ck, cl + d')) \\ &= f(e(a, b), e(c, d)). \end{aligned}$$

Case (b): $a \neq c$ and $k \neq 0$. We have $ak \neq ck$, so

$$\begin{aligned} e(f((a, b), (c, d))) &= e(0, 0) = (0, 0) \\ &= f((ak, al + b'), (ck, cl + d')) \\ &= f(e(a, b), e(c, d)). \end{aligned}$$

Case (c): $a \neq c$ and $k = 0$. We have

$$\begin{aligned} e(f((a, b), (c, d))) &= e(0, 0) = (0, 0) \\ &= f((0, al + b'), (0, cl + d')) \\ &= f((ak, al + b'), (ck, cl + d')) \\ &= f(e(a, b), e(c, d)). \end{aligned}$$

Thus f preserves the endomorphisms of \mathbf{A} . But f is not a term function of \mathbf{A} , as

$$f(((1, 0), (0, 0)) + ((0, 0), (1, 0))) = f((1, 0), (1, 0)) = (m, 0)$$

but

$$f((1, 0), (0, 0)) + f((0, 0), (1, 0)) = (0, 0).$$

Hence \mathbf{A} is not 2-endoprimal. \square

THEOREM 6.4. *Let \mathbf{A} be a non-trivial group in $\mathbb{ISP} \mathbb{Z}$. Then \mathbf{A} is endoprimal if and only if \mathbb{Z}^2 embeds into \mathbf{A} .*

Proof. The ‘only if’ direction follows from Lemma 6.3. To show the ‘if’ direction, let $\mathbf{A} \in \mathbb{ISP} \mathbb{Z}$ and assume that \mathbb{Z}^2 embeds into \mathbf{A} . Then \mathbb{Z}^2 is a retract of \mathbf{A} . As \mathbb{Z} and \mathbb{Z}^2 are the free groups on one and two generators in $\mathbb{ISP} \mathbb{Z}$, it follows by Theorem 1.3 that \mathbb{Z}^2 is endoprimal. Theorem 1.2 now implies that \mathbf{A} is endoprimal. \square

The endoprimal groups in $\mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$ have not been completely classified. We have classified the endoprimal groups in the subquasi-varieties $\mathbb{ISP} \mathbb{Z}$ and $\mathbb{ISP} \mathbb{Z}_m$. However, if $\mathbf{A} \in \mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$ contains both torsion and torsion-free elements, we only know that \mathbf{A} is not endoprimal if \mathbb{Z}^2 does not embed into \mathbf{A} . As \mathbb{Z} is the free group on one generator in $\mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$, we cannot use Theorem 1.3 to find endoprimal groups in $\mathbb{ISP} \{\mathbb{Z}, \mathbb{Z}_m\}$ which are not in $\mathbb{ISP} \mathbb{Z}$.

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