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Annihilators in modular lattices

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Let L be a lattice. For $a, b \in L$ the *annihilator* $\langle a, b\rangle$ and the *dual annihilator* $\langle a, b\rangle_d$ of a relative to b are given by

$$\langle a, b\rangle := \{x \in L \mid x \wedge a \leqslant b\} \quad \text{and} \quad \langle a, b\rangle_d := \{x \in L \mid x \vee a \geqslant b\}.$$

Several authors have studied annihilators in distributive lattices: Mandelker [6], Cornish [2, 3, 4], Beazer [1], Davey [5]. In particular Mandelker proved that L is distributive if and only if $\langle a, b\rangle$ is an ideal for all $a, b \in L$. By working with annihilators instead of ideals we can give a modular-lattice analogue of the fact that a lattice is distributive if and only if every ideal is an intersection of prime ideals.

In any lattice L we have $\langle a, b\rangle = \langle a, a \wedge b\rangle$ and L is distributive if and only if $\langle b, a\rangle_d = \langle a \wedge b, a\rangle_d$ for all $a, b \in L$. An ideal J of a lattice L is prime if and only if $L \backslash J$ is an ideal of L^d. These observations motivate the following definition.

DEFINITION. Let $a, b \in L$. The annihilator $\langle a, b\rangle$ is called *prime* if

(a) $\langle a, b\rangle \cup \langle b, a\rangle_d = L$,

(b) $\langle a, a \wedge b\rangle \cap \langle a \wedge b, a\rangle_d = \varnothing$.

Hence if L is distributive then every prime annihilator is a prime ideal. It should be emphasised that the primeness of $\langle a, b\rangle$ depends upon the elements a and b rather than the set $\langle a, b\rangle$: in a three-element chain, $0 < a < 1$, we have $\langle 1, 0\rangle = \{0\} = \langle a, 0\rangle$ while $\langle a, 0\rangle$ is prime but $\langle 1, 0\rangle$ is not.

In any lattice we have

$$\langle a, a \wedge b\rangle \cap \langle a \wedge b, a\rangle_d = \varnothing \quad \text{implies} \quad a \nleqslant b. \qquad (*)$$

The converse characterizes modularity.

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LEMMA 1. *The following are equivalent*:
(i) L *is modular*;
(ii) $a > b$ *implies* $\langle a, b\rangle \cap \langle b, a\rangle_d = \varnothing$;
(iii) $a \not\leqslant b$ *implies* $\langle a, a \wedge b\rangle \cap \langle a \wedge b, a\rangle_d = \varnothing$.

Proof. This is clear: if $a > b$, then $x \in \langle a, b\rangle \cap \langle b, a\rangle_d$ if and only if $\{a, b, x\}$ generates an N_5-sublattice of L. ■

It follows from Lemma 1 and its dual that if L is modular, then $\langle a, b\rangle$ is a prime annihilator in L if and only if $\langle b, a\rangle_d$ is a prime annihilator in L^d.

LEMMA 2. *Let L be any lattice and let* $a, b, a', b' \in L$.
(i) *Assume* $a > b$. *Then* $\langle a, b\rangle \cup \langle b, a\rangle_d = L$ *if and only if* $a \succ b$.
(ii) *Assume* $a' \leqslant a$ *and* $b \leqslant b'$. *Then* $\langle a, b\rangle \cup \langle b, a\rangle_d = L$ *implies* $\langle a', b'\rangle \cup \langle b', a'\rangle_d = L$.
(iii) *If* $a \succ a \wedge b$ *or* $a \vee b \succ b$, *then* $\langle a, b\rangle \cup \langle b, a\rangle_d = L$.

Proof. We prove only (i) since (ii) is trivial and (iii) follows from (i) and (ii). Assume $a \succ b$ and suppose $x \notin \langle a, b\rangle$. Then $x \wedge a \not\leqslant b$ and hence $b \vee (x \wedge a) = a$ as $a \succ b$. Thus

$$x \vee b = x \vee (x \wedge a) \vee b = x \vee a \geqslant a,$$

whence $x \in \langle b, a\rangle_d$. Thus $\langle a, b\rangle \cup \langle b, a\rangle_d = L$. Conversely assume $a > b$ and suppose there exists $x \in L$ with $a > x > b$. Then $x \notin \langle a, b\rangle \cup \langle b, a\rangle_d$. ■

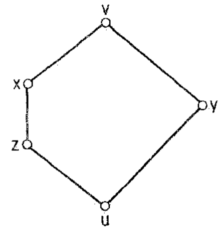
In a modular lattice, (iii) of Lemma 2 yields a particularly simple characterization of prime annihilators.

LEMMA 3. *Let L be a modular lattice and let* $a, b, a', b', \in L$.
(i) *Assume* $a > b$. *Then* $\langle a, b\rangle$ *is prime if and only if* $a \succ b$.
(ii) *Assume* $a' \leqslant a$ *and* $b \leqslant b'$ *with* $a' \not\leqslant b'$. *Then if* $\langle a, b\rangle$ *is prime so is* $\langle a', b'\rangle$.
(iii) $\langle a, b\rangle$ *is prime if and only if* $a \succ a \wedge b$ (*or* $a \vee b \succ b$).

Proof. (i) follows from Lemma 1 and Lemma 2(i), and (ii) follows from Lemma 1 and Lemma 2(ii). By (*) and Lemma 2 (iii) it remains to prove that $\langle a, b\rangle \cup \langle b, a\rangle_d = L$ implies that $a \succ a \wedge b$. Suppose there exists $c \in L$ with $a > c > a \wedge b$. If $c \vee b \geqslant a$, then $\{a, b, c\}$ generates an N_5-sublattice. Hence $c \notin \langle b, a\rangle_d$. But $a \wedge c = c \not\leqslant b$ and thus $c \notin \langle a, b\rangle$. Consequently $\langle a, b\rangle \cup \langle b, a\rangle_d \neq L$. ■

It follows from Lemma 3(i) (iii) that in a modular lattice we can restrict ourselves to prime annihilators $\langle a, b\rangle$ with $a \succ b$: if $\langle a, b\rangle$ is prime, then $a \succ a \wedge b$ and so $\langle a, a \wedge b\rangle$ is prime and moreover $\langle a, b\rangle = \langle a, a \wedge b\rangle$.

We shall say that L satisfies the *Prime–Annihilator Condition* if every annihilator in L is an intersection of prime annihilators. It is easily seen that N_5

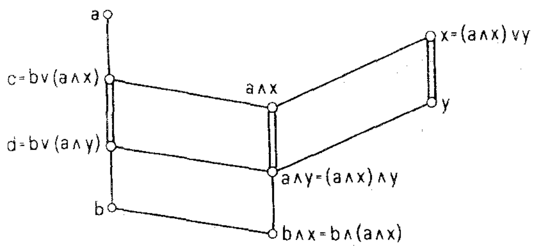


fails the Prime–Annihilator Condition: $\langle x, z\rangle = \{u, y, z\}$ is not prime as $y \in \langle x, z\rangle \cap \langle z, x\rangle_d$ and is contained in no prime annihilator since the only annihilator which contains $\langle x, z\rangle$ is N_5 itself. Since the Prime–Annihilator Condition goes down to sublattices, it follows that any lattice satisfying the Prime–Annihilator Condition must be modular. Clearly the Prime–Annihilator Condition also implies the existence of a profusion of covers.

Recall that L is *weakly atomic* if for all $a, b \in L$ with $a > b$ there exist $c, d \in L$ with $a \geq c \succ d \geq b$.

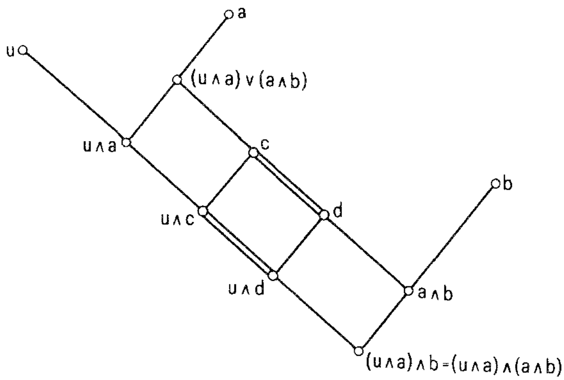
THEOREM 4. *A lattice satisfies the Prime–Annihilator Condition if and only if it is weakly atomic and modular.*

Proof. Assume L satisfies the Prime–Annihilator Condition. Then, as remarked above, L is modular. Let $a, b \in L$ with $a > b$. Since $a \notin \langle a, b\rangle$ there exists a prime annihilator $\langle x, y\rangle$ with $x \succ y$ such that $a \notin \langle x, y\rangle$ and $\langle a, b\rangle \subseteq \langle x, y\rangle$.



Since $a \wedge x \nleq y$ and $x \succ y$ we have $(a \wedge x) \vee y = x$ and thus $a \wedge x \succ a \wedge y$ as L is lower semimodular. Since $b \in \langle a, b\rangle \subseteq \langle x, y\rangle$ we have $b \wedge x \leq y$ and so $a \wedge x \succ a \wedge y \geq b \wedge x$. The isomorphism between the intervals $a \wedge x/b \wedge x$ and $b \vee (a \wedge x)/b$, given by the modularity of L, guarantees that $c := b \vee (a \wedge x) \succ d := b \vee (a \wedge y)$. Thus L is weakly atomic.

Now assume that L is weakly atomic and modular. Let $u \notin \langle a, b \rangle$; then $u \wedge a \not\leqslant b$. Hence $u \wedge a > (u \wedge a) \wedge b$ and so $(u \wedge a) \vee (a \wedge b) > a \wedge b$. Thus there



exist $c, d \in L$ with

$$(u \wedge a) \vee (a \wedge b) \geqslant c \succ d \geqslant a \wedge b.$$

The isomorphism between $(u \wedge a) \vee (a \wedge b)/a \wedge b$ and $u \wedge a/(u \wedge a) \wedge b$ gives $u \wedge c \succ u \wedge d$ and hence $u \wedge c \not\leqslant d$, that is $u \notin \langle c, d \rangle$. Since $c \leqslant a$ and $a \wedge b \leqslant d$ we have

$$\langle a, b \rangle = \langle a, a \wedge b \rangle \subseteq \langle c, d \rangle.$$

Thus $\langle c, d \rangle$ is a prime annihilator which contains $\langle a, b \rangle$ but not u. Consequently $\langle a, b \rangle$ is an intersection of prime annihilators. ■

Since (dually) algebraic lattices are weakly atomic, the first corollary is immediate.

COROLLARY 5. *A lattice which is algebraic or dually algebraic (in particular a lattice satisfying A.C.C. or D.C.C.) is modular if and only if it satisfies the Prime–Annihilator Condition.* ■

Since ideal lattices are algebraic and preserve modularity, by transferring to the ideal lattice we can extend our characterization to arbitrary lattices.

COROLLARY 6. *A lattice is modular if and only if its lattice of ideals satisfies the Prime–Annihilator Condition.* ■

Since L is distributive if and only if every annihilator is an ideal, our final corollary is immediate.

COROLLARY 7. *A modular weakly atomic lattice L is distributive if and only if every prime annihilator in L is an ideal.* ∎

We close by showing that in finite distributive lattices prime annihilators and prime ideals coincide.

LEMMA 8. (i) *If L is distributive then every prime annihilator in L is a prime ideal.*

(ii) *If L is modular and* $m \in L$ *is completely meet irreducible, then the principal ideal* $\downarrow m$ *generated by m is a prime annihilator. Indeed* $\downarrow m = \langle m^*, m \rangle$ *where* m^* *is the unique upper cover of m.*

(iii) *If L is a finite distributive lattice then every prime ideal is a prime annihilator and conversely.*

Proof. The remarks preceding the definition of prime annihilator establish (i). Let L be modular and let $m \in L$ be completely meet irreducible with unique upper cover m^*. That $\downarrow m \subseteq \langle m^*, m \rangle$ is trivial. Suppose that $x \in \langle m^*, m \rangle$ with $x \nleq m$. Then $x \wedge m^* \leq m$ and since $x \vee m > m$ we have $x \vee m \geq m^*$, whence $x \vee m > m^*$ as $x \wedge m^* \neq x$. Hence $\{m^*, m, x\}$ generates an N_5-sublattice, a contradiction. This establishes (ii) and (iii) follows. ∎

REFERENCES

[1] R. BEAZER, *Hierarchies of distributive lattices satisfying annihilator conditions*, J. London Math. Soc. (2) *11* (1975), 216–222.
[2] W. H. CORNISH, *Normal lattices*, J. Austral. Math. Soc. *14* (1972), 200–215.
[3] W. H. CORNISH, *n-Normal lattices*, Proc. Amer. Math. Soc. *45* (1974), 48–54.
[4] W. H. CORNISH, *The multiplier extension of a distributive lattice*, J. Algebra *32* (1974), 339–355.
[5] B. A. DAVEY, *Some annihilator conditions on distributive lattices*, Algebra Universalis *4* (1974), 316–322.
[6] M. MANDELKER, *Relative annihilators in lattices*, Duke Math. J. *40* (1970), 377–386.

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