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Braids and their monotone clones

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Dedicated to Bjarni Jónsson on the occasion of his 70th birthday

Abstract. In this paper we investigate properties of the monotone clones of certain ordered sets known as *braids*. This class of ordered sets arose naturally in the study of how the clone of monotone functions on an ordered set could satisfy, or fail to satisfy, Mal'cev conditions. One version of the main result can be stated as follows. If B is a finite braid with reach $r(B) > 2$ (defined in the text), then the only idempotent order-preserving functions $f: B^n \to B$ are the n projections. It then follows, for example, that no algebra of monotone functions on a finite braid B with $r(B) > 2$ generates a congruence-modular variety.

The class of ordered sets known as *braids* was introduced in Davey [1] as a natural extension of the class of crowns. We begin with a brief résumé of the relevant definitions.

A subset $F = \{a_1, \ldots, a_k\}$ of an ordered set is called a *fence* (from a_1 to a_k) if $a_i < a_{i+1} > a_{i+2}$ or $a_i > a_{i+1} < a_{i+2}$ for all $i = 1, \ldots, k-2$ with no other comparabilities. If $a_1 \leq a_2 > a_3$ and $\{a_2, \ldots, a_k\}$ is a fence, then F is called an *up-fence.* A *down-fence* is defined dually. Define distance functions $d_{\nearrow}$, $d_{\searrow}$ and d on a connected ordered set $\mathbf{P}$ by declaring $d_{\nearrow}(a, b)$, $d_{\searrow}(a, b)$ and $d(a, b)$ to be respectively the number of edges in a minimum sized up-fence, down-fence or fence from a to b. Thus $d_{\nearrow}(a, b)$ and $d_{\searrow}(a, b)$ differ by at most one and $d(a, b) = \min\{d_{\nearrow}(a, b), d_{\searrow}(a, b)\}$. We regard the one-element and two-element chains as degenerate fences; hence $d(a, a) = 0$, and $d(a, b) = 1$ if and only if a and b are comparable. Define $r(\mathbf{P})$, the *reach* of $\mathbf{P}$, to be the supremum of the distances $d(a, b)$ as a and b vary over $\mathbf{P}$. A ordered set $\mathbf{B}$ is called a *braid* if it has at least three elements, is connected, has finite reach, and for all $a \in B$ there exists a unique element $a' \in B$ with $d(a, a') = r(\mathbf{B})$. We refer to a' as the *antipode* of a and for all $a, b \in \mathrm{B}$ a minimum sized fence from a to b is called a *geodesic* from a to b.

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For $n \geq 2$, an *n-crown* is an ordered set $\mathbf{C}_n$ with $2n$ elements $c_1, \ldots, c_{2n}$ such that

$$c_1 < c_2 > c_3 < \cdots > c_{2n-1} < c_{2n} > c_1$$

with no other comparabilities. Clearly C_n is a braid of reach n.

If every element of a braid **B** dominates a minimal element and is dominated by a maximal element, then these extremal elements play an important role in **B**. For starters, in a geodesic every point other than the end-points may be assumed to be extremal. Moreover, the antipode of an extremal element is itself extremal and hence the extremal elements of **B** themselves form a braid of the same reach as **B**. It is easily seen that if $a \in B$ is not extremal then $d(a, a') = d_{\nearrow}(a, a') = d_{\searrow}(a, a')$. [Let $a < u$ in B; then following the edge $a < u$ by a geodesic from u to a' yields an up-fence from a to a' of length $r(\mathbf{P})$, whence $d_{\nearrow}(a, a') = r(\mathbf{P}) = d(a, a')$.] Thus $a \in B$ is extremal if and only if $d_{\nearrow}(a, a') \neq d_{\searrow}(a, a')$.

In Section 1 we present a range of examples of braids. Reach-two braids are easily described: they are just the linear sums of copies of the two-element antichain. We give a complete description of reach-three braids which have enough extremal elements: the extremal elements form an almost-complete-bipartite order and reach three guarantees a particularly tight fit between the extremal elements and the non-extremal elements. If **L** is an atomless Boolean lattice, then $B = L \backslash \{0, 1\}$ provides an example of a reach-three braid with no extremal elements at all. We cannot hope for a reasonable characterization of braids of reach greater than three. Several constructions are given for headband-like braids of height one or two with arbitrary reach. (It was the geometry of these examples which led the first author to the name braids for this class of ordered sets.) An example of a braid of arbitrary height and reach is provided by the "*cyclone fence*". Section 1 closes with some examples from geometry. The ordered set of non-empty, proper flats of the I-cube is dually order-isomorphic to the ordered set of 2-valued, non-empty partial maps on I and is a braid of reach four. Some of the incidence orders on the Platonic solids provide examples of braids: for example the vertex-face incidence order on a cube is a braid of reach four while the vertex-edge incidence order of a dodecahedron is a braid of reach ten.

An algebra $\mathbf{A} = (P; F)$ is said to be *monotone with respect to* $\mathbf{P} = (P; \leq)$ if every operation $f \in F$ is order-preserving. One of the main results of [1] states that if **A** is monotone with respect to a braid of height one, then the variety generated by **A** is not congruence-modular. This was proved by showing that on a braid of height one it is impossible to find idempotent order-preserving, 3-ary maps satisfying the Gumm identities for congruence-modularity. In Sections 2 and 3 we prove the strongest possible extension of this result by showing that for "almost all" braids **B** and for each $n \geq 2$ the only idempotent, order-preserving maps $f: B^n \to B$ are the n projections. In order to explain our results more fully, we require some definitions.

For $n \geq 2$, a map $f: P^n \to P$ is *idempotent* if $f(a, \ldots, a) = a$ for all $a \in P$ and is *strongly idempotent* (or *conservative*) if $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for all $a_1, \ldots, a_n \in P$. We say that the ordered set **P** is *n-taut* if every idempotent, order-preserving map $f: P^n \to P$ is strongly idempotent, and P is *taut* if it is *n-taut* for all $n \geq 2$. (In [1], the word *tight* was used instead of *taut*. We have changed the adjective here to avoid any possible confusion with the concept of a *tight lattice* which occurs in tame congruence theory.) We refer to **P** as *n-idempotent trivial* if the only idempotent, order-preserving maps $f: P^n \to P$ are the projections, and **P** is *idempotent trivial* if it is n-idempotent trivial for all $n \geq 2$. In clone-theoretic terms, **P** is idempotent trivial if the monotone clone of **P** and the idempotent clone on P meet at 0 in the lattice of clones on P. Clearly idempotent triviality implies tautness.

A map $f: P^n \to P$ is called *essentially unary* if there exists i and there exists a map $\sigma: P \to P$ such that $f(\bar{a}) = \sigma(a_i)$ for all $\bar{a} \in P^n$. The *Słupecki clone* on P is defined to be the set of all finitary maps $f: P^n \to P$ for $n \geq 1$ such that f is either non-surjective or essentially unary. We say that an ordered set **P** is *Słupecki* if the monotone clone of **P** is contained in the Słupecki clone on P. It is easily seen that if **P** is Słupecki, then **P** is idempotent trivial.

The interest in ordered sets which are taut, idempotent trivial or Słupecki is (at least) two-fold. Firstly, many important Mal'cev conditions (such as congruence modularity, congruence distributivity, and congruence permutability) are given by the existance of idempotent term functions satisfying certain identities. Hence, if **P** is idempotent trivial or Słupecki and $\mathbf{A} = (P; F)$ is monotone with respect to **P**, then the variety of algebras generated by **A** cannot satisfy such a Mal'cev condition. Secondly, while it is known that there are only finitely many maximal clones on a finite set **P**, it is in general not easy to decide which maximal clones on P contain the monotone clone of **P**. Since the Słupecki clone is maximal, it is of interest to know that **P** is Słupecki.

Section 2 is devoted to tautness and idempotent triviality. The main result of this section states that every braid of reach greater than 2 in which there are enough extremal elements is idempotent trivial. The proof depends upon two lemmas. The first states that every 2-taut braid is 2-idempotent trivial and the second states that every 2-idempotent trivial ordered set with at least three elements is idempotent trivial. The second lemma is a consequence of a purely clone-threoretic lemma which is of independent interest: if C is a clone on A which contain the constant maps and is 2-idempotent trivial but not idempotent trivial, then there exists binary operation $+$ in C such that $(A; +)$ is an abelian group of exponent 2; in particular, if A is finite, then $|A| = 2^n$ for some $n \geq 1$.

In Section 3 we show that the main result of Section 2 can be strengthened if we restrict our attention to finite braids. An immediate corollary of the main result of this section is the fact that every finite braid of reach greater than 2 is Słupecki.

1. Example of braids

(A) Reach two. Let C be a chain and let $\mathbf{T}_\mathbf{C}$ be the linear sum over $\mathbf{C}$ of copies of the two-element antichain. Thus the underlying set of $\mathbf{T}_\mathbf{C}$ is $\{0, 1\} \times C$ with $(\alpha, u) < (\beta, v)$ if and only if $u < v$ in $\mathbf{C}$. Ordered sets of the form $\mathbf{T}_\mathbf{C}$ are referred to as *towers*. If $\mathbf{C}$ is an n-element chain for some finite n, then we write $\mathbf{T}_n$ instead of $\mathbf{T}_\mathbf{C}$. (See Figure 1.) Clearly $\mathbf{T}_\mathbf{C}$ is a braid of reach 2 provided $|C| \geq 2$, with the antipode of $(0, u)$ being $(1, u)$ and vice versa.

PROPOSITION 1.1. *A braid has reach* 2 *if and only if it is of the form* $\mathbf{T}_\mathbf{C}$ *for some chain* $\mathbf{C}$ *with* $|C| \geq 2$.

Proof. Let $\mathbf{B}$ be a braid of reach 2. The uniqueness of the antipode a' of a guarantees that a' is the unique element of B which is non-comparable with a. Thus for all $u \in B\backslash\{a, a'\}$ we have either $u < a, a'$ or $u > a, a'$. Let $\mathbf{C}$ be a maximal chain in $\mathbf{B}$ and define $\varphi\colon T_C \to B$ by $\varphi((0, u)) = u$ and $\varphi((1, u)) = u'$. Then φ is an order-embedding of $\mathbf{T}_\mathbf{C}$ into $\mathbf{B}$ and the maximality of C guarantees that φ is surjective. □

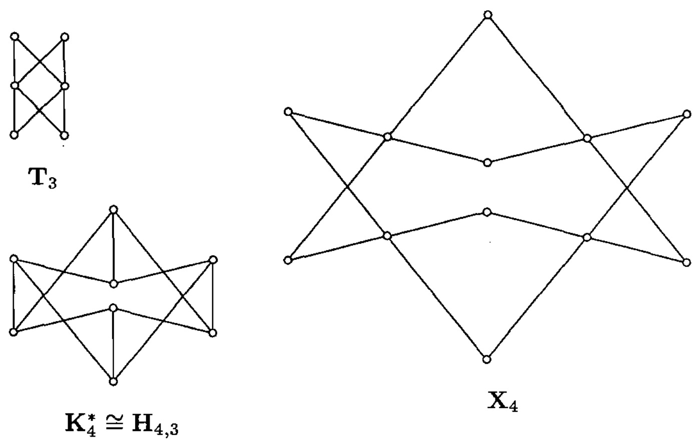


Figure 1

(B) Reach three. Let S be a non-empty set and let $\mathbf{K}_S^*$ be the almost-complete-bipartite ordered set defined on $S \times \{0, 1\}$ by:

$$(s, \alpha) < (t, \beta) \quad \Leftrightarrow \quad \alpha = 0, \beta = 1 \quad \text{and} \quad s \neq t.$$

If $|S| = n$ is finite we write $\mathbf{K}_n^*$ instead of $\mathbf{K}_S^*$. (See Figure 1.) Note that $\mathbf{K}_S^*$ is a braid of reach 3 whenever $|S| \geq 3$; the points $(s, 0)$ and $(s, 1)$ are antipodal.

PROPOSITION 1.2. *Let* $\mathbf{B}$ *be a braid of height* 1. *Then the reach of* $\mathbf{B}$ *is* 3 *if and only if* $\mathbf{B}$ *is of the form* $\mathbf{K}_S^*$ *for some set* S *with* $|S| \geq 3$.

Proof. Let B be a height-one braid of reach 3. Clearly the antipode map is a bijection between the minimal elements and the maximal elements of $\mathbf{B}$. Let S be the set of minimal elements of $\mathbf{B}$. Thus the map $\varphi: K_S^* \to B$, defined by $\varphi((u, 0)) = u$ and $\varphi((u, 1)) = u'$, is a bijection. It remains to see that if u is a minimal element of $\mathbf{B}$ and v is a maximal element of $\mathbf{B}$, then $u < v$ if and only if $v \neq u'$. Since $r(\mathbf{B}) = 3$, this is clear. □

We shall say that an ordered set $\mathbf{P}$ *has enough minimals* if every element dominates a minimal in $\mathbf{P}$. Having enough maximals is defined dually. If $\mathbf{P}$ has enough minimals and enough maximals, then we say that $\mathbf{P}$ *has enough extremals.* Denote the set of minimal elements of $\mathbf{P}$ by D and the set of maximal elements of $\mathbf{P}$ by U, and let ext $\mathbf{P}$ be $U \cup D$ with the order induced from $\mathbf{P}$. Let $\mathbf{B}$ be a braid. As noted earlier, ext $\mathbf{B}$ is closed under the antipode map. If $\mathbf{B}$ has enough extremals, then the distance between two elements in ext $\mathbf{B}$ is the same as in $\mathbf{B}$. Thus ext $\mathbf{B}$ is also a braid and $r(\text{ext } \mathbf{B}) = r(\mathbf{B})$. If the reach of $\mathbf{B}$ is 3, then Proposition 1.2 guarantees that ext $\mathbf{B} \simeq \mathbf{K}_S^*$ for some S with $|S| \geq 3$. To simplify the notation in the following characterization of braids of reach 3, we assume that ext $\mathbf{B} = \mathbf{K}_S^*$.

PROPOSITION 1.3. *Let* $\mathbf{B}$ *be a braid of reach* 3 *which has enough extremals, and assume that* ext $\mathbf{B} = \mathbf{K}_S^*$. *For each* $a \in B \backslash (U \cup D)$ *define* $U_a = \{s \in S \mid a \leq (s, 1)\}$ *and* $D_a = \{s \in S \mid a \geq (s, 0)\}$. *Then for all* $a, b \in B \backslash (U \cup D)$, *we have*

(1) *if* $a \leq b$, *then* $U_a \supseteq U_b$ *and* $D_a \subseteq D_b$,

(2) $U_a \cap D_a = \varnothing$,

(3) $|U_a|, |D_a| \geq 2$,

(4) *if* $b \neq a'$, *then* $U_a \cap U_b \neq \varnothing$ *or* $D_a \cap D_b \neq \varnothing$,

(5) $U_a \cap U_{a'} = \varnothing$ *and* $D_a \cap D_{a'} = \varnothing$.

Conversely, let $\mathbf{M}$ *be an ordered set with a self-map* $a \mapsto a' \in M$ *and set-valued mappings* $a \mapsto U_a \subseteq S$ *and* $a \mapsto D_a \subseteq S$ *which satisfy* (1)–(5) *for all* $a, b \in M$. *Then* $\mathbf{M} \cup \mathbf{K}_S^*$, *with the order between elements of* $\mathbf{M}$ *and elements of* $\mathbf{K}_S^*$ *defined by* $(s, 0) < a < (t, 1)$ *for all* $s \in D_a$ *and all* $t \in U_a$, *is a braid of reach* 3.

Proof. Let $\mathbf{B}$ be a braid of reach 3. It is trivial that (1) is true. Clearly (2) holds since otherwise $(s, 0) \leq a \leq (s, 1)$. If $U_a = \{s\}$, then $d(a, (s, 0)) > 2$, by (2), which is impossible since $a' \neq (s, 0)$. Thus (3) holds. Together, (4) and (5) express the fact that $d(a, a') = 3$ for all $a \in M$: indeed, (4) says that $d(a, b) \leq 2$ for all $a, b \in M$ with $b \neq a'$, while (5) says that $d(a, a') \geq 3$ for all $a \in M$.

For the converse, it is easily seen that (1) and (2) guarantee that the relation $<$ defined on $\mathbf{M} \cup \mathbf{K}_S^*$ is transitive and so is a (strict) order. Condition (3) implies that $d(a, x) \leq 2$ for all $a \in M$ and all $x \in U \cup D$, while (4) yields $d(a, b) \leq 2$ for all $a, b \in M$ with $b \neq a'$. Finally, (5) gives $d(a, a') = 3$ for all $a \in M$. Thus $\mathbf{M} \cup \mathbf{K}_S^*$ is a braid of reach 3. □

This proposition allows us to construct reach-three braids with various degrees of nastiness. The antipode map in the left-hand example in Figure 2 sends the comparable elements a and b to the non-comparable elements a' and b'. In the right-hand example in Figure 2, there is a length-two maximal chain through $3'$ while every maximal chain through 3 has length one.

PROPOSITION 1.4. *Let* $\mathbf{L}$ *be a bounded lattice and assume that* $B = L \backslash \{0, 1\}$ *is connected.*

(i) *If* $\mathbf{B}$ *is a braid, then* $r(\mathbf{B}) \geq 3$ *and for all* $a \in B$ *the antipode* a' *is a complement of* a *in* $\mathbf{L}$.
(ii) $\mathbf{B}$ *is a braid of reach* 3 *if and only if* $\mathbf{L}$ *is uniquely complemented.*
(iii) *If* $\mathbf{L}$ *is modular and* a^* *is a complement of* $a \in B$, *then* $d(a, a^*) = 3$ *in* $\mathbf{B}$.
(iv) *If* $\mathbf{L}$ *is a Boolean lattice, then* $\mathbf{B}$ *is a braid of reach* 3, *and conversely, if* $\mathbf{L}$ *is modular and* $\mathbf{B}$ *is a braid, then* $\mathbf{L}$ *is Boolean.*

Proof. (i) If $\mathbf{B}$ is a braid and $r(\mathbf{B}) = 2$, then $\mathbf{L}$ is not a lattice, by Proposition 1.1. The antipode of a must be a complement of a in $\mathbf{L}$ since $d(a, b) \leq 2$ whenever b is not a complement of a in $\mathbf{L}$.

(ii) If $\mathbf{B}$ is a braid, then $\mathbf{L}$ is complemented by (i). Let a^* be a complement of a; then $d(a, a^*) > 2$ and hence $d(a, a^*) = 3$ as $r(\mathbf{B}) = 3$. Consequently a^* is the antipode of a and thus $\mathbf{L}$ is uniquely complemented. Assume that $\mathbf{L}$ is uniquely complemented. If $b \neq a^*$ (the complement of a), then either $a \wedge b \neq 0$ or $a \vee b \neq 1$

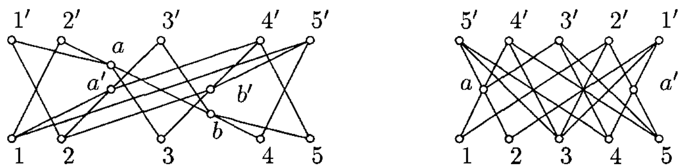


Figure 2. Boolean lattices are a source of reach-three braids.

and hence $d(a, b) \leq 2$. As **B** is connected, there exists $c \in B$ with $c < a$ or $c > a$, say the former. Since $c \wedge a^* = 0$ and since **L** is uniquely complemented, $c \vee a^* \neq 1$ and hence the geodesic $a > c < c \vee a^* > a^*$ demonstrates that $d(a, a^*) = 3$, whence **B** is a braid of reach 3.

(iii) Assume that **L** is modular and again let a^* be a complement of $a \in B$. As before, we have $d(a, a^*) > 2$ immediately, and as **B** is connected, there exists $c \in B$ with $c < a$ or $c > a$, say the former. As **L** is modular, $c \vee a^* \neq 1$ (otherwise $\{0, c, a, a^*, 1\}$ is a pentagon) and hence $a > c < c \vee a^* > a^*$ gives $d(a, a^*) \leq 3$.

(iv) Assume that **L** is Boolean; then B is a braid of reach 3 by (ii). The converse follows by (i), (iii) and (ii) (in that order) since a uniquely complemented modular lattice is Boolean.

(C) Headbands. While a characterization, like Proposition 1.3, of braids of reach greater than 3 is beyond reach (even in the height-one case), we do have a rich array of examples. Let $\mathbf{X}_n$ be obtained by gluing together n copies of the five-element **X** into a circular band as in Figure 3. It is easily seen that $\mathbf{X}_n$ is a braid just when n is even and that $r(\mathbf{X}_{2n}) = n + 1$. Note that $\mathbf{X}_2 \simeq \mathbf{T}_3$.

There is a natural construction for a class of infinite braids of height 1 and reach n. Let **C** be a dense chain (that is, for all $a < b$ in **C** there exists $c \in C$ with $a < c < b$) with at most one bound. Define

$$H_{n,C} = \{0, 1, \ldots, n-1\} \times C \times \{0, 1\}$$

with $(k, u, \alpha) < (\ell, v, \beta)$ if and only if

(i) $\alpha = 0$ and $\beta = 1$, and
(ii) $[\ell = k$ and $u \leq v]$ or $[\ell = k + 1 \pmod n$ and $u > v]$.

Think of $\{k\} \times C \times \{0, 1\}$ as the kth copy of an elastic strip, $C \times \{0, 1\}$, of unit length. (The two copies of **C** lie horizontally with **C** being 1 unit in length.) These n strips are joined to form a circular elastic headband with a point $a = (k, u, 0)$ on the base joined to a point $b = (\ell, v, 1)$ on the top just when b lies directly above a or to the right of a (when viewed from a and facing the centre of the circle) and the

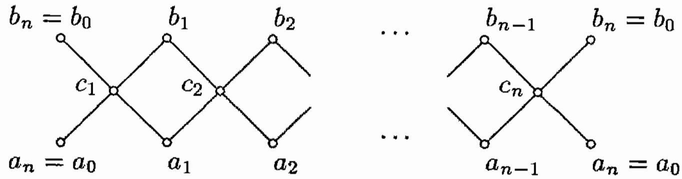


Figure 3

horizontal distance around the circumference from a to b is less than 1. Let $\mathbf{H}^*_{n,\mathbf{C}}$ be obtained from $\mathbf{H}_{n,\mathbf{C}}$ by omitting the relations $(k, u, 0) < (k, u, 1)$. We leave it to the reader to check that

(a) $\mathbf{H}_{1,\mathbf{C}}$ is not a braid, while $\mathbf{H}^*_{1,\mathbf{C}} \simeq \mathbf{K}^*_C$ and so is a braid of reach 3,
(b) for all $n \geq 2$ both $\mathbf{H}_{n,\mathbf{C}}$ and $\mathbf{H}^*_{n,\mathbf{C}}$ are braids of reach $n+2$.

It will quickly become apparent why we require $\mathbf{C}$ to be dense and why $\mathbf{C}$ may have at most one bound. There is a discrete version of this construction. Rather than viewing these finite headbands as made up from strips we regard them simply as two copies of $\{0, 1, \dots, m-1\}$ joined in a uniform way to form a band. More precisely, let $m, k \geq 1$ and define

$$H_{m,k} = \{0, 1, \dots, m-1\} \times \{0, 1\}$$

with $(s, \alpha) < (t, \beta)$ if and only if

(i) $\alpha = 0$ and $\beta = 1$, and
(ii) $t - s \equiv 0, 1, \dots, k-1 \pmod{m}$.

Here a point in the bottom circle is connected to the point above it and to the next $k-1$ points to the right (looking inwards) in the top circle. We leave checking the following claims as straightforward exercises for the reader.

(a) $\mathbf{H}_{m,k}$ is a braid if and only if $m \geq 2, k \geq 2$ and there exists $n \geq 0$ with $m = n(k-1) + 2$ in which case $r(\mathbf{H}_{m,k}) = n + 2$.
(b) For $m \geq 2$ we have $\mathbf{H}_{m,2} \simeq \mathbf{C}_m$.
(c) For $m \geq 1$ we have $\mathbf{H}_{m,m-1} \simeq \mathbf{K}^*_m$.

(D) The cyclone fence. An example of a braid of arbitrary height $h \geq 1$ and reach $r \geq 3$ is provided by the *cyclone fence* $\mathbf{F}_{h,r}$. Let $w = h(r-2) + 1$; then $F_{h,r} = L \cup U \cup D \cup R$, where

$$L = \{(0, i) \mid i = 0, \dots, h\}, \qquad R = \{(w, i) \mid i = 0, \dots, h\}$$
$$U = \{(j, h) \mid j = 0, \dots, w\}, \qquad D = \{(j, 0) \mid j = 0, \dots, w\}.$$

Pictorially, L and R are maximal chains at the left-hand and right-hand ends respectively, while, as before, U and D denote respectively the set of maximal and the set of minimal elements of $\mathbf{F}_{h,r}$. Note that the four corners, $(0, 0)$, $(0, h)$, $(w, 0)$,

(w, h), have been listed twice; thus $|F_{h,r}| = 2(h(r-1)+1)$. The covering relations are as follows:

In L:	$(0, i) -\!\!\!< (0, i+1)$	for $i = 0, \ldots, h-1$;
In R:	$(w, i) -\!\!\!< (w, i+1)$	for $i = 0, \ldots, h-1$;
From L to U:	$(0, i) -\!\!\!< (h-i, h)$	for $i = 0, \ldots, h-1$;
From R to U:	$(w, i) -\!\!\!< (w-h+i, h)$	for $i = 0, \ldots, h-1$;
From D to L:	$(i, 0) -\!\!\!< (0, i)$	for $i = 1, \ldots, h$;
From D to R:	$(w-i, 0) -\!\!\!< (w, i)$	for $j = 1, \ldots, h$;
From D to U:	$(j, 0) -\!\!\!< (j+h, h)$	for $j = 1, \ldots, w-h-1$;
From D to U:	$(j, 0) -\!\!\!< (j-h, h)$	for $j = h+1, \ldots, w-1$.

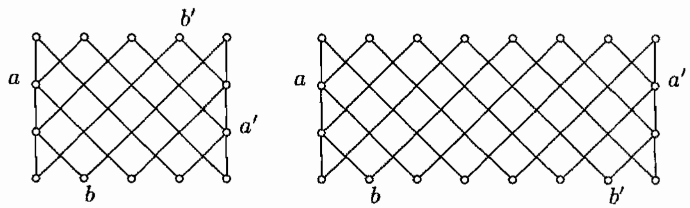
Figure 4 shows $\mathbf{F}_{3,3}$ and $\mathbf{F}_{3,4}$. Note that the width of $F_{h,r}$ is $w + 1 = h(r-2) + 2$. The antipode map is given by:

$$(x, y)' = \begin{cases} (w-x, h-y) & \text{if } r \text{ is odd (rotation through } \pi) \\ (w-x, y) & \text{if } r \text{ is even (reflection in the vertical axis).} \end{cases}$$

Again, we leave the details to the reader.

(E) A domain of partial maps. Let $P_I = (I \multimap 2)$ be the set of all partial maps from i to $\{0, 1\}$, that is, maps from J into $\{0, 1\}$ where $\varnothing \subseteq J \subseteq I$. The order on $\mathbf{P}_I$ is the natural one: $\sigma \leq \tau$ if and only if $\operatorname{dom} \sigma \leq \operatorname{dom} \tau$ and $\sigma(i) = \tau(i)$ for all $i \in \operatorname{dom} \sigma$. The ordered set $\mathbf{P}_I$ is a domain (or algebraic semilattice) and is a sub-domain of $(I \multimap \mathbf{N})$, the domain beloved of theoretical computer scientists (see [2]). Let $B_I = P_I \backslash \{\varnothing\}$. If n is finite, the height of $\mathbf{B}_n$ is $n - 1$. Provided $|I| \geq 2$, the ordered set $\mathbf{B}_I$ is a braid of reach 4. Note that $\mathbf{B}_2 \simeq \mathbf{C}_4$. Let $\varnothing \neq J \subseteq I$; then the antipode of $\sigma: J \to \{0, 1\}$ is $\sigma': J \to \{0, 1\}$ where $\sigma'(j) = 1 - \sigma(j)$ for all $j \in J$.

The braid $\mathbf{B}_I$ has a geometric interpretation. Let $\mathbf{F}_I$ be the lattice of flats of the I-cube, $\{0, 1\}^I$, with the top and bottom removed Then $\mathbf{B}_I$ is dually order-



$\mathbf{F}_{3,3}$ $\mathbf{F}_{3,4}$

Figure 4

isomorphic to $\mathbf{F}_I$ as is witnessed by the map $f\colon B_I \to F_I$ given by

$$f(\sigma) := \{\tau \in \{0, 1\}^I | \tau \restriction \operatorname{dom} \sigma = \sigma\}.$$

(F) The Platonic solids. We conclude our braid parade with some examples from geometry. Consider the ordered set of height 1 consisting of the vertices and faces of a cube ordered by incidence; this yields a braid of reach 4. The same construction using the vertices and edges of a cube gives a braid of reach 6. In both cases antipodes are obtained by taking the opposite vertex, edge or face. Note that, unlike our previous examples, these ordered sets satisfy $|U| \neq |D|$. The edge-faces ordered set for a cube is not a braid.

It is natural to try these constructs for the other regular solids. Since the octahedron is dual to the cube it produces the same ordered sets. For a tetrahedron, only the vertices-faces construction gives a braid, namely $\mathbf{K}_4^*$. While the edges-faces ordered set of a dodecahedron is not a braid, the other two are: vertices-faces gives a braid of reach 6, and vertices-edges give a braid of reach 10. The icosahedron is dual to the dodecahedron and so yields nothing new.

2. Tautness and idempotent triviality

Tautness imposes quite severe restrictions on the geometry of an ordered set. Let $a, b \in P$; we say that a is a *strong lower cover* of b if a is covered by b and $u < b$ implies $u \leq a$ for all $u \in P$. If a is a strong lower cover of b, then it is the unique lower cover of b – in ordered sets satisfying the ascending chain condition this characterizes strong lower covers. A maximal element which has a strong lower cover is called an *up-dangle*. Strong upper covers and down-dangles are defined dually. It is proved in [1] that a 2-taut ordered set $\mathbf{P}$ with at least 3 elements has neither strong upper covers nor strong lower covers; in particular, $\mathbf{P}$ has no dangles. This is strengthened in (iii) of the proposition below. For $a \leq b$ in $\mathbf{P}$, define $\downarrow b = \{x \in P \mid x \leq b\}$ and $[a, b] = \{x \in P \mid a \leq x \leq b\}$.

PROPOSITION 2.1. *Let* $\mathbf{P}$ *be 2-taut with* $|P| > 2$.

(i) $\mathbf{P}$ *is connected.*

(ii) *If* $a < b$ *in* $\mathbf{P}$ *and* a *is a node in* $\downarrow b$ *(that is,* $\downarrow b = \downarrow a \cup [a, b]$*), then* a *is minimal and hence* $\downarrow b = [a, b]$.

(iii) *If* $a < b$ *in* $\mathbf{P}$, *then there exist* $c, d \in P$ with

$$(c < b \;\&\; a \parallel c) \quad \text{and} \quad (a < d \;\&\; b \parallel d).$$

Proof. (i) If $\mathbf{P}$ is an antichain with $|P| > 2$, then clearly $\mathbf{P}$ is not 2-taut. Otherwise there exists a component of $\mathbf{P}$ with at least two elements, $a < b$. Define a binary operation on P by

$$xy = \begin{cases} x & \text{if } x \text{ and } y \text{ are in the same component,} \\ a & \text{otherwise.} \end{cases}$$

Clearly this operation is idempotent, monotone and satisfies $bc = a$ for any c not in the same component as b. Since $\mathbf{P}$ is 2-taut it follows that $\mathbf{P}$ has only one component.

(ii) Assume that $\downarrow b = \downarrow a \cup [a, b]$ and define a binary operation on P by

$$xy = \begin{cases} a & \text{if } x \in [a, b] \text{ and } y \leq a, \\ x & \text{otherwise.} \end{cases}$$

This is clearly idempotent, and, because of the shape of $\downarrow b$ is easily seen to be monotone in each coordinate and therefore monotone. But $bc = a$ for any $c < a$. Thus $\downarrow b = [a, b]$ as $\mathbf{P}$ is 2-taut.

(iii) Let $a < b$ and suppose that every c satisfying $c < b$ is comparable with a. Thus a is a node in $\downarrow b$ and hence a is minimal and $\downarrow b = [a, b]$ by (ii). Suppose that $a \leq c < b$. Define

$$xy = \begin{cases} x & \text{if } x \nleq b \text{ or } y \neq a, \\ c & \text{if } x \in [c, b] \text{ and } y = a, \\ a & \text{otherwise.} \end{cases}$$

This binary operation is trivially idempotent and easily shown to be monotone. As $\mathbf{P}$ is 2-taut and $ba = c$, it follows that $a = c$ and hence a is a lower cover of b. Hence a is a strong lower cover of b (since $\downarrow b = [a, b]$), a contradiction. □

Braids of reach greater than 2 which have enough extremals satisfy the condition stated in 2.1(iii). This follows from the set-representation given in the following proposition. As before, D denotes the set of minimal elements and U denotes the set of maximal elements of $\mathbf{B}$.

PROPOSITION 2.2. *Let* $\mathbf{B}$ *be a braid which has enough extremals and assume that* $r(\mathbf{B}) > 2$.

(i) *For all* $a, b \in B$,

$$\downarrow b \cap D \subseteq \downarrow a \cap D \ \& \ \uparrow b \cap U \subseteq \uparrow a \cap U \quad \Rightarrow \quad a = b.$$

(ii) *The map* $a \to (\downarrow a \cap D, \uparrow a \cap U)$ *is a one-to-one, monotone map op* $\mathbf{P}$ *into* $\mathscr{P}(D) \times \mathscr{P}(U)^{\partial}$.

Proof. Assume that $a \neq b$. If $b \leq a'$, then choose a minimal, d, below b. As $r(\mathbf{B}) > 2$, we have $d \nleq a$ and thus $\downarrow b \cap D \nsubseteq \downarrow a \cap D$. Similarly, $b \geq a'$ implies that $\uparrow b \cap U \nsubseteq \uparrow a \cap U$. If $b \parallel a'$, then without loss of generality, a geodesic from b to a' either begins up via a maximal or down via a minimal. (This again uses the fact that $r(\mathbf{B}) > 2$.) The former gives $\uparrow b \cap U \nsubseteq \uparrow a \cap U$ while the latter gives $\downarrow b \cap D \nsubseteq \downarrow a \cap D$. This proves (i), and (ii) follows at once. □

For some braids the weak representation given in 2.2(ii) is an order-embedding. Moreover, it is sometimes possible to represent a braid via the minimals (or maximals) alone.

PROPOSITION 2.3. (i) *Let* $\mathbf{B}$ *be a braid which has enough minimals and assume that* $r(\mathbf{B}) > 2$. *Then the minimal elements of* $\mathbf{B}$ *separate comparables, i.e. if* $a < b$ *in* $\mathbf{B}$, *then there exists* $d \in D$ *with* $d < b$ *and* $d \nleq a$. *Hence*

$$a \leq b \;\&\; \downarrow b \cap D \subseteq \downarrow a \cap D \quad \Rightarrow \quad a = b.$$

(ii) *Let* $\mathbf{B}$ *be a braid of height* 1 *and assume that* $\mathbf{B} \ncong \mathbf{T}_2$. *Then the map* $a \to \downarrow a \cap D$ *is an order-embedding of* $\mathbf{B}$ into $(\mathscr{P}(D), \subseteq)$.

Proof. (i) This requires nothing more than a simple modification of the proof of 2.2(i). (If $\mathbf{B}$ has enough extremals, then (i) is a corollary of 2.2(i).)

(ii) Assume that $\downarrow a \cap D \subseteq \downarrow b \cap D$ for some $a, b \in B$. Let a be maximal. If $a = b'$, then a geodesic $a = b' > z_1 < \cdots b$ from a to b yields a minimal, z_1 which is below a but not below b (as $r(\mathbf{B}) > 2$). If $a \notin \{b, b'\}$, then a geodesic $a > z_1 < \cdots b'$ from a to b' yields a minimal, z_1, which is below a but not below b. Hence if a is maximal, it follows that $a = b$. If a is minimal, then $\{a\} = \downarrow a \cap D \subseteq \downarrow b \cap D$ implies $a \leq b$. Hence $a \to \downarrow a \cap D$ is an order-embedding of $\mathbf{B}$ into $(\mathscr{P}(D), \subseteq)$. □

Clearly the map $a \to \downarrow a \cap D$ is not even one-to-one on $\mathbf{T}_2$; and while it is one-to-one on the left-hand braid in Figure 2, it is not an order-embedding – consider the images of b' and $1'$.

It is proved in Proposition 17 of [1] that every braid of height one is taut. In fact, the same proof establishes the following sharper result.

PROPOSITION 2.4. *Let* $\mathbf{B}$ *be a braid, let* $n \geq 2$ *and let* $f: B^n \to B$ *be idempotent and monotone.*

(i) If $f(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, *then there exist* k, ℓ *such that*
  (a) $d(a_k, f(a_1, \ldots, a_n)') = d(a_\ell, f(a_1, \ldots, a_n)') = r(\mathbf{B}) - 1$,
  (b) *every geodesic from* a_k *to* $f(a_1, \ldots, a_n)'$ *starts up*,
  (c) *every geodesic from* a_ℓ *to* $f(a_1, \ldots, a_n)'$ *starts down.*
(ii) *If* $f(a_1, \ldots, a_n) \in \operatorname{ext} B$, *then* $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.
(iii) *If* $a_1, \ldots, a_n \in D$, *then* $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.

The following three lemmas lead us to the main result of this section.

LEMMA 2.5. *If a braid* $\mathbf{B}$ *is* 2-*taut, then it is* 2-*idempotent trivial.*

Proof. We establish three claims from which the result will follow. The idea behind the first two claims is to extend a local result to a global result by arguing along a geodesic. Assume that $\mathbf{B}$ is 2-taut and that $r(\mathbf{B}) = n$. Let $(x, y) \mapsto xy$ be a monotone and (strongly) idempotent binary operation on B.

CLAIM 1. *If* $b \in B$ *satisfies* $b'b = b'$ *then* $xb = x$ *for all* $x \in B$.

Subproof. Let $b'b = b'$ and let $x \in B$. Without loss of generality we have $x \notin \{b, b'\}$. Let $b' = z_0, z_1, \ldots, z_m = x$ be a geodesic from b' to x. Thus $1 \le m < n$. Since $d(b', b) = n$, we have $z_k \parallel b$ for $0 \le k < m$. Let $0 \le k < m$ and assume that $z_k b = z_k$. If $z_k > z_{k+1}$, then $z_k = z_k b \ge z_{k+1} b \in \{z_{k+1}, b\}$ and hence $z_{k+1} b = z_{k+1}$ as $z_{k+1} \parallel b$. Similarly, $z_k < z_{k+1}$ implies $z_{k+1} b = z_{k+1}$. Since $z_0 b = b'b = b' = z_0$, it follows that $xb = z_m b = z_m = x$.

CLAIM 2. *If there exists* $b \in B$ *such that* $b'b = b'$, *then* $y'y = y'$ *for all* $y \in B$.

Subproof. Assume that $b'b = b'$ and let $y \in B \backslash \{b\}$. By Claim 1 we have $y'b = y'$. First consider the case where $y \ne b'$. Let $b = z_0, z_1, \ldots, z_m = y$ be a geodesic from b to y. As above, we have $1 \le m < n$. Since $d(y, y') = n$, we have $z_k \parallel y'$ for $0 < k \le m$. Again we can induct along the geodesic to obtain $y'y = y'$. Thus we have proved

$$b'b = b' \quad \Rightarrow \quad y'y = y' \qquad \text{for all } y \ne b'. \tag{$*$}$$

Let $a \in B \backslash \{b, b'\}$. Since $b'b = b'$ by assumption, we have $a'a = a'$ as $a \ne b'$. Then we can apply $(*)$ again with b replaced by a to give $b''b' = b''$ as $b' \ne a'$. Hence $y'y = y'$ for all $y \in B$, as required.

CLAIM 3. *If there exists* $b \in B$ *such that* $b'b = b'$, *then* $xy = x$ *for all* $x, y \in B$.

Subproof. This is an immediate corollary of Claims 1 and 2.

We can now complete the proof of the lemma. Let $b \in B$. Since $\mathbf{B}$ is 2-taut, we have $b'b \in \{b, b'\}$. If $b'b = b'$, then Claim 3 gives $xy = x$ for all $x, y \in B$. If $b'b = b$, then $cc' = c'$ where $c = b'$, whence Claim 3 and symmetry yield $xy = y$ for all $x, y \in B$. □

Let $\mathbf{A} = (A; +)$ be an abelian group of exponent 2 and consider its clone of polynomial functions. The binary polynomials on $\mathbf{A}$ are of the form

$$p(x, y) = \alpha x + \beta y + a, \qquad \text{where } \alpha, \beta \in \{0, 1\} \quad \text{and} \quad a \in A.$$

Hence the only idempotent binary polynomials on $\mathbf{A}$ are the projections. However, for $n \geq 3$, there are nontrivial idempotent n-ary polynomials on $\mathbf{A}$: consider $p(x, y, z) = x + y + z$. Our next lemma, which applies to clones in general and not only to monotone clones, shows that the existence of such a group operation, $+$, is typical of algebras $\mathbf{A} = (A; F)$ whose clone of polynomial functions is 2-idempotent trivial but not idempotent trivial. The first author would like to thank Bernhard Ganter for his generous advice which led to the final form of this lemma.

LEMMA 2.6. *Let $C = (C_1, C_2, \ldots)$ be a clone on A which contains the constant maps. If C is 2-idempotent trivial but not idempotent trivial, then there is a binary operation $+$ in C_2 such that $(A; +)$ is an abelian group of exponent 2.*

In fact, there are idempotent non-projections in C_3; and if $f \in C_3$ is any such map, then

(a) $f(y, x, x) = f(x, y, x) = f(x, x, y) = y$, *for all* $x, y \in A$,

(b) $x + y = f(x, y, e)$ *defines such a group operation for each* $e \in A$.

In particular, if A is finite, then $|A| = 2^n$ for some $n \geq 1$.

Proof. Assume that C is 2-idempotent trivial but not idempotent trivial. Let $f \in C_n$ be idempotent but not a projection with n as small as possible. Thus f becomes a projection whenever any two variables are identified. We claim that $n = 3$.

Suppose that $n \geq 4$. Then there exists i such that for all $\bar{x}$in A^n and all distinct j and k, if $x_j = x_k$ then $f(\bar{x}) = x_i$. (This is a standard result which is simple to prove, for example as a corollary of [4, Lemma 2 on page 206].) For simplicity, we take $i = 1$. Since f is not a projection, there exists $\bar{a} \in A^n$ with $f(\bar{a}) \neq a_1$. Define $g(x, y, z) = f(x, y, z, a_4, a_n)$. Then $g \in C_3$ as C contains the constant maps. Clearly g is idempotent but is not a projection as $g(a_1, a_2, a_2) = a_1$ while $g(a_1, a_2, a_3) \neq a_1$. This contradicts the minimality of n.

Thus we have $n = 3$. Each of $f_1(x, y) = f(y, x, x)$, $f_2(x, y) = f(x, y, x)$ and $f_3(x, y) = f(x, x, y)$ is a projection. Suppose that at least one is the first projection, say $f_1(x, y) = x$. Let $a \in A$ and define $g_a(x, y) = f(a, x, y)$. Then $g_a \in C_2$ and is idempotent, whence g_a is a projection. If g_a is the first projection for some $a \in$ A, then

$$(\forall y) g_a(a, y) = a \quad \Rightarrow \quad (\forall y) f(a, a, y) = a \quad \Rightarrow \quad (\forall y) f_3(a, y) = a.$$

Since f_3 is a projection, we conclude that it must be the first projection. It now follows that g_a is the first projection for all $a \in A$; for if g_b were the second projection, we would have the contradiction

$$y = g_b(b, y) = f(b, b, y) = f_3(b, y) = b.$$

As g_a is the first projection for all $a \in A$, we have $f(a, x, y) = x$ for all $a, x, y \in A$, that is, f is a projection. This contradiction shows that none of f_1, f_2 and f_3 can be the first projection. Thus each is the second projection and consequently f is a minority function, that is,

$$f(y, x, x) = f(x, y, x) = f(x, x, y) = y \qquad \text{for all } x, y \in A.$$

Let $e \in A$ and define $x + y = f(x, y, e)$. Then trivially,

$$x + e = x = e + x \quad \text{and} \quad x + x = e$$

for all $x \in A$. Since $g(x, y) = x + (x + y)$ is idempotent, it is a projection. If $g(x, y) = x$, then $x = g(x, e) = x + (x + e) = e$, which is impossible. Thus $g(x, y) = y$. Hence

$$\text{(i) } x + (x + y) = y, \qquad \text{and similarly (ii) } x + (y + x) = y.$$

These two identities yield commutativity:

$$y + x \overset{\text{(i)}}{=} x + (x + (y + x)) \overset{\text{(ii)}}{=} x + y.$$

To prove associativity, let $a \in A$ and consider the binary map $h(x, y) = (x + a) + (x + (a + y))$. Since h is idempotent it is a projection, but cannot be the first projection since

$$\begin{aligned} h(x, y) = x \quad &\Rightarrow \quad (x + a) + (x + (a + y)) = x \\ &\Rightarrow \quad x + (a + y) = (x + a) + x \\ &\Rightarrow \quad x + (x + (a + y)) = x + (x + (a + x)) \\ &\Rightarrow \quad a + y = a + x \quad \Rightarrow \quad y = x. \end{aligned}$$

Thus $(x + a) + (x + (a + y)) = y$ and consequently

$$x + (a + y) = (x + a) + y.$$

Hence $(A; +)$ is an abelian group of exponent 2. The remainder of the claims now follow. □

LEMMA 2.7. *If an ordered set* **P** *with* $|P| > 2$ *is 2-idempotent trivial, then it is idempotent trivial.*

Proof. Suppose **P** is 2-idempotent trivial but not idempotent trivial. Then, by 2.6, there is a monotone map $f: P^3 \to P$ satisfying

$$f(y, x, x) = f(x, y, x) = f(x, x, y) = y$$

for all $x, y \in P$. Thus

$$x \leq y \quad \Rightarrow \quad y = f(y, x, x) \leq f(y, y, x) = x,$$

whence **P** is an antichain. This contradicts 2.1(i). □

PROPOSITION 2.8. *The towers* $\mathbf{T}_n$, $\mathbf{T}_{\mathbf{N}}$, $\mathbf{T}_{\mathbf{N}^\partial}$ *and* $\mathbf{T}_{\mathbf{Z}}$ *are idempotent trivial while* $\mathbf{T}_{\mathbf{C}}$ *is not 2-taut if* **C** *is a dense chain.*

Proof. Let **B** be one of the towers $\mathbf{T}_n$, $\mathbf{T}_{\mathbf{N}}$, $\mathbf{T}_{\mathbf{N}^\partial}$ or $\mathbf{T}_{\mathbf{Z}}$, and let $\cdot$ be an idempotent, monotone binary operation on B. Since the operation is idempotent and monotone, xy is a lower bound of every upper bound of $\{x, y\}$ and dually. Thus $bb' \in \{b, b'\}$ for all $b \in B$ and $ab \in [a, b]$ if $a < b$. Let $u, v \in B$ and suppose that $uv = b \notin \{u, v\}$. Without loss of generality we may assume that $u < b < v$ and consequently $u < b' < v$. Define $b_0 = b$. Since $ub'_0 \in [u, b'_0]$ and $ub'_0 \leq uv = b_0$, we have $u \leq b_1 = ub'_0 < b_0$. Either $b_1 = u$ or $u < b_1 < b_0$. In the later case, the argument just given yields $u \leq b_2 = ub'_1 < b_1$. Similarly, $b_2 = u$ or $u \leq b_3 = ub'_2 < b_2$, and so on. Since **B** is discrete there exists k such that $u < b_k$ and $u = ub'_k$. Thus $u = ub'_k \geq uu'$ and hence $u = uu'$ as $uu' \in \{u, u'\}$. Let u_1 be a cover of u. Then $u_1u' \in \{u_1, u'\}$ and $u_1u' \geq uu' = u$, whence $u_1u' = u_1$. Since $u_1u'_1 \in \{u_1, u'_1\}$ and $u_1u'_1 \geq u_1u' = u_1$ we conclude that $u_1u'_1 = u_1$. In this way we can work up a covering chain from u to b to give $bb' = b$.

By a dual and symmetric argument we can work up from b to v to give $v'v = v$ and then down a covering chain from v to b' to give $bb' = b''b' = b'$. This contradiction shows that **B** is 2-taut and hence is idempotent trivial via Lemmas 2.5 and 2.7.

Let **C** be a dense chain and let $*$ be a monotone binary operation on C satisfying $x < x * y < y$ and $x < y * x < y$ whenever $x < y$. (For example, let $C = \{c_\alpha \mid \alpha < \gamma\}$ be a well ordering of C and for $x < y$ define $x * y = y * x = c_\beta$ where β is the least α such that $x < c_\alpha < y$.) Define a binary operation on T_C by

$$(x, u) \cdot (y, v) = \begin{cases} (x, u) & \text{if } x = y \text{ and } u = v, \\ (x, 0) & \text{if } x = y \text{ and } u \neq v, \\ (x * y, 0) & \text{if } x \neq y. \end{cases}$$

Since this operation is idempotent and monotone but not strongly idempotent it follows that $\mathbf{T_C}$ is not 2-taut. □

We can now prove the main result of this section.

THEOREM 2.9. *Every braid* **B** *of reach greater than* 2 *which has enough extremals is idempotent trivial.*

Proof. By Lemmas 2.5 and 2.7, it suffices to show that **B** is 2-taut. Let $r(\mathbf{B}) = n$ and let $\cdot$ be an idempotent monotone binary operation on B. We must show that $xy \in \{x, y\}$ for all $x, y \in B$.

First we shall prove that $uv \geq u$ or $uv \geq v$ for all $u, v \in B$. Suppose that for some $u, v \in B$ we have $uv \ngeq u$ and $uv \ngeq v$. Since $uv \notin \{u, v\}$ we know (by 2.4(i)) that

$$d(u, (uv)') = d(v, (uv)') = n - 1$$

and geodesics from u to $(uv)'$ and from v to $(uv)'$ start in opposite directions. Since **B** has enough extremals, we may assume that there is a geodesic $u < s_1 > s_2 \cdots s_{n-1} = (uv)'$ with $s_1, \ldots, s_{n-2} \in \operatorname{ext} B$ and a geodesic $v > t_1 < t_2 \cdots t_{n-1} = (uv)'$ with $t_1, \ldots, t_{n-2} \in \operatorname{ext} B$. Define $u_1 = u$ and $v_1 = t_1$; thus $u_1 v_1 \leq uv$. We claim that $u_1 v_1 \ngeq u_1$ and $u_1 v_1 \ngeq v_1$. If $u_1 v_1 \geq u_1 = u$, then $uv \geq u$ contrary to assumption. If $u_1 v_1 \geq v_1 = t_1$, then $uv \geq t_1 < t_2 > \cdots t_{n-1} = (uv)'$ yields the contradiction $d(uv, (uv)') < n$. Thus we can repeat this construction starting with u_1, v_1 to create u_2, v_2. Since v_1 is minimal the starting directions of the geodesics interchange and we get minimal elements u_2, v_2 with $u_2 v_2 \ngeq u_2$ and $u_2 v_2 \ngeq v_2$. But now 2.4(ii) gives the contradiction $u_2 v_2 \in \{u_2, v_2\}$.

Dually, we have $uv \leq u$ or $uv \leq v$ for all $u, v \in B$. Hence there exist $c, d \in \{u, v\}$ with $c \leq uv \leq d$. So if $u \parallel v$ we conclude that $uv \in \{u, v\}$ immediately.

Let $b \in \mathrm{B}$. Since $b \parallel b'$, we have $b'b \in \{b, b'\}$. We shall show that $b'b = b'$ implies $uv = u$ for all $u, v \in B$. (A symmetric argument shows that $b'b = b$ implies $uv = v$ for all $u, v \in B$.) Assume $b'b = b'$. Let $a \neq b'$ with $a \parallel b$ and look at a geodesic, say $b' < t_1 > \cdots t_k = a$, from b' to a. Now $t_1, \ldots, t_k$ are noncomparable with b. So $t_1 b \geq b'b = b'$ and $t_1 b \in \{t_1, b\}$, which gives $t_1 b = t_1$. Similarly $t_2 b = t_2$, and we can induct along the geodesic to yield $ab = a$. Thus

$$b'b = b' \quad \text{and} \quad a \parallel b \quad \Rightarrow \quad ab = a. \tag{$*$}$$

Assume now that c is comparable with b. Since $r(\mathbf{B}) \geq 3$, we have $c' \parallel b$, so $c'b = c'$ by $(*)$. But $c \geq b$ implies $c'c \geq c'b = c'$ which with $c'c \in \{c', c\}$ gives $c'c = c'$. Likewise $c \leq b$ implies $c'c = c'$. As $\mathbf{B}$ is connected, we conclude that $d'd = d'$ for all $d \in B$, and therefore $ab = a$ whenever $a \parallel b$.

Finally we prove that $uv = u$ when u and v are comparable. Since there exist $c, d \in \{u, v\}$ with $c \leq uv \leq d$, there are two cases: $u < uv \leq v$ and $u > uv \geq v$. Suppose $u < uv \leq v$ and consider a geodesic from u to $(uv)'$. As $u < uv$, the geodesic starts up, and as $\mathbf{B}$ has enough maximals we may assume that the first element, u_1, after u is a maximal. Thus u_1 is maximal in $\mathbf{B}$ and satisfies $u < u_1$ and $u_1 \parallel uv$. Now $u_1 v \geq uv$ implies that $u_1 v \neq u_1$ and hence u_1 and v are comparable. As u_1 is maximal, we obtain the contradiction $u_1 \geq v \geq uv$. Similarly, $u > uv \geq v$ leads to a contradiction. Thus $uv = u$ for all $u, v \in B$. □

Lemma 2.7 can also be used to give examples of idempotent trivial ordered sets which are not braids. For example, the ordered set $\mathbf{P}$ in Figure 5 is idempotent trivial but is not a braid. By Lemma 2.7 it suffices to show that $\mathbf{P}$ is 2-idempotent trivial. Let $\cdot$ be an idempotent, monotone binary operation on P. Since $\mathbf{T}_2$ is idempotent trivial and since $\{b, v\}$ is common to both copies of $\mathbf{T}_2$ in $\mathbf{P}$, it follows that $\cdot$ is a projection (say the first projection) on $\{a, b, u, v\}$ and on $\{b, c, v, w\}$. By symmetry and duality it remains to show that $ac = a$ and $aw = a$. This is left as an easy exercise.

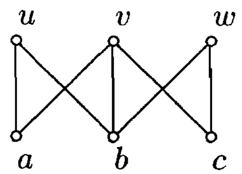


Figure 5

3. The Słupecki property

Recall that the Słupecki clone on a finite set P of order m is the set of all finitary functions on P which are either non-surjective or essentially unary, or equivalently, which preserve the Słupecki relation on P:

$$\eta_P := \{(a_1, \ldots, a_m) \in P^m \mid a_i = a_j \text{ for some } i \neq j\}.$$

By a standard result of clone theory, a finite ordered set **P** is Słupecki (i.e. the monotone clone of **P** is contained in the Słupecki clone on P) if and only if there is some finite ordered set **E** and elements $e_1, \ldots, e_m \in E$ such that

$$\eta_P = \{(\varphi(e_1), \ldots, \varphi(e_m)) \mid \varphi\colon E \to P \text{ is monotone}\},$$

or equivalently, there is some primitive positive formula (i.e. an existential conjunct of atomic formulae in $\leq$), say

$$(\exists y_1 \ldots\ y_\ell)\sigma(x_1, \ldots x_m, y_1, \ldots y_l)$$

such that

$$\eta_P = \{(a_1, \ldots, a_m) \in P^m \mid \mathbf{P} \vDash (\exists y_1, \ldots y_\ell)\sigma(a_1, \ldots, a_m, y_1, \ldots, y_\ell)\}.$$

For example, the four-element tower, $\mathbf{T}_2$, is Słupecki – the required ordered set is shown in Figure 6 and the corresponding primitive positive formula is

$$(\exists x)(\exists y_1 y_2 y_3 y_4)(\exists z_1 z_2 z_3 z_4)\ \&_{i=1}^4\ (y_i \leq x \leq z_i\ \&\ y_i \leq a_i \leq z_i).$$

It is easy to see that every braid of height 1 is Słupecki by constructing a generalization of the ordered set in Figure 6. Indeed, the ordered set constructed in

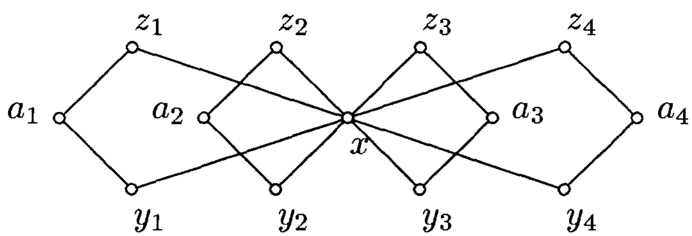


Figure 6

the proof of Theorem 3.3 in [3], which shows that the crown $\mathbf{C}_n$ is Słupecki, works for any reach-n braid of height one.

PROPOSITION 3.1. *The tower* $\mathbf{T}_2$ *is Słupecki but for* $n \geq 3$ *the tower* $\mathbf{T}_n$ *is not Słupecki although it is idempotent trivial.*

Proof. By Proposition 2.8 and the discussion above, it remains to show that for $n \geq 3$ the tower $\mathbf{T}_n$ is not Słupecki. Let $\{a, b\}$ be the maximal elements of T_n, let $\{c, d\}$ be their lower covers and define $A := T_n \backslash \{a, b, c, d\}$. Consider the binary operation defined on T_n by:

$$xy = y \quad \text{if } x \in \{c, d\} \cup A, y \in A,$$

$$xy = d \quad \text{if } x \in \{c, d\} \cup A, y \notin A,$$

$$ay = \begin{cases} c & \text{if } y \in A, \\ a & \text{otherwise,} \end{cases}$$

$$by = d \quad \text{if } y \neq a,$$

$$ba = b.$$

It is readily seen that this operation is monotone, surjective and not essentially unary. Hence $\mathbf{T}_n$ is not Słupecki for $n \geq 3$. □

Our aim in the remainder of this section is to show that, with the exception of the rogue towers T_n with $n \geq 3$, every finite braid is Słupecki. In fact, we can prove somewhat more.

An ordered set $\mathbf{P}$ with $|P| \geq 3$ will be called a *near-braid* if $\mathbf{P}$ is connected, of finite reach, and

(a) $\mathbf{P}$ has enough extremals,
(b) every extremal element has an antipode in $\mathbf{P}$, i.e. for all $a \in \operatorname{ext} P$, there exists a unique element $a' \in P$ with $d(a, a') = r(\mathbf{P})$,
(c) the extremal elements separate the points of P, i.e. for all $a, b \in P$,

$$\downarrow b \cap D \subseteq \downarrow a \cap D \ \& \ \uparrow b \cap U \subseteq \uparrow a \cap U \quad \Rightarrow \quad a = b.$$

By Proposition 2.2, every braid of reach greater than 2 with enough extremals is a near-braid. Obviously, if $\mathbf{P}$ is obtained from a near-braid by deleting some non-extremal elements, then $\mathbf{P}$ is again a near-braid.

THEOREM 3.2. *Every finite near-braid of reach greater than 2 is Slupecki and hence is idempotent trivial.*

Proof. Let $\mathbf{P}$ be a finite near-braid of reach greater than 2. First, observe that if $a \in P \setminus \operatorname{ext} P$, then $d_{\nearrow}(a, y) \leq r(\mathbf{P})$ and $d_{\searrow}(a, y) \leq r(\mathbf{P})$ for all $y \in P$. [Only $y \in P$ with $d(a, y) = r(\mathbf{P})$ could provide a counterexample. Since extremals have unique antipodes, which must again be extremal, it follows that $y \notin \operatorname{ext} P$. Let u be a maximal above a. Since u has a unique antipodal which is not y (as $y \notin \operatorname{ext} P$), we have $d(u, y) < r(\mathbf{P})$ and hence $d_{\nearrow}(a, y) \leq r(\mathbf{P})$.] Denote the non-extremal elements of $\mathbf{P}$ by M (the middle of $\mathbf{P}$) and note that

$$D \cup M = \{x \in P \mid (\forall y \in P) d_{\nearrow}(x, y) \leq r(\mathbf{P})\}$$

and

$$(D \cup M)^n = \{\bar{x} \in P^n \mid (\forall \bar{y} \in P^n) d_{\nearrow}(\bar{x}, \bar{y}) \leq r(\mathbf{P})\}.$$

Let $f: P^n \rightarrow P$ be monotone and surjective. The remainder of the proof is devoted to proving that f is essentially unary. Since f is surjective, the remarks above show that $f((D \cup M)^n) \subseteq D \cup M$, and dually $f((U \cup M)^n) \subseteq U \cup M$, from which it follows that $f(M^n) \subseteq M$. Since f is monotone and surjective, we have $f(D^n) \supseteq D$ and $f(U^n) \supseteq U$. The first major step in the proof will be to show that $f(D^n) = D$ and $f(U^n) = U$.

Define maps $g: D^n \rightarrow D$ and $h: U^n \rightarrow U$ via a choice: $g(\bar{d})$ is any minimal below $f(\bar{d})$ and $h(\bar{u})$ is any maximal above $f(\bar{u})$. Thus if $\bar{d} \in D^n$ and $\bar{u} \in U^n$, then

$$g(\bar{d}) \leq f(\bar{d}), \qquad f(\bar{u}) \leq h(\bar{u}), \quad \text{and} \quad g(\bar{d}) \leq h(\bar{u}) \text{ provided } \bar{d} \leq \bar{u}.$$

Assume, for the moment, that $r(\mathbf{P})$ is odd. Let $\bar{d} \in D^n$ and $\bar{u} \in U^n$ with $d_i \neq u_i'$ for all i. We claim that $g(\bar{d}) \neq h(\bar{u})'$. Since $d_i \neq u_i'$ for all i, we have

$$d(d_i, u_i) = d_{\nearrow}(d_i, u_i) < r(\mathbf{P}) \qquad \text{for all } i,$$

whence

$$d(\bar{d}, \bar{u}) = d_{\nearrow}(\bar{d}, \bar{u}) = \max\{d(d_i, u_i) \mid i = 1, \ldots, n\} < r(\mathbf{P}).$$

Since $g(\bar{d}) \leq f(\bar{d})$ and $f(\bar{u}) \leq h(\bar{u})$, we conclude that

$$d(g(\bar{d}), h(\bar{u})) \leq d(f(\bar{d}), f(\bar{u})) \leq d(\bar{d}, \bar{u}) < r(\mathbf{P}),$$

and consequently $g(\bar{d}) \neq h(\bar{u})'$, as claimed. It follows easily that $g: D^n \to D$ preserves the k-ary relation λ_D (where $k = |D|$) defined on D by

$$(d_1, \ldots, d_k) \in \lambda_D \quad \Leftrightarrow \quad (\exists u \in U)(\forall i) u \neq d_i'.$$

Since the antipode map is a bijection between D and U, it follows that $\lambda_D = \eta_D$ and hence g preserves η_D. Similarly, h preserves η_U.

If $r(\mathbf{P})$ is even, one must modify this argument slightly. First show that if $\bar{d}, \bar{e} \in D^n$ with $d_i \neq e_i'$ for all i, then $g(\bar{d}) \neq g(\bar{e})'$. From this it follows that g preserves the relation λ_D defined by

$$(d_1, \ldots, d_k) \in \lambda_D \quad \Leftrightarrow \quad (\exists e \in D)(\forall i) e \neq d_i'.$$

Since the antipode map restricted to D is a permutation it follows that $\lambda_D = \eta_D$. Hence g preserves η_D and similarly h preserves η_U.

Thus, regardless of the parity of $r(\mathbf{P})$, the map g preserves the Słupecki relation on D and the map h preserves the Słupecki relation on U. As $f(D^n) \supseteq D$ and $f(U^n) \supseteq U$, the maps g and h are surjective and so are essentially unary. Hence there exist i, j and permutations $\sigma \in \mathrm{Sym}(D)$ and $\tau \in \mathrm{Sym}(U)$ such that $g(\bar{d}) = \sigma(d_i)$ and $h(\bar{u}) = \tau(u_j)$ for all $\bar{d} \in D^n$ and all $\bar{u} \in U^n$.

Suppose that $i \neq j$. Let $x \in D, y \in U$ and choose $\bar{d} \in D^n, \bar{u} \in U^u$ with $d_i = x$ $u_j = y$ and $\bar{d} \leq \bar{u}$. Then

$$\bar{d} \leq \bar{u} \quad \Rightarrow \quad g(\bar{d}) \leq h(\bar{u}) \quad \Rightarrow \quad \sigma(x) \leq \tau(y).$$

Since σ and τ are permutations, this implies that $d \leq u$ for all $d \in D$ and all $u \in U$. But this gives the contradiction $r(\mathbf{P}) = r(\mathrm{ext}\,\mathbf{P}) = 2$. Hence $i = j$ and without loss of generality we may assume

$$g(\bar{d}) = \sigma(d_1) \leq f(\bar{d}) \qquad \text{for all } \bar{d} \in D^n$$

and

$$h(\bar{u}) = \tau(u_1) \geq f(\bar{u}) \qquad \text{for all } \bar{u} \in U^n.$$

As σ and τ are permutations, there exists $s \in \mathbf{N}$ such that $\sigma^s = \sigma^{-1}$ and $\tau^s = \tau^{-1}$. It follows that

$$d \leq u \quad \Leftrightarrow \quad \sigma(d) \leq \tau(u) \qquad \text{for all } d \in D \quad \text{and} \quad \text{all } u \in U.$$

Suppose that $\bar{u} \in U^n$ with $f(\bar{u}) \notin U$, i.e. $f(\bar{u}) < h(\bar{u})$. Let $d \in D$ and let $e \in D$ satisfy $d = \sigma(e)$. If $e \leq u_1$, then choose $\bar{d} \in D^n$ with $\bar{d} \leq \bar{u}$ and $d_1 = e$: thus

$$d = \sigma(e) = g(\bar{d}) \leq f(\bar{d}) \leq f(\bar{u}).$$

If $e \nleq u_1$, then

$$d = \sigma(e) \nleq r(u_1) = h(\bar{u}).$$

Consequently, $\downarrow h(\bar{u}) \cap D \subseteq \downarrow f(\bar{u}) \cap D$ and thus $f(\bar{u}) = h(\bar{u})$ since the extremal elements separate the points of P (and therefore the minimal elements separate comparable elements). This contradiction shows that $f(U^n) = U$ and hence $f \upharpoonright U^n = h$. Similarly, $f(D^n) = D$ and $f \upharpoonright D^n = g$. Thus

$$f(\bar{x}) = \begin{cases} \sigma(x_1) & \text{if } \bar{x} \in D^n, \\ \tau(x_1) & \text{if } \bar{x} \in U^n. \end{cases}$$

Define $\ell(\bar{x}) := f(f(\cdots f(f(\bar{x}), x_2, \ldots, x_n), x_2, \ldots, x_n) \cdots)$ iterated often enough so that $\ell(\ell(\bar{x}), x_2, \ldots, x_n) = \ell(\bar{x})$. Since σ and ρ are permutations, it is easily seen that $\ell(\bar{d}) = d_1$ for all $\bar{d} \in D^n$ and $\ell(\bar{u}) = u_1$ for all $\bar{u} \in U^n$.

Given $\bar{a} \in P^{n-1}$, define $\ell_{\bar{a}}: P \to P$ by $\ell_{\bar{a}}(x) = \ell(x, \bar{a})$. Choose any $\bar{y} \in P^{n-1}$ and $\bar{d} \leq \bar{y} \leq \bar{u}$ with $\bar{d} \in D^{n-1}$ and $\bar{u} \in U^{n-1}$. Then $e \leq x \leq v$ with $e \in D$ and $v \in U$ implies

$$e = \ell_{\bar{d}}(e) \leq \ell_{\bar{y}}(x) \leq \ell_{\bar{u}}(v) = v.$$

Thus $\downarrow x \cap D \subseteq \downarrow \ell_{\bar{y}}(x) \cap D$ and $\uparrow x \cap U \subseteq \uparrow \ell_{\bar{y}}(x) \cap U$, whence $\ell_{\bar{y}}(x) = x$ since the extremal elements separate points in **P**. It follows from the definition of ℓ that some power of $f_{\bar{y}}$ (which is defined by $f_{\bar{y}}(x) = f(x, \bar{y})$) is the identity map on P and consequently $f_{\bar{y}}$ is an order-isomorphism.

If $\bar{y} \leq \bar{z}$, then $f_{\bar{y}}(x) \leq f_{\bar{z}}(x)$ for all $x \in P$. Since **P** is finite and both $f_{\bar{y}}$ and $f_{\bar{z}}$ are order-isomorphism, it follows that $f_{\bar{y}} = f_{\bar{z}}$ whenever $\bar{y} \leq \bar{z}$. Since **P** is connected we have $f_{\bar{y}} = f_{\bar{z}}$ for all $\bar{y}, \bar{z} \in P^{n-1}$. Hence f depends only on its first coordinate, as claimed.

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