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From rectangular bands to k-primal algebras

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Abstract

We begin by giving a new proof that every finite rectangular band is naturally dualisable. Motivated by the dualising structure arising from this proof, we call an algebra k*-primal* if it is (isomorphic to) a product of k independent primal algebras. For each $k \geq 2$ we exhibit a strong duality between the quasi-variety generated by a k-primal algebra and the topological quasi-variety $\mathfrak{D}_k$ of Boolean topological k-dimensional diagonal algebras. The category $\mathfrak{D}_2$ is the category of compact, totally disconnected rectangular bands. This duality extends Hu's duality for varieties generated by a primal algebra to the k-dimensional case. We find that Hu's "uniqueness principle" for such varieties also extends to the k-dimensional case, namely, we show that a quasi-variety is equivalent as a category to the quasi-variety generated by a k-primal algebra if and only if it is itself generated by a k-primal algebra.

1. Introduction

Clark and Davey [1, Theorem 4.4.10] give a curious natural duality for the variety $\mathfrak{RB}$ of rectangular bands. A six element set is taken as the "schizophrenic" object for the duality: on one hand it has a non-zero rectangular band operation, while on the other, it has the structure of the commutative ring of integers modulo 6 endowed with the discrete topology. More precisely, letting $R = \{0, \dots, 5\}$, we have $\mathfrak{RB}$ being the quasi-variety generated by $\mathbf{R} = \langle R; * \rangle$ where

$$a * b := 3a + 4b$$

for all pairs $a, b \in R$. Let $\mathfrak{X}$ be the topological quasi-variety generated by

$$\underset{\sim}{\mathbf{R}} := \langle R; +, \cdot, 0, 1, \tau \rangle$$

where τ is the discrete topology and $\langle R; +, \cdot, 0, 1 \rangle$ is the commutative ring $\mathbb{Z}_6$. We then have $\mathfrak{X}$ dually equivalent to $\mathfrak{RB}$ via the natural hom-functors. In particular, every rectangular band may be uniquely represented as the set of continuous ring homomorphisms from some compact, totally disconnected commutative ring in $\mathfrak{X}$ into $\underset{\sim}{\mathbf{R}}$, also, every object in $\mathfrak{X}$ has a unique representation as the set of semigroup homomorphisms from some rectangular band into $\mathbf{R}$. This duality arises from the general theory as a by-product of a previously established strong duality for $\mathbb{Z}_6$ in which the topology resides on the rectangular band side [1, Theorem 4.2.5].

The six-element rectangular band $\mathbf{R}$ used in [1] does not seem the most natural choice as the focus of a duality for $\mathfrak{RB}$—although it generates $\mathfrak{RB}$ as a quasi-variety, it is not the minimal generator. The first main result of this paper is a new duality for $\mathfrak{RB}$, based on the minimal four-element generator. Using a recent result of Davey and Willard [4] and the fact that any non-zero rectangular band generates $\mathfrak{RB}$ as a quasi-variety, one may deduce from either of these dualities that every non-zero rectangular band is naturally dualisable.

In the four-element case, instead of a discretely topologised ring, the dualising structure ends up resembling a Boolean algebra. Both this four-element structure and the six-element structure $\underset{\sim}{\mathbf{R}}$ may be viewed as a product of two independent primal algebras endowed with the discrete topology. We then find it fruitful to approach from the other direction and consider these "bi-primal" (and more generally "k-primal") algebras as the starting point.

A non-trivial algebra $\mathbf{A} = \langle A; F \rangle$ is called *primal* if every finitary operation on the underlying set A is a term function, that is, for each $n \in \mathbb{N}$, every $f : A^n \to A$ may be obtained as a composition of some fundamental operations from F and the projections. Consideration of this property arose from the study of Boolean algebras, and in 1969, T. K. Hu showed that, in a certain sense, primal algebras are indeed very much like Boolean algebras. Hu's duality, in [7], states that the variety (= quasi-variety) generated by a primal algebra is dually equivalent to the category of Boolean spaces, generalising Stone's well known duality for Boolean algebras. As a consequence, the variety generated by any finite primal algebra is equivalent as a category to the variety of Boolean algebras. Hu went on to show in [9] that such varieties are unique in this respect: a variety is equivalent as a category to the variety of Boolean algebras if *and only if* it is the variety generated by a finite primal algebra.

In this paper we extend Hu's duality and uniqueness result to k-primal algebras, for a fixed natural number $k \geq 2$. Instead of the category of Boolean spaces on the topological side of the duality, we end up with a category of "k-dimensional" analogues of compact totally disconnected rectangular bands. Algebraically, these are the k-dimensional diagonal algebras introduced by J. Płonka in [12] and [13].

In the next section we include a brief primer on Natural Duality Theory. Hu's duality may be easily recast in the general setting: a finite primal algebra $\mathbf{M}$ is strongly dualised via the structure $\underset{\sim}{\mathbf{M}} = \langle M; \tau \rangle$ having no relations or (partial) operations in its type where τ denotes the discrete topology. Using the Two-for-One Strong Duality Theorem [1, Theorem 3.3.2], we can flip this arrangement so that a finite algebra $\mathbf{N} = \langle N; \varnothing \rangle$ having no fundamental operations may be strongly dualised by a structure $\underset{\sim}{\mathbf{N}} = \langle N; G, \tau \rangle$ where $\langle N; G \rangle$ is a primal algebra (a concrete choice for $\langle N; G \rangle$ is the $|N|$-element Post algebra).

Recall that algebras $\mathbf{A}$ and $\mathbf{A}'$ on the same underlying set are *term equivalent* if every term function of $\mathbf{A}$ is a term function of $\mathbf{A}'$ and vice-versa. This notion is compatible with duality theory in the sense that a duality for $\mathbf{A}$ also applies to $\mathbf{A}'$. The left-zero band $\langle N; * \rangle$ (where the binary operation $*$ is given by $x * y = x$) is term equivalent to $\mathbf{N} = \langle N; \varnothing \rangle$ since both have only the projections as term functions. Hence we have a strong duality for each left-zero (and similarly right-zero) band via a topologised primal algebra. Using our generalised Hu type duality, we are able to show that products of left and right-zero bands (that is, rectangular bands) are strongly dualised by appropriate products of topologised primal algebras. We achieve this by flipping the strong duality between the quasi-variety generated by a bi-primal algebra and the category of compact, totally disconnected topological rectangular bands.

This paper uses insights obtained from the study of a class of semigroups to develop results within general algebra. Reversing the focus, the new semigroup-theoretic results obtained here may be summarised as follows:

- We exhibit strong dualities for the variety $\mathcal{RB}$ of rectangular bands. Indeed, for any non-zero rectangular band $\mathbf{M}$, we have a strong duality for $\mathcal{RB}$ based on $\mathbf{M}$. There is a overall uniformity to the dual structures—they arise naturally via bi-primal algebras derived from the factorisation of $\mathbf{M}$ into a product of left- and right-zero semigroups. (Corollary 5.9)
- The category $\mathcal{D}_2$ of Boolean topological rectangular bands has exactly the same separation properties as the variety $\mathcal{RB}$, namely, we show that if $\mathbf{X}$ is a Boolean topological rectangular band and $\mathbf{M}$ is finite non-zero rectangular band endowed with the discrete topology, then there are enough continuous homomorphisms from $\mathbf{X}$ into $\mathbf{M}$ to separate points. Consequently, $\mathbf{X}$ is both algebraically and topologically isomorphic to a closed subsemigroup of a power of $\mathbf{M}$. (Corollary 6.4)
- $\mathcal{D}_2$ is dually equivalent to the quasi-variety generated by any bi-primal algebra. (Corollary 5.8)

2. A Primer on Natural Dualities

For a comprehensive account of the theory of natural dualities, we refer the reader to Clark and Davey [1] (which also contains introductory universal algebra in Appendix A). In this section we give a brief overview of the theory to provide the tools needed in this paper.

Let $\mathbf{M}$ be a finite algebra and let $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ be the *quasi-variety* it generates—that is, the class of isomorphic copies of subalgebras of powers (over possibly empty index sets) of $\mathbf{M}$. The set M is to be the underlying set of a "schizophrenic" object that lives both as an algebra in $\mathcal{A}$ and also as a structured compact topological space in a category which we would hope to be dually equivalent to $\mathcal{A}$. Natural Duality Theory begins with the tennant that a reasonable general structure one could impose on M is

$$\underset{\sim}{\mathrm{M}} := \langle M; G, H, R, \tau\rangle,$$

where M has been endowed with:

- a set G of *algebraic operations*—each $g \in G$ is a homomorphism $\mathbf{M}^n \to \mathbf{M}$ for some $n \geq 0$,
- a set H of *algebraic partial operations*—each $h \in H$ is a homomorphism $\mathbf{A} \to \mathbf{M}$ where $\mathbf{A}$ is a subalgebra of some finite power $\mathbf{M}^n$ $(n \geq 1)$,
- a set R of *algebraic relations*—each $r \in R$ is a subalgebra of $\mathbf{M}^n$ for some $n \geq 1$,
- the discrete topology τ.

We call $G \cup H \cup R$ the *type* of $\underset{\sim}{\mathrm{M}}$. For compact structured topological spaces $\mathbf{X}$ and $\mathbf{Y}$ of the same type $G \cup H \cup R$, a continuous map $\alpha : \mathbf{X} \to \mathbf{Y}$ is a *morphism* if it preserves each operation, partial operation and relation in $G \cup H \cup R$ in the usual algebraic sense. The *topological quasi-variety* $\mathcal{X}$ generated by $\underset{\sim}{\mathbf{M}}$ is then the class of isomorphic copies of topologically closed substructures of powers (over non-empty index sets) of $\underset{\sim}{\mathbf{M}}$, written $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$. It turns out that there is enough connection between $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ even at this general level to ensure a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$ as follows (see [1] Chapter 1 for details):

- for each $\mathbf{X} \in \mathcal{X}$, the set $E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ of all morphisms from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ forms a subalgebra of $\mathbf{M}^X$, therefore $E(\mathbf{X}) \in \mathcal{A}$ for all $\mathbf{X} \in \mathcal{X}$,
- for each $\mathbf{A} \in \mathcal{A}$, the set $D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \mathbf{M})$ of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}$ forms a closed structure of $\underset{\sim}{\mathbf{M}}^A$, therefore $D(\mathbf{A}) \in \mathcal{X}$ for all $\mathbf{A} \in \mathcal{A}$,
- for each $\mathbf{A} \in \mathcal{A}$, the evaluation map $e^{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$, given for each $a \in \mathbf{A}$ by
$$e^{\mathbf{A}}(a)(u) := u(a)$$
for each $u \in D(\mathbf{A})$, is an embedding,
- for each $\mathbf{X} \in \mathcal{X}$, the evaluation map $\varepsilon^{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$, given for each $x \in \mathbf{X}$ by
$$\varepsilon(x)(\varphi) := \varphi(x)$$
for each $\varphi \in E(\mathbf{X})$, is an embedding.

If the embedding $e^{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \mathcal{A}$, we say that $\underset{\sim}{\mathbf{M}}$ *yields a natural duality on* $\mathcal{A}$, or, focusing our attention on the generating algebra, that $\mathbf{M}$ is *dualised* by $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$). In this case the natural dual adjunction set up by the contravariant hom-functors D and E is a *dual representation*—each $\mathbf{A} \in \mathcal{A}$ is isomorphic to the set of continuous structure preserving maps from some object in $\mathcal{X}$, namely $D(\mathbf{A})$, to $\underset{\sim}{\mathbf{M}}$. If there is some choice of $G \cup H \cup R$ such that $G \cup H \cup R$ dualises $\mathbf{M}$, we say that $\mathbf{M}$ is *dualisable.* It is a somewhat surprising fact that there exist non-dualisable algebras[1] and the central drive of natural duality theory is to find and characterise the dualisable, and non-dualisable, algebras in some given class.

In the case where there are no partial operations and only finitely many relations in the type of $\underset{\sim}{\mathbf{M}}$, we may completely suppress the category theory and topology and establish that $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$ by verifying an *interpolation condition* at the finite level (see [1, Theorem 2.2.7]):

Theorem 2.1. (IC Duality Theorem) *Suppose* $\mathbf{M}$ *is a finite algebra and* $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \tau\rangle$ *with* R *finite. Then* $\underset{\sim}{\mathbf{M}}$ *dualises* $\mathbf{M}$ *provided the following condition is satisfied:*

(IC) *For each* $n \geq 1$ *and each substructure* $\mathbf{X}$ *of* $\underset{\sim}{\mathbf{M}}^n$*, every morphism* $\mathbf{X} \to \underset{\sim}{\mathbf{M}}$ *extends to a term function* $\mathbf{M}^n \to \mathbf{M}$. ∎

[1] One example (the first known example, see Sections 5 and 7 in Chapter 10 of [1] and the references given there) is the implication algebra on $\{0, 1\}$, its single binary operation $\to$ being given by

$\to$	0	1
0	1	1
1	0	1

When there are no partial operations or relations present in the type of $\underset{\sim}{\mathbf{M}}$, there is a description of the topological quasi-variety $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ analogous to the description for algebraic quasi-varieties (see the Separation Theorem 1.4.4 in [1]):

Theorem 2.2. *Suppose $\underset{\sim}{\mathbf{M}} = \langle M; G, \tau\rangle$ is a structure having no partial operations or relations in its type. Then a compact structured topological space $\mathbf{X}$ (of the same type as $\underset{\sim}{\mathbf{M}}$) is in $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ if and only if for each $x, y \in X$ with $x \neq y$, there is a morphism $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(x) \neq \alpha(y)$.* ∎

Ideally, we would like a natural duality between $\mathcal{A}$ and $\mathcal{X}$ to be a dual equivalence. That is, we would like to choose a system of (partial) operations and relations $G \cup H \cup R$ so that both evaluation maps, $e^{\mathbf{A}}$ for each $\mathbf{A} \in \mathbb{ISP}\mathbf{M}$ and $\varepsilon^{\mathbf{X}}$ for each $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$, are isomorphisms. When this is the case, we say that $\underset{\sim}{\mathbf{M}}$ yields a *full duality* on $\mathcal{A}$. At present, it seems the only tractable path to a general treatment of full duality is through an apparently stronger condition, that of *strong duality*—a full duality with the additional property that $\underset{\sim}{\mathbf{M}}$ is injective in the category $\mathcal{X}$. All existing proofs of full duality actually establish strong duality. Indeed, under the conditions of Theorem 2.1, satisfaction of (IC) gives the injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$ (see the Second Duality Theorem 2.2.7 in [1]). In this paper, we will not need to consider the subtleties of full and strong duality, since we will obtain our strong duality by refining an "off the shelf" result through the process of *entailment*.

We say that a system $G \cup H \cup R$ (or the structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \tau\rangle$) *entails* a finitary algebraic relation or (partial) operation s on $\mathbf{M}$ if for every $\mathbf{A} \in \mathcal{A}$, each continuous map from $D(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$ which preserves the relations and (partial) operations in $G \cup H \cup R$ also preserves s. Let $\underset{\sim}{\mathbf{M}}' = \langle M; G', H', R', \tau\rangle$. If $G \cup H \cup R$ entails s for every s in the system $G' \cup H' \cup R'$, then we say that $G \cup H \cup R$ *entails* $G' \cup H' \cup R'$, or that $\underset{\sim}{\mathbf{M}}$ entails $\underset{\sim}{\mathbf{M}}'$. In this case, if we know that $\underset{\sim}{\mathbf{M}}'$ yields a duality on $\mathcal{A}$, then $\underset{\sim}{\mathbf{M}}$ also yields a duality on $\mathcal{A}$ (see the $\underset{\sim}{\mathbf{M}}$-shift Duality Lemma 2.4.2 in [1]).

Since the definition of entailment quantifies over all $\mathbf{A} \in \mathcal{A}$, it does not provide any obvious way to determine which relations and (partial) operations are entailed by some given system $G \cup H \cup R$. Fortunately, the notion of entailment is compatible with common constructs like intersection of relations and composition of (partial) operations, and it is possible to give a complete list of such constructs[2].

We give the constructs we will use in the entailment theorem below. A thorough discussion, including the complete list of admissible constructs, may be found in Section 2.4 and Chapter 9 of [1].

For strong duality we need the slightly more specialised notion of *strong entailment*. In this paper we will not even give the formal definition but will use the following result which (in (i)) explains why strong entailment is important and (in (ii)—(vii)) explains how to go about proving it (see Lemma 3.2.3 and the constructs 2.4.5 in [1]).

Lemma 2.3. *Consider the structure $\underset{\sim}{\mathbf{M}}' = \langle M; G', H', R', \tau\rangle$.*

[2] In the sense that any s entailed by some $G \cup H \cup R$ may be obtained from a finite number of applications of the constructs in this list.

(i) *If* $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *and* $\underset{\sim}{\mathbf{M}}'$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$, *then* $\underset{\sim}{\mathbf{M}}$ *also yields a strong duality on* $\boldsymbol{\mathcal{A}}$.

(ii) $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *if it is obtained from* $\underset{\sim}{\mathbf{M}}'$ *by*

- (a) *enlarging* G', H' *or* R';
- (b) *deleting members of* G' *or* H' *which can be obtained as composition of the remaining members of* G' *and* H' *and the projection mappings;*
- (c) *deleting a member* h *of* H' *which has an extension among the remaining members of* $G' \cup H'$ *and adding* $\mathrm{dom}(h)$ *to* R'.

(iii) $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *if* $\underset{\sim}{\mathbf{M}}$ *entails* $\underset{\sim}{\mathbf{M}}'$ *and is obtained from* $\underset{\sim}{\mathbf{M}}'$ *by*

- (a) *deleting members of* R', *or*
- (b) *deleting members of* H' *which have an extension in* G' *or* H'.

(iv) *for* n*-ary relations* r *and* s, *the set* $\{r, s\}$ *entails the intersection* $r \cap s$.

(v) *for* n*-ary operations* f *and* g, *the equaliser*

$$\{\, c \in M^n \mid f(c) = g(c) \,\}$$

is entailed by the set $\{f, g\}$. ∎

3. A Duality for Rectangular Bands

We denote the variety of rectangular bands by $\mathcal{RB}$, that is, the subvariety of semigroups satisfying the identity $xyx \approx x$, or equivalently the anti-commutative quasi-identity $xy \approx yx \implies x \approx y$. As is well known, each rectangular band $\mathbf{S} \in \mathcal{RB}$ may be construed as a product $\mathbf{L} \times \mathbf{R}$ where $\mathbf{L}$ is a left-zero band and $\mathbf{R}$ is a right-zero band (a left-zero band satisfies $xy \approx x$, while a right-zero band satisfies the "dual" identity). Accordingly, for a natural number n, we denote by $\mathbf{L}_n$ ($\mathbf{R}_n$) the unique left(right)-zero band on $\{1, \ldots, n\}$ and say that a finite rectangular band is *non-zero* if it is isomorphic to $\mathbf{L}_n \times \mathbf{R}_m$ where both $n, m \geq 2$.

We begin by recalling that subsemigroups of a (finite) rectangular band occur as retracts.

Lemma 3.1. *For all* $n, m \in \mathbb{N}$, *the rectangular band* $\mathbf{L}_n \times \mathbf{R}_m$ *has* $\mathbf{L}_i \times \mathbf{R}_j$ *as a retract for every* $i \leq n$ *and* $j \leq m$.

Proof. Let $n, m, i, j \in \mathbb{N}$ with $i \leq n$ and $j \leq m$. It is immediate that the map $\varphi_L : L_n \to L_i$, given by

$$\varphi_L(a) := \begin{cases} a & \text{if } a \in \{1, \ldots, i\} \\ 1 & \text{otherwise} \end{cases}$$

for all $a \in L_n$, is a retraction of $\mathbf{L}_n$ onto $\mathbf{L}_i$. Similarly define the retraction φ_R of $\mathbf{R}_m$ onto $\mathbf{R}_j$. Then $\varphi : L_n \times R_m \to L_i \times R_j$ given by

$$\varphi(a, b) := (\varphi_L(a), \varphi_R(b))$$

for all $(a, b) \in L_n \times R_m$ is the required retraction of $\mathbf{L}_n \times \mathbf{R}_m$ onto $\mathbf{L}_i \times \mathbf{R}_j$. ∎

In [6], J.A. Gerhard establishes that the only subdirectly irreducibles in $\mathcal{RB}$ are (up to isomorphism) $\mathbf{L}_2$ and $\mathbf{R}_2$. The above lemma shows, in particular, that every non-zero rectangular band has a subsemigroup isomorphic to $\mathbf{L}_2 \times \mathbf{R}_2$, and our next result follows.

Lemma 3.2. *For any non-zero rectangular band* $\mathbf{S}$*, we have* $\mathbb{ISP}(\mathbf{S}) = \mathcal{RB}$*.* ∎

To show that every finite rectangular band is dualisable, in view of the previous two lemmas we may appeal to the following paraphrased result of Davey [2] and reduce the problem to finding a duality for $\mathbf{L}_2 \times \mathbf{R}_2$.

Theorem 3.3. *Suppose that* $\mathbf{M}$ *has a retract* $\mathbf{N}$ *and* $\mathbf{M} \in \mathbb{ISP}(\mathbf{N})$*. If* $\mathbf{N}$ *is dualisable then so is* $\mathbf{M}$*.* ∎

It is worth noting that the duality for $\mathbf{M}$ in the Theorem above is obtained from that of $\mathbf{N}$ by viewing the dualising structure for $\mathbf{N}$ as algebraic relations on $\mathbf{M}$ and adding the endomorphisms of $\mathbf{M}$—details may be found in [2] or Section 7.7 of [1].

Theorem 3.4. *Every finite rectangular band is dualisable.*

Proof. Let $\mathbf{S}$ be a finite rectangular band. In the first case, if $\mathbf{S}$ is isomorphic to $\mathbf{L}_1 \times \mathbf{R}_n$ or $\mathbf{L}_n \times \mathbf{R}_1$ for some n, then $\mathbf{S}$ is term-equivalent to the n-element set, therefore dualisable via a topologised primal algebra as mentioned in the introduction. On the other hand, suppose $\mathbf{S}$ is non-zero. We then have an isomorphic copy of $\mathbf{L}_2 \times \mathbf{R}_2$ as a retract of $\mathbf{S}$ by Lemma 3.1 and $\mathbf{S} \in \mathbb{ISP}(\mathbf{L}_2 \times \mathbf{R}_2)$ by Lemma 3.2. Hence by Theorem 3.3, the dualisability of $\mathbf{S}$ will follow from the dualisability of $\mathbf{L}_2 \times \mathbf{R}_2$.

Consider the $\mathbf{L}_2 \times \mathbf{R}_2$ isomorphic rectangular band $\mathbf{N} := \langle \{a, b, c, d\}; \cdot \rangle$ given by the following table:

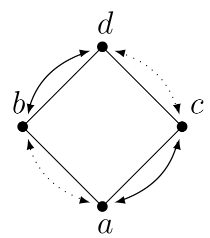
$\cdot$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

We have four 2-element subsemigroups:

$$A := \{a, b\}, \quad B := \{c, d\}, \quad C := \{a, c\}, \quad D := \{b, d\},$$

C and D being left-zero, A and B being right-zero.

We will define a lattice structure on N with additional unary operations $'$ and $^\circ$ according to the depiction below ($'$ dotted, $^\circ$ solid):



That is, $'$ and $^\circ$ are defined by:

$$\begin{array}{c|cccc} ' & a & b & c & d \\ \hline & b & a & d & c \end{array} \qquad \begin{array}{c|cccc} ^\circ & a & b & c & d \\ \hline & c & d & a & b \end{array}$$

It is easily verified that $\vee, \wedge, '$, and $^\circ$ are algebraic over $\mathbf{N}$. Observe that A and B form Boolean algebras under the appropriate restrictions of the operations $\vee, \wedge, '$ and similarly C and D form Boolean algebras under $\vee, \wedge$ and $^\circ$. Let

$$\underset{\sim}{\mathbf{N}} := \langle N; \vee, \wedge, ', ^\circ, a, b, c, d, \tau \rangle.$$

We will show that $\underset{\sim}{\mathbf{N}}$ dualises $\mathbf{N}$. Since $\underset{\sim}{\mathbf{N}}$ has no partial operations in its type, it will suffice to verify (IC) by Theorem 2.1. To this end, let n be a natural number, let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{N}}^n$ and let $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{N}}$ be a morphism. Since we have included the nullary operations a, b, c, d in the type of $\underset{\sim}{\mathbf{N}}$, the constant n-tuples $\underline{h} := (h, \ldots, h)$ are in X for each $h \in N$. Note that for all $x \in X$, we have

$$\underline{b} \wedge x \in X \cap A^n; \quad \underline{c} \vee x \in X \cap B^n; \quad \underline{c} \wedge x \in X \cap C^n; \quad \underline{b} \vee x \in X \cap D^n.$$

Let $Y \in \{A, B, C, D\}$. Since $X \cap Y^n$ is a Boolean algebra under $\vee, \wedge$, the appropriate negation operation ($'$ or $^\circ$) and constants, the restriction of α to $X \cap Y^n$ is a Boolean algebra homomorphism onto the two-element Boolean algebra Y. Therefore $\alpha\restriction_{X \cap Y^n}$ is the characteristic function of the upset of some atom β_Y of $X \cap Y^n$, for instance

$$\alpha\restriction_{X \cap A^n}(x) = \begin{cases} b & \text{if } x \geq \beta_A; \\ a & \text{if } x \ngeq \beta_A, \end{cases}$$

for all $x \in X \cap A^n$, where β_A is some atom of $X \cap A^n$. We may form the set

$$K_A := \{\, i \mid (\beta_A)_i = b \,\}$$

of indices where β_A differs from $\underline{a}$, the bottom of the Boolean algebra $X \cap A^n$. Since β_A is an atom, K_A is non-empty. Similarly, we define K_B, K_C and K_D. Observe that

$$\alpha(\underline{c} \vee \beta_A) = c \vee \alpha(\beta_A) = c \vee b = d.$$

Therefore $\underline{c} \vee \beta_A \geq \beta_B$ since for all $x \in X$, we have $\underline{c} \vee x \in X \cap B^n$. But $\underline{c} \vee \beta_A \geq \beta_B$ if and only if $[(\beta_B)_j = d \implies (\underline{c} \vee \beta_A)_j = d]$, that is, if and only if $[(\beta_B)_j = d \implies (\beta_A)_j = b]$. We have shown that $K_B \subseteq K_A$. Similarly we obtain the reverse inclusion and $K_C = K_D$. Fix a $k \in K_A$ and a $k' \in K_C$. We claim that

$$(*_A) \quad \text{for all } x \in X \cap A^n, \text{ if } x \ngeq \beta_A \text{ then } x_k = a.$$

Suppose to the contrary that there is an $x \in X \cap A^n$ with $x \ngeq \beta_A$ and $x_k = b$. Since $x \ngeq \beta_A$, there is an $i \in K_A$ such that $x_i = a$ (and $i \neq k$). But then $\underline{a} < x \wedge \beta_A < \beta_A$ with $x \wedge \beta_A \in X \cap A^n$, contradicting the fact that β_A is an atom of $X \cap A^n$. We may similarly establish

$$\begin{array}{ll} (*_B) & \text{for all } x \in X \cap B^n, \text{ if } x \ngeq \beta_B \text{ then } x_k = c; \\ (*_C) & \text{for all } x \in X \cap C^n, \text{ if } x \ngeq \beta_C \text{ then } x_{k'} = a; \\ (*_D) & \text{for all } x \in X \cap D^n, \text{ if } x \ngeq \beta_D \text{ then } x_{k'} = b. \end{array}$$

Consider the n-ary term function $t : \mathbf{N}^n \to \mathbf{N}$ given by

$$t(x) := x_{k'} x_k$$

for all $x \in N^n$. Referring to the operation table for $\mathbf{N}$, the following properties are immediate:

$$\begin{aligned}
t(x) = a &\iff (x_{k'} = a \text{ or } x_{k'} = b) \text{ and } (x_k = a \text{ or } x_k = c);\\
t(x) = b &\iff (x_{k'} = a \text{ or } x_{k'} = b) \text{ and } (x_k = b \text{ or } x_k = d);\\
t(x) = c &\iff (x_{k'} = c \text{ or } x_{k'} = d) \text{ and } (x_k = a \text{ or } x_k = c);\\
t(x) = d &\iff (x_{k'} = c \text{ or } x_{k'} = d) \text{ and } (x_k = b \text{ or } x_k = d).
\end{aligned}$$

Now, for all $x \in X$, we have

$$\begin{aligned}
\alpha(x) = a &\iff b \wedge \alpha(x) = a \text{ and } c \wedge \alpha(x) = a\\
&\iff \alpha(\underline{b} \wedge x) = a \text{ and } \alpha(\underline{c} \wedge x) = a\\
&\iff \underline{b} \wedge x \not\geq \beta_A \text{ and } \underline{c} \wedge x \not\geq \beta_C\\
&\iff (\underline{b} \wedge x)_k = a \text{ and } (\underline{c} \wedge x)_{k'} = a \text{ (by } *_A \text{ and } *_C)\\
&\iff (x_k = a \text{ or } x_k = c) \text{ and } (x_{k'} = a \text{ or } x_{k'} = b)\\
&\iff t(x) = a.
\end{aligned}$$

Proceeding similarly, we conclude that t extends α. ■

We will see later that the duality for $\mathbf{L}_2 \times \mathbf{R}_2$ in the above proof is strong, and indeed each non-zero rectangular band may be strongly dualised by an appropriate analogue of the structure $\underset{\sim}{\mathbf{N}}$.

4. Independence and k-Primal Algebras

Consider the dualising structure

$$\underset{\sim}{\mathbf{N}} = \langle N; \vee, \wedge, ', ^\circ, a, b, c, d, \tau \rangle$$

for the rectangular band $\mathbf{N} \cong \mathbf{L}_2 \times \mathbf{R}_2$. By omitting the discrete topology τ we obtain the algebra

$$\langle N; \vee, \wedge, ', ^\circ, a, b, c, d \rangle$$

which may be construed as a product of algebras on $\{0, 1\}$ as follows. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ each be two-element chains with respect to $\vee, \wedge$, and interpret $', ^\circ$ as depicted below ($'$ dotted, $^\circ$ solid):

$\mathbf{B}_1$ $\mathbf{B}_2$

Assigning $0 = a^{\mathbf{B}_1} = b^{\mathbf{B}_1} = a^{\mathbf{B}_2} = c^{\mathbf{B}_2}$ and $1 = c^{\mathbf{B}_1} = d^{\mathbf{B}_1} = b^{\mathbf{B}_2} = d^{\mathbf{B}_2}$, we have $\langle N; \vee, \wedge, ', ^\circ, a, b, c, d\rangle$ isomorphic to $\mathbf{B}_1 \times \mathbf{B}_2$.

Observe that $\mathbf{B}_1$ and $\mathbf{B}_2$ are both term equivalent to the two-element Boolean algebra—in each case we have just added the unary identity map and duplicated the constants. However, note that while $\mathbf{B}_1$ and $\mathbf{B}_2$ are therefore both primal algebras of the same type, they are non-isomorphic. Adjoining the discrete topology to $\mathbf{B}_1$ and $\mathbf{B}_2$ gives distinct dualising structures for the two-element set, therefore also for both $\mathbf{L}_2$ and $\mathbf{R}_2$. In a sense we have obtained a duality for $\mathbf{L}_2 \times \mathbf{R}_2$ by simply forming the product of two distinct dualising structures—one for $\mathbf{L}_2$ and one for $\mathbf{R}_2$. We may in fact identify $\mathbf{L}_2 \times \mathbf{R}_2$ as a term reduct of $\langle N; \vee, \wedge, ', ^\circ, a, b, c, d\rangle$ as follows. For the binary term $t := (x' \wedge y^\circ) \vee (x \wedge y)$, we have $t^{\mathbf{B}_1}(x, y) = x$ and $t^{\mathbf{B}_2}(x, y) = y$ for all $x, y \in \{0, 1\}$. That is, $t^{\mathbf{B}_1} = \pi_1$ and $t^{\mathbf{B}_2} = \pi_2$ where π_1 and π_2 are the binary projection functions. Then the term reduct $\langle \{0, 1\}; t^{\mathbf{B}_1}\rangle \cong \mathbf{L}_2$ while $\langle \{0, 1\}; t^{\mathbf{B}_2}\rangle \cong \mathbf{R}_2$ and we have $\langle \{0, 1\}; t^{\mathbf{B}_1}\rangle \times \langle \{0, 1\}; t^{\mathbf{B}_2}\rangle = \langle B_1 \times B_2; t^{\mathbf{B}_1 \times \mathbf{B}_2}\rangle \cong \mathbf{L}_2 \times \mathbf{R}_2$.

Definition 4.1. Non-trivial algebras $\mathbf{A}$ and $\mathbf{B}$ (of the same type) are said to be *independent* if there is a binary term $*$ such that $*^{\mathbf{A}} = \pi_1$ and $*^{\mathbf{B}} = \pi_2$. We call $*$ an *independence term* for $\mathbf{A}_1$ and $\mathbf{A}_2$. More generally, we say that a finite set $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ of non-trivial algebras is *independent* if there is a k-ary term $*$ such that $*^{\mathbf{A}_i} = \pi_i$ for each $i \in \{1, \dots, k\}$.

Clearly, if $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ is an independent set of algebras via the k-ary term $*$, then each pair $\mathbf{A}_i, \mathbf{A}_j$ is independent for $i \neq j$, with independence term $*(x, \dots, x, y, x, \dots, x)$ where the y appears in the jth coordinate. Similarly, any subset (of size 2 or more) of an independent set of algebras is independent.

Lemma 4.2. *Independent algebras $\mathbf{A}$ and $\mathbf{B}$ are non-isomorphic. Moreover, neither is a homomorphic image of the other. Hence, algebras in an independent set $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ are pairwise non-isomorphic with no $\mathbf{A}_i$ a homomorphic image of any $\mathbf{A}_j$ with $i \neq j$.*

Proof. Suppose that $\iota : \mathbf{A} \to \mathbf{B}$ is an isomorphism. Define $g : \mathbf{A} \times \mathbf{B} \to \mathbf{A} \times \mathbf{B}$ by

$$g(a, b) = (\iota^{-1}(b), \iota(a))$$

for all $(a, b) \in A \times B$. Then g is an automorphism and hence preserves the term function $*^{\mathbf{A}\times\mathbf{B}}$ (where $*$ is an independence term for $\mathbf{A}$ and $\mathbf{B}$). Therefore

$$g((a_1, b_1) *^{\mathbf{A}\times\mathbf{B}} (a_2, b_2)) = g(a_1, b_1) *^{\mathbf{A}\times\mathbf{B}} g(a_2, b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in A \times B$. But the left hand side of this equation becomes

$$g(a_1, b_2) = (\iota^{-1}(b_2), \iota(a_1)),$$

while the right hand side becomes

$$(\iota^{-1}(b_1), \iota(a_1)) *^{\mathbf{A}\times\mathbf{B}} (\iota^{-1}(b_2), \iota(a_2)) = (\iota^{-1}(b_1), \iota(a_2)),$$

giving the contradiction $|A \times B| = 1$.

Now, suppose that $f : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism. Then the quotient algebra $\mathbf{A}/\ker(f)$ is isomorphic to $\mathbf{B}$. We again obtain a contradiction, since $*$ is an independence term for $\mathbf{A}/\ker(f)$ and $\mathbf{B}$. Similarly, $\mathbf{A}$ is not a homomorphic image of $\mathbf{B}$. ∎

The divisive property of an independence term spreads throughout the varieties generated by independent algebras $\mathbf{A}$ and $\mathbf{B}$—we may use the argument in the above proof to show that no non-trivial algebra in $\mathbb{HSP}(\mathbf{A})$ is a homomorphic image of an algebra in $\mathbb{HSP}(\mathbf{B})$ and vice-versa.

For a product $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ of independent algebras $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ with k-ary independence term $*$, consider the term reduct $\langle A; *^{\mathbf{A}} \rangle$. Written out explicitly, we have $*^{\mathbf{A}} : A^k \to A$ given by

$$*^{\mathbf{A}}[(a_1^1, a_2^1, \dots, a_k^1), \dots, (a_1^k, a_2^k \dots, a_k^k)] = (a_1^1, a_2^2, \dots, a_k^k).$$

Observe that $*^{\mathbf{A}}$ is a homomorphism $\mathbf{A}^k \to \mathbf{A}$ since we have

$$*^{\mathbf{A}} = \pi_1 \sqcap \pi_{k+2} \sqcap \pi_{2k+3} \sqcap \cdots \sqcap \pi_{k^2}$$

under the identification of $\mathbf{A}^k$ with the k^2-factor product

$$\mathbf{A}_1 \times \cdots \times \mathbf{A}_k \times \cdots \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k.$$

Evidently, $\langle A; *^{\mathbf{A}} \rangle$ forms a k-dimensional diagonal algebra. Such algebras were studied by Płonka—see [12] and [13]. We will defer for now the characterisation given by Płonka and instead introduce the algebras that turn out to be intimately linked to the k-dimensional diagonal algebras.

Definition 4.3. We say that an algebra $\mathbf{M}$ is *k-primal* if $\mathbf{M} \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ where $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ is an independent set of primal algebras.

One class of examples of bi-primal algebras are the commutative rings with identity $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are distinct primes—in this case an independence term for $\mathbb{Z}_p$ and $\mathbb{Z}_q$ is given by

$$x * y = mx + ny$$

where we choose (by the Chinese Remainder Theorem) m such that $m \equiv 1 \pmod p$ and $m \equiv 0 \pmod q$ and n such that $n \equiv 1 \pmod q$ and $n \equiv 0 \pmod p$.

Another class generalises $\mathbf{B}_1 \times \mathbf{B}_2$ of the previous section, and may be constructed as follows. Recall that an n-element Post algebra

$$\mathbf{P}_n = \langle \{0, \dots, n-1\}; \vee, \wedge, ', D_1, \dots, D_{n-1}, 0, \dots, n-1 \rangle$$

is an n-element chain considered as a lattice, with a unary pseudocomplementation $'$ and unary operations $D_1, \dots, D_{n-1}$ given by

$$a' = \begin{cases} n-1 & \text{if } a = 0, \\ 0 & \text{if } a \neq 0, \end{cases} \quad \text{and} \quad D_i(a) = \begin{cases} n-1 & \text{if } i \leq a, \\ 0 & \text{if } i > a. \end{cases}$$

Let $\mathbf{P}_n$ and $\mathbf{P}_m$ be Post algebras with $n, m \geq 2$, and without loss of generality assume that $n \leq m$. Augment the type of $\mathbf{P}_n$ with $m - n + 1$ unary operation symbols $D_n, \dots, D_{m-1}, ^\circ$, and fill out the nullary symbols to mn. To the type of $\mathbf{P}_m$ we just add $^\circ$ and again consider mn nullaries. We obtain algebras $\widehat{\mathbf{P}}_n$ and $\widehat{\mathbf{P}}_m$ of the same type

$$\langle \vee, \wedge, ', ^\circ, D_1 \dots, D_m, 0, \dots, mn \rangle$$

where $\vee, \wedge$ are interpreted on both as the usual lattice operations. On $\widehat{\mathbf{P}}_n$ we interpret $'$ as pseudo-complementation and $^\circ, D_n, \dots, D_{m-1}$ as the identity with the nullary l interpreted as $l \bmod n$. On $\widehat{\mathbf{P}}_m$ the situation is reversed for $'$ and $^\circ$, that is, $^\circ$ is now pseudo-complementation while $'$ is the identity, and the nullary l is viewed as $l \bmod m$. We have constructed $\widehat{\mathbf{P}}_n$ and $\widehat{\mathbf{P}}_m$ so that they are term equivalent to their respective Post algebras, therefore they are both primal. Using the definition of pseudo-complementation above, it is easily seen that

$$x * y := (x \vee y) \wedge (x^\circ \vee y')$$

is an independence term for $\widehat{\mathbf{P}}_n$ and $\widehat{\mathbf{P}}_m$, hence the product $\widehat{\mathbf{P}}_n \times \widehat{\mathbf{P}}_m$ is a bi-primal algebra. Hence for any $n, m \geq 2$, we may construct a bi-primal algebra of size mn. In fact, given any integer $k > 1$ and an integer n which is a product of k integers, each > 1, we may analogously form a k-primal algebra of size n with each factor term equivalent to a Post algebra.

5. Strong Duality for k-primal Algebras

In this section we aim to produce a strong duality for the quasi-variety generated by a k-primal algebra using only the independence term in the dualising structure. In what follows, we will reserve "k" to mean a fixed, positive integer greater than or equal to 2.

Let $\mathbf{M}$ be a finite k-primal algebra, say $\mathbf{M} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ where $\mathbf{A}_1, \dots, \mathbf{A}_k$ are primal with independence term $*$. We want to show that the structure

$$\underset{\sim}{\mathbf{M}} := \langle M; *^{\mathbf{M}}, \tau \rangle$$

(which, by the remarks following Lemma 4.2, is algebraic over $\mathbf{M}$) yields a strong duality on $\mathbf{M}$.

Since they are primal, $\mathbf{A}_1, \dots, \mathbf{A}_k$ each have have the *ternary discriminator*

$$d(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y \end{cases}$$

as term functions. For each i, suppose s_i is a ternary term such that $s_i^{\mathbf{A}_i}$ is the discriminator on A_i and consider the ternary term

$$p(x, y, z) := *(s_1(x, y, z), \dots, s_k(x, y, z)).$$

Letting $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in M$, a straightforward check ensures that p satisfies

$$p^{\mathbf{M}}(x, y, y) = p^{\mathbf{M}}(x, y, x) = p^{\mathbf{M}}(y, y, x) = x,$$

that is, p is a *Pixley arithmeticity term* for $\mathbf{M}$, showing that each algebra in the variety generated by $\mathbf{M}$ has a distributive lattice of permuting congruences, that is, $\mathbf{M}$ generates an arithmetic variety. There are many results within the general theory of natural dualities that deal with congruence distributivity and arithmeticity—the most conspicuous in this case would seem to be the Arithmetic Strong Duality

Theorem [1, 3.3.11]. However, a hypothesis of this result fails in general for k-primal algebras—for example, $\mathbb{Z}_p \times \mathbb{Z}_q$ has $\mathbb{Z}_p$ as a homomorphic image, however $\mathbb{Z}_p \notin \mathbb{ISP}(\mathbb{Z}_p \times \mathbb{Z}_q)$ when p and q are distinct primes.

Starting from a Pixley arithmeticity term p, if we construct a term m by

$$m(x, y, z) = p(x, p(x, y, z), z),$$

we have

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

That is, m is a *majority term* for $\mathbf{M}$. Hence, we may produce a raw strong duality for $\mathbf{M}$ from the following special case of the NU Strong Duality Theorem [1, 3.3.8], which does not rely on the congruence permutability in $\mathbb{HSP}(\mathbf{M})$.

For an arbitrary finite algebra $\mathbf{N}$, we denote the set of one-element subalgebras of $\mathbf{N}$ by K. For every subalgebra $\mathbf{Q}$ of $\mathbf{N}$, let $\mathrm{irr}(\mathbf{Q})$ be the least number n such that the zero congruence $\perp^{\mathbf{Q}}$ on $\mathbf{Q}$ is a meet of n meet-irreducible congruences. We then define $\mathrm{Irr}(\mathbf{N})$, the *irreducibility index* of $\mathbf{N}$, by

$$\mathrm{Irr}(\mathbf{N}) = \max\{ \mathrm{irr}(\mathbf{Q}) \mid \mathbf{Q} \text{ is a subalgebra of } \mathbf{N} \}.$$

Theorem 5.1. (Majority Strong Duality Theorem) *Assume that $\mathbf{N}$ has a majority term. Let $\mathcal{B}_2$ denote the subalgebras of $\mathbf{N}^2$, let $\mathcal{P}_n$ denote the n-ary algebraic partial operations on $\mathbf{N}$, and define*

$$\underset{\sim}{\mathbf{N}} := \langle N; K, H, \mathcal{B}_2, \tau\rangle \text{ where } H = \bigcup\{ \mathcal{P}_n \mid 1 \le n \le \mathrm{Irr}(\mathbf{N}) \}.$$

Then any structure that strongly entails $\underset{\sim}{\mathbf{N}}$ yields a strong duality on $\mathbf{N}$. ■

A k-primal algebra $\mathbf{M}$ inherits a useful property from its component primal algebras that will allow refinements of the structure $\underset{\sim}{\mathbf{M}}$ arising from the Majority Strong Duality Theorem.

Lemma 5.2. *In a k-primal algebra $\mathbf{M}$, every $a \in M$ is the value of a constant unary term function. Consequently, $\mathbf{M}$ has no proper subalgebras and no endomorphisms other than the identity.*

Proof. Suppose $\mathbf{M} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ with $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ having independence term $*$ and let $a = (a_1, \dots a_k) \in M$. Since each $\mathbf{A}_i$ is primal, for each i there exists a unary term t_i such that $t_i^{\mathbf{A}_i}$ is the constant map distinguishing $a_i \in A_i$. Consider the unary term

$$t(x) := *(t_1(x), \dots, t_k(x)).$$

Then for each $c = (c_1, \dots, c_k) \in M$ we have

$$\begin{aligned} t^{\mathbf{M}}(c) &= *^{\mathbf{M}}(t_1^{\mathbf{M}}(c_1, \dots, c_k), \dots, t_k^{\mathbf{M}}(c_1, \dots, c_k)) \\ &= *^{\mathbf{M}}((t_1^{\mathbf{A}_1}(c_1), \dots, t_1^{\mathbf{A}_k}(c_k)), \dots, (t_k^{\mathbf{A}_1}(c_1), \dots t_k^{\mathbf{A}_k}(c_k))) \\ &= (t_1^{\mathbf{A}_1}(c_1), \dots, t_k^{\mathbf{A}_k}(c_k)) \\ &= (a_1, \dots, a_k) = a, \end{aligned}$$

as required. ■

In terms of the structure $\underset{\sim}{\mathbf{M}}$ above, we now have $K = \varnothing$, the irreducibility index of $\mathbf{M}$ is just $\mathrm{irr}(\mathbf{M})$, and $\widetilde{\mathcal{P}_1} = \{\mathrm{id}_{\mathbf{M}}\}$, the identity map on M.

Incidently, A.F. Pixley [11] has shown that an algebra in an arithmetical variety with no proper subalgebras and no automorphisms other than the identity is *hemi-primal*, that is, the term functions of such an algebra are precisely those finitary operations that preserve all congruences of the algebra. The above lemma together with the previously established arithmeticity of the variety generated by $\mathbf{M}$ therefore establishes the hemi-primality of $\mathbf{M}$. This fact may also be derived from the more general result of Hu [8].

Lemma 5.3. *The congruence lattice of a k-primal algebra $\mathbf{M}$ is Boolean with k atoms. Consequently,* $\mathrm{Irr}(\mathbf{M}) = \mathrm{irr}(\mathbf{M}) = k$.

Proof. Since $\mathbf{M} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ generates a congruence distributive variety (which contains the factors $\mathbf{A}_1, \dots, \mathbf{A}_k$) and such varieties have factorisable congruences (see Fraser and Horn [5]), we have

$$\mathrm{Con}(\mathbf{M}) \cong \mathrm{Con}(\mathbf{A}_1) \times \cdots \times \mathrm{Con}(\mathbf{A}_k).$$

Each $\mathbf{A}_i$ is primal and therefore simple and so each $\mathrm{Con}(\mathbf{A}_i) \cong \mathbf{2}$ where $\mathbf{2}$ denotes the two-element chain. Hence

$$\mathrm{Con}(\mathbf{M}) \cong \mathbf{2}^k.$$

Since the co-atoms are the meet-irreducibles in a Boolean lattice, we have $\mathrm{irr}(\mathbf{M}) = k$. ■

Gathering our findings above, we have a raw duality for $\mathbf{M}$.

Lemma 5.4. *Any structure that strongly entails*

$$\underset{\sim}{\mathbf{M}} = \langle M; H, \mathcal{B}_2, \tau\rangle \textit{ where } H = \bigcup\{\, \mathcal{P}_n \mid 2 \leq n \leq k \,\},$$

yields a strong duality on $\mathbf{M}$. ■

To show that the single k-ary operation $*$ strongly entails the structure $\underset{\sim}{\mathbf{M}}$ above, we will rely on a finite version of Jónsson's Lemma and a result of H. Werner [14], that will exploit respectively the congruence distributivity and permutability present in $\mathbb{HSP}(\mathbf{M})$. The portion of Werner's result we will use is embodied in the following Lemma.

Lemma 5.5. (Werner) *For an algebra $\mathbf{A}$ in a congruence permutable variety, a subset θ of A^2 is a congruence if and only if θ is a reflexive subalgebra.* ■

Theorem 5.6. (Finite Jónsson's Lemma) *Let $\mathbf{B} \leqslant \prod\{\, \mathbf{A}_i \mid i \in I \,\}$ be a subdirect product where I is finite and the congruence lattice of $\mathbf{B}$ is distributive. Suppose $h : \mathbf{B} \to \mathbf{C}$ is a surjective homomorphism where $\mathbf{C}$ is subdirectly irreducible. Then there exists a unique $i \in I$ and a (surjective) homomorphism $g : \mathbf{A}_i \to \mathbf{C}$ such that $h = g \circ \pi_i$ where π_i is the ith projection $\mathbf{B} \to \mathbf{A}_i$.* ■

Theorem 5.7. *The structure $\langle M; *^{\mathbf{M}}, \tau\rangle$ yields a strong duality on* $\mathbf{M}$.

Proof. Let $2 \le n \le k$. We will first show that each partial operation in $\mathcal{P}_n$ is the restriction of a composition of $*^{\mathbf{M}}$ with k n-ary projections. For the purpose of this argument we will denote the n-ary projections $M^n \to M$ by $\rho_1, \dots, \rho_n$. Also, under the identification of $\mathbf{M}^n$ with the nk-fold product

$$\underbrace{\mathbf{A}_1 \times \cdots \times \mathbf{A}_k}_{\mathbf{M}\times\cdots} \times \cdots \times \underbrace{\mathbf{A}_1 \times \cdots \times \mathbf{A}_k}_{\cdots\times\mathbf{M}},$$

we have nk-ary projections

$$\varrho_i : \mathbf{M}^n \to \mathbf{A}_j$$

where $1 \le i \le nk$ and $i \equiv j \pmod{k}$. That is, $\varrho_1, \varrho_{k+1}, \cdots : \mathbf{M} \to \mathbf{A}_1$ while $\varrho_2, \varrho_{k+2}, \cdots : \mathbf{M} \to \mathbf{A}_2$ and so on.

Let $\mathbf{D}$ be a subalgebra of $\mathbf{M}^n$ and let $h : \mathbf{D} \to \mathbf{M}$ be a homomorphism. We have

$$h = h_1 \sqcap \cdots \sqcap h_k$$

where $h_j : \mathbf{D} \to \mathbf{A}_j$ is given by $\pi_j \circ h$ (here, π_j is just the projection $M \to A_j$). We may identify $\mathbf{D}$ as a subdirect product of the nk-fold product $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k \times \cdots \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ since each $\mathbf{A}_j$ has no proper subalgebras by Lemma 5.2. For the same reason, each h_j is surjective.

Let $1 \le j \le k$ and consider the homomorphism $h_j : \mathbf{D} \to \mathbf{A}_j$. By Jónsson's Lemma, Theorem 5.6, there is an $i \in \{1, \dots, nk\}$ and a homomorphism g from one of $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ onto $\mathbf{A}_j$ such that $h_j = g \circ \varrho_i$. But by Lemma 4.2, $\mathbf{A}_j$ may only be a homomorphic image of itself, and by Lemma 5.2, this must be via the identity map. Therefore, in ϱ_i we have $i \equiv j \pmod{k}$ and g is the identity map. That is, $h_j = \varrho_i$ for some i such that $i \equiv j \pmod{k}$.

Hence

$$h = \varrho_{i_1} \sqcap \cdots \sqcap \varrho_{i_k}$$

where each $i_j = m_j k + j$ for some $0 \le m_j \le n-1$. As $n \le k$, some of the m_j may be repeated, so that we recover h via the n-ary composition

$$h = *^{\mathbf{M}}(\rho_{m_1+1}, \rho_{m_2+1}, \dots, \rho_{m_k+1}) \restriction_{\mathbf{D}} .$$

Now, by Lemma 2.3 (b) and (c) and the above, we have the structure

$$\langle M; H, \mathcal{B}_2, \tau\rangle \text{ where } H = \bigcup \{\, \mathcal{P}_n \mid 2 \le n \le k \,\}$$

strongly entailed by

$$\langle M; *^{\mathbf{M}}, \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k, \tau\rangle$$

where $\mathcal{B}_n$ is the set of all n-ary algebraic relations on $\mathbf{M}$. But for each n, the set $\mathcal{B}_n$ is entailed by $\mathcal{B}_2$ (see the NU Duality Theorem 2.3.4 and Duality and Entailment Theorem 2.4.3 in [1]). Hence $\mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is strongly entailed by $\mathcal{B}_2$ (Lemma 2.3 (iii)(a)) which we now show in turn is strongly entailed by just $\{*^{\mathbf{M}}\}$.

As every element of $\mathbf{M}$ is the value of a constant unary term function by Lemma 5.2, each subalgebra of $\mathbf{M}^2$ must contain the diagonal $\{\, (x, x) \mid x \in M \,\}$, that is, each subalgebra must be reflexive. Hence by Lemma 5.5, the subalgebras of $\mathbf{M}^2$ are precisely the congruences of $\mathbf{M}$. But $\mathrm{Con}(\mathbf{M})$ is Boolean (Lemma 5.3) and it is readily seen that its co-atoms are the kernels θ_i of the projections $\pi_i : M \to A_i$ for $i \in \{1, \ldots, k\}$. For each i, we may recover θ_i as the equaliser set (cf. Lemma 2.3 (v))

$$(x, y) \in \theta_i \iff *^{\mathbf{M}}(x, \ldots, x, y, x, \ldots, x) = x$$

where the y appears in the ith position. Then since every proper subalgebra of $\mathbf{M}^2$ is some intersection of the θ_i, by Lemma 2.3 (iv) we have established that $*^{\mathbf{M}}$ (strongly) entails $\mathcal{B}_2$. ■

Corollary 5.8. *Let $\mathbf{M}$ be a bi-primal algebra with independence term $*$. Then the quasi-variety $\mathbb{ISP}(\mathbf{M})$ is dually equivalent to the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ where $\underset{\sim}{\mathbf{M}}$ is the discretely topologised rectangular band $\langle M; *^{\mathbf{M}}, \tau\rangle$.* ■

We will see in the next section (Corollary 6.4) that $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ in the above Corollary is precisely the category of Boolean topological rectangular bands regardless of the choice of $\mathbf{M}$.

If we let $\mathbf{M}$ be the bi-primal algebra $\widehat{\mathbf{P}}_n \times \widehat{\mathbf{P}}_m$ described at the end of the previous section (where $n, m \geq 2$) and apply Corollary 5.8, we have a strong duality via a structure which is nothing but the rectangular band $\mathbf{L}_n \times \mathbf{R}_m$ with the discrete topology adjoined. Applying the Two-for-One Strong Duality Theorem [1, 3.3.2] yields the following.

Corollary 5.9. *The non-zero rectangular band $\mathbf{L}_n \times \underset{\sim}{\mathbf{R}_m}$ is strongly dualised by the structure obtained by adjoining the discrete topology to $\widehat{\mathbf{P}}_n \times \widehat{\mathbf{P}}_m$.* ■

6. Boolean k-dimensional Diagonal Algebras

In 1966, J. Płonka introduced a natural generalisation of the rectangular band. He called an algebra $\mathbf{A}$ having a single k-ary fundamental operation $*$ a *k-dimensional diagonal algebra* if it satisfies the following identities:

$$*(x, \ldots, x) \approx x$$

$$* \begin{pmatrix} *(x_{11}, x_{12}, \ldots, x_{1k}), \\ *(x_{21}, x_{22}, \ldots, x_{2k}), \\ \vdots \\ , *(x_{k1}, x_{k2} \ldots, x_{kk}) \end{pmatrix} \approx *(x_{11}, x_{22} \ldots, x_{kk})$$

These turn out to be equivalent (see [13]) to k-dimensional analogues of associativity and anti-commutativity:

$$*\begin{pmatrix} *(y_1, x_2, \dots, x_k), \\ y_2, \\ \vdots \\ , y_k \end{pmatrix} \approx *\begin{pmatrix} y_1, \\ *(x_2, y_2, x_3, \dots, x_k), \\ y_3, \\ \vdots \\ , y_k \end{pmatrix} \approx \dots$$

$$\dots \approx *\begin{pmatrix} y_1, \\ y_2, \\ \vdots \\ y_{k-2}, \\ *(x_2, \dots, x_{k-1}, y_{k-1}, x_k), \\ y_k \end{pmatrix} \approx *\begin{pmatrix} y_1, \\ y_2, \\ \vdots \\ y_{k-1}, \\ *(x_2, \dots, x_k, y_k) \end{pmatrix}$$

$$*(y, x, \dots, x) \approx *(x, y, x, \dots, x) \approx \cdots \approx *(x, \dots, x, y) \implies x \approx y$$

If $\mathbf{A}$ is a k-dimensional diagonal algebra, for each $i \in \{1, \dots, k\}$ there is a naturally definable equivalence relation θ_i on $\mathbf{A}$ given by

$$(x, y) \in \theta_i :\iff x = *(x, \dots, x, y, x, \dots, x).$$

Note that given $a_1, \dots, a_k \in A$ and a fixed i, in the expression $*(a_1, \dots, a_k)$ we may replace a_i with any element b satisfying $a_i \equiv_{\theta_i} b$ since

$$\begin{aligned} *(a_1, \dots, a_k) &= *(a_1, \dots, a_{i-1}, *(a_i, \dots a_i, b, a_i, \dots, a_i), a_{i+1}, \dots, a_k) \\ &= *(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_k). \end{aligned}$$

It follows that each relation θ_i is in fact a congruence of $\mathbf{A}$.

Theorem 6.1. *A k-dimensional diagonal algebra $\mathbf{A}$ is isomorphic to the product*

$$\mathbf{A}/\theta_1 \times \cdots \times \mathbf{A}/\theta_k$$

via the map

$$\iota : a \mapsto ([a]_{\theta_1}, \dots, [a]_{\theta_k}).$$

In each factor $\mathbf{A}/\theta_i$, the operation $*$ acts as the ith projection.

Proof. We remark that for all $a_1, \dots, a_k \in A$, one has

$$*(a_1, \dots, a_k) \equiv_{\theta_i} a_i$$

for each i.

It follows that for each i, in $\mathbf{A}/\theta_i$ the operation $*^{\mathbf{A}/\theta_i}$ is the ith projection. Indeed, letting $[a_1]_{\theta_i}, \dots, [a_k]_{\theta_i} \in \mathbf{A}/\theta_i$ and applying $*$ gives:

$$*^{\mathbf{A}/\theta_i}([a_1]_{\theta_i}, \dots, [a_k]_{\theta_i}) = [*(a_1, \dots, a_k)]_{\theta_i} = [a_i]_{\theta_i}.$$

The map ι is surjective since we have

$$([a_1]_{\theta_1}, \ldots, [a_k]_{\theta_k}) = \iota(*(a_1, \ldots, a_k))$$

for each $a_1, \ldots, a_k \in A$. Also, the left hand side of this last equation is in fact

$$*^{\mathbf{A}/\theta_1 \times \cdots \times \mathbf{A}/\theta_k}(\iota(a_1), \ldots, \iota(a_k)),$$

so that ι preserves $*$.

Supposing that $\iota(a) = \iota(b)$ gives $a \equiv_{\theta_i} b$ for every i and therefore

$$a = *(a, \ldots, a) = *(b, \ldots, b) = b$$

so that ι is one-to-one. ■

A k-dimensional diagonal algebra $\mathbf{A} = \langle A; * \rangle$ endowed with a compact, totally disconnected topology with respect to which the operation $*$ is continuous will be called a *Boolean k-dimensional diagonal algebra.* The category consisting of all Boolean k-dimensional diagonal algebras (having continuous $*$ preserving maps as morphisms) will be denoted $\boldsymbol{\mathfrak{D}}_k$.

Theorem 6.2. *The factorisation in the previous theorem also holds for Boolean k-dimensional diagonal algebras with the map ι being simultaneously an algebraic and topological isomorphism.*

Proof. Given a Boolean k-dimensional diagonal algebra $\mathbf{A}$, we will show that the algebraic isomorphism ι of the previous Theorem is a homeomorphism where the topology on

$$\mathbf{A}/\theta_1 \times \cdots \times \mathbf{A}/\theta_k$$

is just the product of the quotient spaces $\mathbf{A}/\theta_i$. Note that it follows easily from the facts accumulated in the proof of Theorem 6.1 that for all $([a_1]_{\theta_1}, \ldots, [a_k]_{\theta_k}) \in A/\theta_1 \times \cdots \times A/\theta_k$, we have

$$\iota^{-1}([a_1]_{\theta_1}, \ldots, [a_k]_{\theta_k}) = *(a_1, \ldots, a_k).$$

We will rely on a standard result of topology which states that a continuous map from a compact space to a Hausdorff space is also an open map. Observe that therefore the operation $*$ from $\mathbf{A}^k$ to $\mathbf{A}$ is an open map.

Let B be a basic open set in $A/\theta_1 \times \cdots \times A/\theta_k$. That is, $B = B_1 \times \cdots \times B_k$ where each B_i is open in A/θ_i. But this means that for each i, the union of θ_i-blocks $\cup B_i$ is open as a subset of A, so that $\cup B_1 \times \cdots \times \cup B_k$ is (basic) open as a subset of A^k. Since

$$\iota^{-1}(B) = \{\, *(b_1, \ldots, b_k) \mid [b_i]_{\theta_i} \in B_i \,\} = *(\cup B_1 \times \cdots \times \cup B_k)$$

and $*$ is an open map, we have established that ι is continuous.

To complete the proof, we will show that for each $i \in \{1, \ldots, k\}$, the space A/θ_i is totally disconnected. It will then follow that the product $A/\theta_1 \times \cdots \times A/\theta_k$ is Hausdorff and our continuous map ι is an open map. For each $i \in \{1, \ldots, k\}$ and every $x, y \in A$ we will abbreviate the expression $*(x, \ldots, x, y, x, \ldots, x)$ (where the y appears in the ith position) by $x *_i y$. It follows from the diagonal identities that $*_i$ is a rectangular band operation for each i.

Let $i \in \{1, \ldots, k\}$ and suppose $[a]_{\theta_i}, [b]_{\theta_i} \in A/\theta_i$ are distinct. Then, by definition, we have $a \neq a *_i b$ in A. Since A is totally disconnected, we may choose a clopen subset U such that $a \in U$ while $a *_i b \notin U$. Consider the set

$$V := \{ x \in A \mid (\forall z \in A)[z *_i x \in U \iff z *_i a \in U] \}$$

and observe that $b \notin V$ while $a \in V$. We claim that V is a union of θ_i blocks. Let $x \in V$ and suppose $y \in [x]_{\theta_i}$, that is, $y = y *_i x$. Then for all $z \in A$, we have $z *_i y = z *_i (y *_i x) = z *_i x$ which by hypothesis is in U iff $z *_i a$ is in U, establishing that $y \in V$. Hence $[a]_{\theta_i} \subseteq V$ and $[b]_{\theta_i} \cap V = \varnothing$.

Finally, we claim that V is clopen in A, from which it follows that $[V]_{\theta_i}$ is a clopen subset of A/θ_i separating $[a]_{\theta_i}$ and $[b]_{\theta_i}$.

To see that V is closed, let $r \notin V$. Then there is a $z \in A$ such that the pair $(z *_i r, z *_i a)$ is not in $(U \times U) \cup (A \setminus U \times A \setminus U)$, a clopen subset of $A \times A$. Hence we may contain $(z *_i r, z *_i a)$ in some clopen subset B of $A \times A$ disjoint from $(U \times U) \cup (A \setminus U \times A \setminus U)$. By the continuity of $*_i$ (which follows easily from the continuity of $*$), there are clopen sets R, Z, C containing r, z, a respectively so that $Z *_i R \times Z *_i C \subseteq B$. For each $s \in R$, we then have $(z *_i s, z *_i a) \notin (U \times U) \cup (A \setminus U \times A \setminus U)$, so that R is a clopen set containing r and disjoint from V.

To see that V is open, let $t \in V$. Then for each $z \in A$, we have the pair $(z *_i t, z *_i a) \in (U \times U) \cup (A \setminus U \times A \setminus U)$. Again using the continuity of $*_i$, we may find, for every $z \in A$, clopen subsets T_z, Z_z, C_z containing t, z, a respectively so that $T_z *_i Z_z \times Z_z \times C_z \subseteq (U \times U) \cup (A \setminus U \times A \setminus U)$. Since A is compact, we may choose a finite number of the z so that the Z_z cover A. Let T be the intersection of the corresponding T_z. We have $t \in T$, and for each $s \in T$, the pair $(z *_i s, z *_i a)$ is in $(U \times U) \cup (A \setminus U \times A \setminus U)$ for every $z \in A$, that is, $s \in V$. Hence we have a clopen subset $T \subseteq V$ containing t. ∎

As for rectangular bands, we say that a Boolean k-dimensional diagonal algebra is *non-zero* if each of its factors as per Theorem 6.2 has size ≥ 2.

For any k-primal algebra $\mathbf{M}$, we have a strong duality via the non-zero Boolean k-dimensional diagonal algebra $\underset{\sim}{\mathbf{M}} = \langle M; *^{\mathbf{M}}, \tau \rangle$. Our next result shows that $\underset{\sim}{\mathbf{M}}$ in fact generates the category $\mathfrak{D}_k$ as a topological quasi-variety.

For each $i \in \{1, \ldots, k\}$, let $\mathbf{2}_i$ be the algebra $\langle \{0, 1\}; *^{\mathbf{2}_i} \rangle$ where $*^{\mathbf{2}_i}$ is the k-ary ith projection. Let $\underset{\sim}{\mathbf{2}}$ be the product $\mathbf{2}_1 \times \cdots \times \mathbf{2}_k$ with the discrete topology adjoined. Clearly, We may identify $\underset{\sim}{\mathbf{2}}$ as a substructure of any non-zero Boolean k-dimensional diagonal algebra and hence any $\underset{\sim}{\mathbf{M}}$ that arises from Theorem 5.7.

Theorem 6.3. $\mathfrak{D}_k = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}})$. *Consequently, for any non-zero Boolean k-dimensional diagonal algebra* $\mathbf{B}$, *we have* $\mathfrak{D}_k = \mathbb{IS}_c\mathbb{P}^+(\mathbf{B})$.

Proof. We will show (via Theorem 2.2) that for every Boolean k-dimensional diagonal algebra $\mathbf{A}$ and every $a \neq b$ in A, there exists a morphism $\gamma : \mathbf{A} \to \underset{\sim}{\mathbf{2}}$ such that $\gamma(a) \neq \gamma(b)$.

Write $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ as per Theorem 6.2 and suppose $a \neq b$ in A. Then there is some i such that $a_i \neq b_i$. For this i, choose a clopen subset X of $\mathbf{A}_i$ with $a_i \in X$ but $b_i \notin X$. Define a map $\gamma_i : A_i \to \{0,1\}$ by

$$\gamma_i(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

for each $x \in A_i$. For every other index $j \neq i$, let $\gamma_j : A_j \to \{0,1\}$ be the constant map $x \mapsto 0$. Finally, $\gamma : \mathbf{A} \to \underset{\sim}{\mathbf{2}}$ defined by

$$\gamma(x_1, \ldots, x_k) = (\gamma_1(x_1), \ldots, \gamma(x_k))$$

for each $(x_1, \ldots, x_k) \in A$ is easily shown to be a morphism. ∎

Corollary 6.4. *Let $\underset{\sim}{\mathbf{M}} = \langle M; *, \tau\rangle$ be a finite non-zero rectangular band endowed with the discrete topology. Then $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is the category of Boolean topological rectangular bands.* ∎

7. Category Equivalence

A strong duality between a quasi-variety $\boldsymbol{\mathcal{A}}$ and a topological quasi-variety $\boldsymbol{\mathfrak{X}}$ may be used[3], in some cases, to characterise those quasi-varieties equivalent as a category to $\boldsymbol{\mathcal{A}}$. Any such quasi-variety $\boldsymbol{\mathfrak{B}}$ must be dually equivalent to $\boldsymbol{\mathfrak{X}}$, and the following Theorem (5.6.5 in [1], see also [3]) gives a recipe for distilling this fact into information about $\boldsymbol{\mathfrak{B}}$.

Theorem 7.1. (General Equivalence Theorem) *Assume that $\underset{\sim}{\mathbf{M}}$ yields a strong duality between $\boldsymbol{\mathcal{A}} = \mathbb{ISP}\mathbf{M}$ and $\boldsymbol{\mathfrak{X}} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$, and let $\boldsymbol{\mathfrak{B}}$ be a quasi-variety. Then the following are equivalent:*

(i) $\boldsymbol{\mathcal{A}} \equiv \boldsymbol{\mathfrak{B}}$;

(ii) *there is a finite algebra $\mathbf{N} \in \boldsymbol{\mathfrak{B}}$ and a structure $\underset{\sim}{\mathbf{N}} \in \boldsymbol{\mathfrak{X}}$ (with the same underlying set N) such that*

   (a) *$\underset{\sim}{\mathbf{N}}$ is algebraic over $\mathbf{N}$,*
   (b) *$\boldsymbol{\mathfrak{B}} = \mathbb{ISP}\mathbf{N}$ and $\boldsymbol{\mathfrak{X}} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{N}}$,*
   (c) *$\underset{\sim}{\mathbf{N}}$ yields a strong duality on $\boldsymbol{\mathfrak{B}}$;*

(iii) *there is a finite algebra $\mathbf{N} \in \boldsymbol{\mathfrak{B}}$ and a structure $\underset{\sim}{\mathbf{N}} \in \boldsymbol{\mathfrak{X}}$ (with the same underlying set N) such that*

   (a) *$\underset{\sim}{\mathbf{N}}$ is algebraic over $\mathbf{N}$,*
   (b)′ *$\boldsymbol{\mathfrak{B}} = \mathbb{ISP}\mathbf{N}$,*
   (c)′ *The structures $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{N}}$ are each a retract of a finite power of the other and the following condition holds with respect to $\underset{\sim}{\mathbf{N}}$ and $\mathbf{N}$*
   (CLO) *For every $n \in \mathbb{N}$, each morphism $\alpha : \underset{\sim}{\mathbf{N}}^n \to \underset{\sim}{\mathbf{N}}$ is a term function $\mathbf{N}^n \to \mathbf{N}$.* ∎

[3] Another method has been pioneered by R. McKenzie [10] and does not rely on the presence of a duality.

Combining Theorem 6.3 with our strong duality of Theorem 5.7 and verifying part (ii) of the General Equivalence Theorem 7.1, we have (for a fixed k) the quasi-varieties generated by any two k-primal algebras equivalent as categories. This is one half of the following result:

Theorem 7.2. *Let* $\mathbf{M}$ *be a finite k-primal algebra and let* $\boldsymbol{\mathcal{A}} = \mathbb{ISPM}$ *be the quasi-variety it generates. A quasi-variety* $\boldsymbol{\mathcal{B}}$ *is equivalent as a category to* $\boldsymbol{\mathcal{A}}$ *if and only if there is a finite k-primal algebra* $\mathbf{N}$ *such that* $\boldsymbol{\mathcal{B}} = \mathbb{ISPN}$*.*

Proof. It remains to show the "only if". Let $\boldsymbol{\mathcal{B}}$ be a quasi-variety equivalent to $\boldsymbol{\mathcal{A}} = \mathbb{ISPM}$. By part (iii)(b)$'$ of the General Equivalence Theorem 7.1 there is a finite algebra $\mathbf{N}$ such that $\boldsymbol{\mathcal{B}} = \mathbb{ISPN}$ and by part (iii)(a), there is an algebraic k-dimensional diagonal operation $*$ on $\mathbf{N}$ (so that $\underset{\sim}{\mathbf{N}} = \langle N; *, \tau\rangle$). We must show that $\mathbf{N}$ is k-primal.

As earlier, for each $i \in \{1, \dots, k\}$, define the equivalence relation θ_i on N by:

$$(x, y) \in \theta_i :\iff x = *(x, \dots, x, y, x, \dots, x).$$

Now, since $*$ preserves the fundamental operations of $\mathbf{N}$, each θ_i forms a congruence of $\mathbf{N}$. Furthermore, it is straightforward to check that the map $\iota : N \to N/\theta_1 \times \cdots \times N/\theta_k$ given by

$$\iota(x) = ([x]_{\theta_1}, \dots, [x]_{\theta_k})$$

is an isomorphism $\mathbf{N} \to \mathbf{N}/\theta_1 \times \cdots \times \mathbf{N}/\theta_k$.

Since (CLO) holds for $\underset{\sim}{\mathbf{N}}$ and $\mathbf{N}$ by part (iii)(c)$'$ of Theorem 7.1 and $*$ is itself a $*$-preserving k-ary operation on N (see the comments preceding Definition 2), there is a k-ary term d such that $d^{\mathbf{N}} = *$. Hence, since $*$ is the ith projection on $\mathbf{N}/\theta_i$ as in Theorem 6.1, d is an independence term for $\{\mathbf{N}/\theta_1, \dots, \mathbf{N}/\theta_k\}$.

It remains to show that each factor $\mathbf{N}/\theta_i$ is primal. By part (iii)(c)$'$ of the General Equivalence Theorem 7.1, $\underset{\sim}{\mathbf{M}}$ is a retract of a finite power of $\underset{\sim}{\mathbf{N}}$. But then each factor $\mathbf{N}/\theta_i$ must be non-trivial.

Let $i \in \{1, \dots, k\}$ and let $g : (N/\theta_i)^n \to N/\theta_i$ be an arbitrary n-ary operation on N/θ_i (for some $n \in \mathbb{N}$). Define an n-ary operation $\hat{g}$ on $N/\theta_1 \times \cdots \times N/\theta_k$ as follows. For all $x_1^1, \dots, x_k^1, \dots, x_1^n, \dots, x_k^n \in N$ associate the n-tuple

$$\begin{pmatrix} ([x_1^1]_{\theta_1}, \dots, [x_k^1]_{\theta_k}), \\ \vdots \\ , ([x_1^n]_{\theta_1}, \dots, [x_k^n]_{\theta_k}) \end{pmatrix}$$

of elements of $N/\theta_1 \times \cdots \times N/\theta_k$ with the element

$$([x_1^1]_{\theta_1}, \dots, [x_{i-1}^1]_{\theta_{i-1}}, g([x_i^1]_{\theta_i}, \dots, [x_i^n]_{\theta_i}), [x_{i+1}^1]_{\theta_{i+1}}, \dots, [x_k^1]_{\theta_k}).$$

That is, identifying $(N/\theta_1 \times \cdots \times N/\theta_k)^n$ with $(N/\theta_1)^n \times \cdots \times (N/\theta_k)^n$, we may write

$$\hat{g} = (\pi_1, \dots, \pi_1, g, \pi_1, \dots, \pi_1).$$

It may be verified that $\hat{g}$ preserves $*$ on $\mathbf{N}/\theta_1 \times \cdots \times \mathbf{N}/\theta_k$ and hence there is an n-ary term t such that $\hat{g} = t^{\mathbf{N}/\theta_1 \times \cdots \times \mathbf{N}/\theta_k}$ by (CLO). But, again using the identification

$N^n = (N/\theta_1 \times \cdots \times N/\theta_k)^n = (N/\theta_1)^n \times \cdots \times (N/\theta_k)^n$, we have

$$\hat{g} = (\pi_1, \ldots, \pi_1, g, \pi_1, \ldots, \pi_1) = \\ (t^{\mathbf{N}/\theta_1}, \ldots, t^{\mathbf{N}/\theta_{i-1}}, t^{\mathbf{N}/\theta_i}, t^{\mathbf{N}/\theta_{i+1}}, \ldots, t^{\mathbf{N}/\theta_k}) = t^{\mathbf{N}/\theta_1 \times \cdots \times \mathbf{N}/\theta_k},$$

giving $g = t^{\mathbf{N}/\theta_i}$. Hence $\mathbf{N}/\theta_i$ is primal. ∎

We have defined k-primal algebras in Definition 4.3 in terms of their factorisation properties. Using the above argument, we now have the appropriate primality condition characterising these algebras via their term functions: an algebra $\mathbf{A}$ is k-primal if and only if there is an m-dimensional diagonal operation $*$ (for some $m \geq k$) such that the term functions of $\mathbf{A}$ are precisely the $*$-preserving operations. One direction is given by the implicit satisfaction of (CLO) in the duality of Theorem 5.7 while the other is given by the above proof, noting that we may have trivial factors.

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