NATURAL DUALITIES FOR SEMILATTICE-BASED ALGEBRAS

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Abstract. While every finite lattice-based algebra is dualisable, the same is not true of semilattice-based algebras. We show that a finite semilattice-based algebra is dualisable if all its operations are compatible with the semilattice operation. We also give examples of infinite semilattice-based algebras that are dualisable. In contrast, we present a general condition that guarantees the inherent non-dualisability of a finite semilattice-based algebra. We combine our results to characterise dualisability amongst the finite algebras in the classes of flat extensions of partial algebras and closure semilattices. Throughout, we emphasise the apparent connection between the dualisability of an algebra and the residual character of the variety it generates.

1. Introduction

An algebra is *semilattice based* if it has a semilattice operation amongst its fundamental operations. Semilattice-based algebras have become increasingly important in the study of general algebra. In particular, they are important within the ground-breaking undecidability results of R. McKenzie [20, 21, 22, 23].

Because the class of all semilattice-based algebras is so wide and varied, finding an acceptable characterisation of its dualisable members seems beyond reach. Every finite lattice-based algebra is semilattice based and dualisable [9]. In contrast, every finite non-trivial implication algebra is semilattice based and non-dualisable [6], and every finite non-boolean pseudocomplemented semilattice is non-dualisable [7].

In this paper, we find large classes of dualisable and non-dualisable semilattice-based algebras. First, we prove that a finite semilattice-based algebra must be dualisable if all of its fundamental operations are compatible with the semilattice operation. The class of all such algebras includes all finite modals, for example, where a *modal* is an idempotent, entropic semilattice-based algebra (see the text by Romanowska and Smith [26]). In general, the dualising structures that we obtain involve relations of high arity. Using different techniques, we shall show that many finite semilattices with automorphism are, in fact, strongly self-dualising.

In the negative direction, we show that a semilattice-based algebra must be inherently non-dualisable if one of its fundamental operations fails to be compatible with the semilattice operation in a particular way.

Throughout this paper, we emphasise the apparent link between the dualisability of a finite algebra and the residual character of the variety that it generates. So far,

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every algebra that is known to be dualisable is also known to generate a residually small variety. This leads us to the question:

> *Does every finite dualisable algebra generate a residually small variety?*

The converse does not hold. For example, the two-element implication algebra is non-dualisable and yet generates a residually small variety.

Two of the authors were so bold as to conjecture that the answer to this question is yes [24]. Even if the answer is no, the evidence of our current examples suggests that there may at least be natural conditions on classes of algebras under which dualisability implies residual smallness.

In this paper, we find two large classes of semilattice-based algebras in which the following conditions are equivalent for each finite member $\underline{\mathbf{M}}$:

- $\underline{\mathbf{M}}$ is dualisable;
- each fundamental operation of $\underline{\mathbf{M}}$ is compatible with the semilattice operation;
- the variety generated by $\underline{\mathbf{M}}$ is residually small;
- the variety generated by $\underline{\mathbf{M}}$ is residually very finite.

The classes we present are: flat algebras with an absorbing bottom, which can be obtained by taking the flat one-point extension of a partial algebra; closure semilattices, which are semilattices with an added unary operation that models a closure operator. In passing, we correct a minor error in a result of the second author [14], who initiated the study of varieties of closure semilattices.

The theory of natural dualities can be applied to produce a duality for a class of algebras that is generated by an infinite algebra with a compatible compact topology. Unfortunately, the number of naturally occurring natural dualities based on infinite algebras is very small. The best known example is the duality for abelian groups due to L. Pontrjagin [25], and a less well-known example is the duality for Ockham algebras due to M. S. Goldberg [12]. We end this paper with some examples of infinite semilattices with automorphism that admit 'natural' natural dualities.

2. NATURAL DUALITIES

The Hofmann–Mislove–Stralka duality for meet semilattices with 1 [13] was one of the motivating examples for the general theory of natural dualities. This duality is based on the two-element semilattice $\underline{\mathbf{2}}_1 := \langle \{0,1\}; \wedge, 1\rangle$ and the two-element discrete semilattice $\underset{\sim}{\mathbf{2}}_1 := \langle \{0,1\}; \wedge, 1, \mathcal{T}\rangle$. More generally, every finite semilattice (with or without 0 or 1) is also the base for a natural duality [5].

In this section, we give a brief overview of natural dualities, and refer to the Clark–Davey text [2] for a detailed treatment. Let $\underline{\mathbf{M}} = \langle M; F\rangle$ be a fixed finite algebra. We shall indicate how the theory of natural dualities attempts to provide a class $\boldsymbol{\mathcal{X}}$ of topological structures that is dually equivalent (as a category) to the quasivariety $\boldsymbol{\mathcal{A}}$ generated by $\underline{\mathbf{M}}$.

We build the class $\boldsymbol{\mathcal{X}}$ from a new topological structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ on the same underlying set as the algebra $\underline{\mathbf{M}}$, where G, H and R are (respectively) sets of finitary operations, partial operations and relations on M, and $\mathcal{T}$ is a topology on M. We require some compatibility between $\underset{\sim}{\mathbf{M}}$ and $\underline{\mathbf{M}}$.

For each $n \in \omega\backslash\{0\}$, an n-ary relation on M is called *algebraic over* $\underline{\mathbf{M}}$ if it forms a subalgebra of $\underline{\mathbf{M}}^n$. For each $n \in \omega$, an n-ary (partial) operation on M

is *algebraic over* $\underline{\mathbf{M}}$ if its graph is algebraic over $\underline{\mathbf{M}}$ or, equivalently, if its domain forms a subalgebra of $\underline{\mathbf{M}}^n$ and it is a homomorphism into $\underline{\mathbf{M}}$.

We say that $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ is an *alter ego of* $\underline{\mathbf{M}}$ if the operations, partial operations and relations in $G \cup H \cup R$ are all algebraic over $\underline{\mathbf{M}}$ and the topology $\mathcal{T}$ is discrete. The compatibility between the alter ego $\underset{\sim}{\mathbf{M}}$ and the algebra $\underline{\mathbf{M}}$ guarantees that the category-theoretic aspects of the duality set-up run smoothly.

Now fix an alter ego $\underset{\sim}{\mathbf{M}}$ of the algebra $\underline{\mathbf{M}}$. For each algebra $\mathbf{A}$ in the quasivariety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$, we define the *dual* $\mathrm{D}(\mathbf{A})$ to be the homset $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$, consisting of all homomorphisms from $\mathbf{A}$ into $\underline{\mathbf{M}}$, regarded as a (topologically) closed substructure of $\underset{\sim}{\mathbf{M}}^A$. Thus, the dual of $\mathbf{A}$ lies in the class $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of all isomorphic copies of closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$. Similarly, for each topological structure $\mathbf{X}$ in $\boldsymbol{\mathcal{X}}$, we define the *dual* $\mathrm{E}(\mathbf{X})$ to be the homset $\boldsymbol{\mathcal{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$, consisting of all morphisms from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$, regarded as a subalgebra of $\underline{\mathbf{M}}^X$. (We can extend D and E to contravariant functors between the categories $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, in the obvious way.)

Let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Then there is a natural embedding $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$, from $\mathbf{A}$ into its double dual $\mathrm{ED}(\mathbf{A})$, that associates with each element a of A the *evaluation map* $e_{\mathbf{A}}(a) : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ given by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. If $e_{\mathbf{A}}$ is a surjection, and therefore an isomorphism, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathbf{A}$. If the alter ego $\underset{\sim}{\mathbf{M}}$ yields a duality on every algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{A}}$, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ or, more briefly, that $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$. (In this case, the functors D and E yield a dual equivalence between the category $\boldsymbol{\mathcal{A}}$ and the full subcategory $\mathrm{D}(\boldsymbol{\mathcal{A}})$ of $\boldsymbol{\mathcal{X}}$.) If there is some alter ego that dualises $\underline{\mathbf{M}}$, then we say that the algebra $\underline{\mathbf{M}}$ is *dualisable*.

The theory of natural dualities provides several general results that can be used to prove that a given alter ego $\underset{\sim}{\mathbf{M}}$ dualises the algebra $\underline{\mathbf{M}}$. One that we shall be using involves a natural interpolation condition. We will define this condition in its most general form, in which it describes a relationship between two structures on the same underlying set.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle$ and $\mathbf{M}_2 = \langle M; G_2, H_2, R_2\rangle$ be any pair of structures. We say that $\mathbf{M}_2$ *satisfies the interpolation condition* (IC) *with respect to* $\mathbf{M}_1$ if

(IC) for each $n \in \omega\backslash\{0\}$ and each substructure $\mathbf{X}$ of $(\mathbf{M}_2)^n$, every morphism $\alpha : \mathbf{X} \to \mathbf{M}_2$ extends to an n-ary term function of $\mathbf{M}_1$.

For our applications, we shall only need to interpret this condition where $\mathbf{M}_1$ has no partial operations in its type, and so all term functions of $\mathbf{M}_1$ will be total. Note that it makes no difference if we add the discrete topology to the type of $\mathbf{M}_1$ or $\mathbf{M}_2$.

Since the alter ego $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ is compatible with the algebra $\underline{\mathbf{M}}$, every term function of $\underline{\mathbf{M}}$ is a morphism with respect to $\underset{\sim}{\mathbf{M}}$. So, if $\underset{\sim}{\mathbf{M}}$ satisfies the interpolation condition (IC) with respect to $\underline{\mathbf{M}}$, then it follows that $\underset{\sim}{\mathbf{M}}$ determines the clone of $\underline{\mathbf{M}}$ and that $\underset{\sim}{\mathbf{M}}$ is injective in $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$. In the case that H is empty and R is finite, this is enough to guarantee that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$; this is the Second Duality Theorem (Davey and Werner [9]), stated below.

2.1. **Second Duality Theorem.** [2, 2.2.7] *Let* $\underline{\mathbf{M}}$ *be a finite algebra, and let* $\underset{\sim}{\mathbf{M}}$ *be an alter ego of* $\underline{\mathbf{M}}$ *with no partial operations and only finitely many relations in its type. If* $\underset{\sim}{\mathbf{M}}$ *satisfies the interpolation condition* (IC) *with respect to* $\underline{\mathbf{M}}$, *then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ *and* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$.

We shall also obtain dualities by applying the piggyback techniques of Davey and Werner [10, 11] and Davey and Priestley [8]. The necessary background for these techniques will be presented in Section 7, where it is needed.

The definition of a natural duality is one-sided. Given $\mathbf{X}$ in the dual category $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, there is also a natural embedding $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$, from $\mathbf{X}$ into its double dual $\mathrm{DE}(\mathbf{X})$, that associates with each element x of X the *evaluation map* $\varepsilon_{\mathbf{X}}(x) : \mathrm{E}(\mathbf{X}) \to \underline{\mathbf{M}}$ given by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$, for all $\alpha \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$. If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and $\varepsilon_{\mathbf{X}}$ is a surjection, and therefore an isomorphism, for all $\mathbf{X} \in \mathcal{X}$, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\boldsymbol{\mathcal{A}}$. (In this case, the functors D and E yield a dual category equivalence between $\boldsymbol{\mathcal{A}}$ and $\mathcal{X}$.) If, in addition, the alter ego $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$.

There are several tools available for proving that an alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ yields strong duality on $\boldsymbol{\mathcal{A}}$; see Chapter 3 of the Clark–Davey text [2] and the appendix of the Pitkethly–Davey text [24]. We shall be relying on the following result, which is part of the Two-for-One Strong Duality Theorem [2].

2.2. **Theorem.** [2, 3.3.2] *Let* $\underline{\mathbf{M}}$ *be a finite algebra, and let* $\underset{\sim}{\mathbf{M}}$ *be an alter ego of* $\underline{\mathbf{M}}$ *with no partial operations or relations in its type. Then* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ *if and only if the following conditions hold*:

(1) *every element of* M *that forms a one-element subalgebra of* $\underline{\mathbf{M}}$ *is the value of a nullary term function of* $\underset{\sim}{\mathbf{M}}$;
(2) $\underset{\sim}{\mathbf{M}}$ *satisfies the interpolation condition* (IC) *with respect to* $\underline{\mathbf{M}}$;
(3) $\underline{\mathbf{M}}$ *satisfies the interpolation condition* (IC) *with respect to* $\underset{\sim}{\mathbf{M}}$.

As well as showing that a large class of semilattice-based algebras are dualisable, we shall also present a class of very badly non-dualisable semilattice-based algebras.

Within the study of potentially useful dualities, it is reasonable to restrict to finitary operations, partial operations and relations in the type of our alter egos. But this restriction is not necessary within the general theory. Let κ be any cardinal. We may say that a finite algebra is *κ-dualisable* if it is dualised by an alter ego whose operations, partial operations and relations all have arity less than κ; the usual notion of dualisability corresponds to ω-dualisability.

We now say that the finite algebra $\underline{\mathbf{M}}$ is *inherently non-κ-dualisable* if each finite algebra that has $\underline{\mathbf{M}}$ as a subalgebra is not κ-dualisable. (Our definition of inherent non-κ-dualisability is simpler than but equivalent to the original definition [6, 24].) A finite algebra is as badly behaved as possible, with respect to dualisability, if it is inherently non-κ-dualisable, for all κ.

The following theorem is the main tool for proving inherent non-dualisability.

2.3. **Inherent Non-dualisability Theorem.** [6, Theorem 3] *Let* κ *be an infinite cardinal, let* $\underline{\mathbf{M}}$ *be a finite algebra and let* $\varphi : \omega \to \omega$. *Assume that there is a subalgebra* $\mathbf{A}$ *of* $\underline{\mathbf{M}}^Z$, *for some set* Z, *and a subset* A_0 *of* A *of cardinality at least* κ *such that*

(1) *for every* $n \in \omega$ *and every congruence* θ *on* $\mathbf{A}$ *of index at most* n, *the equivalence relation* $\theta{\restriction}_{A_0}$ *has a unique block of size greater than* $\varphi(n)$,
(2) *the algebra* $\mathbf{A}$ *does not contain the element* g *of* M^Z *defined by* $g(z) := a_z(z)$, *where* a_z *is any element of the unique infinite block of* $\ker(\pi_z){\restriction}_{A_0}$.

Then $\underline{\mathbf{M}}$ *is inherently non-κ-dualisable.*

3. A SEMILATTICE-BASED DUALITY

We shall say that an algebra $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ is *semilattice based* if the reduct $\langle M; \wedge\rangle$ is a meet semilattice. In this section, we present our main positive result. First, we require a couple of definitions.

Let A be a non-empty set and let $\wedge$ be a semilattice operation on A. Let $f : A^n \to A$ be an n-ary operation on A, for some $n \in \omega$. We say that f is *compatible with* $\wedge$ if it is a homomorphism from $\langle A; \wedge\rangle^n$ to $\langle A; \wedge\rangle$. We say that f is *weakly compatible with* $\wedge$ if, for all $\boldsymbol{a} = (a_0, \dots, a_{n-1}) \in A^n$ and all $i \in n$, the unary operation $f_{\boldsymbol{a},i} : A \to A$, defined by

$$f_{\boldsymbol{a},i}(x) := f(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1}),$$

is an endomorphism of $\langle A; \wedge\rangle$.

The idempotence of $\wedge$ guarantees that, for all $n \in \omega$, every n-ary operation that is compatible with $\wedge$ is also weakly compatible with $\wedge$, and that every nullary operation is compatible with $\wedge$. Every weakly $\wedge$-compatible operation preserves the order induced on A by $\wedge$. If $\langle A; \vee, \wedge\rangle$ is a lattice, for example, then $\vee : A^2 \to A$ is weakly compatible with $\wedge$ if and only if the lattice is distributive, but $\vee$ is not compatible with $\wedge$ unless A is trivial.

The proof of the following lemma is a simple induction on the complexity of terms.

3.1. **Lemma.** *Let $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ be a semilattice-based algebra and assume that every operation in F is weakly compatible with $\wedge$. Let $t(x_0, \dots, x_{n-1})$ be an n-ary term of type $\{\wedge\} \cup F$, for some $n \in \omega$. Then $\underline{\mathbf{M}}$ satisfies*

$$t(x_0, \dots, x_{n-1}) \approx \bigwedge_{i \in k} w_i(x_0, \dots, x_{n-1}),$$

for some $k \in \omega\backslash\{0\}$ and n-ary terms $w_0(x_0, \dots, x_{n-1}), \dots, w_{k-1}(x_0, \dots, x_{n-1})$ of type F.

Given an algebra $\underline{\mathbf{M}}$, we let R_n denote the set of all n-ary relations that are algebraic over $\underline{\mathbf{M}}$. The following lemma gives us, as a corollary, the positive dualisability result we seek.

3.2. **Lemma.** *Let $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ be a finite semilattice-based algebra and assume that every operation in F is weakly compatible with $\wedge$. Then the structure $\langle M; \wedge, R_{|M|+1}\rangle$ satisfies the interpolation condition* (IC) *with respect to $\underline{\mathbf{M}}$.*

Proof. Define the structure $\mathbf{M}_0 := \langle M; \wedge, R_{|M|+1}\rangle$. Let $n \in \omega\backslash\{0\}$, let $\mathbf{X}$ be a non-empty substructure of $(\mathbf{M}_0)^n$ and let $\alpha : \mathbf{X} \to \mathbf{M}_0$ be a morphism. We need to show that α extends to a term function of $\underline{\mathbf{M}}$.

Note that the structure $\mathbf{X}$ has a semilattice reduct. For each $a \in \alpha(X) \subseteq M$, let $\widehat{a}$ denote the element $\bigwedge \alpha^{-1}(a)$ of X. Then $\widehat{a}$ is the least element of $\mathbf{X}$ that maps to a under α, for each $a \in \alpha(X)$, since the map α preserves $\wedge$.

Let T_n denote the set of n-ary term functions of the reduct $\langle M; F\rangle$ of $\underline{\mathbf{M}}$. Since M is finite, so is T_n. Moreover, since each operation in F is weakly compatible with $\wedge$, each function in T_n is order preserving. First, we show that there exists $w \in T_n$ such that $a \leqslant w(\widehat{a})$, for all $a \in \alpha(X)$.

Define $\ell := |\alpha(X)|$ and let $a_0, \dots, a_{\ell-1}$ be a list of the elements of $\alpha(X)$. As α preserves $R_{|M|+1}$, it also preserves each algebraic relation on $\underline{\mathbf{M}}$ of arity less than

$|M|+1$. Hence, the map α preserves the ℓ-ary algebraic relation r_α on $\underline{\mathbf{M}}$ generated by the set

$$B := \{\, \big(\widehat{a_0}(i), \dots, \widehat{a_{\ell-1}}(i)\big) \bigm| i \in n \,\}.$$

Since $(\widehat{a_0}, \dots, \widehat{a_{\ell-1}}) \in r_\alpha^{\mathbf{X}}$, we have $(a_0, \dots, a_{\ell-1}) = (\alpha(\widehat{a_0}), \dots, \alpha(\widehat{a_{\ell-1}})) \in r_\alpha$. As r_α is generated by B, there exists an n-ary term function t of the algebra $\underline{\mathbf{M}}$ such that $t(\widehat{a_j}(0), \dots, \widehat{a_j}(n-1)) = a_j$, for all $j \in \ell$. Thus $t(\widehat{a}) = a$, for all $a \in \alpha(X)$. Using Lemma 3.1, there is at least one term function $w \in T_n$ such that $a \leqslant w(\widehat{a})$, for all $a \in \alpha(X)$.

We may now define an n-ary term function t_α of $\underline{\mathbf{M}}$ by

$$t_\alpha(m_0, \dots, m_{n-1}) := \bigwedge \{\, w(m_0, \dots, m_{n-1}) \bigm| w \in T_n \text{ and } \big(\forall a \in \alpha(X)\big)\, a \leqslant w(\widehat{a}) \,\},$$

for all $m_0, \dots, m_{n-1} \in M$. We will show that $t_\alpha{\restriction}_X = \alpha$.

Let $x \in X$ and define $c := \alpha(x)$. Then, by definition, we have $\widehat{c} \leqslant x$. Thus

$$\alpha(x) = c \leqslant t_\alpha(\widehat{c}) \leqslant t_\alpha(x), \tag{1}$$

where the first inequality follows from the definition of t_α, and the second follows from the fact that t_α is order preserving.

Now define r_x to be the $(\ell+1)$-ary algebraic relation on $\underline{\mathbf{M}}$ generated by the set

$$B_x := \{\, \big(\widehat{a_0}(i), \dots, \widehat{a_{\ell-1}}(i), x(i)\big) \bigm| i \in n \,\}.$$

Since α preserves r_x and since $(\widehat{a_0}, \dots, \widehat{a_{\ell-1}}, x) \in r_x^{\mathbf{X}}$, we have

$$(a_0, \dots, a_{\ell-1}, c) = \big(\alpha(\widehat{a_0}), \dots, \alpha(\widehat{a_{\ell-1}}), \alpha(x)\big) \in r_x.$$

As r_x is generated by B_x, there exists an n-ary term function t_x of the algebra $\underline{\mathbf{M}}$ such that $t_x(\widehat{a_j}(0), \dots, \widehat{a_j}(n-1)) = a_j$, for all $j \in \ell$, and $t_x(x(0), \dots, x(n-1)) = c$. Thus, $t_x(\widehat{a}) = a$, for all $a \in \alpha(X)$, and $t_x(x) = c$. By Lemma 3.1, there exist n-ary term functions $w_0, \dots, w_{k-1} \in T_n$, for some $k \in \omega\backslash\{0\}$, such that

$$t_x(m_0, \dots, m_{n-1}) = \bigwedge_{i \in k} w_i(m_0, \dots, m_{n-1}),$$

for all $m_0, \dots, m_{n-1} \in M$.

We now have $a = t_x(\widehat{a}) = w_0(\widehat{a}) \wedge \cdots \wedge w_{k-1}(\widehat{a})$, for all $a \in \alpha(X)$. So $a \leqslant w_i(\widehat{a})$, for all $a \in \alpha(X)$ and all $i \in k$. By the definition of t_α, it follows that $t_\alpha \leqslant t_x$ (pointwise), and therefore

$$t_\alpha(x) \leqslant t_x(x) = c = \alpha(x). \tag{2}$$

By (1) and (2), we have $t_\alpha(x) = \alpha(x)$. Hence $t_\alpha{\restriction}_X = \alpha$, as required. ☐

The Second Duality Theorem 2.1 ensures that, when $\underline{\mathbf{M}}$ is compatible with $\wedge$ in the lemma above, we have a natural duality.

3.3. **Semilattice-Based Duality Theorem.** *Let* $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ *be a finite semilattice-based algebra and assume that each operation in* F *is compatible with* $\wedge$*. Then the alter ego* $\underset{\sim}{\mathbf{M}} := \langle M; \wedge, R_{|M|+1}, \mathfrak{T}\rangle$ *of* $\underline{\mathbf{M}}$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$ *and is injective in the dual category* $\mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$.

A variety is *residually small* if there is a cardinal bound on the sizes of its subdirectly irreducible members, and it is *residually very finite* if there is a finite bound. Now let $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ be a finite semilattice-based algebra. Kearnes and Szendrei [18, 5.1] have shown that, if every operation in F is compatible with $\wedge$, then the variety generated by $\underline{\mathbf{M}}$ is residually very finite. So the Semilattice-Based

Duality Theorem does not provide a negative answer to the question raised in the introduction.

4. A NON-DUALISABILITY THEOREM

Every finite lattice is semilattice based and dualisable, however the operations of a non-trivial lattice are not compatible with each other. Hence the class of finite lattices is a quite dramatic illustration that the converse of the Semilattice-Based Duality Theorem 3.3 does not hold. In this section, we nevertheless provide a weak converse to the Semilattice-Based Duality Theorem, by showing that there is a particular kind of failure of $\wedge$-compatibility that does give rise to non-dualisability. We shall prove a general theorem that will have several natural applications in the following two sections.

First, we set up some notation and give a definition. Let M and Z be non-empty sets. For all $k \in \omega\backslash\{0\}$, all distinct $z_1, \dots, z_k \in Z$ and all $a, b_1, \dots, b_k \in M$, we define the element $a^{b_1 \dots b_k}_{z_1 \dots z_k}$ of M^Z by

$$a^{b_1 \dots b_k}_{z_1 \dots z_k}(z) = \begin{cases} b_i & \text{if } z = z_i, \text{ for some } i \in \{1, \dots, k\}, \\ a & \text{otherwise,} \end{cases}$$

for all $z \in Z$. For each $m \in M$, let $\underline{m}$ denote the constant map in M^Z with value m.

Let us say that a subset I of an algebra $\mathbf{A}$ is *absorbing* if, for each $n \in \omega\backslash\{0\}$ and each non-constant n-ary fundamental operation f of $\mathbf{A}$, we have

$$\{a_0, \dots, a_{n-1}\} \cap I \neq \varnothing \implies f(a_0, \dots, a_{n-1}) \in I,$$

for all $a_0, \dots, a_{n-1} \in A$. For example, a subset of a meet semilattice is absorbing if and only if it is decreasing. But a lattice has no proper, non-empty subsets that are absorbing. Absorbing subsets generalise ideals of semigroups. In particular, if I is an absorbing subset of $\mathbf{A}$, then there is a congruence on $\mathbf{A}$ that collapses I but does not collapse any other elements of A. So an absorbing subset of an algebra can always be collapsed to a single absorbing element, without otherwise altering the structure of the algebra.

We also require the following elementary lemma.

4.1. **Lemma.** *Let S be a set of cardinality greater than κ^n, where κ is any cardinal and $n \in \omega\backslash\{0\}$. Let $\theta_0, \dots, \theta_{n-1}$ be equivalence relations on S, each of index at most κ. Then the equivalence relation $\theta_0 \cap \dots \cap \theta_{n-1}$ on S has a nontrivial block.*

Proof. Define $\theta := \theta_0 \cap \dots \cap \theta_{n-1}$. Then there is a one-to-one map from S/θ to $S/\theta_0 \times \dots \times S/\theta_{n-1}$. So $|S/\theta| \leqslant \kappa^n$, as required. □

Note that, if we apply the previous lemma with an infinite cardinal κ, then we have $\kappa^n = \kappa$, for all $n \in \omega\backslash\{0\}$.

The following theorem gives a condition for a semilattice-based algebra to be non-dualisable, based around the failure of one of its operations to be $\wedge$-compatible. We need to assume a special type of non-compatibility, since the theorem must not apply to non-trivial lattices.

4.2. **Theorem.** *Let $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ be a finite semilattice-based algebra with an absorbing subset I. Assume that there is an operation $f \in F$ (of arity n) and elements $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in M$ such that*

(1) *$f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}) \notin I$, and*

(2) $f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}) \wedge f(a_0 \wedge b_0, \dots, a_{n-1} \wedge b_{n-1}) \in I$.

Then $\underline{\mathbf{M}}$ *is inherently non-*κ*-dualisable, for every infinite cardinal* κ.

Proof. For each $i \in n$, define $c_i := a_i \wedge b_i$. Now define

$$a := f(a_0, \dots, a_{n-1}), \quad b := f(b_0, \dots, b_{n-1}) \quad \text{and} \quad c := f(c_0, \dots, c_{n-1}).$$

Then we can define two elements of M:

$$d := a \wedge b \wedge c \quad \text{(down)} \qquad \text{and} \qquad u := a \wedge b \quad \text{(up)}.$$

Our assumptions guarantee that $d \in I$ and $u \notin I$, with $d < u$.

We will use the Inherent Non-dualisability Theorem 2.3, with the map $\varphi : \omega \to \omega$ given by $\varphi(k) := k^n$. Let κ be an infinite cardinal, let Z be a set of size κ and fix an element $0 \in Z$.

The set $A_0 \subseteq M^Z$, given by

$$A_0 := \{\, u^{dd}_{0z} \mid z \in Z\backslash\{0\} \,\},$$

has cardinality κ. Now define $A \subseteq M^Z$ by

$$A := \{\, x \in M^Z \mid |x^{-1}(I)| \geqslant 2 \,\} \cup \{\, x \in M^Z \mid x^{-1}\big(x(0)\big) \text{ is cofinite in } Z \,\}.$$

Using the fact that I is absorbing, it is straightforward to check that A forms a subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^Z$. Since $d \in I$, we must have $A_0 \subseteq A$. Since $u \notin I$, the element $g := u^d_0$ is not in $\mathbf{A}$.

Assume that θ is a congruence on $\mathbf{A}$ of index at most k, for some $k \in \omega\backslash\{0\}$. It remains to show that $\theta{\restriction}_{A_0}$ has a unique block of size greater than $\varphi(k)$. So suppose that the subsets $\{\, u^{dd}_{0s} \mid s \in S \,\}$ and $\{\, u^{dd}_{0t} \mid t \in T \,\}$ of A_0 are each collapsed by θ, where S and T are disjoint subsets of $Z\backslash\{0\}$ of size $\varphi(k) + 1$. It now suffices to prove that their union, $\{\, u^{dd}_{0z} \mid z \in S \cup T \,\}$, is collapsed by θ.

We shall consider elements of $\mathbf{A}$ of the form $(a_i)^{b_i}_z$, where $i \in n$ and $z \in S \cup T$. We first use Lemma 4.1 to show that θ must collapse certain pairs of these elements. For each $i \in n$, define the equivalence relation θ_i on S by

$$r \equiv_{\theta_i} s \iff (a_i)^{b_i}_r \equiv_\theta (a_i)^{b_i}_s.$$

The set S has size $\varphi(k) + 1 > k^n$. Since θ is of index at most k, the equivalence relations $\theta_0, \dots, \theta_{n-1}$ on S are each of index at most k. So it follows, using Lemma 4.1 and symmetry, that there are distinct elements $\rho, \sigma \in S$ and distinct elements $\tau, \upsilon \in T$ such that

$$(a_i)^{b_i}_\rho \equiv_\theta (a_i)^{b_i}_\sigma \quad \text{and} \quad (a_i)^{b_i}_\tau \equiv_\theta (a_i)^{b_i}_\upsilon,$$

for all $i \in n$.

In the algebra $\mathbf{A}$, we now have

$$\begin{aligned}
\underline{u} &= f(\dots, (a_i)^{b_i}_\sigma, \dots) \wedge \underline{u} \\
&= f(\dots, (a_i)^{b_i}_\sigma \wedge (a_i)^{b_i}_\sigma, \dots) \wedge \underline{u} \\
&\equiv_\theta f(\dots, (a_i)^{b_i}_\rho \wedge (a_i)^{b_i}_\sigma, \dots) \wedge \underline{u} \\
&= f(\dots, (a_i)^{c_i c_i}_{\rho\,\sigma}, \dots) \wedge \underline{u} \\
&= u^{dd}_{\rho\sigma}.
\end{aligned}$$

So $\underline{u} \equiv_\theta u^{dd}_{\rho\sigma}$. Since $d < u$, this gives us

$$u^{dd}_{0\tau} = u^{dd}_{0\tau} \wedge u^{dd}_{0\tau} \equiv_\theta u^{dd}_{0\tau} \wedge u^{dd}_{0\upsilon} = u^{ddd}_{0\tau\upsilon} = \underline{u} \wedge u^{ddd}_{0\tau\upsilon} \equiv_\theta u^{dd}_{\rho\sigma} \wedge u^{ddd}_{0\tau\upsilon} = u^{ddddd}_{0\rho\sigma\tau\upsilon}.$$

By symmetry, we have $u^{dd}_{0\tau} \equiv_\theta u^{ddddd}_{0\rho\sigma\tau\upsilon} \equiv_\theta u^{dd}_{0\sigma}$. Thus $\{\, u^{dd}_{0z} \mid z \in S \cup T \,\}$ is collapsed by θ, as required. □

A variety is *residually large* if there is no cardinal bound on the sizes of its subdirectly irreducible members. We now show that the assumptions of the previous theorem also imply residual largeness, generalising a result of R. Willard [27, 1.2].

4.3. **Theorem.** *Let* $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ *be a finite semilattice-based algebra with an absorbing subset* I. *Assume that there is an operation* $f \in F$ (*of arity* n) *and elements* $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in M$ *such that*

(1) $f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}) \notin I$, *and*
(2) $f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}) \wedge f(a_0 \wedge b_0, \dots, a_{n-1} \wedge b_{n-1}) \in I$.

Then the variety generated by $\underline{\mathbf{M}}$ *is residually large.*

Proof. Again, define

$$\begin{aligned} u &:= f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}) \ \text{ and} \\ d &:= u \wedge f(a_0 \wedge b_0, \dots, a_{n-1} \wedge b_{n-1}) \end{aligned}$$

in M. By our assumptions, we have $d \in I$ and $u \notin I$, with $d < u$.

Let κ be any infinite cardinal. We want to construct a subdirectly irreducible algebra in $\mathrm{Var}(\underline{\mathbf{M}})$ of cardinality greater than κ. Let Z be any set of cardinality greater than κ and define $\mathbf{A} := \underline{\mathbf{M}}^Z$. Define the subset $I_{\mathbf{A}}$ of A by

$$I_{\mathbf{A}} := \{\, x \in A \mid (\exists z \in Z)\ x(z) \in I \,\}.$$

Then $I_{\mathbf{A}}$ is nonempty, as $\underline{d} \in I_{\mathbf{A}}$. Since $u \notin I$, we have $\underline{u} \notin I_{\mathbf{A}}$. It is easy to check that $I_{\mathbf{A}}$ is an absorbing subset of $\mathbf{A}$.

As $I_{\mathbf{A}}$ is absorbing, there is a congruence on $\mathbf{A}$ that collapses $I_{\mathbf{A}}$ but does not collapse any other elements of A. Since $\underline{u} \notin I_{\mathbf{A}}$, there is a congruence θ on $\mathbf{A}$ that is maximal with respect to collapsing $I_{\mathbf{A}}$ and separating $\underline{u}$ from $I_{\mathbf{A}}$. The algebra $\mathbf{A}/\theta$ in $\mathrm{Var}(\underline{\mathbf{M}})$ is subdirectly irreducible. Now suppose, for a contradiction, that $|\mathbf{A}/\theta| \leqslant \kappa$.

For each $i \in n$, define the equivalence relation θ_i on Z by

$$s \equiv_{\theta_i} t \iff (a_i)^{b_i}_s \equiv_\theta (a_i)^{b_i}_t.$$

Since we are supposing that $|\mathbf{A}/\theta| \leqslant \kappa$, the congruence θ is of index at most κ. So the equivalence relations $\theta_0, \dots, \theta_{n-1}$ are each of index at most $\kappa = \kappa^n$. By Lemma 4.1, there are distinct $\rho, \sigma \in Z$ such that $(a_i)^{b_i}_\rho \equiv_\theta (a_i)^{b_i}_\sigma$, for all $i \in n$.

Now, as in the previous proof, it follows that $\underline{u} \equiv_\theta u^{dd}_{\rho\sigma}$. But $u^{dd}_{\rho\sigma} \in I_{\mathbf{A}}$, and so θ does not separate $\underline{u}$ from $I_{\mathbf{A}}$, which is a contradiction. □

The connection between proofs of inherent non-dualisability and residual largeness, illustrated in this section, will be explored more fully in a follow-up paper.

5. Flat algebras

In combination, the Semilattice-Based Duality Theorem 3.3 and Theorem 4.2 produce complete characterisations of dualisability within some very broad and interesting classes of finite algebras. In this section, we use these theorems to give a complete characterisation of dualisability for flat algebras with an absorbing bottom.

A semilattice of height 1 is called a *flat semilattice*, and an algebra with a flat-semilattice reduct is called a *flat algebra*. Let $\mathbf{A} = \langle A; \{\wedge\} \cup F\rangle$ be a flat algebra, and let 0 denote the bottom element of the underlying flat semilattice $\langle A; \wedge\rangle$. We shall say that the flat algebra $\mathbf{A}$ has an *absorbing bottom* if $\{0\}$ is an absorbing subset of $\mathbf{A}$. In other words, for each $n \in \omega\backslash\{0\}$ and each non-constant n-ary operation $f \in F$, we require that

$$0 \in \{a_0, \dots, a_{n-1}\} \implies f(a_0, \dots, a_{n-1}) = 0,$$

for all $a_0, \dots, a_{n-1} \in A$.

There is a very natural way to extend any partial algebra to a flat algebra. To see this, let $\mathbf{A} = \langle A; F\rangle$ be a partial algebra, and assume that $0 \notin A$ and $\wedge \notin F$. Then the *flat* (*one-point*) *extension of* $\mathbf{A}$ is the algebra

$$\flat(\mathbf{A}) := \langle A \cup \{0\}; \{\wedge\} \cup \{\, f_0 \mid f \in F \,\}\rangle,$$

where

- each operation $f \in F$ is extended to a total operation f_0 on $A \cup \{0\}$ by setting all undefined values to 0, and
- the new operation $\wedge$ is defined so that $\langle A \cup \{0\}; \wedge\rangle$ is a flat semilattice with bottom element 0.

Clearly, $\flat(\mathbf{A})$ is a flat algebra with absorbing bottom 0. Indeed, every flat algebra with an absorbing bottom can be obtained in this way.

5.1. **Theorem.** *Let* $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ *be a finite flat algebra with an absorbing bottom. The following are equivalent*:

(1) $\underline{\mathbf{M}}$ *is dualisable*;
(2) *every operation in* F *is compatible with* $\wedge$;
(3) $\mathrm{Var}(\underline{\mathbf{M}})$ *is residually small*;
(4) $\mathrm{Var}(\underline{\mathbf{M}})$ *is residually very finite.*

Moreover, *if* $\underline{\mathbf{M}}$ *is non-dualisable*, *then* $\underline{\mathbf{M}}$ *is inherently non-*κ*-dualisable*, *for every cardinal* κ.

Proof. We immediately have (4) ⇒ (3) and, by the Semilattice-Based Duality Theorem 3.3, we have (2) ⇒ (1). Using the result of Kearnes and Szendrei [18, 5.1], we also have (2) ⇒ (4). We prove that ¬(2) ⇒ ¬(3) and ¬(2) ⇒ ¬(1).

Assume that (2) fails. Then there is an operation $f \in F$ (of arity n) that is not compatible with $\wedge$. Let 0 denote the bottom element of the flat semilattice $\langle M; \wedge\rangle$. The operation f must preserve the order induced by $\wedge$, since $\langle M; \wedge\rangle$ is flat and the bottom element 0 is absorbing in $\underline{\mathbf{M}}$. So there are $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in M$ such that

$$0 = f(a_0 \wedge b_0, \dots, a_{n-1} \wedge b_{n-1}) < f(a_0, \dots, a_{n-1}) \wedge f(b_0, \dots, b_{n-1}).$$

Now define $I := \{0\}$. It follows by Theorems 4.2 and 4.3 that ¬(1) and ¬(3) hold, as does the 'Moreover' claim. □

Flat algebras have been studied in several contexts. For example, let $G = \langle V; E\rangle$ be a finite directed graph (where $E \subseteq V^2$) such that $0 \notin V$. The *flat directed-graph algebra over* G is defined to be the algebra $\mathbf{G}^\flat := \langle V \cup \{0\}; \wedge, \cdot\rangle$, where

- $\langle V \cup \{0\}; \wedge\rangle$ is a flat semilattice with bottom 0, and
- the binary operation $\cdot$ is given by $v \cdot w = v$, if $(v, w) \in E$, and $v \cdot w = 0$, otherwise, for all $v, w \in V \cup \{0\}$.

Equivalently, the algebra $\mathbf{G}^\flat$ is the flat extension of the partial algebra $\langle V; \cdot \rangle$, where $\cdot : E \to V$ is the first projection.

Recall that an algebra $\mathbf{A}$ is *entropic* if, for all $n \in \omega$, every fundamental operation f of $\mathbf{A}$ of arity n is a homomorphism $f : \mathbf{A}^n \to \mathbf{A}$. It is easy to check that, if $\cdot$ is compatible with $\wedge$ in a flat directed-graph algebra, then the algebra is entropic. As a corollary of Theorem 5.1, we obtain a generalisation of a result due to Lampe, McNulty and Willard [19], who looked only at directed graphs with a symmetric edge relation.

5.2. **Corollary.** [19] *A finite flat directed-graph algebra is dualisable if and only if it is entropic.*

The following additional corollary of Theorem 5.1 is easy.

5.3. **Corollary.** *Let* $\underline{\mathbf{G}} = \langle G; * \rangle$ *be a finite non-trivial group. Then*

(1) *the group operation* $*$ *is weakly compatible but not compatible with* $\wedge$ *in* $\flat(\underline{\mathbf{G}})$,
(2) *the variety generated by* $\flat(\underline{\mathbf{G}})$ *is residually large, and*
(3) *the algebra* $\flat(\underline{\mathbf{G}})$ *is inherently non-*κ*-dualisable, for every cardinal* κ.

Generalising part (1) of this result, Ježek, Maróti and McKenzie [17] have shown that a flat extension of a finite (total) binar $\mathbf{A}$ is weakly compatible with $\wedge$ if and only if $\mathbf{A}$ is a quasigroup. Strengthening part (2), M. Jackson [15] has shown that, for a finite non-trivial group $\underline{\mathbf{G}}$, the subdirectly irreducible algebras in $\mathrm{Var}(\flat(\underline{\mathbf{G}}))$ are precisely the flat extensions of the members of $\mathbb{ISP}(\underline{\mathbf{G}})$.

6. Closure semilattices

An algebra $\mathbf{S} = \langle S; \wedge, C \rangle$ is a *closure semilattice* (*CSL*) if $\wedge$ is a semilattice operation and C is a unary operation satisfying the usual axioms for a closure operator:

- $x \leqslant C(x)$ (increasing),
- $x \leqslant y \implies C(x) \leqslant C(y)$ (order preserving),
- $C(C(x)) \approx C(x)$ (idempotent).

If $\mathbf{S}$ is a CSL, then the elements of the form $C(a)$, for some $a \in S$, are referred to as *closed*.

In this section, we prove that a finite CSL is dualisable if and only if it is entropic. We also take this opportunity to correct a minor error that appears in the article, by the second author, in which the class of CSLs is introduced [14].

It is readily seen that the given axioms for a CSL are equivalent to identities, and so the class of all CSLs is a variety. Subvarieties of the variety of all CSLs have been investigated by M. Jackson [14]. We begin by considering the variety of entropic CSLs.

Let $\underline{\mathbf{E}}$ be the three-element entropic CSL shown in Figure 1. The filled-in circles in Figures 1 and 2 denote the closed elements. (This is the reverse of the convention introduced by the second author [14], who was out-voted here.) The closure operation can be reconstructed from the diagram: the closure of an element is the least closed element that is greater than or equal to it.

Since $\underline{\mathbf{E}}$ is entropic, it follows by the Semilattice-Based Duality Theorem 3.3 that $\underline{\mathbf{E}}$ is dualised by the alter ego $\langle E; \wedge, R_4, \mathcal{T} \rangle$. In fact, we can do somewhat better: modulo nullary operations, the CSL $\underline{\mathbf{E}}$ is strongly self-dualising.

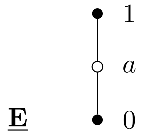

FIGURE 1. The entropic CSL $\underline{\mathbf{E}}$.

6.1. **Theorem.** *Let* $\underline{\mathbf{E}} = \langle E; \wedge, C\rangle$ *be the CSL given in Figure 1, and define* $\boldsymbol{\mathcal{E}} := \mathbb{ISP}(\underline{\mathbf{E}})$. *Then* $\boldsymbol{\mathcal{E}}$ *is the variety of all entropic CSLs, and the alter ego* $\underset{\sim}{\mathbf{E}} := \langle E; \wedge, C, 0, 1, \mathcal{T}\rangle$ *yields a strong duality on* $\boldsymbol{\mathcal{E}}$.

Proof. Let $\mathbf{S}$ be an arbitrary entropic CSL. To show that $\mathbf{S} \in \mathbb{ISP}(\underline{\mathbf{E}})$, it suffices to show that any pair of distinct elements of S can be separated by a homomorphism from $\mathbf{S}$ into $\underline{\mathbf{E}}$.

Let $s, t \in S$ with $s \neq t$. We may assume that $t \nleqslant s$. We want to define the homomorphism $\varphi : \mathbf{S} \to \underline{\mathbf{E}}$ by

$$\varphi(u) := \begin{cases} 1 & \text{if } u \in {\uparrow}t, \\ a & \text{if } u \in C^{-1}({\uparrow}t)\backslash{\uparrow}t, \\ 0 & \text{otherwise.} \end{cases}$$

As C is compatible with $\wedge$, the set $C^{-1}({\uparrow}t)$ is a filter of $\mathbf{S}$. For every $u \in {\uparrow}t$, we have $C(u) \geqslant C(t) \geqslant t$, as C is order preserving and increasing, and therefore $C(u) \in {\uparrow}t$. So the filter ${\uparrow}t$ of $\mathbf{S}$ is contained in the filter $C^{-1}({\uparrow}t)$, which implies that the map φ preserves $\wedge$. Since C is idempotent, we must have $C^{-1}(C^{-1}({\uparrow}t)) = C^{-1}({\uparrow}t)$, and it follows that φ preserves C. Thus $\varphi : \mathbf{S} \to \underline{\mathbf{E}}$ is homomorphism separating s and t, whence $\mathbf{S} \in \mathbb{ISP}(\underline{\mathbf{E}}) = \boldsymbol{\mathcal{E}}$.

Now let $\underset{\sim}{\mathbf{E}} := \langle E; \wedge, C, 0, 1, \mathcal{T}\rangle$. We will use Theorem 2.2 to show that $\underset{\sim}{\mathbf{E}}$ yields a strong duality on $\boldsymbol{\mathcal{E}}$. Since 0 and 1 are in the type of $\underset{\sim}{\mathbf{E}}$, condition (1) of Theorem 2.2 holds. We shall next prove that $\underline{\mathbf{E}}$ satisfies the interpolation condition (IC) with respect to $\underset{\sim}{\mathbf{E}}$.

Let $\mathbf{A} \leqslant \underline{\mathbf{E}}^n$, for some $n \in \omega\backslash\{0\}$, and let $\varphi : \mathbf{A} \to \underline{\mathbf{E}}$ be a homomorphism. We want to show that φ extends to a term function of $\underset{\sim}{\mathbf{E}}$. Since the constants 0 and 1 are term functions of $\underset{\sim}{\mathbf{E}}$, we may assume that $\varphi^{-1}(1)$ is a proper, non-empty subset of A. Define $\widehat{1} := \bigwedge \varphi^{-1}(1)$ in $\mathbf{A}$, and define the term function $t : E^n \to E$ of $\underset{\sim}{\mathbf{E}}$ by

$$t(b_0, \dots, b_{n-1}) := \bigwedge \{\, b_i \mid \widehat{1}(i) = 1 \,\} \wedge \bigwedge \{\, C(b_i) \mid \widehat{1}(i) = a \,\}.$$

Note that at least one of the two big meets must be non-empty, since $\widehat{1}$ is not the constant tuple $\underline{0}$, by assumption.

To see that t extends φ, let $(a_0, \dots, a_{n-1}) \in A$. Then the construction of the term function t guarantees that

$$t(a_0, \dots, a_{n-1}) = 1 \iff (a_0, \dots, a_{n-1}) \geqslant \widehat{1} \iff \varphi(a_0, \dots, a_{n-1}) = 1$$

and

$$\begin{aligned} t(a_0, \dots, a_{n-1}) = 0 &\iff (C(a_0), \dots, C(a_{n-1})) \ngeqslant \widehat{1} \\ &\iff C(\varphi(a_0, \dots, a_{n-1})) \neq 1 \iff \varphi(a_0, \dots, a_{n-1}) = 0. \end{aligned}$$

Thus $t(a_0, \dots, a_{n-1}) = \varphi(a_0, \dots, a_{n-1})$, whence t extends φ.

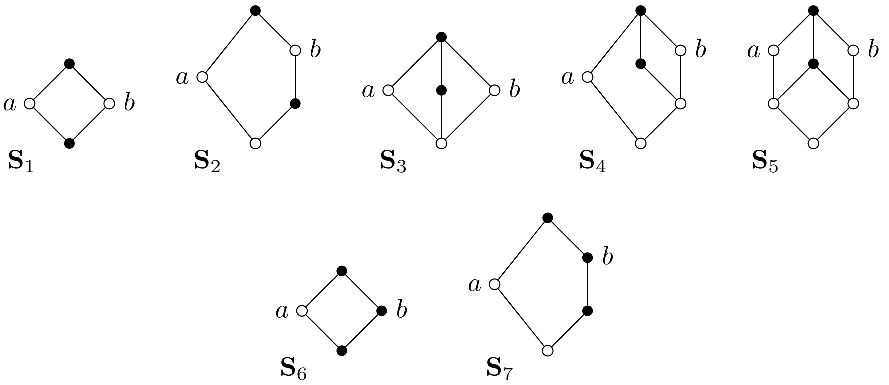

FIGURE 2. The *seven* forbidden subalgebras for entropic CSLs.

We have proved that $\underline{\mathbf{E}}$ satisfies the interpolation condition (IC) with respect to $\underset{\sim}{\mathbf{E}}$, and it follows easily that $\underset{\sim}{\mathbf{E}}$ satisfies the interpolation condition (IC) with respect to $\underline{\mathbf{E}}$. Hence $\underset{\sim}{\mathbf{E}}$ yields a strong duality on $\mathcal{E}$, by Theorem 2.2. $\square$

M. Jackson [14, 4.7] showed that, in the lattice of all CSL varieties, the variety $\mathcal{E}$ of entropic CSLs has exactly two covers, denoted by $\mathcal{B}$ and $\mathcal{C}$. It is further claimed [14, 4.3] that a CSL is entropic if and only if it does not contain any one of a list of six forbidden subalgebras. The claimed proof is by way of a case-study of possible quotients and subalgebras of the (infinite) two-generated free CSL. However, a case was omitted that gives rise to one further forbidden subalgebra. Here we give a short proof of a corrected version of the result, although it is not strictly required for our characterisation of dualisability for CSLs.

The forbidden subalgebras for entropic CSLs are $\mathbf{S}_1$ to $\mathbf{S}_7$, shown in Figure 2. The new forbidden subalgebra is $\mathbf{S}_4$. The CSLs $\mathbf{S}_1$ to $\mathbf{S}_5$ each generate the same variety, $\mathcal{B}$, and the CSLs $\mathbf{S}_6$ and $\mathbf{S}_7$ each generate the same variety, $\mathcal{C}$.

6.2. **Lemma.** *A CSL* $\mathbf{S}$ *is entropic if and only if none of the CSLs* $\mathbf{S}_1$ *to* $\mathbf{S}_7$ *in Figure 2 embeds into* $\mathbf{S}$.

Proof. First, it is easy to check that none of the CSLs $\mathbf{S}_1$ to $\mathbf{S}_7$ is entropic; each satisfies $C(a \wedge b) \neq C(a) \wedge C(b)$.

Now let $\mathbf{S}$ be a non-entropic CSL. We shall find a non-entropic subalgebra $\mathbf{T}$ of $\mathbf{S}$ that is isomorphic to a quotient of $\mathbf{S}_5$ or $\mathbf{S}_7$. A routine examination of the possible non-entropic quotients of these two algebras leads to the list in Figure 2; this is left as an exercise.

As $\mathbf{S}$ is not entropic, there exist $a, b \in S$ with $C(a \wedge b) \neq C(a) \wedge C(b)$. Using symmetry, we need only consider two cases.

Case (1): $C(a \wedge C(b)) \neq C(a) \wedge C(b)$. Let $\mathbf{T}$ be the subalgebra of $\mathbf{S}$ generated by $\{a, C(a) \wedge C(b)\}$. Using the assumption, we have

$$C\big(a \wedge \big(C(a) \wedge C(b)\big)\big) = C\big(a \wedge C(b)\big) \neq C(a) \wedge C(b) = C(a) \wedge C\big(C(a) \wedge C(b)\big),$$

and so $\mathbf{T}$ is not entropic. It is easy to check that

$$T = \big\{a, C(a), a \wedge C(b), C(a \wedge C(b)), C(a) \wedge C(b)\big\},$$

and $\mathbf{T}$ is isomorphic to a quotient of $\mathbf{S}_7$.

Case (2): $C(a \wedge C(b)) = C(a) \wedge C(b) = C(b \wedge C(a))$. Let $\mathbf{T}$ be the subalgebra of $\mathbf{S}$ generated by $\{a \wedge C(b), b \wedge C(a)\}$. Using the assumption, we have

$$C\big(\big(a \wedge C(b)\big) \wedge \big(b \wedge C(a)\big)\big) = C(a \wedge b) \neq C(a) \wedge C(b) = C\big(a \wedge C(b)\big) \wedge C\big(b \wedge C(a)\big),$$

and so $\mathbf{T}$ is not entropic. Using the assumption again, we must have

$$T = \big\{a \wedge b, a \wedge C(a \wedge b), b \wedge C(a \wedge b), C(a \wedge b), a \wedge C(b), b \wedge C(a), C(a) \wedge C(b)\big\},$$

and $\mathbf{T}$ is isomorphic to a quotient of $\mathbf{S}_5$. □

Our next result applies to a class of algebras more general than CSLs.

6.3. **Lemma.** *Let* $\underline{\mathbf{M}} = \langle M; \wedge, f\rangle$ *be a finite semilattice with a unary operation* f*. Assume that* f *is idempotent and preserves the order induced by* $\wedge$*. If* f *is not compatible with* $\wedge$, *then* $\underline{\mathbf{M}}$ *generates a residually large variety and* $\underline{\mathbf{M}}$ *is inherently non-*κ*-dualisable*, *for every cardinal* κ.

Proof. We shall apply Theorems 4.2 and 4.3. Assume that f is not compatible with $\wedge$. Then $f(a_0 \wedge b_0) \neq f(a_0) \wedge f(b_0)$, for some $a_0, b_0 \in M$. Since f is order preserving, we must have $f(a_0 \wedge b_0) < f(a_0) \wedge f(b_0)$. Now define $c := f(a_0 \wedge b_0)$ and $I := {\downarrow}c$. For all $m \in I = {\downarrow}c$, we have $f(m) \leqslant f(c) = f(f(a_0 \wedge b_0)) = f(a_0 \wedge b_0) = c$, as f is order preserving and idempotent, and so $f(m) \in {\downarrow}c = I$. Therefore I is an absorbing subset of $\underline{\mathbf{M}}$, and the two theorems apply. □

6.4. **Remark.** Five of the seven algebras in Figure 2 can be proved to be inherently non-dualisable by a much easier application of the Inherent Non-Dualisability Theorem 2.3 than that required for Theorem 4.2. For algebras $\mathbf{S}_1$ to $\mathbf{S}_4$ we may use an argument based around $A_0 := \{\, a^b_z \mid z \in Z \,\}$, for $\mathbf{S}_6$ we may use an argument based around $A_0 := \{\, b^{aa}_{0z} \mid z \in Z \backslash \{0\} \,\}$, and in each case we may take the bounding function φ to be constant with value 1. These two approaches can be proved to fail for the remaining algebras, $\mathbf{S}_5$ and $\mathbf{S}_7$, and our eventual proof of the inherent non-dualisability of these algebras led to the much more general Theorem 4.2.

The Semilattice-Based Duality Theorem 3.3 and the results of this section combine to give the following.

6.5. **Theorem.** *Let* $\underline{\mathbf{S}}$ *be a finite CSL. The following are equivalent*:

(1) $\underline{\mathbf{S}}$ *is dualisable*;
(2) $\underline{\mathbf{S}}$ *is entropic*;
(3) $\mathrm{Var}(\underline{\mathbf{S}})$ *is residually small*;
(4) $\mathrm{Var}(\underline{\mathbf{S}})$ *is residually very finite.*

Moreover, *if* $\underline{\mathbf{S}}$ *is non-dualisable*, *then it is inherently non-*κ*-dualisable*, *for every cardinal* κ.

7. Piggyback dualities for semilattice-based algebras

The idea behind a piggyback duality is quite simple. We take an algebra that has a reduct in a quasi-variety with a known natural duality, and we attempt to lift this duality up to the quasi-variety generated by our richer algebra. This strategy cannot always work: all algebras, including the non-dualisable ones, have a reduct in the class of sets, which has a natural duality (see [1, 9] and Exercise 4.9 in [2]).

The Piggyback Duality Theorem, first proved by Davey and Werner [10, 11] and generalised by Davey and Priestley [8], provides general sufficient conditions under which a new duality can be obtained by piggybacking on a duality for a reduct. We will present a restricted version of the Piggyback Duality Theorem that applies to semilattice-based algebras with $\wedge$-compatible operations.

As we will apply our theorem to infinite algebras, as well as to finite algebras, we first say a few general words about natural dualities based on infinite algebras. Let $\underline{\mathbf{M}}$ be an algebra, let $\mathcal{T}$ be a compact Hausdorff topology on M, and assume that each fundamental operation of $\underline{\mathbf{M}}$ is continuous with respect to $\mathcal{T}$. Now let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be an alter ego of $\underline{\mathbf{M}}$ using this topology. No compatibility is assumed between $G \cup H \cup R$ and $\mathcal{T}$.

Define the classes $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\boldsymbol{\mathcal{X}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$. It is easily seen that the functors $\mathrm{D} : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}$ and $\mathrm{E} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$ can be defined as usual, as can the embeddings $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$, for $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $\mathbf{X} \in \boldsymbol{\mathcal{X}}$. The basic results of the theory, as they apply to infinite generators, can be found in the following places: Exercise 2.9 of Clark and Davey [2], the appendix of Davey and Werner [9], the piggyback duality papers by Davey and Werner [10, 11] and Davey and Priestley [8], and the precursor to the theory of natural dualities by Davey [4].

Natural dualities based on infinite algebras occur rarely in the literature. The best known example is Pontryagin's duality for abelian groups (see B. A. Davey [4] for a proof in the spirit of the theory of natural dualities). A less well-known example is the natural duality for Ockham algebras proved by M. .S. Goldberg [12], which was reproved using piggyback techniques by Davey and Werner [10, 11].

Now consider a semilattice-based algebra $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$. There are four different semilattice dualities on which we can try to piggyback to produce a duality for $\mathbb{ISP}(\underline{\mathbf{M}})$. The generators and their alter egos are as follows [2, p. 228]:

$$\begin{aligned}
\underline{\mathbf{2}} &:= \langle \{0,1\}; \wedge\rangle && \text{and} \quad \underset{\sim}{\mathbf{2}}_{01} := \langle \{0,1\}; \wedge, 0, 1, \mathcal{T}\rangle,\\
\underline{\mathbf{2}}_0 &:= \langle \{0,1\}; \wedge, 0\rangle && \text{and} \quad \underset{\sim}{\mathbf{2}}_{0} := \langle \{0,1\}; \wedge, 0, \mathcal{T}\rangle,\\
\underline{\mathbf{2}}_1 &:= \langle \{0,1\}; \wedge, 1\rangle && \text{and} \quad \underset{\sim}{\mathbf{2}}_{1} := \langle \{0,1\}; \wedge, 1, \mathcal{T}\rangle,\\
\underline{\mathbf{2}}_{01} &:= \langle \{0,1\}; \wedge, 0, 1\rangle && \text{and} \quad \underset{\sim}{\mathbf{2}} := \langle \{0,1\}; \wedge, \mathcal{T}\rangle.
\end{aligned}$$

Which semilattice duality to try is determined by the definable elements of $\underline{\mathbf{M}}$. We say that $\underline{\mathbf{M}}$ has a *definable bottom* if it has a smallest element that is the value of a constant term function; *definable top* is defined similarly.

As an example, we show how to start piggybacking on the duality given by $\underline{\mathbf{2}}$ and $\underset{\sim}{\mathbf{2}}_{01}$. For each semilattice-based algebra $\mathbf{A} = \langle A; \{\wedge\} \cup F\rangle$ in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$, we define $\mathbf{A}^{\wedge} := \langle A; \wedge\rangle$ to be its semilattice reduct in $\boldsymbol{\mathcal{S}} := \mathbb{ISP}(\underline{\mathbf{2}})$. Now let $\lambda : \underline{\mathbf{M}}^{\wedge} \to \underline{\mathbf{2}}$ be a surjective semilattice homomorphism. For each algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, we can define the map

$$\Phi_\lambda : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to \boldsymbol{\mathcal{S}}(\mathbf{A}^{\wedge}, \underline{\mathbf{2}}) \quad \text{by} \quad \Phi_\lambda(x) := \lambda \circ x, \text{ for all } x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}).$$

This map allows us to transfer between the dual of $\mathbf{A}$ with respect to $\underline{\mathbf{M}}$ and the dual of $\mathbf{A}^{\wedge}$ with respect to $\underline{\mathbf{2}}$. The semilattice homomorphism λ is called a *carrier*, as it provides the means of lifting the duality up from $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{\mathcal{A}}$.

The Piggyback Duality Theorem in its most general form applies to multi-sorted dualities and allows for more than one carrier. The following theorem could be stated in this generality, but we shall state a restricted version that is sufficient for our needs.

Our Semilattice Piggyback Duality Theorem applies to finite algebras that we already know to be dualisable, via the Semilattice-Based Duality Theorem 3.3.

However, the piggyback theorem has two advantages. Firstly, it also applies to infinite algebras. Secondly, it produces an alter ego with relations of arity at most 3, whereas the Semilattice-Based Duality Theorem, when applied to a finite algebra $\underline{\mathbf{M}}$, produces an alter ego with relations of arity $|M|+1$. However, the piggyback theorem does only apply to semilattice-based algebras with a rich supply of endomorphisms.

For any algebra $\underline{\mathbf{M}}$ and map $\lambda : M \to S$, we shall define the set R_λ of binary algebraic relations on $\underline{\mathbf{M}}$ by

$$R_\lambda := \bigl\{\, r \subseteq M^2 \;\big|\; r \text{ is a subuniverse of } \underline{\mathbf{M}}^2 \text{ that is maximal in } \ker(\lambda) \,\bigr\}.$$

For any set G of self-maps of M, we say that $\lambda \circ G$ *separates* M if, for all $a, b \in M$ with $a \neq b$, there exists $g \in G$ such that $\lambda(g(a)) \neq \lambda(g(b))$. We will use $\langle G \rangle$ to denote the submonoid of M^M generated by G.

7.1. **Semilattice Piggyback Duality Theorem.** *Let* $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ *be a semilattice-based algebra such that each operation in* F *is compatible with* $\wedge$*. Let* $\mathcal{T}$ *be a compact Hausdorff topology on* M *such that each operation of* $\underline{\mathbf{M}}$ *is continuous with respect to* $\mathcal{T}$*. Let* $\lambda : \underline{\mathbf{M}}^\wedge \to \underline{\mathbf{2}}$ *be a surjective semilattice homomorphism, and assume that* λ *is continuous with respect to the topology* $\mathcal{T}$ *on* M *and the discrete topology on* $\{0,1\}$*. Assume further that there is a subset* G *of* $\operatorname{End}(\underline{\mathbf{M}})$ *such that* $\lambda \circ \langle G \rangle$ *separates* M*.*

1. *If* $\underline{\mathbf{M}}$ *has a definable bottom and top, then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge\} \cup G, R_\lambda, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$*.*
2. *If* $\underline{\mathbf{M}}$ *has a definable bottom element* 0 *such that* $\{0\}$ *is a subuniverse of* $\underline{\mathbf{M}}$*, then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 0\} \cup G, R_\lambda, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$*.*
3. *If* $\underline{\mathbf{M}}$ *has a definable top element* 1 *such that* $\{1\}$ *is a subuniverse of* $\underline{\mathbf{M}}$*, then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 1\} \cup G, R_\lambda, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$*.*
4. *If* $\underline{\mathbf{M}}$ *has a bottom* 0 *and top* 1 *such that* $\{0\}$ *and* $\{1\}$ *are subuniverses of* $\underline{\mathbf{M}}$*, then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 0, 1\} \cup G, R_\lambda, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$*.*

Proof. A direct application of the existing piggyback theorems [10, 11, 8] would lead to somewhat more complex alter egos than those given above. Nevertheless, the proofs of these theorems are easily modified to produce our result. We sketch the details.

First consider (1). Since $\underline{\mathbf{M}}$ has definable bottom and top, it has a bounded-semilattice term reduct, and we may piggyback on the duality between the class $\boldsymbol{\mathcal{S}}_{01} := \mathbb{ISP}(\underline{\mathbf{2}}_{01})$ of bounded semilattices and the class $\boldsymbol{\mathcal{Y}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}})$ of boolean topological semilattices.

Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. Let $\mathrm{D} : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}$ and $\mathrm{E} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$ be the functors induced by $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$, and let $\mathrm{H} : \boldsymbol{\mathcal{S}}_{01} \to \boldsymbol{\mathcal{Y}}$ and $\mathrm{K} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{S}}_{01}$ be the functors induced by $\underline{\mathbf{2}}_{01}$ and $\underset{\sim}{\mathbf{2}}$. Now let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, and let $|\mathbf{A}|$ denote its underlying bounded semilattice in $\boldsymbol{\mathcal{S}}_{01}$. Let $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ be the natural embedding from $\mathbf{A}$ to its double dual, and let $k_{|\mathbf{A}|} : |\mathbf{A}| \to \mathrm{KH}(|\mathbf{A}|)$ be the natural isomorphism from $|\mathbf{A}|$ to its double dual. We aim to show that $e_{\mathbf{A}}$ is surjective.

It is easy to check that the surjectivity of $k_{|\mathbf{A}|}$ will imply the surjectivity of $e_{\mathbf{A}}$ provided there exists a one-to-one set map $d : \mathrm{ED}(\mathbf{A}) \to \mathrm{KH}(|\mathbf{A}|)$ that commutes with the evaluations, that is, such that $d \circ e_{\mathbf{A}} = k_{|\mathbf{A}|}$. So let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ be a morphism in $\mathrm{ED}(\mathbf{A})$. We now want to define a continuous, $\wedge$-preserving map $d(\alpha) : \boldsymbol{\mathcal{S}}_{01}(|\mathbf{A}|, \underline{\mathbf{2}}_{01}) \to \{0,1\}$ in $\mathrm{KH}(|\mathbf{A}|)$.

First, consider the map $\Phi_\lambda : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to \boldsymbol{\mathcal{S}}_{01}(|\mathbf{A}|, \underline{\mathbf{2}}_{01})$ given by $\Phi_\lambda(x) := \lambda \circ x$. The duality between $\boldsymbol{\mathcal{S}}_{01}$ and $\boldsymbol{\mathcal{Y}}$ is strong, and $\lambda \circ \langle G\rangle$ separates M. Thus, by Proposition 2.4 of Davey and Priestley [8], the map Φ_λ is surjective if and only if the image of Φ_λ forms a subalgebra of $\underset{\sim}{\mathbf{2}}^A$. Since Φ_λ preserves $\wedge$, the image of Φ_λ is certainly a subsemilattice of $\underset{\sim}{\mathbf{2}}^A$. Thus Φ_λ is surjective.

A routine calculation shows that, since α preserves the relations in R_λ, we have $\ker(\Phi_\lambda) \subseteq \ker(\lambda \circ \alpha)$. The maps Φ_λ and $\lambda \circ \alpha$ are continuous and $\wedge$-preserving. Since Φ_λ is surjective, it follows by the Fundamental Homomorphism Theorem for Compact Hausdorff Topological Algebras that there is a continuous, $\wedge$-preserving map $\beta : \boldsymbol{\mathcal{S}}_{01}(|\mathbf{A}|, \underline{\mathbf{2}}_{01}) \to \{0, 1\}$ such that $\beta \circ \Phi_\lambda = \lambda \circ \alpha$. We can define $d(\alpha) := \beta$.

It is easy to show that the map d is one-to-one, using the fact that $\lambda \circ \langle G\rangle$ separates M and α preserves G. Since Φ_λ is surjective, it is also easy to check that $d \circ e_{\mathbf{A}} = k_{|\mathbf{A}|}$. Thus $d : \mathrm{ED}(\mathbf{A}) \to \mathrm{KH}(|\mathbf{A}|)$ is a one-to-one set map that commutes with the evaluations, as required. Hence $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism.

The proof for claim (2) is similar. We piggyback on the duality given by $\underline{\mathbf{2}}_0$ and $\underset{\sim}{\mathbf{2}}_0$, replace $\boldsymbol{\mathcal{S}}_{01}$ and $\boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{S}}_0 := \mathbb{ISP}(\underline{\mathbf{2}}_0)$ and $\boldsymbol{\mathcal{Y}}_0 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_0)$, and declare $|\mathbf{A}|$ to be the $\{\wedge, 0\}$-reduct of $\mathbf{A}$. We require $\{0\}$ to be a subuniverse of $\underline{\mathbf{M}}$ so that 0 can be included in the type of $\underset{\sim}{\mathbf{M}}$. Since λ is surjective, it preserves 0, and it follows that Φ_λ also preserves 0. The proof of (3) is again similar, we piggyback on the duality given by $\underline{\mathbf{2}}_1$ and $\underset{\sim}{\mathbf{2}}_1$. Finally, for claim (4), we piggyback on the duality given by $\underline{\mathbf{2}}$ and $\underset{\sim}{\mathbf{2}}_{01}$. ☐

We next show that, in the previous theorem, the set R_λ of relations can be removed from the dualising alter ego $\underset{\sim}{\mathbf{M}}$ provided the algebra $\underline{\mathbf{M}}$ has a rich supply of unary term functions. We denote the set of all unary term functions of $\underline{\mathbf{M}}$ by $F_{\underline{\mathbf{M}}}(1)$. We denote the diagonal relation on M by Δ_M.

7.2. **Lemma.** *Let $\underline{\mathbf{M}}$ be an algebra and let $\lambda : M \to S$ be a map, for some set S. If $\lambda \circ F_{\underline{\mathbf{M}}}(1)$ separates M, then $R_\lambda = \{\Delta_M\}$.*

Proof. Clearly, Δ_M is a subuniverse of $\underline{\mathbf{M}}^2$ contained in $\ker(\lambda)$. Let r be a subuniverse of $\underline{\mathbf{M}}^2$ that is contained in $\ker(\lambda)$. To prove that $r \subseteq \Delta_M$, let $(a, b) \in r$. Now let t be any unary term function of $\underline{\mathbf{M}}$. As r is a subuniverse of $\underline{\mathbf{M}}^2$, we have $(t(a), t(b)) = t((a, b)) \in r \subseteq \ker(\lambda)$, and therefore $\lambda(t(a)) = \lambda(t(b))$. As $\lambda \circ F_{\underline{\mathbf{M}}}(1)$ separates M, it follows that $a = b$. Thus $(a, b) \in \Delta_M$, as required. ☐

7.3. **Corollary.** *Assume that $\underline{\mathbf{M}}$ and $\lambda : \underline{\mathbf{M}}^\wedge \to \underline{\mathbf{2}}$ satisfy the assumptions of the Semilattice Piggyback Duality Theorem 7.1. If $\lambda \circ F_{\underline{\mathbf{M}}}(1)$ separates M, then the set R_λ of relations can be removed from each of the alter egos given in* (1) *to* (4)*, without destroying the dualities.*

Using this corollary, we can give simple sufficient conditions for a semilattice-based algebra to be self-dualising (modulo nullaries).

7.4. **Semilattice-Based Self-Duality Theorem.** *Let $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ be an entropic semilattice-based algebra such that each operation in F is unary. Let $\mathcal{T}$ be a compact Hausdorff topology on M such that each operation of $\underline{\mathbf{M}}$ continuous with respect to $\mathcal{T}$. Let $\lambda : \underline{\mathbf{M}}^\wedge \to \underline{\mathbf{2}}$ be a surjective semilattice homomorphism, and assume that λ is continuous with respect to the topology $\mathcal{T}$ on M and the discrete topology on $\{0, 1\}$. Assume further that $\lambda \circ \langle F\rangle$ separates M.*

1. *If* $\underline{\mathbf{M}}$ *has a bottom* 0 *and top* 1 *such that* $\{0\}$ *and* $\{1\}$ *are subuniverses of* $\underline{\mathbf{M}}$ *and we define* $\underline{\mathbf{M}}' := \langle M; \{\wedge, 0, 1\} \cup F\rangle$, *then* $\underset{\sim}{\mathbf{M}}' := \langle M; \{\wedge\} \cup F, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}}')$.
2. *If* $\underline{\mathbf{M}}$ *has a definable bottom element* 0 *such that* $\{0\}$ *is a subuniverse of* $\underline{\mathbf{M}}$, *then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 0\} \cup F, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$.
3. *If* $\underline{\mathbf{M}}$ *has a definable top element* 1 *such that* $\{1\}$ *is a subuniverse of* $\underline{\mathbf{M}}$, *then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 1\} \cup F, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$.
4. *If* $\underline{\mathbf{M}}$ *has a bottom* 0 *and top* 1 *such that* $\{0\}$ *and* $\{1\}$ *are subuniverses of* $\underline{\mathbf{M}}$, *then* $\underset{\sim}{\mathbf{M}} := \langle M; \{\wedge, 0, 1\} \cup F, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$.

As a sample application of the Semilattice-Based Self-Duality Theorem above, take the entropic closure semilattice $\underline{\mathbf{E}} = \langle\{0, a, 1\}; \wedge, C\rangle$, shown in Figure 1, and the characteristic function $\chi_{\{1\}} : \underline{\mathbf{E}}^{\wedge} \to \underline{\mathbf{2}}$. Since $\{0\}$ and $\{1\}$ are subuniverses of $\underline{\mathbf{E}}$, we can apply part (4) to conclude that $\underset{\sim}{\mathbf{E}} := \langle\{0, a, 1\}; \wedge, C, 0, 1, \mathcal{T}\rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{E}})$. (In Theorem 6.1, we showed that this duality is strong.)

In general, the set R_λ cannot be removed from the alter egos given by the Semilattice Piggyback Duality Theorem without destroying the dualities. Nevertheless, if $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ is a semilattice-based algebra such that $F \subseteq \mathrm{End}(\underline{\mathbf{M}}^{\wedge})$, then R_λ is particularly simple: it consists of a single binary relation that happens to be a congruence on $\underline{\mathbf{M}}$.

7.5. **Lemma.** *Let* $\underline{\mathbf{M}} = \langle M; \{\wedge\} \cup F\rangle$ *be a semilattice-based algebra such that each member of* F *is a unary operation compatible with* $\wedge$, *and let* $\lambda : \underline{\mathbf{M}}^{\wedge} \to \underline{\mathbf{2}}$ *be a surjective semilattice homomorphism. Then there is a largest subuniverse*, $\ker(\lambda)^{\circ}$, *of* $\underline{\mathbf{M}}^2$ *that is contained in* $\ker(\lambda)$. *Moreover*, $\ker(\lambda)^{\circ}$ *is a congruence on* $\underline{\mathbf{M}}$.

Proof. Define $\ker(\lambda)^{\circ} := \big\{ (a, b) \in M^2 \bigm| \big(\forall f \in \langle F\rangle\big)\, (f(a), f(b)) \in \ker(\lambda) \big\}$. It is easy to check that $\ker(\lambda)^{\circ}$ is the largest subuniverse of $\underline{\mathbf{M}}^2$ contained in $\ker(\lambda)$, and that $\ker(\lambda)^{\circ}$ is an equivalence relation, and therefore a congruence, on $\underline{\mathbf{M}}$. ☐

The congruence $\ker(\lambda)^{\circ}$ is known as the *syntactic congruence* of the equivalence relation $\ker(\lambda)$. (See Clark, Davey, Freese and Jackson [3] for an application of syntactic congruences to the axiomatic description of dual categories that arise in the theory of natural dualities.)

8. Natural dualities for semilattices with automorphism

In this section, we apply the piggyback techniques developed in the previous section to the class of semilattices with automorphism. An algebra $\mathbf{S} = \langle S; \wedge, f, f^{-1}\rangle$ is called a *semilattice with automorphism* if

- the reduct $\mathbf{S}^{\wedge} := \langle S; \wedge\rangle$ is a semilattice, and
- f and f^{-1} are mutually inverse automorphisms of $\mathbf{S}^{\wedge}$.

Clearly, the class $\boldsymbol{\mathcal{A}}$ of all semilattices with automorphism is a variety.

Semilattices with automorphism have been studied by J. Ježek [16], who gave a complete description of the subdirectly irreducible algebras in $\boldsymbol{\mathcal{A}}$. There is a largest subdirectly irreducible in $\boldsymbol{\mathcal{A}}$, which we describe below.

8.1. **Definition.** Let P denote the set $2^{\mathbb{Z}}$ of all functions from the integers into the set $2 := \{0, 1\}$. Define $\wedge$ on P pointwise, relative to the two-element semilattice $\underline{\mathbf{2}} = \langle 2; \wedge\rangle$. Let $s : \mathbb{Z} \to \mathbb{Z}$ be the successor function given by $s(i) := i + 1$, for all

$i \in \mathbb{Z}$, and define $f, f^{-1} : P \to P$ by

$$f(a) := a \circ s \quad \text{and} \quad f^{-1}(a) := a \circ s^{-1},$$

for all $a \in P$. Then $\underline{\mathbf{P}} := \langle P; \wedge, f, f^{-1} \rangle$ is a semilattice with automorphism. The algebra $\underline{\mathbf{P}}$ comes equipped with a natural compact Hausdorff topology $\mathcal{T}$: the product topology coming from the discrete topology on $\{0, 1\}$. It is easy to see that the operations of $\underline{\mathbf{P}}$ are continuous with respect to $\mathcal{T}$.

This algebra $\underline{\mathbf{P}}$ is subdirectly irreducible, with the monolith $\mathrm{Cg}_{\underline{\mathbf{P}}}(\chi_{\{0\}}, \underline{0})$, and J. Ježek [16] proved that every subdirectly irreducible semilattice with automorphism embeds into $\underline{\mathbf{P}}$. So $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{P}})$.

Every finite semilattice with automorphism is dualisable, by the Semilattice-Based Duality Theorem 3.3. We will apply the Semilattice-Based Self-Duality Theorem 7.4 to prove that many topologically closed subalgebras of $\underline{\mathbf{P}}$ are self-dualising (modulo nullaries). The result will apply, for example, to every finite subalgebra of $\underline{\mathbf{P}}$, and therefore to every finite subdirectly irreducible semilattice with automorphism. It will also apply to various natural infinite closed subalgebras of $\underline{\mathbf{P}}$, in particular to $\underline{\mathbf{P}}$ itself and to the flat algebra $\underline{\mathbf{F}}$ that consists of the constant map $\underline{0}$ and all the atoms of $\underline{\mathbf{P}}$.

Before we state the theorem, we mention some facts we need concerning arbitrary non-trivial topologically closed subalgebras $\underline{\mathbf{Q}}$ of $\underline{\mathbf{P}}$.

- *The constant map $\underline{0}$ belongs to $\underline{\mathbf{Q}}$.* Since $\underline{\mathbf{Q}}$ is non-trivial, there is a function $a \in Q \subseteq 2^{\mathbb{Z}}$ with $0 \in a(\mathbb{Z})$. By using f, f^{-1} and $\wedge$, we can generate functions in $\underline{\mathbf{Q}}$ that send any given finite subset of $\mathbb{Z}$ to 0. As $\underline{\mathbf{Q}}$ is topologically closed, we must have $\underline{0} \in Q$. (More generally, every compact Hausdorff topological meet semilattice has a smallest element; see B. A. Davey [4, 4.2.3], for example.)
- *If $\underline{\mathbf{Q}}$ has a top, then $\underline{1}$ belongs to $\underline{\mathbf{Q}}$.* As the flat subalgebra $\underline{\mathbf{F}}$ of $\underline{\mathbf{P}}$ illustrates, the algebra $\underline{\mathbf{Q}}$ need not have a largest element. But, if it does, then its largest element must be a fixpoint of the automorphism f, and so must be $\underline{1}$.
- *The constant map $\underline{1}$ is not definable in $\underline{\mathbf{Q}}$ and, if $\underline{1} \in Q$, then $\underline{0}$ is not definable.* Since $\{\underline{0}\}$ is a subuniverse of $\underline{\mathbf{Q}}$, the algebra $\underline{\mathbf{Q}}$ cannot have a definable top. Likewise, if $\underline{1} \in Q$, then $\underline{\mathbf{Q}}$ does not have a definable bottom.
- *If $\underline{\mathbf{Q}}$ is finite and $\underline{1} \notin Q$, then $\underline{0}$ is definable in $\underline{\mathbf{Q}}$.* Assume $\underline{\mathbf{Q}}$ is finite. There exists $n \in \omega \backslash \{0\}$ such that, for all $a \in Q$, we have $f^n(a) = a$. Define the unary term function t of $\underline{\mathbf{Q}}$ by $t(a) := \bigwedge_{\ell \in n} f^{\ell}(a)$. Then, for each $a \in Q$, the element $t(a)$ is a fixpoint of f. So, if $\underline{1} \notin Q$, then t is constant with value $\underline{0}$, whence $\underline{0}$ is definable.

8.2. **Theorem.** *Let $\underline{\mathbf{Q}}$ be a non-trivial topologically closed subalgebra of the semilattice with automorphism $\underline{\mathbf{P}}$, defined in 8.1, and let $\mathcal{T}$ be the relative topology on $\underline{\mathbf{Q}}$.*

1. *If $\underline{1} \in Q$, then $\underset{\sim}{\mathbf{Q}} := \langle Q; \wedge, f, f^{-1}, \underline{0}, \underline{1}, \mathcal{T} \rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{Q}})$.*
2. *If $\underline{1} \in Q$ and we define $\underline{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0}, \underline{1} \rangle$, then $\underset{\sim}{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \mathcal{T} \rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{Q}}')$.*
3. *The algebra $\underline{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0} \rangle$ is self-dualising, that is, the alter ego $\underset{\sim}{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0}, \mathcal{T} \rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{Q}}')$.*

Proof. The result follows immediately from the Semilattice-Based Self-Duality Theorem 7.4, with $F := \{f, f^{-1}\}$ and $\lambda : \underline{\mathbf{Q}}^{\wedge} \to \underline{\mathbf{2}}$ chosen to be the projection π_0. Since $\lambda(f^i(a)) = f^i(a)(0) = a(i)$, for all $a \in Q$ and $i \in \mathbb{Z}$, the algebra $\underline{\mathbf{Q}}$ satisfies

$$\big((\forall i \in \mathbb{Z})\ \lambda(f^i(a)) = \lambda(f^i(b))\big) \iff a = b,$$

for all $a, b \in Q$. Hence $\lambda \circ \langle F \rangle$ separates Q. □

Part (1) of the previous theorem applies to $\underline{\mathbf{P}}$ itself. Part (3) tells us that the flat subalgebra $\underline{\mathbf{F}}$ of $\underline{\mathbf{P}}$ is self-dualising: for all $a \in F$, we have $a \wedge f(a) = \underline{0}$, and so $\underline{0}$ is already definable in $\underline{\mathbf{F}}$.

We shall next show that, in the case that the subalgebra $\underline{\mathbf{Q}}$ is finite, each of the dualities given in the previous theorem is strong. We begin with a general lemma about the finite subalgebras of $\underline{\mathbf{P}}$. For notational convenience, we assume $\mathbb{Z} \supseteq \omega$.

8.3. **Lemma.** *Let $\underline{\mathbf{Q}}$ be a finite non-trivial subalgebra of the semilattice with automorphism $\underline{\mathbf{P}}$. Define $\mu := \chi_{m\mathbb{Z}} \in 2^{\mathbb{Z}}$, where m is the number of atoms of $\underline{\mathbf{Q}}$.*

1. *The set of atoms of $\underline{\mathbf{Q}}$ is $\{\, f^i(\mu) \mid i \in m \,\}$.*
2. *For all $q \in Q$ and $i \in \mathbb{Z}$, we have $\mu \leqslant f^i(q)$ if and only if $q(i) = 1$.*
3. *Every element of $\underline{\mathbf{Q}}$ is a join of atoms of $\underline{\mathbf{Q}}$.*
4. *The algebra $\underline{\mathbf{Q}}$ satisfies the identity $f^m(x) \approx x$.*
5. *For all $p, q \in Q$, we have $p \leqslant q$ if and only if $p{\restriction}_m \leqslant q{\restriction}_m$.*
6. *The algebra $\underline{\mathbf{Q}}$ is subdirectly irreducible.*

Proof. As $\underline{\mathbf{Q}}$ is non-trivial, it has at least one atom, say a. Since $a > \underline{0}$, there exists $j \in \mathbb{Z}$ such that $a(j) = 1$. Hence $b := f^j(a)$ is the unique atom of $\underline{\mathbf{Q}}$ such that $b(0) = 1$. (For any atom c of $\underline{\mathbf{Q}}$ with $c(0) = 1$, we have $(b \wedge c)(0) \neq \underline{0}(0)$, which implies that $b = c$.) Thus b belongs to the orbit (under f) of each atom of $\underline{\mathbf{Q}}$. Since f is an automorphism, the set of m atoms of $\underline{\mathbf{Q}}$ must be equal to the orbit of b.

To establish (1), it remains to prove that $b = \mu$. The orbit of b has size m. So, for all $i \in \mathbb{Z}$, we have $i \in m\mathbb{Z} \iff f^i(b) = b \iff f^i(b)(0) = 1 \iff b(i) = 1$, using the uniqueness property of the atom b. Thus $b = \chi_{m\mathbb{Z}} = \mu$, and (1) holds.

To prove (2), let $q \in Q$ and $i \in \mathbb{Z}$. If $\mu \leqslant f^i(q)$, then $q(i) = f^i(q)(0) \geqslant \mu(0) = 1$. If $q(i) = 1$, then $f^i(q) \wedge \mu \neq \underline{0}$, which implies that $\mu \leqslant f^i(q)$, as μ is an atom. Thus (2) holds.

Note that, as $\underline{\mathbf{Q}}$ is finite, every subset of Q that has an upper bound in $\underline{\mathbf{Q}}$ has a join in $\underline{\mathbf{Q}}$. For (3), let $p, q \in Q$ with $p \nleqslant q$. There exists $i \in \mathbb{Z}$ such that $p(i) = 1$ and $q(i) = 0$. So $\mu \leqslant f^i(p)$ and $\mu \nleqslant f^i(q)$, by (2). Thus $f^{-i}(\mu)$ is an atom of $\underline{\mathbf{Q}}$ satisfying $f^{-i}(\mu) \leqslant p$ and $f^{-i}(\mu) \nleqslant q$. It follows easily that each element of $\underline{\mathbf{Q}}$ is a join of atoms of $\underline{\mathbf{Q}}$, whence (3) holds.

The atoms of $\underline{\mathbf{Q}}$ form an orbit of f, by (1). So every atom of $\underline{\mathbf{Q}}$ satisfies the identity $f^m(x) \approx x$. Since f preserves all existing joins in $\underline{\mathbf{Q}}$, part (3) guarantees that every element of $\underline{\mathbf{Q}}$ satisfies $f^m(x) \approx x$. Hence (4) holds. This implies that every element of $Q \subseteq 2^{\mathbb{Z}}$ has period m, and therefore (5) also holds.

Finally, we prove (6). Let θ be a non-trivial congruence on $\underline{\mathbf{Q}}$. Then there exist $p, q \in Q$, with $p \nleqslant q$, such that $p \equiv_\theta q$. There is $i \in \mathbb{Z}$ with $p(i) = 1$ and $q(i) = 0$. By (2), we have $\mu \leqslant f^i(p)$ and $\mu \nleqslant f^i(q)$. So $\mu = \mu \wedge f^i(p) \equiv_\theta \mu \wedge f^i(q) = \underline{0}$, as μ is an atom of $\underline{\mathbf{Q}}$. Thus $\underline{\mathbf{Q}}$ is subdirectly irreducible, with the monolith $\mathrm{Cg}_{\underline{\mathbf{Q}}}(\underline{0}, \mu)$. □

The following lemma does the bulk of our work for us in establishing strong dualities for finite subalgebras of $\underline{\mathbf{P}}$.

8.4. **Lemma.** *Let* $\underline{\mathbf{Q}}$ *be a finite non-trivial subalgebra of the semilattice with automorphism* $\underline{\mathbf{P}}$*, defined in 8.1. Define the enrichment* $\mathbf{Q}_K := \langle Q; \wedge, f, f^{-1}, K\rangle$ *of* $\underline{\mathbf{Q}}$*, where* $K := Q \cap \{\underline{0}, \underline{1}\}$*. Then* $\underline{\mathbf{Q}}$ *satisfies the interpolation condition* (IC) *with respect to* $\mathbf{Q}_K$*.*

Proof. Let $k \in \omega\backslash\{0\}$, let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{Q}}^k$, and let $\varphi : \mathbf{A} \to \underline{\mathbf{Q}}$ be a homomorphism. We must show that φ extends to a term function of $\mathbf{Q}_K$. If φ is constant, then it must have value $\underline{0}$ or $\underline{1}$, since these are the only fixpoints of f in $\underline{\mathbf{P}}$. Thus we can assume that φ is not constant.

Let m and n be the numbers of atoms of $\underline{\mathbf{Q}}$ and $\varphi(\mathbf{A}) \leqslant \underline{\mathbf{Q}}$, respectively. Then $\mu := \chi_{m\mathbb{Z}}$ is an atom of $\underline{\mathbf{Q}}$ and $\nu := \chi_{n\mathbb{Z}}$ is an atom of $\varphi(\mathbf{A})$, by Lemma 8.3(1). For each $q \in \varphi(A)$, define $\widehat{q} := \bigwedge \varphi^{-1}(q)$ in $\mathbf{A} \leqslant \underline{\mathbf{Q}}^k$.

Now define the term function $t : Q^k \to Q$ of $\mathbf{Q}_K$ by

$$t(q_0, \dots, q_{k-1}) := \bigwedge \left\{ f^{\ell}(q_j) \;\middle|\; j \in k \text{ and } \ell \in m \text{ and } \widehat{\nu}(j)(\ell) = 1 \right\}.$$

Note that this meet must be non-empty, since $\widehat{\nu} \neq (\underline{0}, \dots, \underline{0})$ and so $\widehat{\nu}(j){\restriction}_m \neq \underline{0}{\restriction}_m$, for some $j \in k$, by Lemma 8.3(5).

We will show that t is an extension of φ. We begin by proving two easy facts.

(i) For all $q \in \varphi(A)$, we have $\mu \leqslant q \iff \nu \leqslant q$.

(ii) For all $q \in \varphi(A)$ and all $a \in A$, we have $\widehat{q} \leqslant a \iff q \leqslant \varphi(a)$.

To prove (i), let $q \in \varphi(A) \subseteq Q$. By applying Lemma 8.3(2) first in $\underline{\mathbf{Q}}$ and then in $\varphi(\mathbf{A})$, we have $\mu \leqslant q \iff q(0) = 1 \iff \nu \leqslant q$. Hence (i) holds.

For (ii), let $q \in \varphi(A)$ and let $a \in A$. If $\widehat{q} \leqslant a$, then $q = \varphi(\widehat{q}) \leqslant \varphi(a)$, since φ preserves $\wedge$. If $q \leqslant \varphi(a)$, then $\varphi(\widehat{q} \wedge a) = \varphi(\widehat{q}) \wedge \varphi(a) = q \wedge \varphi(a) = q$ and so $\widehat{q} \leqslant \widehat{q} \wedge a \leqslant a$. Thus (ii) holds.

Now let $a \in A$. For each $i \in \mathbb{Z}$, we find that

$$\begin{aligned}
& t(a)(i) = 1 & \\
\iff\; & (\forall j \in k)(\forall \ell \in m)\ \widehat{\nu}(j)(\ell) = 1 \Rightarrow f^{\ell}(a(j))(i) = 1 && \text{by the definition of } t \\
\iff\; & (\forall j \in k)(\forall \ell \in m)\ \widehat{\nu}(j)(\ell) = 1 \Rightarrow f^{i}(a(j))(\ell) = 1 && \\
\iff\; & (\forall j \in k)\ \widehat{\nu}(j) \leqslant f^{i}(a(j)) = f^{i}(a)(j) && \text{by Lemma 8.3(5)} \\
\iff\; & \widehat{\nu} \leqslant f^{i}(a) && \\
\iff\; & \nu \leqslant \varphi(f^{i}(a)) && \text{by (ii)} \\
\iff\; & \mu \leqslant \varphi(f^{i}(a)) = f^{i}(\varphi(a)) && \text{by (i)} \\
\iff\; & \varphi(a)(i) = 1 && \text{by Lemma 8.3(2).}
\end{aligned}$$

Thus $t(a) = \varphi(a)$, whence t extends φ. □

It follows very easily from this lemma that all the finite-generator dualities given by Theorem 8.2 are strong.

8.5. **Theorem.** *Let* $\underline{\mathbf{Q}}$ *be a finite non-trivial subalgebra of the semilattice with automorphism* $\underline{\mathbf{P}}$*, defined in 8.1.*

(1) *If* $\underline{1} \in Q$*, then* $\underset{\sim}{\mathbf{Q}} := \langle Q; \wedge, f, f^{-1}, \underline{0}, \underline{1}, \mathcal{T}\rangle$ *yields a strong duality on* $\mathbb{ISP}(\underline{\mathbf{Q}})$*.*

(2) *If* $\underline{1} \in Q$ *and we define* $\underline{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0}, \underline{1}\rangle$, *then* $\underset{\sim}{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \mathcal{T}\rangle$ *yields a strong duality on* $\mathbb{ISP}(\underline{\mathbf{Q}}')$.

(3) *The algebra* $\underline{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0}\rangle$ *is strongly self-dualising, that is, the alter ego* $\underset{\sim}{\mathbf{Q}}' := \langle Q; \wedge, f, f^{-1}, \underline{0}, \mathcal{T}\rangle$ *yields a strong duality on* $\mathbb{ISP}(\underline{\mathbf{Q}}')$.

Proof. We shall apply Theorem 2.2. The only one-element subuniverses of $\underline{\mathbf{P}}$ are $\{\underline{0}\}$ and $\{\underline{1}\}$. So, in all of the situations above, if $a \in Q$ forms a one-element subalgebra of the generator, then the corresponding alter ego includes a as the value of a nullary operation.

We first consider part (1). Assume that $\underline{1} \in Q$. We know that $\underline{\mathbf{Q}}$ satisfies the interpolation condition (IC) with respect to $\underset{\sim}{\mathbf{Q}} := \langle Q; \wedge, f, f^{-1}, \underline{0}, \underline{1}, \mathcal{T}\rangle$, by the previous lemma. For the other direction, let $n \in \omega\backslash\{0\}$, let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{Q}}^n$ and let $\alpha : \mathbf{X} \to \underline{\mathbf{Q}}$ be a morphism. We have included $\underline{0}$ and $\underline{1}$ in the type of $\underset{\sim}{\mathbf{Q}}$. So the morphism α cannot be constant. Thus we can use the previous lemma again to deduce that α extends to an n-ary term function of $\underline{\mathbf{Q}}$. Hence $\underset{\sim}{\mathbf{Q}}$ satisfies the interpolation condition (IC) with respect to $\underline{\mathbf{Q}}$, and it follows by Theorem 2.2 that $\underset{\sim}{\mathbf{Q}}$ yields a strong duality on $\mathbb{ISP}(\underline{\mathbf{Q}})$.

We have proven (1). The argument for (2) is identical to that for (1), as the topology played no role, and the proof of (3) is similar. □

We have in fact shown that each finite subalgebra of the semilattice with automorphism $\underline{\mathbf{P}}$ is injective in the quasi-variety it generates. To see this, fix a finite $\underline{\mathbf{Q}} \leqslant \underline{\mathbf{P}}$ and define $\mathcal{Q} := \mathbb{ISP}(\underline{\mathbf{Q}})$. As $\underline{\mathbf{Q}}$ is entropic, every term function of $\underline{\mathbf{Q}}$ is a homomorphism. Hence Lemma 8.4 implies that $\underline{\mathbf{Q}}$ is injective in $\mathcal{Q}_{\text{fin}}$ and so is injective in $\mathcal{Q}$ [2, 2.2.10].

We close by briefly looking again at the infinite flat semilattice with automorphism $\underline{\mathbf{F}}$, which consists of the atoms of $\underline{\mathbf{P}}$ along with $\underline{0}$. Since $\underline{\mathbf{F}}$ has a definable bottom, the alter ego $\underset{\sim}{\mathbf{F}} := \langle F; \wedge, f, f^{-1}, \underline{0}, \mathcal{T}\rangle$ yields a natural duality on $\mathbb{ISP}(\underline{\mathbf{F}})$, by Theorem 8.2. In every known example of a natural duality based on a finite algebra, the generating algebra turns out to have a finite basis of identities. While the reasons for this apparent trend are unclear, it may be an artefact of the fact that all known dualisable algebras generate residually finite varieties. It is a famous conjecture due to Jónsson and Park that every finite algebra that generates a residually finite variety has a finite basis for its identities. With this in mind, the following observation is of interest.

8.6. **Lemma.** *The class* $\mathbb{ISP}(\underline{\mathbf{F}})$ *is a variety with no finite basis of identities.*

Proof. For all $i \in \mathbb{Z}\backslash\{0\}$ and $a \in F$, we have $f^i(a) \wedge a = \underline{0}$. So $\underline{\mathbf{F}}$ satisfies the set of identities $\Sigma := \{\, f^i(x) \wedge x \approx f(y) \wedge y \mid i \in \mathbb{Z}\backslash\{0\}\,\}$. We will show that $\underline{\mathbf{F}}$ is the only non-trivial subalgebra of the semilattice with automorphism $\underline{\mathbf{P}}$ that satisfies Σ.

Let $\underline{\mathbf{Q}}$ be a non-trivial subalgebra of $\underline{\mathbf{P}}$ that satisfies Σ, and let $q \in Q$. For all $i, j, k, \ell \in \mathbb{Z}\backslash\{0\}$ with $i \neq j$ and $k \neq \ell$, we have $f^i(q) \wedge f^j(q) = f^k(q) \wedge f^\ell(q)$ and therefore $q(i) \wedge q(j) = q(k) \wedge q(\ell)$. So $q \in \{\underline{0}, \underline{1}\}$ or q is an atom of $\underline{\mathbf{P}}$. Since $\underline{\mathbf{Q}}$ is non-trivial, it follows that $\underline{\mathbf{Q}} = \underline{\mathbf{F}}$. So $\underline{\mathbf{F}}$ is the only non-trivial subalgebra of $\underline{\mathbf{P}}$ that satisfies Σ.

We know that every subdirectly irreducible semilattice with automorphism embeds into $\underline{\mathbf{P}}$ [16]. Thus $\underline{\mathbf{F}}$ is, up to isomorphism, the only subdirectly irreducible algebra in $\operatorname{Var}(\underline{\mathbf{F}})$. Hence $\operatorname{Var}(\underline{\mathbf{F}}) = \mathbb{ISP}(\underline{\mathbf{F}})$.

Since every variety is determined by its subdirectly irreducible members, we have in fact proved that $\mathrm{Var}(\underline{\mathbf{F}})$ is determined by Σ within the variety of semilattices with automorphism. Now let Σ' be a finite subset of Σ. Then the subalgebra of $\underline{\mathbf{P}}$ generated by $\chi_{m\mathbb{Z}}$, for a sufficiently large $m \in \omega$, satisfies Σ' but not Σ. The Compactness Theorem now tells us there is no finite basis for the identities of $\underline{\mathbf{F}}$. $\square$

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