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A SCHIZOPHRENIC OPERATION WHICH AIDS THE EFFICIENT TRANSFER OF STRONG DUALITIES

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Abstract. We show that, in many cases, if $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras which generate the same quasi-variety $\mathfrak{D}$, then a strong duality for $\mathfrak{D}$ based on $\underline{\mathbf{D}}$ may be transferred to a strong duality for $\mathfrak{D}$ based on $\underline{\mathbf{M}}$ by the addition some endomorphisms of $\underline{\mathbf{M}}$ and just one further partial operation. This additional operation exhibits the schizophrenia so typical of the theory of natural dualities. We show how the result may be applied to yield an efficient strong duality in the case when $\underline{\mathbf{M}}$ is a distributive lattice, a semilattice or an abelian group.

1. Introduction

A **full duality** for the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{D}})$ generated by a finite algebra $\underline{\mathbf{D}} = \langle D; F\rangle$ is a dual category equivalence between $\boldsymbol{\mathcal{A}}$ and the category $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$ where $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ is a (discrete) topological structure allowing sets R of relations, G of operations and H of partial operations in its type. Here $\mathbb{I}$, $\mathbb{S}$ $\mathbb{P}$ (arbitrary index sets) and $\mathbb{P}^+$ (non-empty index sets), denote the usual class operators while $\mathbb{S}_c$ denotes "*closed substructure of*". The duality is called **strong** if it is full and moreover $\underset{\sim}{\mathbf{M}}$ is injective in $\boldsymbol{\mathfrak{X}}$. Strong dualities provide a powerful tool for the study of the quasi-variety $\boldsymbol{\mathcal{A}}$. Since a quasi-variety may have many different generating algebras it is useful to know how to take a known strong duality for $\boldsymbol{\mathcal{A}}$ based on a finite generating algebra $\underline{\mathbf{D}}$ and transfer it to a strong duality based on another finite generating algebra $\underline{\mathbf{M}}$. Our concern here is not the *existence* of a strong duality based on $\underline{\mathbf{M}}$: that is known—see Davey and Willard [7] and Saramago [9] for the existence of a duality and Hyndman [8]

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for the existence of a strong duality. The aim of this paper is to give conditions under which we obtain a particularly efficient and user friendly transfer of a strong duality for $\boldsymbol{\mathcal{A}}$ based on $\underline{\mathbf{D}}$ to a strong duality based on $\underline{\mathbf{M}}$.

We shall motivate our Strong Duality Transfer Theorem by considering an example which has been studied intensively. Let $\underline{\mathbf{D}} = \langle\{0,1\}; \vee, \wedge, 0, 1\rangle$ be the two-element bounded lattice. Priestley duality (see Davey and Priestley [6]) tells us that $\underset{\sim}{\mathbf{D}} := \langle\{0,1\}; \leqslant, \mathcal{T}\rangle$, where $\leqslant$ is the usual order and $\mathcal{T}$ is the discrete topology, yields a strong duality between the (quasi-)variety $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices and the category $\mathcal{P} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$ of Priestley spaces. Now take $\underline{\mathbf{M}} = \langle\{0, d, 1\}; \vee, \wedge, 0, 1\rangle$ to be the three-element chain. Note that $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$ and denote the two endomorphisms of $\underline{\mathbf{M}}$ which map into $\underline{\mathbf{D}}$ by e, with $e^{-1}(1) = \{1\}$, and f, with $f^{-1}(0) = \{0\}$. It is proved in Davey, Haviar and Priestley [4] (see also Davey [3]) that

$$\underset{\sim}{\mathbf{M}}^{(1)} := \langle\{0, d, 1\}; e, f, \leqslant_D, \mathcal{T}\rangle,$$

where $\leqslant_D := \{00, 01, 11\}$, yields a duality on $\mathfrak{D}$ (based on $\underline{\mathbf{M}}$ rather than $\underline{\mathbf{D}}$) and that the relation $\leqslant_D$ may be removed without destroying the duality, so that

$$\underset{\sim}{\mathbf{M}}^{(2)} := \langle\{0, d, 1\}; e, f, \mathcal{T}\rangle,$$

also yields a duality. Unfortunately, the resulting dualities are neither full nor strong.

The NU Strong Duality Theorem (see 3.3.8 in [2]) tells us that these dualities may be upgraded to strong dualities by adding all binary partial homomorphisms. In Davey, Haviar and Priestley [5] (see also Section 5 in Chapter 9 of [2]), a computer program was used to find all 308 binary partial homomorphisms and to check that they were all generated by the endomorphisms e and f and one further partial operation $h : M^2\backslash\{10\} \to M$, given by $h^{-1}(1) = {\uparrow}1d$, $h^{-1}(d) = [0d, d1]$ and $h^{-1}(0) = {\downarrow}d0$. Thus,

$$\underset{\sim}{\mathbf{M}}^{(3)} := \langle\{0, d, 1\}; e, f, h, \mathcal{T}\rangle$$

yields a strong duality on $\mathfrak{D}$ based on $\underline{\mathbf{M}}$. It turns out that we may replace the partial operation h by a more natural one of which h is an extension. Since e and f separate the points of M, the map

$$\omega := e \sqcap f : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^2 \leqslant \underline{\mathbf{M}}^2$$

given by $\omega(a) := e(a)f(a)$, for all $a \in M$, is an embedding with image

$$M_r := \omega(M) = \{00, 01, 11\}.$$

(The fact that $M_r = \leqslant_D$ in this case is a coincidence and an unfortunate distraction: in general, M_r will have the same size as M while $\leqslant_D$ remains a fixed three-element set.) Let $\sigma : M_r \to M$ be the inverse of ω regarded as a binary partial operation on $\underline{\mathbf{M}}$. It is easily checked that, for $ab \in M^2 \backslash \{10\}$,

$$\sigma(e(a), f(b)) = h(ab) \text{ and hence } \sigma(e \circ \rho_1, f \circ \rho_2) = h,$$

where $\rho_i : M^2\backslash\{10\} \to M$ is the restriction of the ith projection to $M^2\backslash\{10\}$. Thus, the partial clone generated by $\{e, f, \sigma\}$ contains h and consequently, since $\underset{\sim}{\mathbf{M}}^{(3)}$ yields a strong duality for $\mathfrak{D}$ based on $\underline{\mathbf{M}}$, the structure

$$\underset{\sim}{\mathbf{M}}^{(4)} := \langle \{0, d, 1\}; e, f, \sigma, \mathcal{T} \rangle$$

does also. Note that, in a typically schizophrenic way, the partial operation σ, which guarantees that this duality based on $\underline{\mathbf{M}}$ is strong, is an isomorphism between the subalgebra $\mathbf{M}_r$ of $\underline{\mathbf{M}}^2$ and $\underline{\mathbf{M}}$ itself. It is somewhat unsatisfactory that this argument that $\underset{\sim}{\mathbf{M}}^{(4)}$ yields a strong duality relies on computer-based counting. Our main theorem generalises this strong duality and thereby provides the first purely algebraic proof that both $\underset{\sim}{\mathbf{M}}^{(3)}$ and $\underset{\sim}{\mathbf{M}}^{(4)}$ yield strong dualities on $\mathfrak{D}$ based on the three-element chain.

We shall not revise in detail the foundations of the theory of natural dualities. Rather we refer the reader to the monograph by Clark and Davey [2]: strong dualities are discussed in detail in Chapter 3 and our results are closely related to those presented in Sections 7 and 8 of Chapter 7 and Section 5 of Chapter 9. To aid the reader, our notation is consistent with the notation used in [2].

2. The Transfer Theorem

Let $\underline{\mathbf{D}}$ be a finite algebra, let $\underline{\mathbf{M}}$ be a finite algebra in $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ which has $\underline{\mathbf{D}}$ as a subalgebra. It follows at once that $\mathfrak{D} = \mathbb{ISP}(\underline{\mathbf{M}})$. Let r be an n-ary algebraic relation on $\underline{\mathbf{D}}$. Since $\mathbf{r} \leqslant \underline{\mathbf{D}}^n$ and $\underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n$ we may view r as an n-ary algebraic relation on $\underline{\mathbf{M}}$. Following the notation used in both [3] and [2], we shall denote this relation *on* $\underline{\mathbf{M}}$ by r_D. Similarly, if g is a total (or partial) operation on D then g_D denotes g regarded as a *partial* operation on M. If $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$ is algebraic over $\underline{\mathbf{D}}$, then the corresponding sets of partial operations and relations on $\underline{\mathbf{M}}$ are denoted by $G_D := \{ g_D \mid g \in G \}$, $H_D := \{ h_D \mid h \in H \}$ and $R_D := \{ r_D \mid r \in R \}$. The first step in transferring a strong duality for $\mathfrak{D}$ based on $\underline{\mathbf{D}}$ to one based on $\underline{\mathbf{M}}$ is provided by the following theorem which shows how to transfer a duality. (The special case of this result in which one of the endomorphisms ω_i is a retraction of $\underline{\mathbf{M}}$ onto the subalgebra

$\underline{\mathbf{D}}$ was proved in [3]. See also Section 7 of Chapter 7 in [2].) The ability to transfer a duality based on $\underline{\mathbf{D}}$ to one based on $\underline{\mathbf{M}}$ relies on the existence of a set $\Omega = \{\omega_1, \dots, \omega_n\}$ of endomorphisms of $\underline{\mathbf{M}}$ which satisfy:

(E1) $\omega_i(M) \subseteq D$ for all i, and
(E2) $\omega_1, \dots, \omega_n$ separate the points of M.

Note that the assumptions that $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$ and that $\underline{\mathbf{M}}$ belongs to $\mathfrak{D}$ guarantee the existence of such a set of endomorphisms. Indeed, if $\omega : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n$ is an embedding and $\pi_i : \underline{\mathbf{D}}^n \to \underline{\mathbf{D}}$ is the ith projection, then $\{\pi_1 \circ \omega, \dots, \pi_n \circ \omega\}$ is such a set.

Duality Transfer Theorem. (Saramago [9]) *Let* $\underline{\mathbf{D}}$ *be a finite algebra, let* $\underline{\mathbf{M}}$ *be a finite algebra in* $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ *which has* $\underline{\mathbf{D}}$ *as a subalgebra and let* $\Omega = \{\omega_1, \dots, \omega_n\}$ *be a set of endomorphisms of* $\underline{\mathbf{M}}$ *satisfying* (E1) *and* (E2) *above. If the structure* $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ *yields a duality on* $\mathfrak{D}$ *based on* $\underline{\mathbf{D}}$*, then the structure*

$$\underset{\sim}{\mathbf{M}} := \langle M; \Omega, G_D \cup H_D, R_D \cup \{D\}, \mathcal{T}\rangle$$

yields a duality on $\mathfrak{D}$ *based on* $\underline{\mathbf{M}}$*.*

In general, the duality for $\mathfrak{D}$ based on $\underline{\mathbf{M}}$ given by this theorem will not be strong even if the original duality based on $\underline{\mathbf{D}}$ is. It is rather surprising that in order to transfer a strong duality for $\mathfrak{D}$ based on $\underline{\mathbf{D}}$ to a strong duality based on $\underline{\mathbf{M}}$ only one further partial operation needs to be added to the structure $\underset{\sim}{\mathbf{M}}$ defined in the Duality Transfer Theorem. Let $\underline{\mathbf{D}}$, $\underline{\mathbf{M}}$, $\mathfrak{D}$ and $\omega_1, \dots, \omega_n$ be as above. Let

$$\omega := \omega_1 \sqcap \cdots \sqcap \omega_n : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n$$

so that $\omega(a) = (\omega_1(a), \dots, \omega_n(a))$. Since the maps $\omega_1, \dots, \omega_n$ separate the points of M, the homomorphism ω is an embedding. Let $M_r := \omega(M) \subseteq D^n \subseteq M^n$ and let $\sigma : \mathbf{M}_r \to \underline{\mathbf{M}}$ be the inverse of ω regarded as an n-ary algebraic partial operation on $\underline{\mathbf{M}}$. Hence,

$$\sigma(\omega_1(a), \dots, \omega_n(a)) = a \text{ for all } a \in M.$$

This additional partial operation on $\underline{\mathbf{M}}$ is all that we require.

Strong Duality Transfer Theorem. *Let* $\underline{\mathbf{D}}$ *be a finite algebra, let* $\underline{\mathbf{M}}$ *be a finite algebra in* $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ *which has* $\underline{\mathbf{D}}$ *as a subalgebra and let* $\Omega = \{\omega_1, \dots, \omega_n\}$

be a set of endomorphisms of $\underline{\mathbf{M}}$ *satisfying* (E1) *and* (E2) *above. If the structure* $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ *yields a strong duality on* $\mathfrak{D}$ *based on* $\underline{\mathbf{D}}$*, then the structure*

$$\underset{\sim}{\mathbf{M}} := \langle M; \Omega, G_D \cup H_D \cup \{\sigma\}, R_D \cup \{D\}, \mathcal{T}\rangle$$

yields a strong duality on $\mathfrak{D}$ *based on* $\underline{\mathbf{M}}$.

Proof. We know from the Duality Transfer Theorem that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathfrak{D}$ based on $\underline{\mathbf{M}}$. In order to show that the duality is strong it suffices to show that, for every non-empty set S, each closed substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^S$ is **term-closed**, that is, if $y \in M^S \backslash X$, then there exist S-ary term functions t_1 and t_2 such that $t_1{\restriction}_X = t_2{\restriction}_X$ but $t_1(y) \neq t_2(y)$. (In fact, this is the usual definition of a strong duality. See the First Strong Duality Theorem in Chapter 3 of [2] for a proof that this is equivalent to the definition given above.)

Let $\mathbf{X}$ be a closed substructure of $\underset{\sim}{\mathbf{M}}^S$ and let $y \in M^S \backslash X$. As the atomic formula

$$\sigma(\omega_1(v), \dots, \omega_n(v)) = v$$

holds in $\underset{\sim}{\mathbf{M}}$, it also holds in $\mathbf{X}$. (See the Preservation Theorem in Chapter 1 of [2].) Thus, if $\omega_1(y), \dots, \omega_n(y) \in X$, then $y = \sigma(\omega_1(y), \dots, \omega_n(y)) \in X$, a contradiction. Hence, $\omega_j(y) \notin X$ for some j. Let $Y := X \cap D^S$. As $\mathbf{X}$ is closed under h_D for all $h \in G \cup H$, it follows that Y is closed under h for all $h \in G \cup H$. Thus, $\mathbf{Y}$ is a closed substructure of $\underset{\sim}{\mathbf{D}}^S$ and consequently is term-closed in $\underset{\sim}{\mathbf{D}}^S$ since $\underset{\sim}{\mathbf{D}}$ yields a strong duality on $\mathfrak{D}$. Since ω_j maps into D, we have $\omega_j(y) \in D^S \backslash Y$, whence there are S-ary term functions t_1 and t_2 such that

$$t_1{\restriction}_Y = t_2{\restriction}_Y \text{ but } t_1(\omega_j(y)) \neq t_2(\omega_j(y)).$$

Since ω_j is a homomorphism and therefore commutes with the term functions t_1 and t_2, it follows at once that $t_1(y) \neq t_2(y)$. It remains to prove that $t_1{\restriction}_X = t_2{\restriction}_X$. Let $x \in X$; then, since $\omega_i(x) \in Y$ for all i,

$$\begin{aligned} t_1(x) &= t_1(\sigma(\omega_1(x), \dots, \omega_n(x))) \\ &= \sigma(t_1(\omega_1(x)), \dots, t_1(\omega_n(x))) \\ &= \sigma(t_2(\omega_1(x)), \dots, t_2(\omega_n(x))) \\ &= t_2(\sigma(\omega_1(x), \dots, \omega_n(x))) \\ &= t_2(x). \end{aligned}$$

This completes the proof. □

3. Applications

The Strong Duality Transfer Theorem may be applied to produce a plethora of new strong dualities. We shall illustrate its application to three important examples. All of the strong dualities presented below are new except for the special case of the three-element distributive lattice which was discussed in the introduction.

Distributive lattices. Let $\mathfrak{D}$ be the variety of bounded distributive lattices, let $\underline{\mathbf{D}} = \langle\{0,1\}; \vee, \wedge, 0, 1\rangle$ be the two-element bounded lattice and let $\underline{\mathbf{M}}$ be any finite bounded distributive lattice. Let $\omega : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n$ be an embedding (with n minimal, for efficiency). We may take $\omega_i := \pi_i \circ \omega$, in which case σ is just ω^{-1} regarded as an n-ary algebraic partial operation on $\underline{\mathbf{M}}$. Since $\underset{\sim}{\mathbf{D}} = \langle\{0,1\}; \leqslant, \mathcal{T}\rangle$ yields a strong duality on $\mathfrak{D}$, the Strong Duality Transfer Theorem tells us that

$$\underset{\sim}{\mathbf{M}} := \langle M; \omega_1, \dots, \omega_n, \sigma, \leqslant_D, \{0,1\}, \mathcal{T}\rangle$$

yields a strong duality based on $\underline{\mathbf{M}}$. Since ω_i is a retraction of $\underline{\mathbf{M}}$ onto $\underline{\mathbf{D}}$, the unary relation $\{0,1\}$, which is the fixpoint set of ω_i, may be deleted from the type of $\underset{\sim}{\mathbf{M}}$ without destroying the strong duality. If $\underline{\mathbf{M}}$ is non-Boolean, then, by the results of [4] (see also [3]), $\leqslant_D$ may also be removed and hence

$$\underset{\sim}{\mathbf{M}}' := \langle M; \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$$

yields a strong duality on $\mathfrak{D}$ based on $\underline{\mathbf{M}}$. If $\underline{\mathbf{M}}$ is Boolean, then [4] tells us that $\leqslant_D$ cannot be removed (though it can be replaced by any non-Boolean algebraic relation). Since Boolean lattices are injective in $\mathfrak{D}$, the partial operation σ extends to a total operation $\widehat{\sigma} : \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$ and consequently the total structure

$$\underset{\sim}{\mathbf{M}}'' := \langle M; \omega_1, \dots, \omega_n, \widehat{\sigma}, \leqslant_D, \mathcal{T}\rangle$$

yields a strong duality based on $\underline{\mathbf{M}}$.

Semilattices. Let $\underline{\mathbf{D}}_K = (\{0,1\}; \vee, K)$ be the two-element semilattice with bounds $K \subseteq \{0,1\}$, let $\mathfrak{S}_K := \mathbb{ISP}(\underline{\mathbf{D}}_K)$ and let $\underline{\mathbf{S}}$ be a finite semilattice in $\mathfrak{S}_K$. As in the case of distributive lattices, let $\omega : \underline{\mathbf{S}} \to \underline{\mathbf{D}}^n \leqslant \underline{\mathbf{S}}^n$ be an embedding, take $\omega_i := \pi_i \circ \omega$, so that σ is just ω^{-1} regarded as an n-ary algebraic partial operation on $\underline{\mathbf{S}}$. We then have the following strong dualities.

(i) $\underset{\sim}{\mathbf{S}} := \langle S; \vee, 0, 1, \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$ yields a strong duality on $\mathfrak{S}$ based on the semilattice $\underline{\mathbf{S}} = \langle S; \vee\rangle$.

(ii) $\underset{\sim}{\mathbf{S}}_0 = \langle S; \vee, 0, \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$ yields a strong duality on $\mathfrak{S}_0$ based on the semilattice with zero $\underline{\mathbf{S}}_0 = \langle S; \vee, 0\rangle$.

(iii) $\underset{\sim}{\mathbf{S}}_1 = \langle S; \vee, 1, \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{S}_1$ based on the semilattice with one $\underline{\mathbf{S}}_1 = \langle S; \vee, 1\rangle$.

(iv) $\underset{\sim}{\mathbf{S}}_{01} = \langle S; \vee, \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{S}_{01}$ based on the bounded semilattice $\underline{\mathbf{S}}_{01} = \langle S; \vee, 0, 1\rangle$.

We shall explain how to derive the first of these results from the Strong Duality Transfer Theorem as the other three follow analogously. The structure $\underset{\sim}{\mathbf{D}} := \langle\{0,1\}; \vee, 0, 1, \mathcal{T}\rangle$ yields a duality on the variety $\mathcal{S}$ of semilattices based on $\underline{\mathbf{D}} = \langle\{0,1\}; \vee\rangle$ (see [2] page 228). Thus, by the Strong Duality Transfer Theorem, the structure

$$\underset{\sim}{\mathbf{S}}' := \langle S; \omega_1, \dots, \omega_n, 0, 1, \sigma, \vee_{\{0,1\}}, \{0,1\}, \mathcal{T}\rangle$$

yields a strong duality on $\mathcal{S}$ based on $\underline{\mathbf{S}}$. Once again, the unary relation $\{0,1\}$ can be removed as it is the fixpoint set of ω_i. The partial operation $\vee_{\{0,1\}}$ can be replaced by the total operation $\vee$ as $\vee$ is a binary homomorphism on $\underline{\mathbf{S}}$ and is an extension of $\vee_{\{0,1\}}$. (Here we are using the concept of strong entailment of one structure by another. See [2] for a detailed discussion of entailment.)

Abelian groups. Let $\underline{\mathbf{M}} = \langle M; \cdot, ^{-1}, 1\rangle$ be a finite abelian group. Then there is a cyclic subgroup $\underline{\mathbf{D}}$ of $\underline{\mathbf{M}}$ such that $\underline{\mathbf{D}}$ is a direct factor of $\underline{\mathbf{M}}$ and such that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ generate the same quasi-variety $\mathcal{D}$. Let ω_1 be the retraction of $\underline{\mathbf{M}}$ onto $\underline{\mathbf{D}}$ and let $\omega_2, \dots, \omega_n$ be enough further homomorphisms of $\underline{\mathbf{M}}$ into $\underline{\mathbf{D}}$ to guarantee that $\omega_1, \dots, \omega_n$ separate the points of M. Let

$$\omega := \omega_1 \sqcap \cdots \sqcap \omega_n : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n$$

and let σ be the inverse of ω regarded as an n-ary algebraic partial operation on $\underline{\mathbf{M}}$. Since $\underset{\sim}{\mathbf{D}} = \langle D; \cdot, ^{-1}, 1, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{D}$ based on $\underline{\mathbf{D}}$, we may apply the Strong Duality Transfer Theorem to obtain a strong duality for $\mathcal{D}$ based on $\underline{\mathbf{M}}$. An argument similar to the one above for semilattices shows that we may delete the unary relation D (the fixpoint set of ω_1) and replace the partial operations $\cdot_D$ and $^{-1_D}$ by the corresponding total maps. Thus

$$\underset{\sim}{\mathbf{M}} := \langle M; \cdot, ^{-1}, 1, \omega_1, \dots, \omega_n, \sigma, \mathcal{T}\rangle$$

yields a strong duality on $\mathcal{D}$ based on $\underline{\mathbf{M}}$. If $\omega : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n$ is an isomorphism, then we may omit σ provided we add some extra endomorphisms: indeed,

$$\underset{\sim}{\mathbf{M}} := \langle M; \cdot, ^{-1}, 1, \mathrm{End}(\underline{\mathbf{M}}), \mathcal{T}\rangle$$

yields a strong duality on $\underline{\mathbf{M}}$. To see this, define endomorphisms $e_1, \dots, e_n$ of $\underline{\mathbf{M}}$ by

$$e_i(a) := \sigma(1, \dots, 1, \omega_1(a), 1, \dots, 1),$$

with $\omega_1(a)$ in the ith position. Then for all $(a_1, \dots, a_n) \in \mathrm{dom}(\sigma) = D^n$,

$$\sigma(a_1, \dots, a_n) = e_1(a_1) \cdot \ldots \cdot e_n(a_n).$$

Strong dualities were introduced and studied intensively by Clark and Davey in [1]. Strong dualities were obtained there for quasi-varieties generated by certain abelian groups. The authors of [1] observed that their general results could not be used to obtain a strong duality for the quasi-variety generated by an arbitrary finite abelian group and they remarked that a general method for obtaining such a strong duality "*might very well be the stepping stone to a new wave of strong duality theorems*." The authors of this paper offer it as a step along the way.

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