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Applications of Priestley Duality in Transferring Optimal Dualities

Abstract. This paper illustrates how Priestley duality can be used in the transfer of an optimal natural duality from a minimal generating algebra for a quasi-variety to other generating algebras. Detailed calculations are given for the quasi-variety $\mathbb{ISP}(\underline{\mathbf{4}})$ of Kleene algebras and the quasi-varieties $\mathbf{\mathfrak{B}}_n$ of pseudocomplemented distributive lattices ($n \geqslant 1$).

Keywords: Natural duality, optimal duality, Priestley duality, endodualisability, endoprimality, entailment, retraction

1. Introduction

Over the past 30 years Priestley duality and the theory of natural dualities have developed into powerful tools in the study of finitely generated quasi-varieties of algebras that have an underlying distributive lattice structure. These two tools are particularly potent when used in tandem. If we have an effective method for converting the restricted Priestley dual into the natural dual then we have both the logarithmic advantages of Priestley duality along with the simple descriptions of free algebras and coproducts provided by the natural duality.

Applications of Priestley duality grew out of Priestley's seminal paper [25] in 1972. Priestley [27] gives a survey of applications of the duality up to 1984 as well as a useful dictionary for converting objects and morphisms in the algebraic quasi-variety into the corresponding objects and morphisms in the restricted Priestley dual. A list of 239 references related to Priestley duality is given by Adams and Dziobiak at the end of the 1996 *Studia Logica* special issue on Priestley duality [1].

The theory of natural dualities was developed in Davey and Werner [18] as a refinement of the approach put forward by Davey [4] in 1978. In [18], and in [4] before it, the authors set out to provide a common framework for

Special issue of Studia Logica: "Algebraic Theory of Quasivarieties"
Presented by **M. E. Adams, K. V. Adaricheva, W. Dziobiak, and A. V. Kravchenko**;
Received August 5, 2002, *Accepted* January 29, 2003

Studia Logica **78**: 213–236, 2004.
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Pontryagin duality for abelian groups [22, 23, 24], Stone duality for Boolean algebras [34] and Priestley duality for bounded distributive lattices.

Assume that we wish to study a finitely generated quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$, where $\underline{\mathbf{M}}$ is a finite algebra that has an underlying bounded-distributive-lattice structure. It is quite often the case that we have a description of the restricted Priestley duality but know very little about possible natural dualities other than the fact that the set of all binary algebraic relations does yield a natural duality. There are several papers that illustrate how knowledge of the restricted Priestley duality allows us to obtain a small set of binary algebraic relations that yields a natural duality. In some cases, the translation process in both directions between the natural dual of an algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{A}}$ and the restricted Priestley dual of $\mathbf{A}$ has been completely described. (See, for example, [14, 28, 29, 30, 11] and Chapter 7 of [2].) This paper is a further contribution in this direction.

The time is ripe to collect together all of these examples of the interplay between restricted Priestley dualities and natural dualities and to develop the general theory behind the translation process.

2. Transferring optimal dualities

Why are we interested in transferring an optimal duality from a minimal generating algebra $\underline{\mathbf{D}}$ for a quasi-variety $\boldsymbol{\mathcal{D}} := \mathbb{ISP}(\underline{\mathbf{D}})$ to an optimal duality for another finite generating algebra $\underline{\mathbf{M}}$ in $\boldsymbol{\mathcal{D}}$? There are several reasons.

- The transferred duality allows us to obtain useful information about the algebra $\underline{\mathbf{M}}$. For example, it immediately gives us a description of the term functions of $\underline{\mathbf{M}}$.
- A judicious choice of $\underline{\mathbf{M}}$ can lead to a duality for $\boldsymbol{\mathcal{D}}$ based on $\underline{\mathbf{M}}$ for which the alter ego $\underset{\sim}{\mathbf{M}}$ that induces the duality is 'simpler' than the alter ego $\underset{\sim}{\mathbf{D}}$ that induced the duality based on $\underline{\mathbf{D}}$.
- The algebra $\underline{\mathbf{M}}$ and its alter egos may have properties, not shared by $\underline{\mathbf{D}}$ and its alter egos, that are valuable in the study of the general theory of natural dualities.

To illustrate, we consider Priestley duality for bounded distributive lattices. Let $\boldsymbol{\mathcal{D}} := \mathbb{ISP}(\underline{\mathbf{D}})$ be the category of bounded distributive lattices where $\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ is the two-element bounded lattice. Then Priestley duality is obtained via the alter ego $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$: the two-element discrete ordered space. The dual category is the category $\boldsymbol{\mathcal{P}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$

of Priestley spaces, and the duality between $\mathcal{D}$ and $\mathcal{P}$ is given by the hom-functors

$$H := \mathcal{D}(-, \underline{\mathbf{D}}) : \mathcal{D} \to \mathcal{P} \text{ and } K := \mathcal{P}(-, \underset{\sim}{\mathbf{D}}) : \mathcal{P} \to \mathcal{D}.$$

See Priestley [25] and Section 1.2 of Clark and Davey [2]. It is proved in Davey, Haviar and Priestley [9] that Priestley duality can be transferred to an optimal duality on the three-element bounded distributive lattice

$$\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle.$$

The alter ego that yields the duality is

$$\underset{\sim}{\mathbf{M}} = \langle \{0, a, 1\}; f, g, \mathcal{T} \rangle,$$

where f and g are the endomorphisms of $\underline{\mathbf{M}}$ given by $f(0) = f(a) = 0$, $f(1) = 1$ and $g(0) = 0$, $g(a) = g(1) = 1$. (This result has been reproved several times in different contexts: see [7, 13, 8].) An immediate application of this transferred duality is that $\underline{\mathbf{M}}$ is **endoprimal** (that is, for all $n \in \mathbb{N}$, a map $t : M^n \to M$ is a term function of $\underline{\mathbf{M}}$ if and only if t preserves the operations f and g), a result first proved by Márki and Pöschel [21]. This duality for $\mathcal{D}$, based on $\underline{\mathbf{M}}$ rather that $\underline{\mathbf{D}}$, is at the heart of a recent paper by Davey, Haviar and Willard [12] which gives a partial solution to one of the oldest problems in the theory of natural dualities.

In [8] we studied the theory behind the transferral of an optimal duality based on a finite minimal generating algebra $\underline{\mathbf{D}}$ for a quasi-variety $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ to an optimal duality based on another finite generating algebra $\underline{\mathbf{M}}$ in $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$. We gave applications to Kleene algebras with a fixpoint, semilattices, distributive lattices, and Stone algebras. These applications were obtained by purely algebraic means.

As the minimal generating algebra $\underline{\mathbf{D}}$ for a quasi-variety and the relations on $\underline{\mathbf{D}}$ become larger, the purely algebraic methods illustrated in [8] become more difficult to apply. This paper shows how the logarithmic property of Priestley duality can be used in such situations for quasi-varieties of distributive-lattice-based algebras. We first use it in transferring an optimal natural duality from the four-element Kleene algebra to an arbitrary finite non-Boolean Kleene algebra in the quasi-variety $\mathbb{ISP}(\underline{\mathbf{4}})$, and then we apply it in the varieties $\mathcal{B}_n$ of pseudocomplemented distributive lattices ($n \geqslant 1$).

We assume that the reader is familiar with the basics of natural duality theory as presented in the first two chapters of [2]. In this section we present a few results from the general theory which we need in our investigations in the subsequent sections. We also recall some results from [8] regarding the transferral of optimal dualities. For more results on the theory of transferring optimal dualities and motivations on this topic we refer to [8].

Let $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ be a quasi-variety generated by a finite algebra $\underline{\mathbf{D}}$. Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ having $\underline{\mathbf{D}}$ as a subalgebra so that $\mathcal{D} = \mathbb{ISP}(\underline{\mathbf{M}})$. Let G, H and R be sets of finitary operations, partial operations and relations on D which are **algebraic** over $\underline{\mathbf{D}}$, meaning that the relations in R and the graphs of all (partial) operations in $G \cup H$ are subalgebras of appropriate powers of $\underline{\mathbf{D}}$. Any n-ary algebraic relation r on $\underline{\mathbf{D}}$ may be viewed as an n-ary algebraic relation on $\underline{\mathbf{M}}$ since $\mathbf{r} \leqslant \underline{\mathbf{D}}^n$ and $\underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n$. Following the notation used in [7] and [2], this relation *on* $\underline{\mathbf{M}}$ is denoted by r_D. Similarly, if g is an algebraic total (or partial) operation on D then g_D denotes g regarded as an algebraic *partial* operation on M. The corresponding sets of algebraic partial operations and relations on $\underline{\mathbf{M}}$ are denoted by G_D, H_D and R_D.

We start the collection of general results needed in our later investigations with the following simplified version of a result from Samarago [32].

Theorem 2.1. [32, Proposition 2.1] *Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ having $\underline{\mathbf{D}}$ as a subalgebra. If $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$ yields a duality on $\mathcal{D}$ based on $\underline{\mathbf{D}}$, then*

$$\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), G_D \cup H_D, R_D \cup \{D\}, \mathcal{T} \rangle$$

yields a duality on $\mathcal{D}$ based on $\underline{\mathbf{M}}$.

As in [8], we shall henceforth drop the subscript from r_D when regarding an algebraic relation r on $\underline{\mathbf{D}}$ as a relation on $\underline{\mathbf{M}}$. We note that Theorem 2.1 remains valid if we replace the assumption that $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$ by the assumption that there is an embedding $\nu : \underline{\mathbf{D}} \to \underline{\mathbf{M}}$. All our other results that assume that $\underline{\mathbf{M}}$ has $\underline{\mathbf{D}}$ as a subalgebra also remain valid under the weaker assumption that $\underline{\mathbf{M}}$ has a subalgebra isomorphic to $\underline{\mathbf{D}}$.

Our aim in this paper is to develop a method for transferring optimal dualities, in quasi-varieties of distributive-lattice-based algebras, by means of Priestley duality. We recall that a structure $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$ yields an **optimal duality** on $\mathcal{D}$ if $\underset{\sim}{\mathbf{D}}$ yields a duality on $\mathcal{D}$ and if the removal of any element of $G \cup H \cup R$ gives a structure which does not yield a duality on $\mathcal{D}$. For an algebra $\underline{\mathbf{M}}$ which also generates the quasi-variety $\mathcal{D}$ we aim to convert the structure $\underset{\sim}{\mathbf{D}}$, yielding an optimal duality on $\mathcal{D}$ based on $\underline{\mathbf{D}}$, to a structure $\underset{\sim}{\mathbf{M}}$ yielding an optimal duality on $\mathcal{D}$ based on $\underline{\mathbf{M}}$. As we are usually happy to include (a generating set for) the endomorphism monoid on $\underline{\mathbf{M}}$ in the dualising structure for $\underline{\mathbf{M}}$, we concentrate on dualities which are **optimal modulo endomorphisms**, that is, we seek structures $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), S, \mathcal{T}\rangle$ yielding a duality on $\mathfrak{D}$ based on $\underline{\mathbf{M}}$ such that, if any member of S is removed, the resulting structure does not yield a duality on $\mathfrak{D}$.

Methods for transferring a duality from a usually small algebra $\underline{\mathbf{D}}$ up to a bigger algebra $\underline{\mathbf{M}} \in \mathfrak{D}$ were presented in Davey [7] and Samarago [32]. Nevertheless, it was far from clear which reducts of the resulting dualising structure $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), S, \mathcal{T}\rangle$ yield optimal (modulo endomorphisms) dualities on $\mathfrak{D}$ based on $\underline{\mathbf{M}}$. The only thing known for sure was that a relation $s \in S$ could be removed without destroying the duality provided the corresponding algebra $\mathbf{s}$ was a retract of $\underline{\mathbf{M}}$.

To state this result more precisely, recall that given a set $R \cup \{s\}$ of finitary algebraic (partial) operations and relations on $\underline{\mathbf{D}}$ and an algebra $\mathbf{A} \in \mathfrak{D}$, it is said that R **entails** s **on** $D(\mathbf{A}) = \mathfrak{D}(\mathbf{A}, \underline{\mathbf{D}})$ if every continuous map $\alpha : D(\mathbf{A}) \to D$ which preserves the (partial) operations and relations in R also preserves s; we say that R **entails** s if R entails s on $D(\mathbf{A})$ for all $\mathbf{A} \in \mathfrak{D}$. We also recall that an algebra $\mathbf{B}$ is a **retract** of an algebra $\mathbf{A}$ if there are homomorphisms $u : \mathbf{B} \to \mathbf{A}$ and $v : \mathbf{A} \to \mathbf{B}$ such that $v \circ u = \mathrm{id}_B$, in which case v is called a **retraction** and u is called a **coretraction**. In the case that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and u is simply the inclusion map, we say that $\mathbf{B}$ is a **subretract** of $\mathbf{A}$.

LEMMA 2.2. [17, Lemmas 4.1, 4.2] *Let $\underline{\mathbf{D}}$ be a finite algebra and let $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$.*

(i) $\mathrm{End}(\underline{\mathbf{D}})$ *yields a duality* (*based on* $\underline{\mathbf{D}}$) *on the algebra* $\underline{\mathbf{D}}$.

(ii) *If* $\underset{\sim}{\mathbf{D}}$ *yields a duality* (*based on* $\underline{\mathbf{D}}$) *on an algebra* $\mathbf{A} \in \mathfrak{D}$, *then* $\underset{\sim}{\mathbf{D}}$ *yields a duality on every retract of the algebra* $\mathbf{A}$.

(iii) $\mathrm{End}(\underline{\mathbf{D}})$ *entails every finitary algebraic relation* s *on* $\underline{\mathbf{D}}$ *such that* $\mathbf{s}$ *is a retract of* $\underline{\mathbf{D}}$.

Part (iii) of the lemma above follows from parts (i) and (ii) via the Test Algebra Lemma which will be used often throughout this paper.

LEMMA 2.3. (**Test Algebra Lemma**) [16, Lemma 2.3, Proposition 2.5] *Let $R \cup \{s\}$ be a family of finitary algebraic* (*partial*) *operations and relations on a finite algebra $\underline{\mathbf{D}}$ and let $\mathfrak{D} = \mathbb{ISP}(\underline{\mathbf{D}})$.*

(i) *R entails s if and only if R entails s on $D(\mathbf{s}) = \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}})$.*

(ii) *If R yields a duality* (*based on* $\underline{\mathbf{D}}$) *on the algebra* $\mathbf{s}$, *then R entails s.*

(iii) *Assume that* R *yields a duality on* $\mathfrak{D}$ (*based on* $\underline{\mathbf{D}}$) *and let* $r \in R$. *Then the following are equivalent:*

(a) $R \setminus \{r\}$ *yields a duality on* $\mathfrak{D}$*;*
(b) $R \setminus \{r\}$ *yields a duality on the algebra* $\mathbf{r}$*;*
(c) $R \setminus \{r\}$ *entails* r *on* $D(\mathbf{r}) = \mathfrak{D}(\mathbf{r}, \underline{\mathbf{D}})$.

The Test Algebra Lemma says that a relation r can be removed from a dualising set R of relations without destroying the duality provided $R \setminus \{r\}$ yields a duality on the single algebra $\mathbf{r}$. This algebra has therefore been called a **test algebra**. As in [8], we slightly modify this concept and say that an algebra $\mathbf{s}$ is a **test algebra** (**for showing that** S **does not yield a duality on** $\mathfrak{D}$) if S does not yield a duality on the algebra $\mathbf{s}$.

The following Retraction Test Algebra Lemma from Haviar and Priestley [19] tells us that a known test algebra $\mathbf{s}$ can be replaced by any algebra $\mathbf{t}$ which retracts onto $\mathbf{s}$.

Lemma 2.4. **(Retraction Test Algebra Lemma)** [19, Lemma 4] (See also [6, Lemma 6.3] and [17, Lemma 4.2].) *Let* $\underline{\mathbf{D}}$ *be a finite algebra, let* $R \cup \{s, t\}$ *be a set of finitary algebraic relations on* $\underline{\mathbf{D}}$. *Assume that* $\mathbf{s}$ *is a test algebra which shows that* R *does not yield a duality on* $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$. *If* $\mathbf{t}$ *has* $\mathbf{s}$ *as a retract, then* $\mathbf{t}$ *is also a test algebra showing that* R *does not yield a duality on* $\mathfrak{D}$.

We will be able to apply our results to derive information concerning endodualisable and endoprimal algebras. We recall that $\underline{\mathbf{D}}$ is **endodualisable** if $\underset{\sim}{\mathbf{D}} = \langle D; \mathrm{End}(\underline{\mathbf{D}}), \mathcal{T}\rangle$ yields a duality on $\mathfrak{D}$, and $\underline{\mathbf{D}}$ is **almost endodualisable** (with **extra relation** s) if $\underset{\sim}{\mathbf{D}}' = \langle D; \mathrm{End}(\underline{\mathbf{D}}), s, \mathcal{T}\rangle$ yields a duality on $\mathfrak{D}$ which is optimal modulo endomorphisms. Finally, $\underline{\mathbf{D}}$ is k**-endoprimal** (resp. **endoprimal**) if $\underset{\sim}{\mathbf{D}} = \langle D; \mathrm{End}(\underline{\mathbf{D}}), \mathcal{T}\rangle$ yields a duality on the k-generated free algebra $\mathbf{F}\mathfrak{D}(k)$ (resp. on all $\mathbf{F}\mathfrak{D}(k)$ for $k \geqslant 1$).

The following corollary of the Retraction Test Algebra Lemma is obtained by considering the test algebra $\mathbf{t}$ to be the k-generated free algebra $\mathbf{F}\mathfrak{D}(k)$.

Corollary 2.5. [19, Corollary 5] *Let* $\underline{\mathbf{D}}$ *be a finite algebra and define* $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$. *Assume that* $\underline{\mathbf{D}}$ *is not endodualisable and that this is shown by a test algebra* $\mathbf{s}$. *If* $\mathbf{s}$ *is a retract of the* k*-generated free algebra* $\mathbf{F}\mathfrak{D}(k)$ (*or equivalently, if* $\mathbf{s}$ *is a* k*-generated projective algebra in* $\mathfrak{D}$), *then* $\underline{\mathbf{D}}$ *is not* k*-endoprimal.*

We define the **brute force** set of algebraic relations on $\underline{\mathbf{D}}$ to be

$$\mathcal{B}(\underline{\mathbf{D}}) := \{\, r \mid \ r \text{ is an } n\text{-ary algebraic relation on } \underline{\mathbf{D}} \text{ for some } n \in \mathbb{N}\,\}.$$

A subset U of $\mathcal{B}(\underline{\mathbf{D}})$ is called a **failset of** s if $s \in U$ and there exists a map $\gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ such that the set

$$\mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma') := \{\, r \in \mathcal{B}(\underline{\mathbf{D}}) \mid \gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D \text{ fails to preserve } r\,\}$$

is equal to U. By the Test Algebra Lemma, R does not entail s if and only if there exists a map $\gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ such that the failset $\mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma')$ contains s but contains no member of R.

We now recall the necessary concepts and results from [8]. Let R be a set of finitary algebraic relations on $\underline{\mathbf{D}}$ and let $s \in R$. We say that s is **needed in** R if $R \setminus \{s\}$ does not entail s. By the Test Algebra Lemma 2.3, the relation s is needed in R if and only if $R \setminus \{s\}$ does not entail s on $D(\mathbf{s})$, that is, there exists a map $\gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ such that

(N1) γ' does not preserve s, that is, $s \in \mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma')$,

(N2) γ' preserves each relation $r \in R\setminus\{s\}$, that is, $\mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma') \cap (R\setminus\{s\}) = \varnothing$.

Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathfrak{D}$. We shall say that the relation s **avoids** $\underline{\mathbf{M}}$ **relative to** R if there exists a map $\gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ which satisfies (N1), (N2) and

(A) γ' preserves every finitary algebraic relation $r \in \mathcal{B}(\underline{\mathbf{D}})$ such that $\mathbf{r} \cong \underline{\mathbf{M}}$, that is, $\mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma') \cap \{\, r \in \mathcal{B}(\underline{\mathbf{D}}) \mid \mathbf{r} \cong \underline{\mathbf{M}}\,\} = \varnothing$.

If $S \subseteq R$ and each $s \in S$ avoids $\underline{\mathbf{M}}$ relative to R then we say that S **avoids** $\underline{\mathbf{M}}$ **relative to** R. In the case that $S = R$, we say simply that R **avoids** $\underline{\mathbf{M}}$. We often prove that s avoids $\underline{\mathbf{M}}$ relative to R by establishing a stronger minimality condition on s. We say that s is **as small as possible relative to** R if there exists a map $\gamma' : \mathfrak{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ which satisfies (N1), (N2) and

(S) γ' preserves every relation $s_1 \in \mathcal{B}(\underline{\mathbf{D}})$ such that $\mathbf{s}$ is not a retract of $\mathbf{s}_1$, that is, $\mathrm{Fail}^{\underline{\mathbf{D}}}_{\mathbf{s}}(\gamma') \subseteq \{\, s_1 \in \mathcal{B}(\underline{\mathbf{D}}) \mid \mathbf{s} \text{ is a retract of } \mathbf{s}_1\,\}$.

The following three results are amongst the theoretical tools developed in [8].

Lemma 2.6. [8, Lemma 2.3] *Let* $\underline{\mathbf{D}}$ *be a finite algebra, let* $\mathfrak{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ *and assume that* $\underline{\mathbf{M}} \in \mathfrak{D}$ *is finite and has* $\underline{\mathbf{D}}$ *as a subalgebra. Let* S *be a set of finitary algebraic relations on* $\underline{\mathbf{D}}$ *and let* $s \in S$*. The following conditions are related by* (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*:*

(1) *s (as a relation on $\underline{\mathbf{D}}$) avoids $\underline{\mathbf{M}}$ relative to* $\mathrm{End}(\underline{\mathbf{D}}) \cup S$*;*

(2) *s (as a relation on $\underline{\mathbf{M}}$) is needed in* $\mathrm{End}(\underline{\mathbf{M}}) \cup S \cup \{D\}$*;*

(3) *$\mathbf{s}$ is not a retract of $\underline{\mathbf{M}}$.*

THEOREM 2.7. **(Optimal Duality Transfer Theorem)** [8, Theorem 2.4] *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{M}} \in \mathcal{D}$ is finite and has $\underline{\mathbf{D}}$ as a subalgebra. Let S be a set of finitary algebraic relations on $\underline{\mathbf{D}}$ and assume that the structure $\underset{\sim}{\mathbf{D}} = \langle D; \mathrm{End}(\underline{\mathbf{D}}), S, \mathcal{T}\rangle$ yields a duality on $\mathcal{D}$ based on $\underline{\mathbf{D}}$ which is optimal modulo endomorphisms. Consider the set $S^\circ := \{\, s \in S \mid \mathbf{s}$ is not a retract of $\underline{\mathbf{M}}\,\}$. If S° avoids $\underline{\mathbf{M}}$ relative to* $\mathrm{End}(\underline{\mathbf{D}}) \cup S^\circ$*, then, with the possible exception that D may not be needed,*

$$\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), S^\circ \cup \{D\}, \mathcal{T}\rangle$$

yields a duality on $\mathcal{D}$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Moreover, if $\underline{\mathbf{D}}$ is a subretract of $\underline{\mathbf{M}}$, then D is not needed and consequently

$$\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), S^\circ, \mathcal{T}\rangle$$

yields a duality on $\mathcal{D}$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.

LEMMA 2.8. [8, Lemma 2.5] *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{M}}$ is a finite algebra in $\mathcal{D}$. Let S be a set of finitary algebraic relations on $\underline{\mathbf{D}}$, let $s \in S$ and consider the following conditions:*

(1) *$\{\, s_1 \in \mathcal{B}(\underline{\mathbf{D}}) \mid \mathbf{s}$ is a retract of $\mathbf{s}_1 \,\}$ contains a $\underline{\mathbf{D}}$-failset of s;*

(2) *s is as small as possible relative to S;*

(3) *s is as small as possible relative to* $\mathrm{End}(\underline{\mathbf{D}}) \cup S$*;*

(4) *s avoids $\underline{\mathbf{M}}$ relative to* $\mathrm{End}(\underline{\mathbf{D}}) \cup S$.

If $\mathbf{s}$ is not a retract of $\underline{\mathbf{M}}$, then (3) $\Rightarrow$ (4)*. If $\mathbf{s}$ is not a retract of $\underline{\mathbf{D}}$, then* (2) $\Rightarrow$ (3)*. If $\mathbf{s}$ is not a retract of $\mathbf{r}$ for all $r \in S \setminus \{s\}$, then* (1) $\Rightarrow$ (2).

We can sometimes establish a condition which lies between "s avoids $\underline{\mathbf{M}}$ relative to R" and "s is as small as possible relative to R".

LEMMA 2.9. *Let S be a set of finitary algebraic relations on $\underline{\mathbf{D}}$ and let $s \in S$. Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathcal{D} := \mathbb{ISP}(\underline{\mathbf{D}})$. Assume there is a finitary algebraic relation t on $\underline{\mathbf{D}}$ such that $\mathbf{t}$ is a retract of $\mathbf{s}$ and $\mathbf{t}$ is not a retract of $\mathbf{r}$, for all $r \in S \setminus \{s\}$. Moreover, assume that $\{\, r \in \mathcal{B}(\underline{\mathbf{D}}) \mid \mathbf{t}$ is a retract of $\mathbf{r} \,\}$ contains a failset of s.*

(i) *If* $\underline{\mathbf{M}}$ *does not have* $\mathbf{t}$ *as a retract, then* s *avoids* $\underline{\mathbf{M}}$ *relative to* S.

(ii) *If neither* $\underline{\mathbf{D}}$ *nor* $\underline{\mathbf{M}}$ *has* $\mathbf{t}$ *as a retract, then* s *avoids* $\underline{\mathbf{M}}$ *relative to* $\mathrm{End}(\underline{\mathbf{D}}) \cup S$.

Proof. Let U be a failset of s such that $\mathbf{t}$ is a retract of $\mathbf{r}$ for all $r \in U$. Since $\mathbf{t}$ is not a retract of $\mathbf{r}$ for all $r \in S \setminus \{s\}$, we have $U \cap S = \{s\}$, that is, (N1) and (N2) hold with respect to $R := S$. Assume that $\mathbf{t}$ is not a retract of $\underline{\mathbf{M}}$ and let $r \in \mathcal{B}(\underline{\mathbf{D}})$ with $\mathbf{r} \cong \underline{\mathbf{M}}$. Thus, $\mathbf{t}$ is not a retract of $\mathbf{r}$ and consequently $r \notin U$, that is, (A) holds. Hence, s avoids $\underline{\mathbf{M}}$ relative to S. If, moreover, $\mathbf{t}$ is not a retract of $\underline{\mathbf{D}}$, then U contains no relation r with $\mathbf{r} \cong \underline{\mathbf{D}}$. In particular, U contains no (graph of an) endomorphism of $\underline{\mathbf{D}}$, that is, (N1) and (N2) hold with respect to $R := \mathrm{End}(\underline{\mathbf{D}}) \cup S$. Consequently, s avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{D}}) \cup S$. ■

3. Transferral from the four-element non-Boolean Kleene algebra

Consider the quasi-variety $\boldsymbol{\mathcal{K}}_4 = \mathbb{ISP}(\underline{\mathbf{4}})$ of Kleene algebras generated by the four-element algebra

$$\underline{\mathbf{4}} = \langle \{0, a, b, 1\}; \vee, \wedge, \neg, 0, 1 \rangle$$

such that $0 < a < b < 1$ and $\neg 0 = 1$, $\neg 1 = 0$, $\neg a = b$, $\neg b = a$. In Davey, Haviar and Priestley [11] it is shown that

$$\begin{aligned} \underset{\sim}{\mathbf{4}}_1 &= \langle \{0, a, b, 1\}; e, f, c, m, \mathcal{T} \rangle, \\ \underset{\sim}{\mathbf{4}}_2 &= \langle \{0, a, b, 1\}; e, k, m, \mathcal{T} \rangle, \text{ and} \\ \underset{\sim}{\mathbf{4}}_3 &= \langle \{0, a, b, 1\}; e, \ell, m, \mathcal{T} \rangle \end{aligned}$$

yield dualities on $\boldsymbol{\mathcal{K}}_4$ which are optimal modulo endomorphisms, where e is the unique non-identity endomorphism of $\underline{\mathbf{4}}$ given by

$$e(0) = 0, \ e(a) = 0, \ e(b) = 1, \ e(1) = 1$$

and the binary algebraic relations c, f, k, ℓ and m are

$$\begin{aligned} c &= \{(0,0), (a,0), (a,a), (b,b), (b,1), (1,1)\}, \\ f &= \{(0,0), (a,a), (a,b), (b,a), (b,b), (1,1)\}, \\ k &= \{(0,0), (a,0), (a,a), (a,b), (b,a), (b,b), (b,1), (1,1)\}, \\ \ell &= \{(0,0), (a,0), (b,0), (a,a), (a,b), (b,a), (b,b), (a,1), (b,1), (1,1)\}, \\ m &= 4^2 \setminus \{(0,1), (1,0)\}. \end{aligned}$$

Moreover, these are the only dualities for $\mathcal{K}_4$, given by binary algebraic relations on $\underline{\mathbf{4}}$, which are optimal modulo endomorphisms. We note that the algebras $\mathbf{f}$ and $\mathbf{c}$ are isomorphic to the 1-generated free Kleene algebra $\mathbf{F}\mathcal{K}(1)$ and the six-element chain algebra $\mathbf{6}$, respectively, while $\mathbf{k}$ and $\boldsymbol{\ell}$ are isomorphic to $\mathbf{2} \oplus (\mathbf{2} \times \mathbf{2}) \oplus \mathbf{2}$ and $\mathbf{1} \oplus (\mathbf{2} \times \mathbf{4}) \oplus \mathbf{1}$, respectively. (In [11] the relation c was denoted by g. We have changed the notation here to avoid any confusion with the order-reversing map g which occurs in the restricted Priestley duality for Kleene algebras—see below.) We also note the following result from [11].

Theorem 3.1. [11, Lemma 5.4] *A finite non-Boolean algebra $\underline{\mathbf{M}} \in \mathcal{K}_4$ is* 1-*endoprimal if and only if $\underline{\mathbf{M}}$ has the* 1-*generated free algebra* $\mathbf{F}\mathcal{K}(1)$ *as a retract.*

As mentioned at the start of Section 2, Priestley duality between the categories $\mathcal{D}$ of bounded distributive lattices and the category $\mathcal{P}$ of Priestley spaces is given by the functors $H : \mathcal{D} \to \mathcal{P}$ and $K : \mathcal{P} \to \mathcal{D}$. If $\mathcal{A}$ is a (not necessarily full) subcategory of $\mathcal{D}$ then the corresponding (not necessarily full) subcategory $\mathcal{Y} := \mathbb{I}(H(\mathcal{A}))$ of $\mathcal{P}$ is called the **restricted Priestley dual category for $\mathcal{A}$**.

Assume that $\underline{\mathbf{M}} = \langle M; \{\vee, \wedge, 0, 1\} \cup F\rangle$, where $\langle M; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, let $S \subseteq \mathcal{B}(\underline{\mathbf{M}})$ be a set of algebraic relations on $\underline{\mathbf{M}}$ and let $G \subseteq \mathrm{End}(\underline{\mathbf{M}})$ be a set of endomorphisms of $\underline{\mathbf{M}}$. Let $\mathcal{Y}$ be the restricted Priestley dual category for $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$. Let the restricted Priestley dual of $\underline{\mathbf{M}}$ be $\mathbf{Y}_M$ and, for $\mathbf{A} \in \mathcal{A}$, let $\mathbf{Y}_A$ be its restricted Priestley dual. We shall show how to interpret S and G on $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$. For each $s \in S$, let $\mathbf{Y}_s$ be the restricted Priestley dual of $\mathbf{s}$ and let $\sigma_1, \dots, \sigma_n : \mathbf{Y}_M \to \mathbf{Y}_s$ be the jointly surjective maps corresponding to the projections $\rho_1, \dots, \rho_n : \mathbf{s} \to \underline{\mathbf{M}}$.

(i) Each relation $s \in S$ can be interpreted on $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$ by declaring that, for all maps $\Psi_1, \dots, \Psi_n \in \mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$, we have $(\Psi_1, \dots, \Psi_n) \in s$ if and only if $(\Psi_1, \dots, \Psi_n)$ factors in $\mathcal{Y}$ through $(\sigma_1, \dots, \sigma_n)$, that is, there exists a (necessarily unique) morphism $\mu : \mathbf{Y}_s \to \mathbf{Y}_A$ such that $\Psi_j = \mu \circ \sigma_j$ for $j = 1, \dots, n$. Note that the converse also holds: every morphism $\mu \in \mathcal{Y}(\mathbf{Y}_s, \mathbf{Y}_A)$ gives rise to maps $\Psi_1, \dots, \Psi_n$ in $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$ with $(\Psi_1, \dots, \Psi_n) \in s$, namely $\Psi_1 := \mu \circ \sigma_1, \dots, \Psi_n := \mu \circ \sigma_n$. See Figure 1.

(ii) Each endomorphism $e \in G$ may be interpreted on $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$ via composition, that is, $e(\Psi) := \Psi \circ H(e)$, for all $\Psi \in \mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$, where the map $H(e) \in \mathrm{End}(\mathbf{Y}_M)$ is the restricted Priestley dual of the endomorphism e.

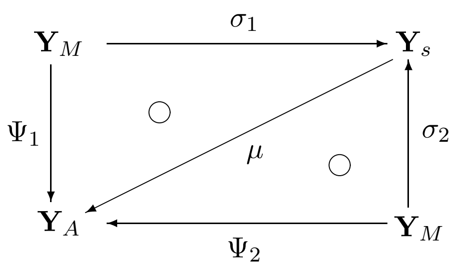

Figure 1. (Ψ_1, Ψ_2) factors in $\mathcal{Y}$ through (σ_1, σ_2)

We shall use the following result repeatedly throughout this section and the next. It is easy to prove and is part of the folklore of restricted Priestley dualities.

LEMMA 3.2. *With the relations $s \in S$ and endomorphisms $e \in G$ interpreted on $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$ as indicated above, the natural bijection between $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ and $\mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A)$ is an isomorphism between the structures $\langle \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}); G, S\rangle$ and $\langle \mathcal{Y}(\mathbf{Y}_M, \mathbf{Y}_A); G, S\rangle$.*

As shown in Cornish and Fowler [3] (see also [11]), the restricted Priestley dual of the quasi-variety $\mathcal{K}_4$ is the category $\mathcal{Y}$ of Priestley spaces $\mathbf{Y}$ with an order-reversing homeomorphism g of order 2 such that every point $y \in Y$ is comparable to a g-fixpoint. Morphisms in the category $\mathcal{Y}$ are order- and g-preserving continuous maps.

The duals in $\mathcal{Y}$ of the algebras $\underline{\mathbf{4}}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{k}$, $\boldsymbol{\ell}$ and $\mathbf{m}$ are given in Figure 2. In each case, 1 and $\widetilde{1}$, represented by slightly larger circles, are g-fixpoints while

$$g(0) = 2,\ g(\widetilde{0}) = \widetilde{2},\ g(2) = 0,\ g(\widetilde{2}) = \widetilde{0}.$$

For each $s \in \{c, f, k, \ell, m\}$, the jointly surjective maps $\sigma_1^s, \sigma_2^s : \mathbf{Y}_4 \to \mathbf{Y}_s$ corresponding to the projections $\rho_1, \rho_2 : \mathbf{s} \to \underline{\mathbf{4}}$ are given by the labelling of Y_s, namely, $\sigma_1^s(i) = i$ and $\sigma_2^s(i) = \widetilde{i}$ for $i = 0, 1, 2$. Note that $(\sigma_1^s, \sigma_2^s) \in s$ in $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_s)$. When the context is clear, we shall often abbreviate σ_i^s to σ_i.

For each relation s, we replace $D(\mathbf{s}) := \mathcal{K}(\mathbf{s}, \underline{\mathbf{4}})$ by $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_s)$ and, since there is a natural isomorphism in $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{4}})$ between $\underset{\sim}{\mathbf{4}}$ and $D(\mathbf{F}\mathcal{K}(1)) := \mathcal{K}(\mathbf{F}\mathcal{K}(1), \underline{\mathbf{4}})$, we replace $\underset{\sim}{\mathbf{4}}$ itself by $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$. To apply the Optimal Duality Transfer Theorem, we would like to know which of the relations c, f, k, ℓ and m are as small as possible relative to which subsets of $\{c, f, k, \ell, m\}$. In order to apply Lemma 2.8, for each $s \in \{c, f, k, \ell, m\}$, we look for a map $\Phi_s : \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_s) \to \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$ which does not preserve s but which preserves all n-ary algebraic relations $s_1 \subseteq 4^n$ $(n \geqslant 1)$ such that

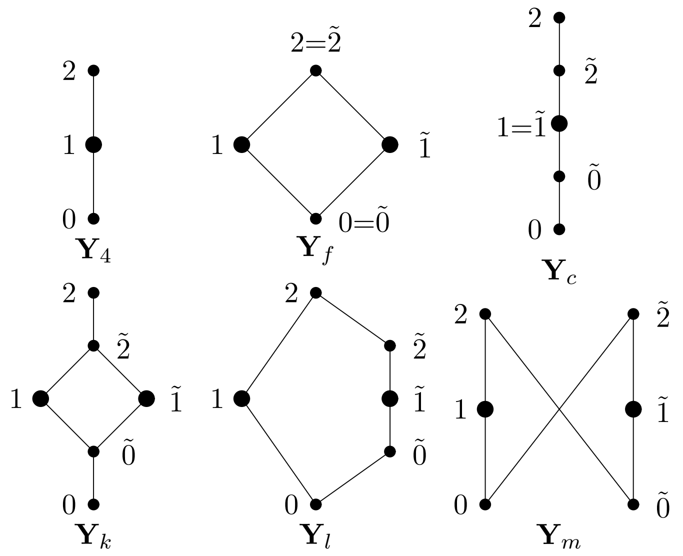

Figure 2. Priestley duals of the algebras $\underline{\mathbf{4}}$, $\mathbf{f}$, $\mathbf{c}$, $\mathbf{k}$, $\boldsymbol{\ell}$ and $\mathbf{m}$

$\mathbf{s}_1$ does not have $\mathbf{s}$ as a retract. We shall define the map Φ_s by declaring that it maps each $\Psi \in \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_s)$ to $\varphi_s \circ \Psi \in \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$ where $\varphi_s : Y_s \to Y_f$ will be a map defined separately in each case. This useful device was first utilised in Davey and Priestley [15]—see the discussion before Lemma 4.1 below.

Lemma 3.3. *Let $s \in \{c, f, k, \ell, m\}$ and let $\varphi_s : Y_s \to Y_f$ be a map such that*

$$\Phi_s := \varphi_s \circ - : \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_s) \to \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$$

is well defined. If φ_s is not order-preserving, then $(\varphi_s \circ \sigma_1, \varphi_s \circ \sigma_2) \notin s$ on $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$, and hence Φ_s does not preserve s.

Proof. Let $y < z$ in Y_s with $\varphi_s(y) \nleqslant \varphi_s(z)$ in Y_f. Suppose, by way of contradiction, that $(\varphi_s \circ \sigma_1, \varphi_s \circ \sigma_2) \in s$ on $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$. Then, by the definition of s on $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$, there exists $\mu \in \mathcal{Y}(\mathbf{Y}_s, \mathbf{Y}_f)$ such that

$$\varphi_s \circ \sigma_j = \mu \circ \sigma_j \text{ for } j = 1, 2.$$

The maps $\sigma_1, \sigma_2 : Y_4 \to Y_s$ are jointly surjective, thus there are $y_0, z_0 \in Y_4$ such that $\sigma_k(y_0) = y$ and $\sigma_l(z_0) = z$, for some $k, l \in \{1, 2\}$. Hence, from $\sigma_k(y_0) = y < z = \sigma_l(z_0)$ and the fact that μ is order-preserving, it follows that

$$(\mu \circ \sigma_k)(y_0) = \mu(y) \leqslant \mu(z) = (\mu \circ \sigma_l)(z_0),$$

whence

$$\varphi_s(y) = (\varphi_s \circ \sigma_k)(y_0) \leqslant (\varphi_s \circ \sigma_l)(z_0) = \varphi_s(z),$$

a contradiction which completes the proof. ■

Lemma 3.4.

(i) *The set* $\{ c_1 \in \mathcal{B}(\underline{\mathbf{4}}) \mid \mathbf{c}$ *is a retract of* $\mathbf{c}_1 \}$ *has a subset* U *which is a failset of* c*,* k *and* ℓ*.*

(ii) *The relation* c *is as small as possible relative to* $\{c, f, m\}$*.*

(iii) *Let* $\underline{\mathbf{M}} \in \mathcal{K}_4$ *and assume that* $\mathbf{c}$ *is not a retract of* $\underline{\mathbf{M}}$*.*

   (a) *The relation* c *avoids* $\underline{\mathbf{M}}$ *relative to* $\mathrm{End}(\underline{\mathbf{4}}) \cup \{c, f, m\}$*.*

   (b) *The relation* k *avoids* $\underline{\mathbf{M}}$ *relative to* $\mathrm{End}(\underline{\mathbf{4}}) \cup \{k, m\}$*.*

   (c) *The relation* ℓ *avoids* $\underline{\mathbf{M}}$ *relative to* $\mathrm{End}(\underline{\mathbf{4}}) \cup \{\ell, m\}$*.*

Proof. We commence by showing that $\{ c_1 \in \mathcal{B}(\underline{\mathbf{4}}) \mid \mathbf{c}$ is a retract of $\mathbf{c}_1 \}$ contains a failset of c. Let $\varphi_c : Y_c \to Y_f$ be the map shown in Figure 3.

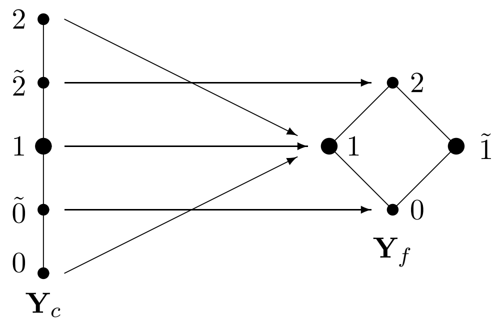

Figure 3. The map $\varphi_c : Y_c \to Y_f$

Note that φ_c is g-preserving but not order-preserving and that $0 < \widetilde{0}$ and $\widetilde{2} < 2$ are the only order relations in Y_c which φ_c fails to preserve. It is very easy to check that Φ_c is well defined.

From Lemma 3.3 it follows that Φ_c does not preserve c. We now show that Φ_c preserves all n-ary algebraic relations $r \subseteq 4^n$ $(n \geqslant 1)$ such that $\mathbf{r}$ does not have $\mathbf{c}$ as a retract. Let $\mathbf{Y}_r$ be the dual of $\mathbf{r}$, let $\sigma_1, \dots, \sigma_n : \mathbf{Y}_4 \to \mathbf{Y}_r$ be the maps corresponding to the projections $\rho_1, \dots, \rho_n : \mathbf{r} \to \underline{\mathbf{4}}$ and let $\Psi_1, \dots, \Psi_n \in \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_c)$ with $(\Psi_1, \dots, \Psi_n) \in r$. Then there exists a morphism $\mu : \mathbf{Y}_r \to \mathbf{Y}_c$ such that $\Psi_j = \mu \circ \sigma_j$ for $j = 1, \dots, n$. To see that $(\varphi_c \circ \Psi_1, \dots, \varphi_c \circ \Psi_n) \in r$, that is, that $(\varphi_c \circ \mu \circ \sigma_1, \dots, \varphi_c \circ \mu \circ \sigma_n) \in r$, it suffices to show that the map $\varphi_c \circ \mu : Y_r \to Y_f$ is order-preserving. Suppose that $\varphi_c \circ \mu$ is not order-preserving. Thus, without loss of generality, there exists $x < y$ in Y_r with $\mu(x) = \widetilde{2}$ and $\mu(y) = 2$. Let z be a g-fixpoint less than x. Then $Y_c' := \{g(y), g(x), z, x, y\}$ is a g-closed chain in $\mathbf{Y}_r$. The isomorphism η of $\mathbf{Y}_c$ onto $\mathbf{Y}_c' \leqslant \mathbf{Y}_r$ satisfies $\mu \circ \eta = \mathrm{id}_{\mathbf{Y}_c}$, contradicting the fact that $\mathbf{c}$ is not a retract of $\mathbf{r}$. Hence, $\varphi_c \circ \mu$ is order-preserving. Thus, modulo the obvious bijections between $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_c)$ and $\mathcal{K}(\mathbf{c}, \underline{\mathbf{4}})$ and between $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$ and $\underset{\sim}{\mathbf{4}}$, the set $U := \mathrm{Fail}^{\underline{\mathbf{4}}}_{\mathbf{c}}(\Phi_c)$ is the required failset of c and to prove (i) it only remains to show that U is also a failset of both k and ℓ. Note from Figure 2 that $\mathbf{Y}_c$ is a retract of both $\mathbf{Y}_k$ and $\mathbf{Y}_\ell$ and consequently $\mathbf{c}$ is a retract of both $\mathbf{k}$ and $\boldsymbol{\ell}$.

First, we prove that $k, \ell \in U$. Let $s \in \{k, \ell\}$ and let $\mu : \mathbf{Y}_s \to \mathbf{Y}_c$ be the unique retraction. Thus $\Psi_i := \mu \circ \sigma_i^s$ satisfy $(\Psi_1, \Psi_2) \in s$ on $\mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_c)$. But

$$(\Phi_c(\Psi_1), \Phi_c(\Psi_2)) = (\varphi_c \circ \mu \circ \sigma_1^s, \varphi_c \circ \mu \circ \sigma_2^s) = (\underline{1}, \sigma_1^f),$$

where $\underline{1} : \mathbf{Y}_4 \to \mathbf{Y}_f$ is the constant map onto $\{1\}$. It is clear that there is no morphism $\mu' \in \mathcal{Y}(\mathbf{Y}_s, \mathbf{Y}_f)$ such that $\mu' \circ \sigma_1^s = \underline{1}$ and $\mu' \circ \sigma_2^s = \sigma_1^f$. Hence, $(\underline{1}, \sigma_1^f) \notin s$ and consequently $s \in U$. Now let $u : \mathbf{c} \to \mathbf{k}$ be a coretraction. It is an easy exercise (see Lemma 2.4 in [16] or Exercise 8.9 in [2]) to prove that $U = \mathrm{Fail}^{\underline{\mathbf{4}}}_{\mathbf{k}}(\Phi_k)$, where $\Phi_k := \Phi_c \circ D(u)$, whence U is a failset of k. In the same way, it follows that U is a failset of ℓ. Since $\underline{\mathbf{4}}$, $\mathbf{f}$ and $\mathbf{m}$ do not have $\mathbf{c}$ as a retract while $\mathbf{k}$ and $\boldsymbol{\ell}$ do, parts (ii) and (iii) follow by Lemmas 2.8 and 2.9. ∎

Lemma 3.5.

(i) $\{\, m_1 \in \mathcal{B}(\underline{\mathbf{4}}) \mid \mathbf{m}$ *is a retract of* $\mathbf{m}_1 \,\}$ *contains a failset of* m.

(ii) *The relation* m *is as small as possible relative to* $\{c, f, k, \ell, m\}$.

(iii) *Let* $\underline{\mathbf{M}} \in \mathcal{K}_4$ *and assume that* $\mathbf{m}$ *is not a retract of* $\underline{\mathbf{M}}$. *Then* m *avoids* $\underline{\mathbf{M}}$ *relative to* $\mathrm{End}(\underline{\mathbf{4}}) \cup \{c, f, k, \ell, m\}$.

Proof. Since $\underline{\mathbf{4}}$, $\mathbf{f}$, $\mathbf{c}$, $\mathbf{k}$ and $\boldsymbol{\ell}$ do not have $\mathbf{m}$ as a retract, by Lemma 2.8 it suffices to prove (i). We define the map $\varphi_m : Y_m \to Y_f$ as indicated in Figure 4.

It is easy to see that φ_m is g-preserving but not order-preserving and that $0 < \widetilde{2}$ and $\widetilde{0} < 2$ are the only order relations in $\mathbf{Y}_m$ which φ_m fails to preserve. Again, it is easy to check that Φ_m is well defined.

The fact that the map Φ_m does not preserve m follows again from Lemma 3.3. We shall show that Φ_m preserves all n-ary algebraic relations

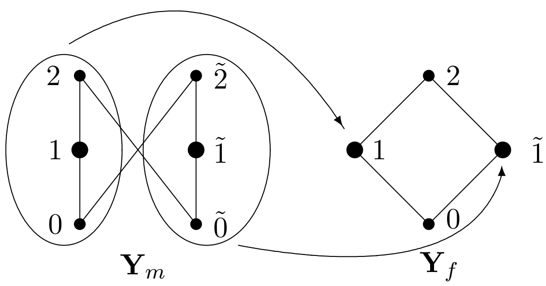

Figure 4. The map $\varphi_m : Y_m \to Y_f$

$r \subseteq 4^n$ ($n \geqslant 1$) such that $\mathbf{r}$ does not have $\mathbf{m}$ as a retract. As in the previous proof, let $\mathbf{Y}_r$ be the dual of $\mathbf{r}$, let $\sigma_1, \dots, \sigma_n : \mathbf{Y}_4 \to \mathbf{Y}_r$ be the jointly surjective maps corresponding to the projections and let $\Psi_1, \dots, \Psi_n \in \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_m)$ with $(\Psi_1, \dots, \Psi_n) \in r$. Then there exists a morphism $\mu : \mathbf{Y}_r \to \mathbf{Y}_m$ such that $\Psi_j = \mu \circ \sigma_j$ for $j = 1, \dots, n$. Once again, to prove that $(\varphi_m \circ \Psi_1, \dots, \varphi_m \circ \Psi_n) \in r$, that is, that $(\varphi_m \circ \mu \circ \sigma_1, \dots, \varphi_m \circ \mu \circ \sigma_n) \in r$, it suffices to show that the map $\varphi_m \circ \mu : Y_r \to Y_f$ is order-preserving. Suppose that $\varphi_m \circ \mu$ is not order-preserving. Then, without loss of generality, there exist $x < y$ in Y_r such that $\mu(x) = 0$ and $\mu(y) = \widetilde{2}$. Since μ is a morphism and since $\mathbf{Y}_r \in \mathcal{Y}$, there exist distinct fixpoints z and z' in $\mathbf{Y}_r$ such that $x < z < g(x)$ and $g(y) < z' < y$. Hence,

$$Y'_m := \{x, z, g(x), g(y), z', y\}$$

is a g-closed subset of Y_r with $\mathbf{Y}'_m$ isomorphic to $\mathbf{Y}_m$. Thus, $\mu : \mathbf{Y}_r \to \mathbf{Y}_m$ is a coretraction and consequently $\mathbf{m}$ is a retract of $\mathbf{r}$, a contradiction. It follows that $\varphi_m \circ \mu$ is order-preserving, as required. Thus, $\mathrm{Fail}^{\mathbf{4}}_{\mathbf{m}}(\Phi_m)$ is the required failset of m. ■

Remark 3.6. It is impossible to use a similar argument to prove that the relation f is as small as possible relative to $\{c, f, m\}$. It is easy to see that, if $\varphi_f : Y_f \to Y_f$ is a map such that $\Phi_f := \varphi_f \circ - : \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f) \to \mathcal{Y}(\mathbf{Y}_4, \mathbf{Y}_f)$ is well defined, then φ_f must be order-preserving.

Next we consider which of the algebras $\mathbf{c}$, $\mathbf{k}$, $\boldsymbol{\ell}$ and $\mathbf{m}$ are retracts of the free Kleene algebra $\mathbf{F}\mathcal{K}(n)$ ($n \geqslant 1$), which is equivalent to asking which are n-generated and projective in $\mathcal{K}$. We could utilise the description of finite projective Kleene algebras given in Jalali [20], but, as that paper is difficult to obtain and, for these algebras, the proofs are easy, we shall argue directly instead.

Lemma 3.7.

(i) *The algebras* $\mathbf{c}$, $\mathbf{k}$ *and* $\boldsymbol{\ell}$ *are not retracts of the free algebra* $\mathbf{F}\mathcal{K}(1)$ *but are retracts of the free algebra* $\mathbf{F}\mathcal{K}(2)$.

(ii) *The algebra* $\mathbf{m}$ *is not a retract of any free Kleene algebra.*

Proof. We shall use the description of the restricted Priestley dual of $\mathbf{F}\mathcal{K}(2)$ given in Davey and Priestley [14] (see also [2, Section 7.5]). For any set S, extend the order and the map g on $\mathbf{Y}_f$ pointwise to the power $\mathbf{Y}_f^S$. Then the set $Y_{(S)} := \{\, y \in Y_f^S \mid y \leqslant g(y) \text{ or } y \geqslant g(y)\,\}$ determines a closed substructure of $\mathbf{Y}_f^S$. With the induced order and map g, we obtain a Kleene space $\mathbf{Y}_{(S)}$ isomorphic to the restricted Priestley dual, $H(\mathbf{F}\mathcal{K}(S))$, of the S-generated free Kleene algebra $\mathbf{F}\mathcal{K}(S)$ (see [2, 7.5.7]).

It is clear, on grounds of size, that $\mathbf{c}$, $\mathbf{k}$ and $\boldsymbol{\ell}$ are not a retracts of $\mathbf{F}\mathcal{K}(1)$. The restricted Priestley dual of $\mathbf{F}\mathcal{K}(2)$ is $\mathbf{Y}_{(2)}$, where $Y_{(2)}$ is the set of all pairs $y \in Y_f^2$ such that y is comparable with $g(y)$. It is easy to see that only the pairs $(0, 2)$ and $(2, 0) = g((0, 2))$ in Y_f^2 do not satisfy this condition. The Kleene space $\mathbf{Y}_{(2)}$ is depicted in Figure 5: the four elements of height 2 are the g-fixpoints and otherwise $g(y)$ is the point either vertically above or below y. The large circles denote a copy of $\mathbf{Y}_\ell$ and ovals denote the equivalence classes of $\ker(\varphi)$, where $\varphi : \mathbf{Y}_{(2)} \to \mathbf{Y}_\ell$ is a retraction of the Kleene space $\mathbf{Y}_{(2)}$ onto the Kleene space $\mathbf{Y}_\ell$. Hence the algebra $\boldsymbol{\ell}$ is a retract of $\mathbf{F}\mathcal{K}(2)$. We leave it as an easy exercise for the reader to find the corresponding diagrams which show that $\mathbf{c}$ and $\mathbf{k}$ are retracts of $\mathbf{F}\mathcal{K}(2)$. This proves (i).

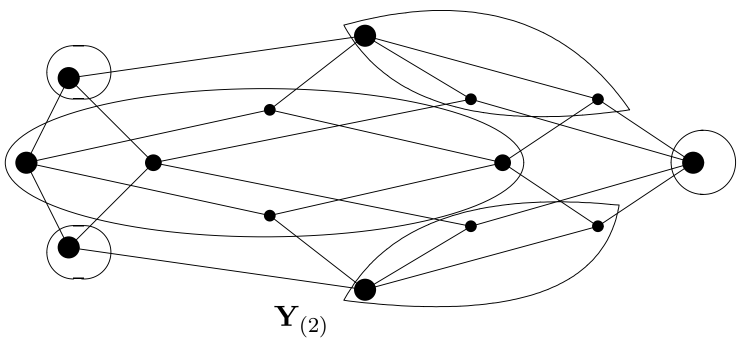

Figure 5. Priestley dual of the free Kleene algebra $\mathbf{F}\mathcal{K}(2)$

To prove that $\mathbf{m}$ is not a retract of any free Kleene algebra it suffices to show that $\mathbf{m}$ is not projective in $\mathcal{K}$. Thus it suffices to show that there is a finite Kleene space $\mathbf{Y}$ which contains $\mathbf{Y}_m$ as a substructure but such that there is no retraction of $\mathbf{Y}$ onto $\mathbf{Y}_m$. Clearly such a Kleene space $\mathbf{Y}$ may be obtained by adjoining a new bottom, $\perp$, and a new top, $\top$, to $\mathbf{Y}_m$ and defining $g(\perp) = \top$ and $g(\top) = \perp$. Hence, (ii) holds. ■

We can now bring our analysis together and derive an array of optimal dualities within the quasi-variety $\mathcal{K}_4$. These results refine and extend some of the results of [11].

THEOREM 3.8. *Let $\underline{\mathbf{M}}$ be a finite non-Boolean Kleene algebra in the quasi-variety $\mathcal{K}_4 := \mathbb{ISP}(\underline{\mathbf{4}})$ which has $\underline{\mathbf{4}}$ as a subretract.*

(1) *If $\underline{\mathbf{M}}$ has either $\mathbf{k}$ and $\mathbf{m}$ or $\boldsymbol{\ell}$ and $\mathbf{m}$ as retracts, then $\underline{\mathbf{M}}$ is endodualisable.*

(2) *If $\underline{\mathbf{M}}$ does not have $\mathbf{m}$ as a retract but has either $\mathbf{k}$ or $\boldsymbol{\ell}$ as a retract, then $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.*

(3) *Let $s \in \{k, \ell\}$. Assume that $\underline{\mathbf{M}}$ does not have $\mathbf{s}$ as a retract but has $\mathbf{m}$ as a retract.*

   (3.1) *If $\underline{\mathbf{M}}$ does not have $\mathbf{c}$ as a retract, then $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Moreover, $\underline{\mathbf{M}}$ is not 2-endoprimal.*

   (3.2) *Assume that $\underline{\mathbf{M}}$ has $\mathbf{c}$ as a retract.*

      (3.2.1) *If $\underline{\mathbf{M}}$ has $\mathbf{f}$ as a retract, then $\underline{\mathbf{M}}$ is endodualisable.*

      (3.2.2) *If $\mathbf{f}$ is not a retract of $\underline{\mathbf{M}}$, then $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Moreover, $\underline{\mathbf{M}}$ is not 1-endoprimal.*

(4) *Let $s \in \{k, \ell\}$. Assume that $\underline{\mathbf{M}}$ has neither $\mathbf{s}$ nor $\mathbf{m}$ as retracts.*

   (4.1) *If $\underline{\mathbf{M}}$ does not have $\mathbf{c}$ as a retract, then the structure $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Moreover, the algebra $\underline{\mathbf{M}}$ is not 2-endoprimal.*

   (4.2) *Assume that $\underline{\mathbf{M}}$ has $\mathbf{c}$ as a retract.*

      (4.2.1) *If $\underline{\mathbf{M}}$ has $\mathbf{f}$ as a retract, then $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.*

      (4.2.2) *If $\underline{\mathbf{M}}$ does not have $\mathbf{f}$ as a retract, then the structure $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), f, m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Moreover, $\underline{\mathbf{M}}$ is not 1-endoprimal.*

Proof. Assume that $\underline{\mathbf{M}}$ is a finite non-Boolean Kleene algebra in the quasivariety $\mathcal{K}_4 := \mathbb{ISP}(\underline{\mathbf{4}})$ which has $\underline{\mathbf{4}}$ as a subretract. It follows that $\underline{\mathbf{M}}$ has no subalgebra isomorphic to $\underline{\mathbf{3}}$. Since $\underset{\sim}{\mathbf{4}}_1$, $\underset{\sim}{\mathbf{4}}_2$ and $\underset{\sim}{\mathbf{4}}_3$ yield dualities on $\mathcal{K}_4$, Theorem 2.1 implies that

$$\begin{aligned}\underset{\sim}{\mathbf{M}}_1 &:= \langle M; \mathrm{End}(\underline{\mathbf{M}}), f, c, m, \mathcal{T}\rangle,\\ \underset{\sim}{\mathbf{M}}_2 &:= \langle M; \mathrm{End}(\underline{\mathbf{M}}), k, m, \mathcal{T}\rangle \text{ and}\\ \underset{\sim}{\mathbf{M}}_3 &:= \langle M; \mathrm{End}(\underline{\mathbf{M}}), \ell, m, \mathcal{T}\rangle\end{aligned}$$

yield dualities on $\mathcal{K}_4$. We remark that if $u : \underline{\mathbf{M}} \to \underline{\mathbf{D}}$ is a subretraction, then $u \in \mathrm{End}(\underline{\mathbf{M}})$ with $\mathrm{fix}(u) = D$, whence u entails D.

(1) Assume that $\underline{\mathbf{M}}$ has either $\mathbf{k}$ and $\mathbf{m}$ or $\boldsymbol{\ell}$ and $\mathbf{m}$ as retracts. Then, by Lemma 2.2(iii), the relations k, l and m, may be removed from the dualising structures $\underset{\sim}{\mathbf{M}}_2$ and $\underset{\sim}{\mathbf{M}}_3$ and consequently $\underline{\mathbf{M}}$ is endodualisable.

(2) Assume that $\underline{\mathbf{M}}$ does not have $\mathbf{m}$ as a retract but has either $\mathbf{k}$ or $\boldsymbol{\ell}$ as a retract. Since, by Lemma 3.5, the relation m avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{m\}$, the Optimal Duality Transfer Theorem 2.7 applied to $\underset{\sim}{\mathbf{4}}_2$ and $\underset{\sim}{\mathbf{4}}_3$ shows that $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.

(3) Let $s \in \{k, \ell\}$. Assume that $\underline{\mathbf{M}}$ does not have $\mathbf{s}$ as a retract but has $\mathbf{m}$ as a retract. To prove (3.1), assume that $\mathbf{c}$ is not a retract of $\underline{\mathbf{M}}$. Then by Lemma 3.4(iii)(b)(c), the relation s avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{s\}$. Hence, the Optimal Duality Transfer Theorem 2.7, applied to the optimal dualities given by $\underset{\sim}{\mathbf{4}}_2$ and $\underset{\sim}{\mathbf{4}}_3$, shows that $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. As this duality is optimal modulo endomorphisms, the non-endodualisability of $\underline{\mathbf{M}}$ is shown by the test algebra $\mathbf{s}$ using Lemma 2.3. Since, by Lemma 3.7, $\mathbf{s}$ is a retract of $\mathbf{F}\mathcal{K}(2)$, it follows from Corollary 2.5 that $\underline{\mathbf{M}}$ is not 2-endoprimal.

We now prove (3.2). Assume that $\underline{\mathbf{M}}$ has $\mathbf{c}$ as a retract. For (3.2.1), assume that $\underline{\mathbf{M}}$ also has $\mathbf{f}$ as a retract. Since $\underset{\sim}{\mathbf{M}}_1$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ and $\underline{\mathbf{M}}$ has $\mathbf{m}, \mathbf{f}$ and $\mathbf{c}$ as retracts, $\underline{\mathbf{M}}$ is endodualisable by Lemmas 2.2 and 2.3.

To prove (3.2.2), assume that $\mathbf{f}$ is not a retract of $\underline{\mathbf{M}}$. Then $\underline{\mathbf{M}}$ is not 1-endoprimal, by Theorem 3.1. Since $\mathbf{s}$ has $\mathbf{f}$ as a retract, by the Retraction Test Algebra Lemma 2.4, $\mathbf{s}$ also serves as a test algebra showing the non-endodualisability of $\underline{\mathbf{M}}$. As m can be removed from the dualising structure by Lemmas 2.2 and 2.3, we find that $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.

It remains to prove (4). Again let $s \in \{k, \ell\}$ and assume that $\underline{\mathbf{M}}$ has neither $\mathbf{m}$ nor $\mathbf{s}$ as a retract. For (4.1), assume that $\mathbf{c}$ is not a retract of $\underline{\mathbf{M}}$. By Lemmas 3.4 and 3.5, s and m avoid $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{s, m\}$.

Hence, by the Optimal Duality Transfer Theorem, applied to the optimal dualities given by $\underset{\sim}{\mathbf{4}}_2$ and $\underset{\sim}{\mathbf{4}}_3$, the structure $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), s, m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. Since s avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{s, m\}$, the algebra $\mathbf{s}$ serves as a test algebra showing that $\underline{\mathbf{M}}$ is not endodualisable. Once again, Lemma 3.7 and Corollary 2.5 combine to show that $\underline{\mathbf{M}}$ is not 2-endoprimal. This proves (4.1).

To prove (4.2), assume that $\underline{\mathbf{M}}$ has $\mathbf{c}$ as a retract. If $\underline{\mathbf{M}}$ also has $\mathbf{f}$ as a retract, then, since m avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{m\}$ by Lemma 3.5, the Optimal Duality Transfer Theorem, applied to the optimal duality given by $\underset{\sim}{\mathbf{4}}_1$, shows that $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. This establishes (4.2.1).

Finally, we prove (4.2.2). Again, $\underset{\sim}{\mathbf{M}}_1 = \langle M; \mathrm{End}(\underline{\mathbf{M}}), f, c, m, \mathcal{T}\rangle$ is the dualising structure from which c can be removed by Lemmas 2.2 and 2.3. Thus $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), f, m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$. Since m avoids $\underline{\mathbf{M}}$ relative to $\mathrm{End}(\underline{\mathbf{4}}) \cup \{f, m\}$, by Lemma 3.5, m is needed in $\mathrm{End}(\underline{\mathbf{M}}) \cup \{f, m\}$, by Lemma 2.6. We now prove that, provided $\underline{\mathbf{M}}$ does not have $\mathbf{f}$ as a retract, f is needed in $\mathrm{End}(\underline{\mathbf{M}}) \cup \{f, m\}$. Since

$$D(\mathbf{f}) = \mathcal{K}(\mathbf{f}, \underline{\mathbf{M}}) \cong \mathcal{K}(\mathbf{F}\mathcal{K}(1), \underline{\mathbf{M}}) \cong \underset{\sim}{\mathbf{M}},$$

it suffices to find a map $\gamma : M \to M$ which preserves $\mathrm{End}(\underline{\mathbf{M}}) \cup \{m\}$ but does not preserve f. The map we need is defined and used in the proof of Lemma 5.4 in Davey, Haviar and Priestley [11], where it is denoted by u. We do not need the explicit definition of γ. We need to note only that (a) γ is well defined provided $\underline{\mathbf{M}}$ does not have $\mathbf{f} \cong \mathbf{F}\mathcal{K}(1)$ as a retract, (b) γ preserves $\mathrm{End}(\underline{\mathbf{M}})$, (c) γ satisfies $\gamma(x) = 0$, for all $x \in M^\wedge$, and $\gamma(x) = \neg x$, for all $x \in M^\vee$, and (d) $1 \notin \gamma(M)$. (Here $M^\wedge$ denotes the ideal $\{\, y \wedge \neg y \mid y \in M \,\}$ and $M^\vee$ denotes the filter $\{\, y \vee \neg y \mid y \in M \,\}$.) It remains to show that γ preserves m and does not preserve f. By (c), γ satisfies $\gamma(4) \subseteq 4^\wedge$. Since $1 \notin \gamma(M)$, it is now clear that γ preserves m. To see that γ does not preserve f, it suffices to observe that $a \in M^\wedge$ and $b \in M^\vee$, whence $(a, b) \in f$ but $(\gamma(a), \gamma(b)) = (0, \neg b) = (0, a) \notin f$. We conclude that $\underset{\sim}{\mathbf{M}} := \langle M; \mathrm{End}(\underline{\mathbf{M}}), f, m, \mathcal{T}\rangle$ yields a duality on $\mathcal{K}_4$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms. ∎

4. Pseudocomplemented distributive lattices.

The variety $\mathcal{B}_n$ of pseudocomplemented distributive lattices equals $\mathbb{ISP}(\underline{\mathbf{P}}_n)$, where $\underline{\mathbf{P}}_n$ is obtained by adjoining a new top element to the n-atom Boolean lattice. Let $n \geqslant 1$. We shall use the restricted Priestley duality for the varieties $\mathcal{B}_n$. As set out in Priestley [26] (see [15, Proposition 3.1]), there is a contravariant category equivalence given by functors $H : \mathbfcal{B}_n \to \mathcal{Y}_n$ and $K : \mathcal{Y}_n \to \mathbfcal{B}_n$ between $\mathbfcal{B}_n$ and the category $\mathcal{Y}_n$ of Priestley spaces $\mathbf{Y}$ with the property that for each clopen up-set U in $\mathbf{Y}$, the set

$$\downarrow U := \{z \in Y \mid z \leqslant y \text{ for some } y \in U\}$$

is clopen and each point in Y is majorized by at most n maximal points. For $\mathbf{Y}, \mathbf{Z} \in \mathcal{Y}_n$, a map $\Psi : Y \to Z$ is a morphism in the category $\mathcal{Y}_n$ if and only if Ψ is continuous and order-preserving, and

$$\Psi(\max(y)) = \max(\Psi(y)) \text{ for all } y \in Y,$$

where $\max(z)$ denotes the set of maximal points above z.

We follow the terminology and notation used in [15]. Given a k-ary algebraic relation $s \subseteq P_n^k$, the ordered set $\mathbf{Y}_s := H(\mathbf{s})$ is called the *shape* of s. Let the Priestley dual of $\underline{\mathbf{P}}_n$ be $V_n = \{0, 1, 2, \dots, n\}$, let the dual of the free algebra $\mathbf{F}\mathbfcal{B}_n(1) \cong (\mathbf{2} \times \mathbf{3}) \oplus \mathbf{1}$ be $F = \{\bot, b, d, \top\}$ and let $Y_l = \{0, \widetilde{0}, 1, \dots, l, l+1, \dots, n\}$, with the order relations as indicated in Figure 6. Since the ordered set $\mathbf{Y}_l$ is uniquely determined by the number l, we say that a binary relation r with $\mathbf{Y}_r \cong \mathbf{Y}_l$ has *shape* l. If r is a binary relation of shape l, then we replace $\mathbfcal{B}_n(\mathbf{r}, \underline{\mathbf{P}}_n)$ by $\mathcal{Y}_n(\mathbf{V}_n, \mathbf{Y}_l)$ and we replace $\underset{\sim}{\mathbf{P}}_n$ itself by $\mathcal{Y}_n(\mathbf{V}_n, \mathbf{F})$. We refer the reader to [15] for more details.

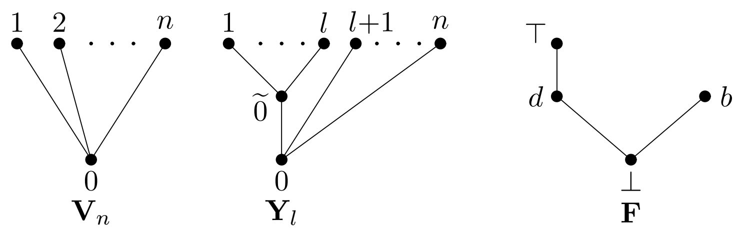

Figure 6. The Priestley spaces $\mathbf{V}_n := H(\underline{\mathbf{P}}_n)$, $\mathbf{Y}_l$ and $\mathbf{F} := H(\mathbf{F}\mathbfcal{B}_n(1))$

In Davey and Priestley [15, Theorem 5.1], it is shown that if R is a set of binary algebraic relations on $\underline{\mathbf{P}}_n$, then $\underset{\sim}{\mathbf{P}}_n = \langle P_n; \mathrm{End}(\underline{\mathbf{P}}_n), R, \mathcal{T}\rangle$ yields a duality on $\mathbfcal{B}_n$ which is optimal modulo endomorphisms if and only if $R = \{r_1, \dots, r_n\}$ where r_l is a relation of shape l for $1 \leqslant l \leqslant n$. To apply the Optimal Duality Transfer Theorem 2.7, let us consider a binary relation r of shape l, with $l \in \{1, \dots, n\}$. To obtain the desired failset we require a map $\Phi : \mathbfcal{B}_n(\mathbf{r}, \underline{\mathbf{P}}_n) \to P_n$ which preserves $\mathrm{End}(\underline{\mathbf{P}}_n)$ and $\{r_1, \dots, r_n\} \setminus \{r\}$, but not r. In [15, Proposition 4.3], the map Φ is defined dually as $\Phi : \mathcal{Y}_n(\mathbf{V}_n, \mathbf{Y}_l) \to \mathcal{Y}_n(\mathbf{V}_n, \mathbf{F})$ by declaring that Φ maps each element Ψ in $\mathcal{Y}_n(\mathbf{V}_n, \mathbf{Y}_l)$ to $\varphi \circ \Psi$ in $\mathcal{Y}_n(\mathbf{V}_n, \mathbf{F})$, where $\varphi : Y_l \to F$ is given by

$$\varphi(y) = \begin{cases} \top & \text{if } y \text{ is maximal in } Y_l, \\ d & \text{if } y = \widetilde{0}, \\ \top & \text{if } y = 0. \end{cases}$$

By [15, Proposition 4.3], Φ preserves the action of every endomorphism $e \in \mathcal{Y}_n(\mathbf{V}_n, \mathbf{V}_n)$, and preserves a binary algebraic relation s on $\underline{\mathbf{P}}_n$ if and only if $\mathbf{s} \not\cong \mathbf{r}$ (or, equivalently, if s does not have the same shape as r).

LEMMA 4.1. *Let $R = \{r_1, \dots, r_n\}$ consist of exactly one binary algebraic relation on $\underline{\mathbf{P}}_n$ of shape l for each $l \in \{1, \dots, n\}$. Then each $r \in R$ is as small as possible relative to* $\mathrm{End}(\underline{\mathbf{P}}_n) \cup R$.

PROOF. Assume that $r = r_l$. Let $s \subseteq P_n^k$ be an algebraic k-ary relation such that $\mathbf{s}$ does not have $\mathbf{r}$ as a retract. It follows from what we have said above about the map Φ, that it only remains to show that Φ preserves s. Let $\mathbf{Y}_s$ be the Priestley dual of $\mathbf{s}$ and let $\sigma_1, \dots, \sigma_k : \mathbf{V}_n \to \mathbf{Y}_s$ be the jointly surjective maps corresponding to the projections $\rho_1, \dots, \rho_k : \mathbf{s} \to \underline{\mathbf{P}}_n$.

Assume that $\Psi_1, \dots, \Psi_k \in \mathcal{Y}_n(\mathbf{V}_n, \mathbf{Y}_l)$ with $(\Psi_1, \dots, \Psi_k) \in s$. By the definition of s on $\mathcal{Y}_n(\mathbf{V}_n, \mathbf{Y}_l)$, there exists a morphism $\mu : \mathbf{Y}_s \to \mathbf{Y}_l$ such that

$$\Psi_i = \mu \circ \sigma_i \text{ for } i = 1, \dots, k.$$

To show that $(\varphi \circ \Psi_1, \dots, \varphi \circ \Psi_n) \in s$, it suffices to prove that the map $\varphi \circ \mu : Y_s \to F$ is a morphism and so it suffices to show that $\varphi \circ \mu$ is order-preserving. Suppose that $\varphi \circ \mu$ is not order-preserving. Since the only order relation in $\mathbf{Y}_r$ not preserved by φ is $0 < \widetilde{0}$, it follows that there exist $y, z \in Y_s$ with $y < z$ such that $\mu(y) = 0$ and $\mu(z) = \widetilde{0}$. Then $\max(0) = \max(\mu(y)) = \mu(\max(y))$ and $\max(\widetilde{0}) = \max(\mu(z)) = \mu(\max(z))$. Thus the fact that $Y_r = \{0, \widetilde{0}\} \cup \max(0) \cup \max(\widetilde{0})$ implies that μ is surjective. Define $Z := \{y, z\} \cup \max(y) \cup \max(z)$. It is easy to see that the inclusion of Z into Y_s is an embedding, and, as y can be majorized in Y_s by at most n maximal points, we have that $|\max(y)| = n$ and $\mu{\restriction}_Z : \mathbf{Z} \to \mathbf{Y}_l$ is an isomorphism. Hence $\mu : \mathbf{Y}_s \to \mathbf{Y}_l$ is a retraction, contradicting the fact that $\mathbf{r}$ is not a retract of $\mathbf{s}$. ∎

The Optimal Duality Transfer Theorem 2.7 and Lemma 2.8 now yield optimal dualities (modulo endomorphisms) on every finite algebra in $\mathcal{B}_n$ that has $\underline{\mathbf{P}}_n$ as a subretract.

Theorem 4.2. *Let $n \geqslant 1$ and let $\underline{\mathbf{M}}$ be a finite algebra in $\mathcal{B}_n$ which has $\underline{\mathbf{P}}_n$ as a subretract. Let $R = \{r_1, \ldots, r_n\}$ consist of exactly one binary algebraic relation of shape l for each $l \in \{1, \ldots, n\}$. Define R° to be the subset of R consisting of the relations $r \in R$ such that $\mathbf{r}$ is not a retract of $\underline{\mathbf{M}}$. Then $\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), R^\circ, \mathcal{T}\rangle$ yields a duality on $\mathcal{B}_n$ based on $\underline{\mathbf{M}}$ which is optimal modulo endomorphisms.*

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