

THE SYNTAX AND SEMANTICS OF ENTAILMENT IN DUALITY THEORY

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Abstract. Both syntactic and semantic solutions are given for the entailment problem of duality theory. The test algebra theorem provides both a syntactic solution to the entailment problem in terms of primitive positive formulae and a new derivation of the corresponding result in clone theory, viz. the syntactic description of $\text{Inv}(\text{Pol}(R))$ for a given set R of finitary relations on a finite set. The semantic solution to the entailment problem follows from the syntactic one, or can be given in the form of an algorithm. It shows, in the special case of a purely relational type, that duality-theoretic entailment is describable in terms of five constructs, namely trivial relations, intersection, repetition removal, product, and retractive projection. All except the last are concrete, in the sense that they are described by a quantifier-free formula. It is proved that if the finite algebra \underline{M} generates a congruence-distributive variety and all subalgebras of \underline{M} are subdirectly irreducible, then concrete constructs suffice to describe entailment. The concept of entailment appropriate to strong dualities is also introduced, and described in terms of coordinate projections, restriction of domains, and composition of partial functions.

§1. Introduction. Our principal aim in this paper is to solve the entailment problem of duality theory. This problem, previously called the generation problem, was first posed in [10], pp. 140–142, and restated ten years later in [5], p. 89. Our approach to this problem relies on the concept of entailment (*alias* generation; see [5], pp. 89–90). Entailment and its role in duality theory are analysed in detail in [9].

We begin by recalling our basic definitions. Assume that we are given a quasi-variety $\mathcal{A} = \text{ISP}(\underline{M})$ of algebras generated by a finite algebra \underline{M} . We denote by \mathcal{P} the set $\bigcup_{n \geq 1} \mathcal{S}(\underline{M}^n)$ of finitary algebraic relations on \underline{M} . An n -ary relation $s \in \mathcal{P}$ may be regarded as an algebra, \underline{s} , in \mathcal{A} . We write $\underline{s} \leq \underline{M}^n$ when we wish to think of s as an element of $\mathcal{S}(\underline{M}^n)$. We let \mathcal{D} be the set of all partial or total algebraic operations on \underline{M} ; that is, e belongs to \mathcal{D} if and only if e is a homomorphism from D to \underline{M} for some subalgebra D of some finite power \underline{M}^n of \underline{M} . We assume we have a structure $\underline{M} := (M; G, H, R, \tau)$ on the underlying set M of \underline{M} such that

- (i) G is a set of (total) operations on M such that if $g \in G$ is nullary then $\{g\}$ is a subalgebra of \underline{M} , and if g is n -ary for $n \geq 1$ then $g : \underline{M}^n \rightarrow \underline{M}$ is a homomorphism;
- (ii) H is a set of partial operations on M (of arity at least 1) such that if $h \in H$ is n -ary, then $\text{dom } h$, the domain of h , is a (nonempty) proper subalgebra of \underline{M}^n and $h : \text{dom } h \rightarrow \underline{P}$ is a homomorphism;

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- (iii) R is a subset of \mathcal{B} ;
- (iv) τ is the discrete topology on M .

Conditions (i) and (ii) simply spell out in detail the requirement that G is a set of total operations on M and H is a set of partial operations on M such that $G \cup H \subseteq \mathcal{P}$.

We encompass clones as follows. Let M be a finite set and let R be a family of finitary relations on M . Then $\text{Pol}(R)$ denotes the clone of all finitary functions $f: M^n \rightarrow M$ ($n \geq 1$) which preserve the relations in R . The set of all finitary relations s which are invariant under all functions $f \in \text{Pol}(R)$ is denoted by $\text{Inv}(\text{Pol}(R))$; clearly $R \subseteq \text{Inv}(\text{Pol}(R))$. We may define an algebra $\underline{M} = (M; F)$, where $F = \text{Pol}(R)$. Then $\text{Inv}(\text{Pol}(R))$ is just the familiar family \mathcal{B} of finitary algebraic relations on \underline{M} .

In duality theory, it is appropriate to allow for operations as well as relations. In particular, we often wish to include partial endomorphisms in the type of \underline{M} . For any $e \in G \cup H$, the graph of e is an algebraic relation. In certain contexts we may replace the members of $G \cup H$ by their graphs, thereby obtaining a purely relational topological structure $\underline{M}^\bullet := (M; R^\bullet, \tau)$, where $R^\bullet = R \cup \{\text{graph}(e) \mid e \in G \cup H\}$. However the difference in type of \underline{M} and \underline{M}^\bullet is sometimes important; see §4.

We let $\mathcal{Z} := \text{IS}_c\mathbb{P}(\underline{M})$ be the topological quasivariety generated by \underline{M} , namely the class of isomorphic copies of closed substructures of powers of \underline{M} . Given $s \in \mathcal{P} \cup \mathcal{B}$ and $Z \in \mathcal{Z}$, we write s_Z to denote s acting pointwise on Z . For each $A \in \mathcal{A}$ we define the dual of A to be $D(A) := \mathcal{A}(A, \underline{M}) \in \mathcal{Z}$. The dual of $Z \in \mathcal{Z}$ is $E(Z) := \mathcal{Z}(Z, \underline{M})$, *qua* subalgebra of \underline{M}^Z . When defined to act on morphisms by composition, D and E are functors setting up a dual adjunction between \mathcal{A} and \mathcal{Z} . The structure \underline{M} (or the set $G \cup H \cup R$) is said to *yield a duality on* \mathcal{A} if, for each $A \in \mathcal{A}$, the map $e_A: a \mapsto e_a$ from A to its second dual $ED(A)$ is an isomorphism, where $e_a: x \mapsto x(a)$ ($x \in D(A)$) is the evaluation map. For further details, see [10] or [5].

The set $G \cup H \cup R$ is said to *entail* a relation s (in symbols, $(G \cup H \cup R) \vdash s$) if, for every $A \in \mathcal{A}$, every continuous map $\varphi: D(A) := \mathcal{A}(A, \underline{M}) \rightarrow M$ which preserves every element of $G \cup H \cup R$ also preserves s . Locally, for a fixed $A \in \mathcal{A}$, we say $G \cup H \cup R$ *entails* s on $Z = D(A)$ if every continuous map $\varphi: Z \rightarrow M$ preserving $G \cup H \cup R$ also preserves s . Note that when $G \cup H \cup R$ yields a duality on \mathcal{A} we have $(G \cup H \cup R) \vdash s$ for every $s \in \mathcal{B}$, since an evaluation map preserves every algebraic relation. Note that $(G \cup H \cup R) \vdash s$ if and only if $R^\bullet \vdash s$. We may now state our problem.

ENTAILMENT PROBLEM. Find an intrinsic description of the relations entailed by $G \cup H \cup R$.

When this problem first arose, it was envisaged that the solution would be a semantic one in terms of a preservation theorem: a list of finitary constructs which preserve entailment and such that if $(G \cup H \cup R) \vdash s$ then s would be obtainable from $G \cup H \cup R$ via a finite sequence of the constructs. As it turned out, our semantic solution arises as a direct application of a syntactic solution: a description of relations entailed by $G \cup H \cup R$ in terms of the first-order formulæ

of the language with equality, $\mathcal{L}_{\underline{M}}$, associated with \underline{M} . A major step towards the solution was the recognition that on a given set Ω of finitary algebraic relations on \underline{M} the map

$$R \longmapsto \bar{R} := \{s \in \Omega \mid R \vdash s\}$$

is a closure operator (entailment closure), and that this closure operator is algebraic, in the sense that the closure of any set R is the union of the closures of its finite subsets (so that the lattice of closed sets is algebraic). This provides indirect evidence for a positive solution to the entailment problem. In [9] we deduced that entailment closure is algebraic as a corollary of the test algebra lemma (Lemma 2.3 of [9]). This lemma has as a corollary the key Proposition 2.2 of [8], which foreshadowed it. It asserts that $R \vdash s$ if and only if $R \vdash s$ on $D(\underline{s})$, whence we refer to \underline{s} as the *test algebra* corresponding to the relation s . In this paper we extend and amplify [9]’s test algebra lemma, upgrading it to the test algebra theorem (Theorem 2.3 below). This theorem states *inter alia* that $(G \cup H \cup R) \vdash s$ if and only if there is a primitive positive formula Φ in the language $\mathcal{L}_{\underline{M}}$ such that

$$D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$$

and

$$s = \{(c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n)\},$$

where ρ_1, \dots, ρ_n are the restrictions to $\underline{s} \leq \underline{M}^n$ of the projection maps. Indeed, we may take Φ to be the primitive positive type of ρ_1, \dots, ρ_n in $D(\underline{s})$. Through the test algebra theorem, we are able to convert our syntactic solution to the entailment problem to a semantic solution, so obtaining a set of constructs sufficient to describe entailment. In case $G \cup H = \emptyset$, the list may be taken to be: trivial relations, repetition removal, intersection, product, and retractive projection (in which the natural projection map is required to be a retraction (see §3)). Applying our results in the clone setting, we derive the well-known fact (see, for example, [13], Chapter 2, or [1]), that $\text{Inv}(\text{Pol}(R))$ can be obtained from R by a finite number of applications of trivial relations, intersection, repetition removal, product and projection. In duality theory, R entails s on the powers of \underline{M} (which are the duals of free algebras in the associated quasivariety \mathcal{A} ; see, for example, [10]) if and only if s can be obtained from R using the constructs listed for the clone-theoretic case. However, as is well known, arbitrary projection is not necessarily an allowable construct on structures of the form $D(A) = \mathcal{A}(A, \underline{M})$. If it were, we could form the relational product of two relations, which is not guaranteed to lift to structures $D(A)$ which are not full powers. This explains why a set R of algebraic relations on \underline{M} which determines the clone of term functions on \underline{M} will not necessarily yield a duality on \mathcal{A} . This is illustrated in [5], p.102, in case \mathcal{A} is the variety \mathbf{K} of Kleene algebras; for a more extended discussion, see [9], §5.

Each of the constructs originally recognised as permissible by Davey and Werner (which include products, intersections, domains, equalisers, kernels, and the relational product of a function e and of a relation r) has the special property that it is specified by a formula Φ which is quantifier-free. We shall call such constructs

concrete. They have the following property: if s is concretely constructed from $r_1, \dots, r_k \in R$, then the relationship $R \vdash s$ depends only upon the sets r_1, \dots, r_k, s and not upon the algebras $\underline{r}_1, \dots, \underline{r}_k, \underline{s}$.

If \underline{M} has an underlying lattice structure, then the NU duality theorem ([10], 1.19; [5], 2.8) implies that the set $R := \mathbb{S}(\underline{M}^2)$ of all binary algebraic relations on \underline{M} yields a duality on \mathcal{A} . Prior to this paper, in every example in which entailment was used to obtain a workable duality via a small subset of $\mathbb{S}(\underline{M}^2)$, it was found that concrete constructs were sufficient. In §4 we explain why this was so: we show in particular that if \underline{M} generates a congruence-distributive variety and every subalgebra of \underline{M} is subdirectly irreducible, then concrete constructs suffice provided the language $\mathcal{L}_{\underline{M}}$ includes a (partial) function symbol for each (partial) endomorphism of \underline{M} . As it happened, each of the lattice-ordered examples previously analysed had the property that every subalgebra of \underline{M} was subdirectly irreducible!

Let $Z \in \mathcal{A}$. The set Z is called *hom-closed* in M^S if and only if for every set $I \neq \emptyset$, and every homomorphism $h: D \rightarrow M$, where D is a subalgebra of \underline{M}^I , the set Z is closed under h . Since $D(A)$ is always hom-closed in M^A , hom-closed sets play an important role in duality theory especially when considering full dualities and strong dualities (see Clark and Davey [2]). Since hom-closed sets are closed under all the (partial) operations in $G \cup H$ we could define entailment with respect to the class of all hom-closed sets rather than the subclass consisting of sets of the form $D(A)$. We shall show that these two definitions are equivalent, thereby linking our work to that of L. Zadori [14].

In §5 we discuss strong duality and introduce a notion of entailment amongst partial functions, called *hom-entailment*, appropriate to strong dualities. We show that the constructs coordinate projection, restriction of domain and composition of partial functions are adequate to provide a semantic description of hom-entailment.

Finally, in §6 we present an analysis of entailment and hom-entailment amongst the binary algebraic relations and partial functions on the 3-element chain regarded as a bounded distributive lattice. Here concrete constructs do not suffice, and it was our computer analysis of this example which commenced the chain of results which eventually led to the theorems presented in §§2 and 3.

§2. The syntax of entailment. We take $\underline{M} = (M; G, H, R, \tau)$ as in the preceding section. Because of the presence of partial operations in our setting, which is not customary in standard treatments of relational structures, we include a description of the first-order language with equality, $\mathcal{L}_{\underline{M}}$, associated with \underline{M} . This language is set up in the same manner as in [4].

We let Ξ be a fixed set of variables. For each relation or operation in $G \cup H \cup R \cup \{=\}$ we take a corresponding function or relation symbol of the same arity. We define terms recursively:

- (i) each variable and each nullary operation symbol of G is a term, and
- (ii) if t_1, t_2, \dots, t_n are terms and $e \in G \cup H$ is n -ary, then $e(t_1, t_2, \dots, t_n)$ is a term.

An atomic formula is an expression of either of the forms $t_1 = t_2$ or $r(t_1, t_2, \dots, t_n)$, where t_1, t_2, \dots, t_n are terms and $r \in R$ is n -ary. Nonatomic formulæ are built recursively from the atomic formulæ in the expected way. We shall require only the primitive positive formulæ, in which the only connectives are \wedge and \exists .

We provide these syntactic expressions with a semantic interpretation as follows. Let $Z \in \mathcal{Z}$. A variable assignment $v: \Xi \rightarrow Z$ extends to terms, assigning a value $v[t]$ in Z to each term t for which the necessary partial operations are defined, according to the following rules:

- (i) $v[x] = v(x)$ for each variable x ;
- (ii) $v[c] = c_Z$ if c is a nullary operation symbol;
- (iii) if $e \in G \cup H$ is n -ary ($n \geq 1$), then

$$v[e(t_1, t_2, \dots, t_n)] = e_Z(v[t_1], v[t_2], \dots, v[t_n])$$

provided that $v[t_1], v[t_2], \dots, v[t_n]$ are all defined and that their values are in the domain of e_Z ; otherwise $v[e(t_1, t_2, \dots, t_n)]$ is undefined. Each atomic formula is assigned a truth value by v as follows (and the assignment v is then extended to nonatomic formulæ in the standard way):

- (i) $t_1 = t_2$ is assigned T if $v[t_1]$ and $v[t_2]$ are both defined and are equal; otherwise it is assigned F;
- (ii) $r(t_1, t_2, \dots, t_n)$ is assigned T provided $v[t_1], v[t_2], \dots, v[t_n]$ are all defined and $(v[t_1], v[t_2], \dots, v[t_n])$ is in r_Z , otherwise it is assigned F.

For a formula Φ we write $Z \models \Phi$ if $v[\Phi]$ is assigned value T for every truth assignment $v: \Xi \rightarrow Z$.

We denote by π_i ($i = 1, \dots, n$) the natural projection maps from \underline{M}^n to \underline{M} , and, for a given subalgebra \underline{s} of \underline{M}^n , let $\rho_i := \pi_i \upharpoonright_{\underline{s}}$ (the domain will always be clear from the context). Note that $\rho_1, \dots, \rho_n \in D(\underline{s})$. One further item of notation: given a nonempty set T and maps $x_1, \dots, x_n: T \rightarrow \underline{M}$, we let $x_1 \sqcap \dots \sqcap x_n: T \rightarrow \underline{M}^n$ be given by

$$(\forall a \in T) (x_1 \sqcap \dots \sqcap x_n)(a) := (x_1(a), \dots, x_n(a)).$$

LEMMA 2.1. *Let $s \in \mathcal{B}$ be n -ary, let T be a nonempty set, let Z be a nonempty subset of M^T , and let $z_1, \dots, z_n \in Z$. Then $(z_1, \dots, z_n) \in s_Z$ if and only if $(z_1 \sqcap \dots \sqcap z_n)(T) \subseteq s$.*

PROOF. Assume that $(z_1, \dots, z_n) \in s_Z$. Then, by the definition of s_Z , we have $(z_1 \sqcap \dots \sqcap z_n)(t) = (z_1(t), \dots, z_n(t)) \in s$ for all $t \in T$, and hence $(z_1 \sqcap \dots \sqcap z_n)(T) \subseteq s$. The converse is equally obvious. \square

LEMMA 2.2. *Let Z be a hom-closed subset of M^T for some nonempty set T and let $\underline{s} \leq \underline{M}^n$.*

- (a) $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$.
- (b) *Let $z_1, \dots, z_n \in Z$. Then the following are equivalent (and may be interpreted as asserting that $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$ is a universal instance of the relation s in the class $\{Z \subseteq M^T \mid T \neq \emptyset \text{ and } Z \text{ is hom-closed}\}$):*
 - (i) $(z_1, \dots, z_n) \in s_Z$;
 - (ii) *there is a (necessarily unique) map $\gamma: T \rightarrow \underline{s}$, namely $\gamma = z_1 \sqcap \dots \sqcap z_n$, such that $z_i = \rho_i \circ \gamma$ for all i ;*

- (iii) there is a (necessarily unique) map $v: D(\underline{s}) \rightarrow Z$ such that v preserves every finitary algebraic relation on \underline{M} and $v(\rho_i) = z_i$ for all i ;
- (iv) there is a (necessarily unique) map $v: D(\underline{s}) \rightarrow Z$ such that v preserves every algebraic n -ary partial operation with domain s and satisfies $v(\rho_i) = z_i$ for all i ;
- (v) there is an s -preserving map $v: D(\underline{s}) \rightarrow Z$ such that $v(\rho_i) = z_i$ for all i .

PROOF. For (a) simply observe that $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$ if and only if, for all $a \in \underline{s}$, $(\rho_1 \sqcap \dots \sqcap \rho_n)(a) \in s$. This holds because $a = (\rho_1 \sqcap \dots \sqcap \rho_n)(a)$ by definition.

We now prove (b). Assume that $(z_1, \dots, z_n) \in s_Z$. Then, by Lemma 2.1, $(z_1 \sqcap \dots \sqcap z_n)(T) \subseteq s$ and hence the map $\gamma := z_1 \sqcap \dots \sqcap z_n: T \rightarrow \underline{s}$ is well defined and satisfies $z_i = \rho_i \circ \gamma$ for all i . The categorical definition of product asserts that γ is the unique map satisfying $z_i = \rho_i \circ \gamma$ for all i . Thus (i) implies (ii).

Now assume (ii) and define $v: D(\underline{s}) \rightarrow M^T$ by $v(y) := y \circ \gamma$ for all $y \in D(\underline{s})$. For $y \in D(\underline{s})$, let \hat{y} be y regarded as an n -ary partial operation on \underline{M} . Then

$$v(y) = y \circ \gamma = y \circ (z_1 \sqcap \dots \sqcap z_n) = \hat{y}(z_1, \dots, z_n) \in Z,$$

since Z is hom-closed. Thus $v: D(\underline{s}) \rightarrow Z$ is well defined and satisfies $v(\rho_i) = z_i$ for all i . We now prove that v preserves every finitary algebraic relation on \underline{M} . Let $r \subseteq \underline{M}^k$ and let $(y_1, \dots, y_k) \in r_{D(\underline{s})}$. Thus $y_1 \sqcap \dots \sqcap y_k: \underline{s} \rightarrow r$ is a well-defined homomorphism (by Lemma 2.1 with T and s replaced by s and r respectively). To prove that $(v(y_1), \dots, v(y_k)) \in r_Z$ it suffices to show that $(v(y_1) \sqcap \dots \sqcap v(y_k))(T) \subseteq r$ (by Lemma 2.1 with s replaced by r). But

$$\begin{aligned} v(y_1) \sqcap \dots \sqcap v(y_k) &= y_1 \circ (z_1 \sqcap \dots \sqcap z_n) \sqcap \dots \sqcap y_k \circ (z_1 \sqcap \dots \sqcap z_n) \\ &= (y_1 \sqcap \dots \sqcap y_k) \circ (z_1 \sqcap \dots \sqcap z_n), \end{aligned}$$

whence $(v(y_1) \sqcap \dots \sqcap v(y_k))(T) \subseteq r$, as the image of $y_1 \sqcap \dots \sqcap y_k$ is a subset of r . Thus v preserves r . We now prove that v is determined by its values on $\{\rho_1, \dots, \rho_n\}$. Let $y \in D(\underline{s})$ and again let \hat{y} be y regarded as an n -ary partial operation on \underline{M} . Thus, as v preserves all finitary algebraic relations on \underline{M} , the map v preserves \hat{y} . Since $\hat{y}(\rho_1, \dots, \rho_n) = y$, we have $v(y) = v(\hat{y}(\rho_1, \dots, \rho_n)) = \hat{y}(v(\rho_1), \dots, v(\rho_n)) = \hat{y}(z_1, \dots, z_n)$, whence v is indeed determined by its values on $\{\rho_1, \dots, \rho_n\}$. Thus (ii) implies (iii) and (iii) implies (iv).

To prove that (iv) implies (v) note that the map v guaranteed by (iv) preserves the map $\rho_1: s \rightarrow M$ by assumption and hence preserves its domain, namely s .

Finally we show that (v) implies (i). Let v be the map given by (v). Since $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$ by (a), it follows immediately that

$$(z_1, \dots, z_n) = (v(\rho_1), \dots, v(\rho_n)) \in s_Z. \quad \square$$

Note that since, by definition, a hom-closed subset Z of M^T is closed under every algebraic (partial) operation, it makes sense to consider entailment on Z . Since $D(A)$ is hom-closed in M^A for all $A \in \mathcal{A}$, entailment defined with respect to the class of all hom-closed subsets of powers of M would appear to be stronger

than entailment defined with respect to the class $\{D(A) \mid A \in \mathcal{A}\}$. In fact, it is one of the conclusions of the test algebra theorem that these two entailment concepts are equivalent. The latter concept is that used by L. Zadori in [14].

The test algebra theorem is a strengthened version of the test algebra lemma, Lemma 2.3 of [9].

We remark that since $(G \cup H \cup R) \vdash s$ if and only if $R^* \vdash s$, we could without loss of generality restrict attention to purely relational structures in (a)–(e) below; the choice of type becomes significant only in (f).

THEOREM 2.3 (The Test Algebra Theorem). *Let \underline{M} be a finite algebra, let $\underline{M} = (M; G, H, R, \tau)$, where $G \cup H \subseteq \mathcal{P}$ and $R \subseteq \mathcal{B}$, and let $s \in \mathcal{B}$. Then the following are equivalent:*

- (a) $G \cup H \cup R$ entails s on each hom-closed subset of every power of M ;
- (b) $G \cup H \cup R$ entails s ;
- (c) $G \cup H \cup R$ entails s on $D(\underline{s})$;
- (d) some finite subset of $G \cup H \cup R$ entails s on $D(\underline{s})$;
- (e) $s = \{ (u(\rho_1), \dots, u(\rho_n)) \mid u: D(\underline{s}) \rightarrow M \text{ preserves } G \cup H \cup R \}$;
- (f) $\Phi(x_1, \dots, x_n)$ in the language $\mathcal{L}_{\underline{M}}$ such that
 - (i) $D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$ and
 - (ii) $s = \{ (c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n) \}$.

PROOF. It is trivial that (a) implies (b) and (b) implies (c). Assume that $G \cup H \cup R$ entails s on $D(\underline{s})$. Let Z be a hom-closed subset of M^T and let $u: Z \rightarrow M$ be continuous and preserve $G \cup H \cup R$. Let $(z_1, \dots, z_n) \in s_Z$. By Lemma 2.2 there exists a $(G \cup H \cup R)$ -preserving map $v: D(\underline{s}) \rightarrow Z$ which satisfies $v(\rho_i) = z_i$ for each i . Hence $u \circ v: D(\underline{s}) \rightarrow M$ is $(G \cup H \cup R)$ -preserving and so preserves s . Thus

$$(u(z_1), \dots, u(z_n)) = ((u \circ v)(\rho_1), \dots, (u \circ v)(\rho_n)) \in s,$$

whence u preserves s . Consequently $G \cup H \cup R$ entails s on Z , whence (c) implies (a).

Assume (c). For each $c \in s$ the evaluation map $e_c: D(\underline{s}) \rightarrow M$ preserves $G \cup H \cup R$, and hence

$$c = (c_1, \dots, c_n) = (\rho_1(c), \dots, \rho_n(c)) = (e_c(\rho_1), \dots, e_c(\rho_n)) \in s^b,$$

where

$$s^b = \{ (u(\rho_1), \dots, u(\rho_n)) \mid u: D(\underline{s}) \rightarrow M \text{ preserves } G \cup H \cup R \}.$$

Thus $s \subseteq s^b$. For the reverse inclusion, let $u: D(\underline{s}) \rightarrow M$ preserve $G \cup H \cup R$, so that we have $(u(\rho_1), \dots, u(\rho_n)) \in s^b$. But $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$ and consequently $(u(\rho_1), \dots, u(\rho_n)) \in s$, as $G \cup H \cup R$ entails s on $D(\underline{s})$, by (c). Thus $s = s^b$, whence (e) holds. Since there are only a finite number of maps from $D(\underline{s})$ to M which do not preserve $G \cup H \cup R$, we have (e) \Rightarrow (d), and (d) \Rightarrow (c) is trivial. Hence (a)–(e) are equivalent.

We now derive (f) from (a)–(e). Let $K \subseteq G \cup H \cup R$ be such that K is finite and K entails s on $D(\underline{s})$. Let $D(\underline{s}) = \{\rho_1, \dots, \rho_n, \tau_1, \dots, \tau_m\}$. For each relation $r \in R \cup \{=\}$, there is a formula $\Phi_r(x_1, \dots, x_n, y_1, \dots, y_m)$ which is a conjunct of atomic formulæ and which precisely describes the relation r on $D(\underline{s})$

via the correspondence $x_i \leftrightarrow \rho_i$ and $y_j \leftrightarrow \tau_j$. In a similar way, there is a formula $\Phi_e(x_1, \dots, x_n, y_1, \dots, y_m)$ describing each $e \in G \cup H$ on $D(\underline{s})$. Define $\Phi(x_1, \dots, x_n)$ to be

$$(\exists y_1 \cdots y_m) \&\mathcal{L}\{\Phi_k(x_1, \dots, x_n, y_1, \dots, y_m) \mid k \in K \cup \{=\}\}.$$

By construction, $D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$. Define

$$s^\sharp := \{(c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n)\}.$$

We shall show that $s = s^\sharp$. Let $(u(\rho_1), \dots, u(\rho_n)) \in s^b$ with $u: D(\underline{s}) \rightarrow M$ a $(G \cup H \cup R)$ -preserving map. Since $D(\underline{s})$ satisfies $\Phi(\rho_1, \dots, \rho_n)$, it follows at once that M satisfies $\Phi(u(\rho_1), \dots, u(\rho_n))$, whence $(u(\rho_1), \dots, u(\rho_n)) \in s^\sharp$. Now let $(c_1, \dots, c_n) \in s^\sharp$. Thus there exist $d_1, \dots, d_m \in M$ such that M satisfies $\Phi_k(c_1, \dots, c_n, d_1, \dots, d_m)$ for each $k \in K \cup \{=\}$. It follows that $\rho_i \mapsto c_i$ and $\tau_j \mapsto d_j$ yields a well-defined $(G \cup H \cup R)$ -preserving map from $D(\underline{s})$ to M , and hence $(c_1, \dots, c_n) \in s^b$. Thus, by (e), we have $s = s^b = s^\sharp$, whence (f) holds.

Finally, assume (f). Let $u: D(\underline{s}) \rightarrow M$ be an $(G \cup H \cup R)$ -preserving map. To prove (c), we must show that u preserves s . Let $z_1, \dots, z_n \in D(\underline{s})$, with $(z_1, \dots, z_n) \in s_{D(\underline{s})}$. By Lemma 2.2, there exists a $(G \cup H \cup R)$ -preserving map $v: D(\underline{s}) \rightarrow D(\underline{s})$ with $v(\rho_i) = z_i$ for all i . Thus $w := u \circ v: D(\underline{s}) \rightarrow M$ is a $(G \cup H \cup R)$ -preserving map satisfying $w(\rho_i) = u(z_i)$ for all i . Since $D(\underline{s})$ satisfies $\Phi(\rho_1, \dots, \rho_n)$, it follows that M satisfies $\Phi(w(\rho_1), \dots, w(\rho_n))$, whence (by (f)) $(u(z_1), \dots, u(z_n)) \in s$. Hence u preserves s , and consequently $G \cup H \cup R$ entails s on $D(\underline{s})$. \square

We turn now to the clone-theoretic application.

THEOREM 2.4. *Let R be a family of finitary relations on a finite set M and let $s \subseteq M^n$. Then the following are equivalent:*

- (a) $s \in \text{Inv}(\text{Pol}(R))$;
- (b) R entails s on M^s ;
- (c) $s = \{(u(\rho_1), \dots, u(\rho_n)) \mid u: M^s \rightarrow M \text{ preserves } R\}$;
- (d) there is some finite structure Z of the same type as $(M; R)$ and elements z_1, \dots, z_n in Z such that $s = \{(u(z_1), \dots, u(z_n)) \mid u: Z \rightarrow M \text{ preserves } R\}$;
- (e) $s = \{(c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n)\}$ for some primitive positive formula $\Phi(x_1, \dots, x_n)$ (in the language of the relational structure $(M; R)$).

PROOF. By Theorem 2.3 we have (b) \Leftrightarrow (c). Of course, (c) \Rightarrow (d) is trivial, while (c) \Rightarrow (e) is a weakening of the conclusion of (e) \Rightarrow (f) in Theorem 2.3. That (e) implies (a) is a standard (and easy) argument. We shall prove (d) \Rightarrow (a).

Assume (d), let $c^1, \dots, c^k \in s$, and let $f: M^k \rightarrow M$ be R -preserving. By (d), there exist R -preserving maps $u_1, \dots, u_k: Z \rightarrow M$ with $c^j = (u_j(z_1), \dots, u_j(z_n))$ for $j = 1, \dots, k$. Define $v: Z \rightarrow M$ by $v = f \circ (u_1 \sqcap \cdots \sqcap u_k)$, and note that v is R -preserving. Then

$$\begin{aligned} f(c^1, \dots, c^k) &= (f(u_1(z_1), \dots, u_k(z_1)), \dots, f(u_1(z_n), \dots, u_k(z_n))) \\ &= (v(z_1), \dots, v(z_n)), \end{aligned}$$

whence $f(c^1, \dots, c^k) \in s$ by (d). Thus $s \in \text{Inv}(\text{Pol}(R))$.

Thus it remains to prove that (a) \Rightarrow (b), that is, that every $s \in \text{Inv}(\text{Pol}(R))$ is entailed by R on M^s . Let $s \in \text{Inv}(\text{Pol}(R))$ be n -ary, let $u: M^s \rightarrow M$ preserve R , and assume that $z_1, \dots, z_n \in M^s$ with $(z_1, \dots, z_n) \in s$ on M^s . Let $s = \{s_1, \dots, s_k\}$; then there exists an R -preserving map $f: M^k \rightarrow M$ such that for all $z \in M^s$ we have $u(z) = f(z(s_1), \dots, z(s_k))$. Thus

$$\begin{aligned} (u(z_1), \dots, u(z_n)) &= (f(z_1(s_1), \dots, z_1(s_k)), \dots, f(z_n(s_1), \dots, z_n(s_k))) \\ &= f((z_1(s_1), \dots, z_n(s_1)), \dots, (z_1(s_k), \dots, z_n(s_k))) \\ &= f(t_1, \dots, t_k), \end{aligned}$$

with $t_1, \dots, t_k \in s$ as $(z_1, \dots, z_n) \in s$ on M^s . Hence $(u(z_1), \dots, u(z_n)) \in s$, since $s \in \text{Inv}(\text{Pol}(R))$ and $f \in \text{Pol}(R)$. Consequently R entails s on M^s , as required. \square

§3. Syntax versus semantics. It is a consequence of the test algebra theorem that if s is an algebraic relation derived from the set $G \cup H \cup R$ by means of a primitive positive formula Φ satisfying

- (i) $D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$ and
- (ii) $s = \{(c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n)\}$,

then $(G \cup H \cup R) \vdash s$. If s is determined by a primitive positive formula such that (i) holds as well as (ii), we may say that s is obtained by an *admissible construct*. We have called a construct *concrete* if it is described by a quantifier-free formula Φ . Every such construct is admissible because, whenever Φ is quantifier free,

$$s = \{(c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n)\}$$

implies

$$D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$$

(in fact, Φ exactly describes s on $D(\underline{s})$).

For ease of reference we list together, in three groups, all our constructs, giving the associated formula in selected cases (the others are easily stated too). We shall subsequently show that these constructs suffice, redundantly, to describe any relation entailed by $G \cup H \cup R$.

(I) Concrete constructs for a relational type.

Trivial relations: Given an equivalence relation θ on $\{1, \dots, n\}$, construct the trivial relation $\Delta^\theta := \{(c_1, \dots, c_n) \in M^n \mid i \theta j \Rightarrow c_i = c_j\}$. The associated formula Φ_{Δ^θ} is $\&L_{(i,j) \in \theta} (x_i = x_j)$. For example, the diagonal $\Delta := \{(c, c) \mid c \in M\}$ is just $\Delta^=$.

Subscript manipulation: Given $\varepsilon: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ and an m -ary relation r , construct the n -ary relation $r^\varepsilon := \{(c_1, \dots, c_n) \in M^n \mid (c_{\varepsilon(1)}, \dots, c_{\varepsilon(m)}) \in r\}$, provided r^ε is nonempty.

Suppose we are interested only in relations which are at most binary, so we can restrict to $m, n \in \{1, 2\}$. Then, given a relation r of arity at most 2, there are 6 possibilities other than simply writing down r again. If r is unary, we can construct the binary relations $r \times M$ and $M \times r$. If r is binary, we can construct the unary relation $r^1 := \pi_1(r \cap \Delta)$ and the binary relations $r^1 \times M$, $M \times r^1$, and the converse r^- of r .

Specialising, we have three constructs worthy of explicit mention.

Trivial expansion: The case of subscript manipulation in which $m < n$ and ϵ is one-to-one and order-preserving, for example, the construction of the binary relations $r \times M$ and $M \times r$ from the unary relation r .

Permutation: The case of subscript manipulation in which $m = n$ and ϵ is a permutation, for example, the construction of r^\sim from the binary relation r .

Repetition removal: Given an m -ary relation r and a subscript j such that there exists $i \neq j$ with $c_i = c_j$ whenever $(c_1, \dots, c_m) \in r$, construct

$$r'_j := \{(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_m) \in M^{m-1} \mid (c_1, \dots, c_m) \in r\}.$$

If repetition removal is applied successively on the elements j_1, j_2, \dots, j_q of the set $J := \{j_1, \dots, j_q\}$, starting with the relation r , then we write r'_J for the resulting relation.

Intersection: Given n -ary relations r and s , construct $r \cap s$.

Product: Given an n -ary relation r and an m -ary relation s , construct

$$r \times s := \{(c_1, \dots, c_n, d_1, \dots, d_m) \in M^{n+m} \mid (c_1, \dots, c_n) \in r \text{ and } (d_1, \dots, d_m) \in s\}.$$

The associated formula, $\Phi_{r \times s}(x_1, \dots, x_n, y_1, \dots, y_m)$, is

$$r(x_1, \dots, x_n) \wedge s(y_1, \dots, y_m).$$

(II) Concrete constructs involving partial operations.

Nullary operations: If c is a nullary operation, construct the subalgebra $\{c\}$ of \underline{M} ; the associated formula Φ_c is just the function symbol c .

Domains: From a partial operation $e: \text{dom } e \subseteq \underline{M}^n \rightarrow \underline{M}$ ($n \geq 1$), construct its domain, $\text{dom } e$. The associated formula, $\Phi_{\text{dom } e}(x_1, \dots, x_n)$, is just $e(x_1, \dots, x_n) = e(x_1, \dots, x_n)$. To see that this interprets in the required way, note that $D(\text{dom } e)$ satisfies $e(\rho_1, \dots, \rho_n) = e(\rho_1, \dots, \rho_n)$, as this says simply that

$$(\rho_1(c), \dots, \rho_n(c)) \in \text{dom } e \quad \text{for all } c \in \text{dom } e,$$

which is true since $\rho_1 \cap \dots \cap \rho_n$ is the inclusion map of $\text{dom } e$ into M^n . It is also trivial that $\text{dom } e = \{(c_1, \dots, c_n) \in M^n \mid e(c_1, \dots, c_n) = e(c_1, \dots, c_n)\}$, since $e(c_1, \dots, c_n) = e(c_1, \dots, c_n)$ says just that $e(c_1, \dots, c_n)$ is defined.

Domains are a special case of the next construct.

Equalisers: From partial operations $e_1: \text{dom } e_1 \subseteq \underline{M}^n \rightarrow \underline{M}$ and $e_2: \text{dom } e_2 \subseteq \underline{M}^n \rightarrow \underline{M}$ ($n \geq 1$), construct

$$\text{eq}(e_1, e_2) := \{c \in \text{dom } e_1 \cap \text{dom } e_2 \mid e_1(c_1, \dots, c_n) = e_2(c_1, \dots, c_n)\}.$$

Joint kernels: Given partial operations $e_1: \text{dom } e_1 \subseteq \underline{M}^n \rightarrow \underline{M}$ and $e_2: \text{dom } e_2 \subseteq \underline{M}^m \rightarrow \underline{M}$ ($n, m \geq 1$), construct

$$\begin{aligned} \ker(e_1, e_2) := & \{(c_1, \dots, c_{n+m}) \in \text{dom } e_1 \\ & \times \text{dom } e_2 \mid e_1(c_1, \dots, c_n) = e_2(c_{n+1}, \dots, c_{n+m})\}. \end{aligned}$$

Graphs: From a partial operation $e: \text{dom } e \subseteq \underline{M}^n \rightarrow \underline{M}$ ($n \geq 1$), construct the graph of e :

$$\text{graph}(e) := \{(c_1, \dots, c_n, e(c_1, \dots, c_n)) \in M^{n+1} \mid (c_1, \dots, c_n) \in \text{dom } e\}.$$

Composition: Let $K \subseteq \mathcal{P}$. The *partial clone* generated by K is the smallest subset $[K]$ of \mathcal{P} which satisfies

- (a) for all $n \geq 1$, each projection $\pi_i: M^n \rightarrow M$ belongs to $[K]$;
- (b) if $e \in [K]$ is n -ary and $g_i \in [K]$ is m -ary for $i = 1, \dots, n$ and

$$D := \{ (c_1, \dots, c_m) \in \bigcap_{i=1}^n \text{dom } g_i \mid (g_1(c_1, \dots, c_m), \dots, g_n(c_1, \dots, c_m)) \in \text{dom } e \}$$

is nonempty, then $e \circ (g_1 \sqcap \dots \sqcap g_n): D \rightarrow M$ belongs to $[K]$. If $D \neq \emptyset$, we shall say that $e(g_1, \dots, g_n)$ is defined and equals the map $e \circ (g_1 \sqcap \dots \sqcap g_n)$. Of course, the elements of the partial clone $[G \cup H]$ are exactly the interpretations on their domains of the terms of the language \mathcal{L}_M .

Term manipulation: Given an m -ary relation r and n -ary terms t_1, \dots, t_m , we may consider the formula

$$\Phi_{r(t_1, \dots, t_m)}(x_1, \dots, x_n) = r(t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)).$$

We call the associated construct *term manipulation*. In case the terms t_1, \dots, t_m are merely variables (as must happen when $G \cup H = \emptyset$), term manipulation reduces to subscript manipulation.

Term manipulation also subsumes the next construct.

Action by a partial endomorphism: Given an m -ary relation s and a partial endomorphism e of M , we may construct

$$e \cdot s := \{ (c_1, \dots, c_m) \in M^m \mid (e(c_1), c_2, \dots, c_m) \in s \}.$$

Note that when $m = 1$ we have $e \cdot s := \{ c \in \text{dom } e \mid e(c) \in s \} = e^{-1}(s)$. If s is itself the graph of an endomorphism, then $e \cdot s$ is the composition of e and s *qua* maps, with e done first, that is $s \circ e$.

(III) Admissible nonconcrete constructs. On full powers of M , projection and relational product are allowable constructs. In our more general setting we have to impose restrictions.

Retractive projection: Given an m -ary relation r and an injection $\eta: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ ($n \leq m$) create the relation

$$r_\eta = \{ (c_1, \dots, c_n) \in M^n \mid (\exists d_1 \dots d_m) (d_1, \dots, d_m) \in r \text{ and } c_i = d_{\eta(i)} (1 \leq i \leq n) \}$$

(alternatively denoted $P_{\eta(1), \dots, \eta(n)}(r)$). We say $s := r_\eta$ is a *retractive projection* of r if the natural projection map $p: \underline{r} \rightarrow \underline{s}$ is a retraction, that is, there exists a homomorphism $q: \underline{s} \rightarrow \underline{r}$ such that $p \circ q = \text{id}_s$. It is a *bijjective projection* (as introduced by L. Zadori [14]) if also $q \circ p = \text{id}_r$. A retractive projection derived from an injection of $\{1, \dots, m - 1\}$ into $\{1, \dots, m\}$ is called a *1-step retractive projection* of r .

For notational convenience assume now that η is the map such that $\eta(i) = i$ for $i = 1, \dots, n$, so that we project onto the first n coordinates, in order. We claim that s is a retractive projection of r if and only if $D(\underline{s}) \models \Phi_s$, where $\Phi_s(x_1, \dots, x_n)$

is the formula $(\exists y_{n+1} \dots y_m) r(x_1, \dots, x_n, y_{n+1}, \dots, y_m)$. An analogous statement holds in the case of projection onto an arbitrary set of coordinates.

To prove the claim, first assume that s is a retractive projection of r . Define homomorphisms $\tau_i := \rho_i \circ q$ for $i = n + 1, \dots, m$. Then

$$D(\underline{s}) \models r(\rho_1, \dots, \rho_n, \tau_{n+1}, \dots, \tau_m).$$

For the converse assume there exist homomorphisms $\tau_i : \underline{s} \rightarrow \underline{M}$ ($i = n + 1, \dots, m$) such that $D(\underline{s}) \models r(\rho_1, \dots, \rho_n, \tau_{n+1}, \dots, \tau_m)$. Then the map q required for s to be a retractive projection of r is

$$q := \rho_1 \sqcap \dots \sqcap \rho_n \sqcap \tau_{n+1} \sqcap \dots \sqcap \tau_m.$$

Homomorphic relational product: Given an n -ary relation r and an m -ary relation s , construct the relational product

$$r \cdot s := \{ (c_1, \dots, c_{n+m-2}) \in M^{n+m-2} \mid (\exists c \in M)(c_1, \dots, c_{n-1}, c) \in r \text{ and } (c, c_n, \dots, c_{n+m-2}) \in s \}.$$

We say $r \cdot s$ is a *homomorphic relational product* if there exists an $(n + m - 2)$ -ary homomorphism $f : r \cdot s \rightarrow \underline{M}$ such that $(c_1, \dots, c_{n-1}, f(c_1, \dots, c_{n+m-2})) \in r$ and $(f(c_1, \dots, c_{n+m-2}), c_n, \dots, c_{n+m-2}) \in s$. Observe that $t := r \cdot s$ is a homomorphic relational product precisely when

$$D(\underline{t}) \models (\exists \tau)(r(\rho_1, \dots, \rho_{n-1}, \tau) \wedge s(\tau, \rho_n, \dots, \rho_{n+m-2})).$$

Assume that $e : \text{dom } e \rightarrow \underline{M}$ is a partial endomorphism and s an m -ary relation. When we identify e with its graph (a binary relation), $e \cdot s$ as a relational product coincides with $e \cdot s$ as the action of a partial endomorphism on a relation. Further, $t := e \cdot s$ is a homomorphic relational product. To see this, note that $\tau := e \circ \rho_1 \in D(\underline{t})$, so that

$$D(\underline{t}) \models (\exists \tau)(e(\rho_1, \tau) \wedge s(\tau, \rho_2, \dots, \rho_m)),$$

as this is just a restatement of the fact that

$$D(\underline{t}) \models s(e(\rho_1), \rho_2, \dots, \rho_m).$$

It is convenient to have all the above constructs available, but a more restricted list will suffice. We record the following proposition, parts of which are well known. At the end of the section we consider the relationship between retractive projection and homomorphic relational product.

PROPOSITION 3.1. (a) *Products can be obtained from trivial expansions and intersections.*

(b) *Trivial expansions can be obtained from trivial relations, repetition removal, intersections and products.*

(c) *Permutations can be obtained from trivial relations, repetition removal, intersections and products.*

(d) *Subscript manipulation can be obtained from trivial relations, repetition removal, intersections and products.*

(In (b)–(d), *trivial relations and intersections can be replaced by the single unary construct “intersection with a trivial relation”*).

PROOF. For (a) we use the fact that $r \times s = (r \times M^m) \cap (M^n \times s)$, for an n -ary relation r and an m -ary relation s .

Now let r be n -ary and let ε be the transposition $(i j)$, where $i, j \in \{1, \dots, n\}$. Define θ to be the equivalence relation on $\{1, \dots, 2n\}$ for which $k \theta (k + n)$ for $k \notin \{i, j\}$, $i \theta (j + n)$ and $j \theta (i + n)$. Then r^ε can be obtained from $(r \times r) \cap \Delta^\theta$ by repeated applications of repetition removal. Part (c) follows from this. The proof of (b) is similar, but simpler.

To prove (d) we use the fact that an arbitrary map $\varepsilon: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ factor through $\{1, \dots, m\} / \ker \varepsilon$. If ε is injective, then r^ε is obtained from r by permutation and trivial expansion. If ε is surjective, then r^ε can be obtained from r by permutation, intersection with trivial relations, and repetition removal. The required result now follows from (b) and (c). \square

PROPOSITION 3.2. *Assume that $s \in \mathcal{B}$ can be obtained from $R \subseteq \mathcal{B}$ by concrete constructs. Then s can be obtained from R by a finite number of applications of product, intersection, trivial relations, and repetition removal.*

More generally, assume that s can be obtained from $G \cup H \cup R$ by concrete constructs. Then s can be obtained from $G \cup H \cup R$ by a finite number of applications of product, intersection, trivial relations, repetition removal, and the constructs listed under (II) above.

PROOF. We shall show, using a syntactic description, how s can be derived from R (or, more generally, from $G \cup H \cup R$).

Assume first that $G \cup H = \emptyset$. Let the atomic formula Φ which determines s be

$$\Phi = s_l(x_{i_{l1}}, \dots, x_{i_{lm}}) \wedge \dots \wedge s_k(x_{i_{k1}}, \dots, x_{i_{kn}}).$$

Then $s = \bigcap_{j=1}^k s_j^{\sigma_j}$, where $\sigma_j: \{1, \dots, n_j\} \rightarrow \{1, \dots, n\}$ is given by $\sigma_j(l) := i_{jl}$. Thus we get s from R via subscript manipulation applied to relations in R and intersection. We may now appeal to Proposition 3.1(d).

In the general case, atomic formulæ not involving $=$ take the form $\Phi_{r(t_1, \dots, t_m)}$, while the remaining ones are simply of the form $t_1 = t_2$. The corresponding relations are obtained using, respectively, the constructs in (II) above. \square

We wish to show that our listed constructs suffice to build any relation s entailed by $G \cup H \cup R$. We do this in two stages. First, we show that the second dual $ED(\underline{s})$ can be concretely constructed from $G \cup H \cup R$, whether or not $G \cup H \cup R$ entails s . Second, we show that if $G \cup H \cup R$ entails s , then s can be obtained from $ED(\underline{s})$ by a retractive projection, which is a bijective projection in the case when $G \cup H \cup R$ yields a duality on \underline{s} .

Take G, H and R as before and let $Z = \{z_1, \dots, z_k\}$ be a finite substructure of M^T , for some nonempty set T . By the *graph of $E(Z)$* (with respect to $G \cup H \cup R$) we shall mean the relation

$$G[E(Z)] := \{(u(z_1), \dots, u(z_k)) \in M^k \mid u: Z \rightarrow M \text{ preserves } G \cup H \cup R\}.$$

Thus the graph of $E(Z)$ is simply $E(Z)$, given a fixed labelling of Z .

LEMMA 3.3. *Let Z be a finite hom-closed subset of M^T for some nonempty set T . Then the relation $G[E(Z)]$ can be concretely constructed from $G \cup H \cup R$.*

PROOF. As above, write $Z = \{z_1, \dots, z_k\}$. Then

$$G[E(Z)] = \{(u(z_1), \dots, u(z_k)) \mid u: Z \rightarrow M \text{ preserves } G \cup H \cup R\}.$$

This set equals

$$\{(c_1, \dots, c_k) \in M^k \mid M \models \Psi(c_1, \dots, c_k)\},$$

where Ψ is the conjunct of all atomic formulæ true in Z (given the above labelling of the elements of Z). Since Ψ is quantifier free, the proof is complete. \square

Now suppose $s \in \mathcal{B}$ is n -ary. Let $Z = D(\underline{s})$ and enumerate its elements as $\{\rho_1, \dots, \rho_n, \tau_1, \dots, \tau_m\}$. Let

$$G[\underline{s}] := \{(\rho_1(a), \dots, \rho_n(a), \tau_1(a), \dots, \tau_m(a)) \in M^{n+m} \mid a \in s\}$$

encode the evaluation maps from $D(\underline{s})$ to M . Of course, $G[\underline{s}]$ is in bijective correspondence with s itself. If $G \cup H \cup R$ yields a duality on \underline{s} , then $G[ED(\underline{s})]$ coincides with $G[\underline{s}]$. In general the relation $G[ED(\underline{s})] \setminus G[\underline{s}]$ can be thought of as a measure of how far $G \cup H \cup R$ is from yielding a duality on $D(\underline{s})$.

LEMMA 3.4. *Let $\underline{s} \leq \underline{M}^n$, and let $G \cup H \cup R$ entail s . Then s is a retractive projection of the graph $G[ED(\underline{s})]$ of $ED(\underline{s})$.*

PROOF. By the test algebra theorem, s is the retractive projection of $G[ED(\underline{s})]$ onto its first n coordinates, where $D(\underline{s})$ is labelled as above. \square

We deduce a number of consequences. The first is immediate from Lemma 3.3, applied with $T = s$ and $Z = D(\underline{s})$, and Lemma 3.4.

THEOREM 3.5 (The Semantic Entailment Theorem). *Let $s \in \mathcal{B}$, and let $(G \cup H \cup R) \vdash s$. Then s can be obtained from $G \cup H \cup R$ by a finite number of applications of the constructs listed in Proposition 3.2, followed by one application of retractive projection. In case $G \cup H = \emptyset$, product, intersection, trivial relations, repetition removal and retractive projection suffice.*

If a subset R of \mathcal{B} is such that $R \vdash s$ for every $s \in \mathcal{B}$, then we say that R is *entailment-dense* in \mathcal{B} .

THEOREM 3.6 (cf. Zadori [14], Theorem 3.1). *Let $R \subseteq \mathcal{B}$ and let $s \in \mathcal{B}$.*

- (a) *If R yields a duality on \underline{s} , then s can be constructed from R by a finite number of applications of product, intersection, trivial relations, repetition removal and bijective projection.*
- (b) *The following are equivalent:*
 - (i) *R yields a duality on every finite algebra in \mathcal{A} ;*
 - (ii) *R is entailment-dense in \mathcal{B} ;*
 - (iii) *every relation $s \in \mathcal{B}$ can be constructed from R by a finite number of applications of product, intersection, trivial relations, repetition removal and bijective projection.*

PROOF. By Proposition 3.2 and Lemma 3.3, the graph, $G[ED(\underline{s})]$, of $ED(\underline{s})$ with respect to R can be constructed from R via a finite number of applications of product, intersection, trivial relations, and repetition removal. If R yields a duality on \underline{s} , we have that $G[ED(\underline{s})]$ (identified with the algebra $ED(\underline{s})$) is isomorphic to

\underline{s} , and hence the retractive projection which yields s from $G[ED(\underline{s})]$ is a bijective projection. This proves (a).

Now consider (b). The implication (i) \Rightarrow (iii) follows at once from (a) while (iii) \Rightarrow (ii) is trivial since all of the listed constructs are admissible. That (ii) implies (i) is simply a reformulation of the brute force duality theorem: see 1.15 in [10] or 2.7 in [5]. Note that the implication (i) \Rightarrow (ii) also follows from Proposition 2.2 of [8], which states that if $s \in \mathcal{B}$ and R yields a duality on \underline{s} , then $R \vdash s$. \square

In the context of this result we should note that L. Zadori ([14], Theorem 3.4) and, independently, R. Willard (private communication) have proved the important compactness result that if $G \cup H \cup R$ is finite and is entailment-dense in \mathcal{B} , then $G \cup H \cup R$ yields a duality on \mathcal{A} .

Specialising in the same way as we did to derive Theorem 2.4, we obtain from Theorem 3.5 the following well-known result for clones (see, for example, [1], [13]).

THEOREM 3.7. *Let R be a set of finitary relations on a set M . Then every element of $\text{Inv}(\text{Pol}(R))$ can be obtained from R by a finite number of applications of product, intersection, trivial relations, repetition removal, and projection.*

The characterisation of $\text{Inv}(\text{Pol}(R))$ originally obtained by Bodnarčuk et al. [1] used a semantic approach. The same matrix approach in our more general setting yields, in an algorithmic way, a result equivalent to Theorem 3.5. We include the proof since the clone theorists' procedure is not easily accessible. For simplicity, we restrict ourselves to purely relational structures.

We fix a finite nonempty set $T = \{b_1, \dots, b_k\}$. Let $Z = \{z_1, \dots, z_m\}$ be a substructure of M^T and let u_1, \dots, u_ℓ be an enumeration of all maps $u: Z \rightarrow M$ preserving R . By the *matrix form of $G[E(Z)]$* we shall mean the matrix $(P \mid Q)$ defined in the following way:

(i) $P = (p_{ij})$ is the $m \times k$ matrix in which

$$p_{ij} = z_i(b_j), \quad i = 1, \dots, m, \quad j = 1, \dots, k;$$

(ii) $Q = (q_{ij})$ is the $m \times l$ matrix in which

$$q_{ij} = u_j(z_i), \quad i = 1, \dots, m, \quad j = 1, \dots, l.$$

Thus the left-hand block P serves to encode, by rows, the elements of Z as elements of M^k (where $k = |T|$). The right-hand block Q encodes the relation $G[E(Z)]$, the columns of Q being the m -tuples in this relation and each of its rows representing the values of the maps u_j on the element given by the corresponding row in P . See Figure 1 (next page). Note also that if $Z = D(\underline{s})$ and R yields a duality on \underline{s} , then, up to a permutation of the columns, we have $P = Q$.

PROPOSITION 3.8. *Let $R \subseteq \mathcal{B}$ and let Z be a substructure of M^T for some nonempty set T . Then the graph $G[E(Z)]$ of $E(Z)$ (with respect to R) can be constructed algorithmically from the relations in R by a finite number of applications of product, intersection, trivial relations, and repetition removal.*

PROOF. Proof By the test algebra theorem we may assume without loss of generality that R is finite. Let $R = \{r^1, \dots, r^p\}$, where r^t is an n_t -ary relation, for $t = 1, \dots, p$. We construct the matrix $(P \mid Q)$, defined as above, by obtaining successive approximations to it. Our initial approximation is the matrix form

is satisfied; that is, when $(P^{[1]} | Q^{[1]})$ is such that

$$(MCC) \quad p_{is}^{[1]} = p_{js}^{[1]}, \quad s = 1, \dots, k \implies q_{is} = q_{js}, \quad s = 1, \dots, \ell.$$

Let θ be the equivalence relation on the set $\{1, \dots, N\}$ such that $i \theta j \leftrightarrow w_i = w_j$, and take the trivial relation

$$\Delta^\theta = \{(c_1, \dots, c_N) \in M^N \mid i \theta j \implies c_i = c_j\}.$$

Put $G^{[2]} := G^{[1]} \cap \Delta^\theta$ as the second approximation to $G[E(Z)]$.

Step 3. We shall remove from $P^{[2]}$ all redundant rows, so keeping just one copy of each element of Z as a row. We remove the corresponding rows from $Q^{[2]}$. Let I be a transversal of the equivalence classes of the congruence θ from Step 2. We then repeatedly apply repetition removal to $G^{[2]}$ to get the $|I|$ -ary relation

$$G^{[3]} := [((r^1)^{m_1} \times \dots \times (r^p)^{m_p}) \cap \Delta^\theta]_{I'}^J,$$

where $J := \{1, \dots, N\} \setminus I$. We then take $(P^{[3]} | Q^{[3]})$ to be the matrix form for $G^{[3]}$.

Step 4. Let $y_{|I|+1}, \dots, y_m$ be those elements of Z which did not participate in any n_t -tuple $(z_1, \dots, z_{i_{n_t}}) \in (r^t)_Z$ for some r^t . Add to the matrix $P^{[3]}$ a set $m - |I|$ rows corresponding to these elements. Since we have to construct all possible maps $u : Z \rightarrow M$ preserving R , we must assign all possible values to the corresponding rows of the right-hand block. Hence we take $G^{[4]} := G^{[3]} \times M^{m-|I|}$. By construction,

$$G^{[4]} = \{(u(z_1), \dots, u(z_{i_m})) \in M^m \mid u : Z \rightarrow M \text{ preserves } R\},$$

where $(i_1, \dots, i_m) = (\sigma(1), \dots, \sigma(m))$ is some permutation of $(1, \dots, m)$. So we have reached our goal to within a permutation.

Step 5. Put

$$G^{[5]} := G^{[4]^\sigma} = \{(u(z_1), \dots, u(z_m)) \in M^m \mid (u(z_{i_1}), \dots, u(z_{i_m})) \in G^{[4]}\}.$$

Then, clearly, $G^{[5]} = G[E(Z)]$. □

In the final section we shall consider an example in which concrete constructs do not suffice. However, in this instance adding homomorphic relational product gives a sufficient set of constructs. We are led to investigate homomorphic relational product and its relationship to retractive projection.

Let $\underline{r} \leq \underline{M}^n$, and $\underline{s} \leq \underline{M}^m$ be algebraic relations on \underline{M} , and let $t = r \cdot s$ be the homomorphic relational product of r and s . Since t is entailed by $\{r, s\}$, it can be built from $\{r, s\}$ using concrete constructs and retractive projection. We show explicitly how this can be done.

PROPOSITION 3.9. *A homomorphic relational product can be obtained from a finite number of applications of trivial relations, repetition removal, intersection, product, and 1-step retractive projection.*

PROOF. Let η be the natural embedding of $\{1, \dots, n-1, n+2, \dots, n+m\}$ into $\{1, \dots, n+m\}$. It is clear that $t = ((r \times s) \cap \Delta^\theta)_\eta$, where θ is the equivalence

relation on $\{1, \dots, n + m\}$ having $\{n, n + 1\}$ as its only nontrivial class. Since $t = r \cdot s$ is a homomorphic relational product, there exists $w \in D(\underline{t})$ such that

$$(\rho_1, \dots, \rho_{n-1}, w) \in r_{D(\underline{t})} \quad \text{and} \quad (w, \rho_n, \dots, \rho_{n+m-2}) \in s_{D(\underline{t})},$$

whence

$$(\rho_1, \dots, \rho_{n-1}, w, w, \rho_n, \dots, \rho_{n+m-2}) \in [(r \times s) \cap \Delta^\theta]_{D(\underline{t})}.$$

So there exists $w \in D(\underline{t})$ such that

$$(\rho_1, \dots, \rho_{n+m-2}, w, w) \in [(r \times s) \cap \Delta^\theta]^\sigma_{D(\underline{t})},$$

where $[(r \times s) \cap \Delta^\theta]^\sigma$ is a permutation of $(r \times s) \cap \Delta^\theta$. Hence t is obtained from $[(r \times s) \cap \Delta^\theta]^\sigma$ by repetition removal followed by a 1-step retractive projection. \square

Conversely, we can show that every 1-step retractive projection can be obtained using concrete constructs plus homomorphic relational product. In §6 we give an example which supports the conjecture that not every retractive projection can be so obtained.

PROPOSITION 3.10. *Let $\underline{r} \leq \underline{M}^n$, and let $s = r_\eta$ be a 1-step retractive projection of r . Then s can be obtained from $\{r\}$ by a finite number of applications of trivial relations, repetition removal, intersection, product, and homomorphic relational product.*

PROOF. Since permutation can always be obtained using products, intersection with trivial relations, and repetition removal, we can assume that $s = r_\eta$, where η is the natural embedding of $\{1, \dots, n - 1\}$ into $\{1, \dots, n\}$. That is, s is obtained from r by cancelling the n th component. Let $p: \underline{r} \rightarrow \underline{s}$ be the natural projection map. Let

$$r^\sigma := \{(c_1, \dots, c_n) \in \underline{M}^n \mid (c_n, \dots, c_1) \in r\}$$

be obtained by permutation from r . Then

$$s = r_\eta = [(r \cdot r^\sigma) \cap \Delta^\theta]^\nu_J,$$

where θ is the equivalence relation on $\{1, \dots, 2n - 2\}$ with classes $\{1, 2n - 2\}$, $\{2, 2n - 3\}, \dots, \{n - 1, n\}$ and $J := \{n, n + 1, \dots, 2n - 2\}$ is the set of coordinates deleted by applying repetition removal $(n - 1)$ times. We aim to show that $t := r \cdot r^\sigma$ is a homomorphic relational product; that is, that there exists $z \in D(\underline{t})$ such that

$$(\rho_1, \dots, \rho_{n-1}, z) \in r_{D(\underline{t})} \quad \text{and} \quad (z, \rho_n, \dots, \rho_{2n-2}) \in (r^\sigma)_{D(\underline{t})}.$$

Equivalently, we must find $z \in D(\underline{t})$ such that

$$(\forall a = (a_1, \dots, a_{2n-2}) \in t)((a_1, \dots, a_{n-1}, z(a)) \in r \wedge (z(a), a_n, \dots, a_{2n-2}) \in r^\sigma).$$

Because s is a retractive projection of r , there exists a homomorphism $q: \underline{s} \rightarrow \underline{r}$ such that $p \circ q = \text{id}_s$ and $(\rho_1, \dots, \rho_{n-1}, \rho_n \circ q) \in r_{D(\underline{s})}$. Therefore, if $p': \underline{t} \rightarrow \underline{s}$ is the natural projection map and $z := \rho_n \circ q \circ p'$, then $z \in D(\underline{t})$, and for any $a = (a_1, \dots, a_{2n-2}) \in t$ we have

$$z(a) = (\rho_n \circ q)((a_1, \dots, a_{n-1})),$$

whence $(a_1, \dots, a_{n-1}, z(a)) \in r$. It remains to show that $(z(a), a_n, \dots, a_{2n-2}) \in r^\sigma$.

Let $p'' : \underline{t} \rightarrow \underline{s}$ be the homomorphism given by $p''(a_1, \dots, a_{2n-2}) = (a_{2n-2}, \dots, a_n)$. Now, for any $a = (a_1, \dots, a_{2n-2}) \in t$, we have

$$(q \circ p'')(a) = q(a_{2n-2}, \dots, a_n) = (a_{2n-2}, \dots, a_n, z(a)) \in r,$$

whence $(z(a), a_n, \dots, a_{2n-2}) \in r^\sigma$, as required. □

§4. Concrete entailment. In this section we seek conditions under which concrete constructs alone suffice to describe entailment closure, and in such circumstances to show quite explicitly how entailment operates.

We recall that s is said to be obtained from $G \cup H \cup R$ by a concrete construction if there exists Φ , which is a conjunct of atomic formulæ in the language $\mathcal{L}_{\underline{M}}$, such that

$$s = \{ (c_1, \dots, c_n) \in M^n \mid M \models \Phi(c_1, \dots, c_n) \}.$$

If we elect to work with purely relational structures, we substitute \underline{M}^\bullet (as defined in §1) for $\underline{M} = (M; G, H, R, \tau)$ and encompass partial endomorphisms by replacing them by their graphs. With $\mathcal{L}_{\underline{M}^\bullet}$ as our language, the relational product of a partial endomorphism e (*qua* relation) and a relation r is a nonconcrete construct. However if we allow operations *per se*, then the relational product of e and r can be built using an atomic formula in the appropriate language, $\mathcal{L}_{\underline{M}}$.

The next proposition gives a sufficient condition for a relation s to be constructed from $G \cup H \cup R$ by concrete constructs only. We recall that the definition of the partial clone $[K]$ generated by a subset K of \mathcal{P} was given in the preceding section. The partial clone is partially ordered by defining $e_1 \leq e_2$ in $[K]$ if and only if e_1 and e_2 are of the same arity and $\text{graph}(e_1) \subseteq \text{graph}(e_2)$.

PROPOSITION 4.1. *Let $G \cup H \subseteq \mathcal{P}$, $R \subseteq \mathcal{B}$, and $s \in \mathcal{B}$. Assume that $(G \cup H \cup R) \vdash s$. If $D(\underline{s}) \subseteq [G \cup H]$, then s can be constructed concretely from $G \cup H \cup R$. If the maximal elements of \mathcal{P} lie in $[G \cup H]$, then any s entailed by $G \cup H \cup R$ can be concretely constructed from $G \cup H \cup R$.*

PROOF. Because $D(\underline{s}) \subseteq [G \cup H]$, we can express each element of $D(\underline{s})$ as the interpretation of a term in $\mathcal{L}_{\underline{M}}$. Consequently any formula describing s on $D(\underline{s})$ is given by a quantifier-free formula. □

There are important special cases in which the preceding proposition applies. Note first that $D(\underline{s})$ is contained in the partial clone generated by the partial endomorphisms of \underline{M} whenever the following condition holds for $\underline{s} \leq \underline{M}^n$.

- (H) Every homomorphism from \underline{s} to \underline{M} is of the form $e \circ \rho_i$ for some $i \in \{1, \dots, n\}$, where e is a partial endomorphism of \underline{M} and $\text{im } \rho_i \subseteq \text{dom } e$.

There is one important class of examples for which \underline{s} satisfies (H) for every s in \mathcal{B} .

LEMMA 4.2. *Assume that \underline{M} generates a congruence-distributive variety and that every subalgebra of \underline{M} is subdirectly irreducible. Then condition (H) holds for every subalgebra of \underline{M}^n .*

PROOF. Let $\underline{s} \leq \underline{M}^n$ and let $x : \underline{s} \rightarrow \underline{M}$ be a homomorphism. Since $\text{im } x$ is subdirectly irreducible, the standard Pixley-Jónsson argument ([11], Théorem 2.5,

[12]) says that $\ker \rho_i \leq \ker x$ for some i . Hence there is a partial endomorphism $e: \text{im } \rho_i \rightarrow \underline{M}$ such that $x = e \circ \rho_i$, as required. \square

Every congruence of a finite algebra \underline{N} is a meet of meet-irreducible congruences on \underline{N} , and the number $\text{irr}(\underline{N})$, defined to be the least p such that the zero congruence $\mathbf{0}$ on \underline{N} is the meet of p meet-irreducible congruences, gives a measure of how far \underline{N} is from being subdirectly irreducible. The *irreducibility index* of \underline{M} is then defined to be

$$\text{Irr}(\underline{M}) := \max\{\text{irr}(\underline{N}) \mid \underline{N} \leq \underline{M}\}.$$

(Note that the irreducibility index of \underline{M} is denoted by $Z(\underline{M})$ in [2], where it was introduced.)

Lemma 4.2 asserts that if the variety generated by \underline{M} is congruence-distributive and such that $\text{Irr}(\underline{M}) = 1$, then every subalgebra of \underline{M}^n satisfies (H) for all n . We have a partial converse.

PROPOSITION 4.3. *If every subalgebra of \underline{M}^2 satisfies (H), then $\text{Irr}(\underline{M}) = 1$.*

PROOF. Let $\underline{N} \leq \underline{M}$, and let $\theta_1, \theta_2 \in \text{Con } \underline{N}$ with $\theta_1 \wedge \theta_2 = \mathbf{0}$. By assumption, the relational product $\theta_1 \cdot \theta_2$ satisfies (H). Since $\theta_1 \cap \theta_2 = \mathbf{0}$, for all $(a, b) \in \theta_1 \cdot \theta_2$ there is a unique $c \in \underline{N}$ such that $a \theta_1 c \theta_2 b$. Thus we may define $h: \theta_1 \cdot \theta_2 \rightarrow \underline{M}$ by $h(a, b) := c$. Since $\theta_1 \cdot \theta_2$ satisfies (H), without loss of generality we may assume that h depends only on the first coordinate. Now

$$\begin{aligned} (a, b) \in \theta_1 &\implies a \theta_1 a \theta_2 a \text{ and } a \theta_1 b \theta_2 b \\ &\implies h(a, a) = a \text{ and } h(a, b) = b \\ &\implies a = b. \end{aligned}$$

Thus $\theta_1 = \mathbf{0}$, whence \underline{N} is subdirectly irreducible. Since every subalgebra of \underline{M} is subdirectly irreducible, we have $\text{Irr}(\underline{M}) = 1$. \square

In [2], $\text{Irr}(\underline{M})$ was introduced in order to allow the authors to derive theorems on strong dualities; see §5 below. For our purposes, it is possible to extract from the proof of Lemma 4.5 of [2] a result which is not stated explicitly there and which generalises Lemma 4.2.

LEMMA 4.4. *Assume that \underline{M} generates a congruence-distributive variety and that $\text{Irr}(\underline{M}) = k$. Then for $s \in \mathcal{B}$ the following condition holds for \underline{s} :*

(H_k) *any homomorphism $f: \underline{s} \rightarrow \underline{M}$ belongs to the partial clone generated by the homomorphisms $g: D \rightarrow \underline{M}$, where $D \in \bigcup_{m=1}^k \mathcal{S}(\underline{M}^m)$.*

We deduce the following corollary of Proposition 4.1.

COROLLARY 4.5. *Assume that \underline{M} generates a congruence-distributive variety and that $\text{Irr}(\underline{M}) = k$. Assume that $s \in \mathcal{B}$, and suppose that $(\mathcal{P}_k \cup R) \vdash s$, where \mathcal{P}_k consists of the partial operations in \mathcal{P} of arity not greater than k . Then s can be concretely constructed from $\mathcal{P}_k \cup R$.*

Of course, this result has something nontrivial to say only when the arity of s is greater than k . Indeed, if $\underline{s} \leq \underline{M}^n$ for some $n \leq k$, then the projection $\rho_1: s \rightarrow \underline{M}$ belongs to \mathcal{P}_k and $s = \text{dom } \rho_1$.

We next reveal how the syntactic solution to the entailment problem provided by the test algebra theorem translates into an explicit semantic solution when concrete constructs suffice. At the end of this section we take a purely semantic route to the same solution.

Because it is notationally simplest, and most important in practice, we consider principally the case in which condition (H) holds.

Note that we may regard Theorem 4.6 as giving an explicit description of a relation entailed by a set R in terms of the action of partial endomorphisms, permutation of subscripts and m -to- n subscript manipulation of relations in R .

THEOREM 4.6 (The Concrete Entailment Theorem). *Assume that $s \in \mathcal{B}$.*

(a) *Assume that \underline{s} satisfies (H) and $(G \cup H \cup R) \vdash s$. Then s is a finite intersection of relations of the following forms:*

(D) *the trivial expansion of the domain of a nonextendable partial endomorphism of \underline{M} ,*

(E) *the trivial expansion of the equaliser of two nonextendable partial endomorphisms of \underline{M} ,*

(K) *the trivial expansion of the joint kernel of two nonextendable partial endomorphisms of \underline{M} ,*

(T) *a relation t , where*

$$t = ((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon \\ = \{(c_1, \dots, c_n) \in (\text{dom } e_1 \times \dots \times \text{dom } e_m)^\varepsilon \mid (e_1(c_{\varepsilon(1)}), \dots, e_m(c_{\varepsilon(m)})) \in r\},$$

for some m -ary relation $r \in R$, map $\varepsilon: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, and nonextendable partial endomorphisms e_1, \dots, e_m of \underline{M} .

Assume that $[G \cup H]$ contains the set of all nonextendable partial endomorphisms of \underline{M} . Then (D), (E), (K) and (T) are concrete admissible constructs.

Assume that $[G \cup H]$ contains the set of all nonextendable partial endomorphisms of \underline{M} and that \underline{s} satisfies (H). Then $(G \cup H \cup R) \vdash s$ if and only if s can be obtained from $G \cup H \cup R$ via intersection and the concrete constructs (D), (E), (K) and (T).

PROOF. By condition (H), every element of $D(\underline{s})$ is of the form $e \circ \rho_i$, where e is a nonextendable partial endomorphism and $\text{im } \rho_i \subseteq \text{dom } e$. From the test algebra theorem and Proposition 4.1, we have a formula $\Phi(x_1, \dots, x_n)$ in the language $\mathcal{L}_{\underline{M}}$, which is a conjunct of atomic formulæ, such that $D(\underline{s}) \models \Phi(\rho_1, \dots, \rho_n)$ and $s = \{(c_1, \dots, c_n) \mid M \models \Phi(c_1, \dots, c_n)\}$. In $D(\underline{s})$, atomic formulæ may take the following forms:

- (a) $e \circ \rho_i = e \circ \rho_i$, where e is a nonextendable partial endomorphism,
- (b) $e \circ \rho_i = e \circ \rho_j$, where e is a nonextendable partial endomorphism and $i \neq j$,
- (c) $e \circ \rho_i = f \circ \rho_j$, where e, f are nonextendable partial endomorphisms,
- (d) $r(e_1 \circ \rho_{\varepsilon(1)}, \dots, e_m \circ \rho_{\varepsilon(m)})$, where r is an m -ary relation, e_1, \dots, e_m are nonextendable partial endomorphisms, and ε is a map from $\{1, \dots, m\}$ to $\{1, \dots, n\}$.

When interpreted in M , formulæ of types (a)–(d) yield relations of types (D)–(T) in the statement of the theorem, so (a) follows immediately from the test algebra theorem.

Part (b) is trivial, and (c) follows immediately from (a) and (b). □

We may elaborate on the form of the relations of type (T) obtained by term manipulation. Let σ be the cycle $(m \dots 2 1)$. It is easily seen that

$$e \cdot r = (e \times \text{id}_M \times \cdots \times \text{id}_M)^{-1}(r)$$

and that

$$(e_1 \times \cdots \times e_m)^{-1}(r) = (e_m \cdots (e_2 \cdot (e_1 \cdot r)^\sigma)^\sigma \cdots)^\sigma.$$

Thus $(e_1 \times \cdots \times e_m)^{-1}(r)$ can be constructed from r via the action of the partial endomorphisms e_1, \dots, e_m together with a permutation of subscripts, and consequently $((e_1 \times \cdots \times e_m)^{-1}(r))^\varepsilon$ can be obtained from r via the action of the partial endomorphisms e_1, \dots, e_m along with subscript manipulation.

In very many important examples \underline{M} has an underlying lattice structure and generates a quasivariety for which the NU duality theorem ([10], 1.19; [5], 2.8) supplies a duality employing relations which are at most binary. Accordingly, we record a specialisation of the concrete entailment theorem appropriate to this situation.

THEOREM 4.7. *Let $s \in \mathcal{S}(\underline{M}) \cup \mathcal{S}(\underline{M}^2)$ satisfy (H), let $R \subseteq \mathcal{S}(\underline{M}) \cup \mathcal{S}(\underline{M}^2)$, and assume that $[G \cup H]$ contains the nonextendable partial endomorphisms of \underline{M} . Assume that $G \cup H \cup R \vdash s$.*

Assume s is unary. Then s is an intersection of relations of the forms

- (a) $\text{dom } e$, for some nonextendable partial endomorphism e of \underline{M} ,
- (b) $\text{eq}(e, f)$, for some nonextendable partial endomorphisms e and f of \underline{M} ,
- (c) $e^{-1}(r)$, for some nonextendable partial endomorphism e of \underline{M} and some unary relation $r \in R$,
- (d) $(e \boxtimes f)^{-1}(r)$, for some nonextendable partial endomorphisms e and f of \underline{M} and some binary relation $r \in R$.

In (d),

$$(e \boxtimes f)^{-1}(r) := \{c \in M \mid c \in \text{dom } e \cap \text{dom } f \text{ and } (e(c), f(c)) \in r\}.$$

Assume s is binary. Then s is an intersection of relations of the forms

- (e) $r \times M$ or $M \times r$, where r is a unary relation of one of the types described in (a)–(d) above,
- (f) $\text{ker}(e, f)$, for some nonextendable partial endomorphisms e and f of \underline{M} ,
- (g) $(e \times f)^{-1}(r)$ or $(e \times f)^{-1}(r)$, for some nonextendable partial endomorphisms e and f of \underline{M} and some binary relation $r \in R$.

PROOF. Assume s is unary. Conditions (D) and (E) of the concrete entailment theorem correspond to parts (a) and (b) in the above statement. Condition (K) does not apply since s is unary, while condition (T) yields (c) and (d) above (note that if r is binary and $\varepsilon: \{1, 2\} \rightarrow \{1\}$, then $((e_1 \times e_2)^{-1}(r))^\varepsilon = (e_1 \boxtimes e_2)^{-1}(r)$).

Now assume s is binary. We need to analyse relations of the types described in (D)–(T) of the concrete entailment theorem in the special case when s is binary and r is at most binary. Relations of types (D), (E) or (K) are covered by (e) and (f). Thus we need to analyse $((e_1 \times e_2)^{-1}(r))^\varepsilon$, where $r \in R$ is binary and $\varepsilon: \{1, 2\} \rightarrow \{1, 2\}$, and to analyse $e_1^{-1}(r)^\varepsilon$, where r is unary and $\varepsilon: \{1\} \rightarrow \{1, 2\}$. The latter case yields $e_1^{-1}(r) \times M$ and its converse, which are covered by (e). Consider the former case. If $\varepsilon = \text{id}_{\{1,2\}}$ or $\varepsilon = (1\ 2)$, then (g) applies. If ε is a constant map, with constant value 1, say, then

$$\begin{aligned} ((e_1 \times e_2)^{-1}(r))^\varepsilon &= \{ (c_1, c_2) \in (\text{dom } e_1 \times \text{dom } e_2)^\varepsilon \mid (e_1(c_1), e_2(c_1)) \in r \} \\ &= \{ (c_1, c_2) \in (\text{dom } e_1 \cap \text{dom } e_2) \times M \mid (e_1(c_1), e_2(c_2)) \in r \} \\ &= (e_1 \boxtimes e_2)^{-1}(r) \times M. \end{aligned}$$

Thus the cases where ε is a constant map are covered by (e). □

It is instructive to consider an alternative approach to concrete entailment, based on the theory in [9]. We need to recall one definition. Let $s \in \mathcal{B}$ and let $u: D(\underline{s}) \rightarrow M$ be any map. Define

$$U = \text{Fail}_{\underline{s}}(u) := \{ r \in \mathcal{B} \mid u \text{ fails to preserve } r \};$$

such a set is called a *failset (of s)* if it contains s . If u is an evaluation map, say $u = e_{\underline{s}}(c)$ for some $c \in \underline{s}$, then the set U is empty. In general, U gives a measure of how far u is from being an evaluation map. A detailed discussion of failsets can be found in §3 of [9].

The technical Lemma 4.8 below is proved in [9] (Lemma 6.7); it is used there to prove the failset theorem, a key result of [9].

LEMMA 4.8. *Let $s \in \mathcal{B}$ be n -ary and assume that \underline{s} satisfies (H). Let s^* be the intersection of all n -ary algebraic relations r such that $s \subseteq r$ and r has one of the following forms:*

- (D) r is a trivial expansion of the domain of a nonextendable partial endomorphism,
- (E) r is a trivial expansion of the equaliser $\text{eq}(e, f)$ of two nonextendable partial endomorphisms e and f of \underline{M} ,
- (K) r is a trivial expansion of the joint kernel $\text{ker}(e, f)$ of two nonextendable partial endomorphisms e and f of \underline{M} .

Choose $(a_1, \dots, a_n) \in \underline{M}^n$ and “define” $u: D(\underline{s}) \rightarrow M$ by $u(x) = e(a_i)$ whenever $x = e \circ \rho_i$ for some nonextendable partial endomorphism e such that $\text{im } \rho_i \subseteq \text{dom } e$. Then u is well defined if and only if $(a_1, \dots, a_n) \in s^*$. Moreover, if $(a_1, \dots, a_n) \in s^*$ and r is an m -ary relation in $\text{Fail}_{\underline{s}}(u)$, then there exist a map $\varepsilon: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ and nonextendable partial endomorphisms e_1, \dots, e_m of \underline{M} such that

$$s \subseteq ((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon$$

and

$$(a_1, \dots, a_n) \notin ((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon.$$

From this lemma we can derive a slight variant of Theorem 4.6. We shall prove that if s satisfies (H) and $R \vdash s$, then s is a finite intersection of relations of types (D), (E), (K) and (T). Define s^* as in Lemma 4.8. If $s^* = s$, there is nothing to prove. If there exists $(a_1, \dots, a_n) \in s^* \setminus s$, then $u(e \circ \rho_i) := e(a_i)$ gives a well-defined map. Since $(\rho_1, \dots, \rho_n) \in s_{D(\underline{s})}$, and since $(u(\rho_1), \dots, u(\rho_n)) = (a_1, \dots, a_n) \notin s$, we have $s \in \text{Fail}_{\underline{s}}(u)$. As $R \vdash s$, there exists $r \in R$ with $r \in \text{Fail}_{\underline{s}}(u)$. Hence, by the last part of Lemma 4.8, there exist nonextendable partial endomorphisms e_1, \dots, e_m such that $s \subseteq ((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon$ and $(a_1, \dots, a_n) \notin ((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon$. Since $((e_1 \times \dots \times e_m)^{-1}(r))^\varepsilon$ includes s and excludes (a_1, \dots, a_n) and is a relation

of the type described in (T), it is now clear that the intersection of all relations of the types (D)–(T) which contain s is equal to s , as required.

§5. Hom-entailment. Let $\underline{M} = (M; G, H, R, \mathcal{F})$ as before. If \underline{M} yields a duality on \mathcal{A} and moreover for all $Z \in \mathcal{Z}$ the natural evaluation map $\varepsilon: Z \rightarrow DE(Z) = \mathcal{A}(\mathcal{Z}(Z, \underline{M}), \underline{M})$ is an isomorphism in \mathcal{Z} , we say that \underline{M} yields a *full duality* on \mathcal{A} . Let Z be a subset of M^S for some set S . Then Z is called *term-closed* in M^S if Z is an intersection of equalisers of S -ary term functions or, equivalently, if for all $y \in M^S \setminus Z$ there exist S -ary term functions $\sigma, \tau: M^S \rightarrow M$ such that $\sigma|_Z = \tau|_Z$ and $\sigma(y) \neq \tau(y)$. It can be shown (see, for example, [2], §2) that term-closure is equivalent to hom-closure. We say that \underline{M} yields a *strong duality* on \mathcal{A} if the equivalent conditions of the following theorem hold:

THEOREM 5.1 (Clark and Davey, [2], Theorem 3.2). *Assume that \underline{M} yields a duality on \mathcal{A} . Then the following are equivalent:*

- (a) \underline{M} yields a full duality on \mathcal{A} and \underline{M} is injective in \mathcal{Z} ;
- (b) for each nonempty set S , every closed substructure Z of M^S is term-closed in M^S ;
- (c) for each nonempty set S every closed substructure Z of \underline{M}^S is hom-closed in M^S .

No example is yet known of a full duality which is not strong.

It is natural to introduce a concept of entailment based upon hom-closure. Let $K \subseteq \mathcal{P}$ and let $e \in \mathcal{P}$. We shall say that K *hom-entails* e , and write $K \Vdash e$, if, for all $m \geq 1$, each subset Z of M^m which is closed under the partial operations in K is also closed under e . Define $\widehat{K} := \{e \in \mathcal{P} \mid K \Vdash e\}$. Then $K \mapsto \widehat{K}$ is a closure operator on \mathcal{P} , and we refer to \widehat{K} as the *hom-entailment closure* of K .

Just as the entailment-density of $G \cup H \cup R$ is a necessary condition for $\underline{M} = (M; G, H, R, \mathcal{F})$ to yield a duality on \mathcal{A} , by Theorem 5.1 we have $\widehat{G \cup H} = \mathcal{P}$ as a necessary condition for \underline{M} to yield a strong duality on \mathcal{A} . By comparison with the problem of giving a sufficient set of constructs to describe the entailment closure of $G \cup H \cup R$, it is surprisingly easy to give an internal description of the hom-entailment closure of $G \cup H$.

THEOREM 5.2. *Let $K \subseteq \mathcal{P}$ and let $e \in \mathcal{P}$.*

- (a) $K \Vdash e$ if and only if there exists $k \in [K]$ such that $e = k|_{\text{dom } e}$.
- (b) $\widehat{K} = \{k|_D \mid k \in [K] \text{ and } D \leq \text{dom } k\}$.

PROOF. It is easy to see that if $k \in [K]$ then $K \Vdash k$ and that $k \Vdash k|_D$ for each subalgebra D of $\text{dom } k$. Assume that $K \Vdash e$ with $e: D \rightarrow M$, where $D \leq \underline{M}^n$. Let $X := \{k|_D: D \rightarrow M \mid k \text{ is } n\text{-ary}, k \in [K] \text{ and } D \leq \text{dom } k\}$. If $k_1|_D, \dots, k_m|_D \in X$ and $h \in K$ is an m -ary partial map such that $h(k_1, \dots, k_m)$ is defined, then $h(k_1, \dots, k_m)|_D$ is defined and $h(k_1|_D, \dots, k_m|_D) = h(k_1, \dots, k_m)|_D$. Thus X is closed under K , and so is closed under e since K hom-entails e . Now

$$\rho_1 = \pi_1|_D, \dots, \rho_n = \pi_n|_D \in X$$

and $(\rho_1, \dots, \rho_n) \in \text{dom } e_X$, whence $e = e(\rho_1, \dots, \rho_n) \in X$. Thus $e = k \upharpoonright_D$ for some $k \in [K]$. This proves both (a) and (b). \square

We may regard the projections $\pi_i: M^n \rightarrow M$ as nullary constructs for hom-entailment just as the trivial relations are nullary constructs for entailment. The next result is an immediate consequence of Theorem 5.2.

THEOREM 5.3. *The projections $\pi_i: M^n \rightarrow M$, restriction of domain (viz. $k \mapsto k \upharpoonright_D$ for $D \leq \text{dom } k$), and composition of partial functions (viz. $(h, g_1, \dots, g_n) \mapsto h(g_1, \dots, g_n)$, where defined) form a complete set of constructs for hom-entailment.*

Note that Theorem 5.2 exhibits the schizophrenia typical of duality theory: the set X is to e on the operational side as $D(\underline{s})$ is to s on the relational side.

§6. An example. In this section we illustrate our results by presenting the example which helped us to derive them. As we have already pointed out, concrete constructs suffice to describe entailment whenever \underline{M} generates a congruence-distributive variety and each subalgebra of \underline{M} is subdirectly irreducible. Of course, if \underline{M} itself is not subdirectly irreducible, then the latter condition is not satisfied.

We shall take \underline{M} to be $\mathbf{3} = (\{0, d, 1\}; \vee, \wedge, 0, 1)$, the 3-element chain $0 < d < 1$ in the variety \mathbf{D} of bounded distributive lattices. The variety \mathbf{D} has the 2-element chain $0 < 1$ as its only subdirectly irreducible algebra. Certainly $\mathbf{D} = \text{ISP}(\underline{M})$. The NU duality theorem ensures that $R = \mathbb{S}(\underline{M}^2)$ yields a duality on \mathbf{D} . The lattice $\mathbb{S}(\underline{M}^2)$ has 49 elements. The meet-irreducibles are the following subalgebras, and their converses:

$$V_1 = \mathbf{2} \times \mathbf{3}, \quad V_2 = \{(0, 0), (0, d), (0, 1), (d, 1), (1, 1)\},$$

$$V_3 = \mathbf{3}^2 \setminus \{(1, 0), (1, d)\}, \quad V_4 = \mathbf{3}^2 \setminus \{(d, 0), (1, 0)\}, \quad V_5 = \mathbf{3}^2 \setminus \{(1, 0)\}.$$

The only proper subalgebra of \underline{M} is $\underline{N} := \{0, 1\}$; in line with the notation in [9] we denote $\{(0, 0), (1, 1)\}$ by Δ_N .

In [7] (see also [8]) we showed that the endomorphism monoid $\text{End } \underline{L}$ of a finite distributive lattice \underline{L} together with the order $s = \{(0, 0), (0, 1), (1, 1)\}$ on $\{0, 1\} \leq \underline{L}$ always yields a duality on \mathbf{D} . Further, $\text{End } \underline{L}$ yields a duality on \mathbf{D} if and only if \underline{L} is not Boolean. Thus, in particular, $\text{End } \underline{M}$ yields a duality on \mathbf{D} . This endomorphism monoid consists of the identity map and two retractions, e and f , onto $\{0, 1\}$, given by $e^{-1}(1) = \{1\}$ and $f^{-1}(0) = \{0\}$. The general theory tells us that $\{e, f\} \vdash s$.

Note that $D(\underline{s}) = \{\rho_1, \rho_2, \tau\}$, where $\tau((0, 1)) = d$. Since $e(\tau) = e \circ \tau = \rho_1$ and $f(\tau) = f \circ \tau = \rho_2$, it follows that $D(\underline{s}) \models (\exists y)(e(y) = \rho_1 \ \& \ f(y) = \rho_2)$ and, moreover,

$$s = \{(e_1, c_2) \in \mathbf{3}^2 \mid (\exists c_3)(e(c_3) = c_1 \ \& \ f(c_3) = c_2)\}.$$

Thus, $s = \text{graph}(e) \cdot \text{graph}(f)$, and this is a homomorphic relational product.

We now show that s cannot be expressed in terms of e and f using concrete constructs only. If s were given concretely from e and f , then there would be a (quantifier free) conjunct of atomic formulæ, $\Phi(x_1, x_2)$, such that

$$s = \{(a, b) \in \mathbf{3}^2 \mid \mathbf{3} \models \Phi(a, b)\}.$$

In particular, $\Phi(0, 1)$ holds but $\Phi(1, 0)$ fails. We shall show that this is impossible.

Since composition of maps on $\{e, f\}$ is simply a right zero semigroup (as each is a retraction onto $\{0, 1\}$), there are only 15 atomic formulæ which are not of the form $t = t$ for some term t , namely,

$$\begin{aligned} x_1 = x_2, \quad e(x_1) = x_1, \quad e(x_2) = x_2, \quad e(x_1) = x_2, \quad e(x_2) = x_1, \quad f(x_1) = x_1, \\ f(x_2) = x_2, \quad f(x_1) = x_2, \quad f(x_2) = x_1, \quad e(x_1) = e(x_2), \quad f(x_1) = f(x_2), \\ e(x_1) = f(x_1), \quad e(x_2) = f(x_2), \quad e(x_1) = f(x_2), \quad f(x_1) = e(x_2). \end{aligned}$$

Since $\Phi(x_1, x_2)$ is a conjunct of some of these formulæ and since $\Phi(0, 1)$ holds, $\Phi(x_1, x_2)$ must be a conjunct of some of

$$\begin{aligned} e(x_1) = x_1, \quad e(x_2) = x_2, \quad f(x_1) = x_1, \quad f(x_2) = x_2, \\ e(x_1) = f(x_1), \quad e(x_2) = f(x_2). \end{aligned}$$

But then $\Phi(1, 0)$ also holds; a contradiction. Thus any formula describing s in terms of e and f must involve at least one quantifier.

A computer analysis based on the theory in [9] has allowed us to determine all optimal dualities for \mathbf{D} using subsets of $\mathbb{S}(\mathbf{3}) \cup \mathbb{S}(\mathbf{3}^2)$. This analysis reveals that the single relation

$$t = \{(0, 0), (0, d), (d, 1), (1, 1)\}$$

yields a duality on \mathbf{D} . The intersection $t \cap t^\sim$ is Δ_N . Thus V_1 may be obtained from t by concrete constructs. The following formulæ express the remaining meet-irreducibles in $\mathbb{S}(\mathbf{3}^2)$ in terms of t (in each case the relational products involved are homomorphic relational products):

$$V_2 = t \cdot t, \quad V_3 = t^\sim \cdot (t \cdot t), \quad V_4 = t \cdot (t \cdot t^\sim), \quad V_5 = (t^\sim \cdot (t \cdot t)) \cdot t^\sim.$$

The formulæ are not unique: for example, V_5 is also given by

$$V_5 = t^\sim \cdot (t \cdot (t \cdot t^\sim)).$$

We observe also that the order on the 3-element chain is $V_3 \cap V_4$.

It is not the case that all relational products in $\mathbb{S}(\mathbf{3}^2)$ are homomorphic. Take $r_1 = \{(0, 0), (0, d), (d, d), (1, 1)\}$ and $r_2 = \mathbf{3}^2 \setminus \{(0, 1)\}$. Then $\underline{M}^2 = \mathbf{3}^2 = r_1 \cdot r_2$. However, this is not a homomorphic relational product. To see this, first note that $D(\underline{M}^2)$ has 6 elements, $g \circ \pi_i$ ($g \in \text{End } \underline{M}$ and $i = 1, 2$). On $D(\underline{M}^2)$ the relation \underline{M}^2 is $D(\underline{M}^2)^2$. We need to show that there is no $z \in D(\underline{M}^2)$ for which $(\rho_1, z) \in r_1$ and $(z, \rho_2) \in r_2$ on $D(\underline{M}^2)$. The only possible choice for z if we are to have $(\rho_1, z) \in r_1$ is $z = \rho_1$. However $(\rho_1, \rho_2) \notin r_2$ on $D(\underline{M}^2)$.

How can we upgrade the duality for \mathbf{D} to a strong duality? Since $\mathbf{3}$ is not injective in \mathbf{D} , the total structure theorem of [3] (see Theorem 3.13 in [5]) tells us that the type of \underline{M} must include some proper partial operations. Moreover, since the irreducibility index of $\mathbf{3}$ is 2, the NU strong duality theorem of [2] (see Theorem 3.13 in [5]) implies that if \underline{M} is known to yield a duality then a strong duality may be obtained by adding the unary and binary algebraic partial operations on \underline{M} to the type of \underline{M} . Our Theorem 5.2 tells us that it is sufficient to add to the type of \underline{M} a set K of partial unary and binary algebraic operations

such that the partial clone generated by K includes the nonextendable (partial) unary and binary algebraic operations on \underline{M} .

Let $D = \theta_a \cdot \theta_f = V_3$, and define $h: D \rightarrow M$ by declaring $h(a, b)$ to be the unique element $c \in M$ such that $a\theta_e c \theta_f b$. Note that h is an extension of the map $y: s \rightarrow M$ defined above. We claim that $\underline{M}' = \langle \{0, a, 1\}; e, f, h, \tau \rangle$ yields a strong duality on \mathbf{D} . By the remarks above, it remains to prove that every nonextendable (partial) binary algebraic operation on \underline{M} is in the partial clone generated by $\{e, f, h\}$. A computer check revealed that there are 308 (partial) binary homomorphisms on \underline{M} and that, up to symmetry, only 7 of them are nonextendable. These 7 nonextendable (partial) binary homomorphisms are the total maps $\pi_1, e \circ \pi_1, f \circ \pi_1$ and the proper partial maps $h, h(\text{id}, e), h(f, \text{id})$, and $h(f, e)$, all of which belong to the partial clone generated by $\{e, f, h\}$.

In the example we have been considering we have used concrete constructs together with homomorphic relational product. We have proved that in general concrete constructs together with retractive projection suffice. We return finally to the question raised in §4, namely whether retractive projection can always be replaced by homomorphic relational product.

Let us take \underline{M} to be the 4-element chain $0 < a < b < 1$ as a bounded distributive lattice, and let

$$r := \{(1, 1, 1, 1), (1, b, 1, 1), (1, b, 1, 0), (a, b, 1, 1), (a, b, 1, 0), (a, 0, 1, 0), (a, b, 0, 0), (a, 0, 0, 0), (0, 0, 0, 0)\}.$$

It is easy to see that the projection

$$P_{3,4}(r) = s = \{(1, 1), (1, 0), (0, 0)\}$$

is retractive, but neither of the 1-step projections $P_{2,3,4}(r)$ or $P_{1,3,4}(r)$ is retractive. It does not appear to be possible to construct s from r by any other sequence of retractive 1-step projections, together with trivial relations, repetition removal, intersections and products. This supports the conjecture that homomorphic relational product, or alternatively 1-step retractive projection, cannot be substituted for retractive projection in the last part of Theorem 3.5.

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